



**HAL**  
open science

# Around the physical Church-Turing thesis: cellular automata, formal languages, and the principles of quantum theory.

Gilles Dowek

► **To cite this version:**

Gilles Dowek. Around the physical Church-Turing thesis: cellular automata, formal languages, and the principles of quantum theory.. 2012. hal-04075081v2

**HAL Id: hal-04075081**

**<https://inria.hal.science/hal-04075081v2>**

Preprint submitted on 21 Apr 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Around the physical Church-Turing thesis: cellular automata, formal languages, and the principles of quantum theory

Gilles Dowek

INRIA, 23 avenue d'Italie, CS 81321, 75214 Paris Cedex 13, France.  
gilles.dowek@inria.fr, <http://www-roc.inria.fr/who/Gilles.Dowek/>

**Abstract.** The physical Church-Turing thesis explains the Galileo thesis, but also suggests an evolution of the language used to describe nature. It can be proved from more basic principle of physics, but it also questions these principles, putting the emphasis on the principle of a bounded density of information. This principle itself questions our formulation of quantum theory, in particular the choice of a field for the scalars and the origin of the infinite dimension of the vector spaces used as state spaces<sup>1</sup>.

## 1 The Church-Turing thesis and its various forms

### 1.1 Why a thesis?

It is a quite common situation in mathematics, that a notion, first understood intuitively, receives a formal definition at some point. For instance, the notion of a real number has been understood intuitively in geometry, for instance as the length of a segment, before it has been formally defined in the 19th century, by Cauchy and Dedekind. Another example is the notion of an algorithm, that has been understood intuitively for long, before a formal definition of the notion of a *computable function* has been given in the 30s, by Gödel and Herbrand, Church, Turing, and others.

There is however a difference between these two notions. When the notion of a real number was defined, there seems to have been no real discussion of the fact that the formal definition of Cauchy or Dedekind was indeed a formalization of our intuitive notion of a real number. In the same way, there is no discussion of the fact that the formal definitions of the notions of a triangle, a square, or a circle are indeed formalizations of our intuitive notions of a triangle, a square, or a circle. Even for the formal definitions of the notions of a distance and of orthogonality, based on the notion of an inner product, we finally agree that they correspond to our intuitive notion of a distance and of orthogonality.

---

<sup>1</sup> These tutorial notes owe a lot to Pablo Arrighi with whom most of this research has been done and to Maël Pégny with whom I gave a course on the Church-Turing thesis in the PUC Rio in 2011, on the invitation of Luiz Carlos Pereira. I also want to thank Olivier Bournez, José Félix Costa, Nachum Dershowitz, Jean-Baptiste Joinet, Giuseppe Longo, and Thierry Paul for many discussion on this thesis.

The situation is different with the notion of a computable function, as the formal definition of this notion came with a thesis that this formal definition indeed captures our intuitive notion of computability.

This peculiarity of the notion of a computable function can be explained by its history. Before the modern definition, another had been given: the notion of a primitive recursive function, and it was then noticed that there existed functions, such as the Ackermann function, that were intuitively computable, but not primitive recursive. This has raised a reasonable doubt that history may repeat itself and that another counterexample may be given to this new definition.

## 1.2 The physical Church-Turing thesis

Stated as such, the epistemological status of the Church-Turing thesis is unclear. It cannot be proved, as it uses a notion that is not formally defined: that of a function computable in the intuitive sense. It cannot be experimentally tested as it is not a statement about nature. But it could be falsified, if anyone came with a counterexample, *i.e.* a function that is not computable in the formal sense, but such that everyone agrees that it is computable in the intuitive sense, as it happened with the Ackermann function. Yet, this requirement of a consensus on the computability, in the intuitive sense, of this function puts it at the border of falsifiability. Thus, stated as such, the Church-Turing thesis is hardly a thesis, and several attempts have been made to replace this notion of computability in the intuitive sense by a more precise notion. These attempts go at least in three directions.

- The first is to axiomatize a computability predicate and to prove the equivalence of this implicitly defined notion with that defined by Gödel and Herbrand, Church, Turing, *etc.* This is the way taken by Dershowitz and Gurevich [19], and others.
- The second is to define a notion of computability based on the analysis of the operations executed by a human being performing a computation and to prove the equivalence of this notion with that defined by Gödel and Herbrand, Church, Turing, *etc.* This is the way taken by Turing himself, and others.
- The third is to define a notion of computability based on the analysis of the operations executed by a machine performing a computation and to prove the equivalence of this notion with that defined by Gödel and Herbrand, Church, Turing, *etc.* This is the way taken by Gandy [26], and others.

The thesis expressing that any function computable by a machine is computable in the sense of Gödel and Herbrand, Church, Turing, *etc.* is called the *physical Church-Turing thesis*.

Giving a meaning to this thesis requires to give a definition of the notion of a machine. A *machine* is any physical system, equipped with a interaction protocol defining a way to communicate some information to the system by choosing some

parameters  $a = \langle a_1, \dots, a_n \rangle$ , when preparing the system, and to extract some information from the system, by measuring some values  $b = \langle b_1, \dots, b_p \rangle$ .

Such a machine defines a relation: the relation  $R$  such that  $a R b$  if  $b$  is a possible result for the measurements, when the chosen parameters are  $a$ . We say that this relation  $R$  is *realized* by this machine. For instance, if one applies an electrical tension  $U$  to a conductor of resistance  $R$  and measures the current  $I$  passing through this conductor, then  $a = \langle U, R \rangle$  and  $b = I$  and the realized relation is that relating  $\langle U, R \rangle$  and  $I$  when  $U = RI$ . In the same way, if one lets a body freely falling in vacuum for some time  $t$  and measures the position  $x$  of this body, the realized relation is that relating  $t$  and  $x$  when  $x = \frac{1}{2}gt^2$ .

The physical Church-Turing thesis expresses that relations realized by a machine are computable.

The physical Church-Turing thesis is obviously a thesis about nature. Thus its correctness depends on the world we are in. It is easy to imagine worlds where this thesis is valid and worlds where it is not. In particular this thesis is not valid in worlds where accelerating machines or infinitely parallel machines can be built. The theory of hypercomputability studies such worlds.

Yet, not everyone agrees on the epistemological status of this thesis. According to some, such as Deutsch [20], this thesis — or its negation — is a principle of physics, like the homogeneity of space or the homogeneity of time. According to others, such as Gandy, this thesis — or its negation — must be established as a consequence of other principles of physics. There is no real contradiction between these two points of view, as we know that the choice of axioms in a theory is somewhat arbitrary. For instance, in set theory, we may chose a particular form of the axiom of choice and prove the others. Yet, some propositions are simple enough to be taken as axioms or principles, and others are not.

## 2 Beyond natural numbers

Computability is primarily defined for the functions from the set of natural numbers to itself, but many functions we use when discussing the physical Church-Turing thesis are defined on other domains. Thus, the notion of computability must be defined beyond the set of natural numbers.

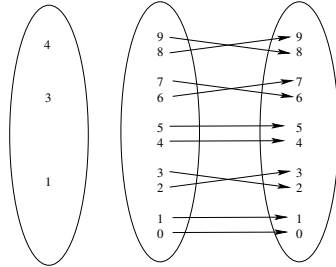
### 2.1 Indexings

The first way to define a notion of computability on a set  $S$  different from the natural numbers is to transport the notion of computability from the set of real numbers to the set  $S$  with the use of an *indexing*, *i.e.* a bijection  $\ulcorner \cdot \urcorner$  between  $S$  and a subset of  $\mathbb{N}$ .

Then, a function  $f$  from  $S$  to  $S$  is said to be computable, if there exists a computable function  $\hat{f}$  from  $\mathbb{N}$  to  $\mathbb{N}$  such that for all  $x$  in  $S$ ,  $\ulcorner f(x) \urcorner = \hat{f}(\ulcorner x \urcorner)$ .

An objection to this method, due to Montague [28], is that the defined notion of computability depends on the choice of the indexing. To see why, let us consider an undecidable set  $U$ , that does not contain 0 and the function  $D$  mapping

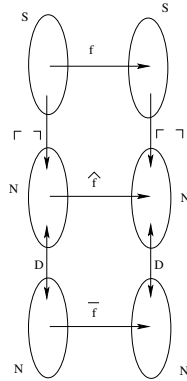
- $2n$  to  $2n + 1$  and  $2n + 1$  to  $2n$  if  $n \in U$ ,
- $2n$  to  $2n$  and  $2n + 1$  to  $2n + 1$  otherwise.



This function is a non computable bijection from  $\mathbb{N}$  to  $\mathbb{N}$ . Moreover it is involutive, *i.e.*  $D^{-1} = D$ ,  $D(0) = 0$ , and  $D(1) = 1$ .

Then, consider a set  $S$ , an indexing  $\ulcorner \cdot \urcorner$  of this set, elements  $u$  and  $v$  in  $S$  such that  $\ulcorner u \urcorner = 0$  and  $\ulcorner v \urcorner = 1$ , and the function  $f$  from  $S$  to  $S$  defined by  $f(x) = u$  if  $\ulcorner x \urcorner$  is even and  $f(x) = v$  otherwise. As the function  $\hat{f}$  from  $\mathbb{N}$  to  $\mathbb{N}$  defined by  $\hat{f}(n) = 0$  if  $n$  is even and  $\hat{f}(n) = 1$  otherwise is computable, the function  $f$  is computable *with respect to the indexing*  $\ulcorner \cdot \urcorner$ .

But, the function  $D \circ \ulcorner \cdot \urcorner$  is also an indexing of  $S$



and the function  $\bar{f} = D \circ \hat{f} \circ D$  is not computable. Indeed  $\bar{f}(2n) = 1$  if and only if  $D(\hat{f}(D(2n))) = 1$  if and only if  $D(2n)$  is odd if and only if  $n \in U$ . Thus, the function  $f$  is not computable *with respect to the indexing*  $D \circ \ulcorner \cdot \urcorner$ . The computability of  $f$  therefore depends on the chosen indexing.

Thus, unless we have a canonical way to represent the parameters  $a$  and the measured values  $b$  by natural numbers, the formulation of the physical Church-Turing thesis is relative to the choice of an indexing.

Three answers have been given to this objection.

**Articulation** A list  $a_1, a_2, \dots, a_k$  of natural numbers has a *canonical index*

$$\ulcorner a_1, a_2, \dots, a_k \urcorner = (a_1; (a_2; (\dots; (a_k; 0))))$$

where  $\gamma$  is the bijection between  $\mathbb{N}^2$  and  $\mathbb{N} \setminus \{0\}$  defined by  $(n; p) = (n+p)(n+p+1)/2 + p + 1$ . And a tree indexed by natural numbers  $p(t_1, \dots, t_k)$  has a *canonical index*

$$\ulcorner p(t_1, \dots, t_k) \urcorner = (p; \ulcorner t_1 \urcorner; \dots; \ulcorner t_k \urcorner; 0)$$

This permits to define indexings for all articulated sets of trees, where a set of tree is *articulated* if it is  $n$ -articulated for some  $n$ , *0-articulated* if all its elements are labeled in a finite set and  $(n+1)$ -*articulated* if all its elements are labeled in a  $n$ -articulated set of trees. For instance, proofs are trees labeled by sequents, that are trees labeled by formulae that are trees labeled by function symbols, predicate symbols, and variables, that are trees labeled in a finite set. Thus the set of proofs is articulated. In a similar way, all the sets of syntactic objects, formulae, proofs, programs, *etc.*, are articulated.

Let  $S$  be an articulated set of trees, ultimately labeled in the finite set  $E$ . Using the canonical indexing of trees above, we can transform any indexing  $\ulcorner \cdot \urcorner^0$  of the set  $E$  into an indexing  $\ulcorner \cdot \urcorner^n$  of  $S$  by

$$\ulcorner f(t_1, \dots, t_k) \urcorner^{r+1} = (\ulcorner f \urcorner^r; \ulcorner t_1 \urcorner^{r+1}; \dots; \ulcorner t_k \urcorner^{r+1}; 0)$$

As all finite functions are computable, the choice of the indexing  $\ulcorner \cdot \urcorner^0$  is immaterial and all the indexings based on this canonical tree indexing define the same set of computable functions.

**Finite generation** This argument has been generalized by Rabin [30] to all finitely generated algebraic structures.

In mathematics, we are not interested in sets *per se*, but in *structures*, *i.e.* sets equipped with operations:  $\langle \mathbb{N}, S \rangle$ ,  $\langle \mathbb{N}, +, \times \rangle$ ,  $\langle L, :: \rangle$ ,  $\langle L, @ \rangle$ , *etc.* As their name suggests, these operations must be computable. Thus, when indexing a structure, we must restrict to the *admissible* indexings: those that make these operations computable.

A structure is said to be *finitely generated* when all its elements can be built from a finite number of elements with the operations of the structure, for instance the set of natural numbers equipped with addition, the set of lists of elements of a finite set equipped with concatenation, the set of rational numbers equipped with addition, multiplication and division, *etc.* are finitely generated structures.

When a structure is finitely generated, if  $i$  and  $j$  are two admissible indexings, then there exists a computable function  $f$  such that  $j = f \circ i$ . Thus, all admissible indexings define the same notion of computability: computability is *stable*.

In particular computability is stable on all the finite extensions of the field of rational numbers, all the finite-dimensional vector spaces over such fields, *etc.* [3].

**Computability of sets of functions** Programs in a given programming language are lists of letters in a finite alphabet. We have seen that, as the set of lists of elements of a finite set equipped with concatenation @ is finitely generated, all its admissible indexings define the same set of computable functions, and the

characteristic function  $\chi$  of the set of terminating programs is non computable, relatively to any admissible indexing. In other words, for any indexing, if the function  $@$  is computable, then the function  $\chi$  is not.

Boker and Dershowitz [11] have suggested that this result could be stated without focusing on the concatenation operation  $@$ , as the fact that the functions  $@$  and  $\chi$  cannot be both computable, with respect to the same indexing. Calling a set of functions *computable*, if there exists an indexing with respect to which all the elements of the set are computable, the set  $\{ @, \chi \}$  is non computable, in an absolute sense.

The set  $\{ \chi \}$  is computable as it is possible to cheat on the indexing and associate an even number to terminating programs and an odd number to non terminating ones, but the price to pay is to make concatenation non computable.

Another way to formulate this result is that the set  $C$  of functions computable with respect to an admissible indexing is a maximal computable set of functions. Indeed, let  $\phi$  be a function that is not in  $C$  and  $i$  an indexing such that  $\phi$  is computable relatively to  $i$ . Then,  $i$  is not admissible and the concatenation operation is not computable with respect to  $i$ . Thus, there exists a function of  $C$  that is not computable with respect to  $i$  and the set of functions computable with respect to  $i$  is not a proper superset of  $C$ .

In other words, changing the indexing can change the set of computable function, but it cannot extend it.

As a consequence, if we formulate the physical Church-Turing thesis as the fact that the set of functions realized by a machine is the set of computable functions, we obtain a thesis that depends on the way we interpret the physical parameters and the measured values. And this thesis can be invalidated by building a machine that computes the parity of a natural number and interpreting it as a machine deciding the halting problem, by cheating on the indexing.

But if we formulate this thesis as the fact that the set of functions realized by a machine is a strict superset of the set of computable functions for no indexing, then we have a thesis that is independent on the choice of an indexing.

## 2.2 Real numbers

The use of an indexing permits to transport the notion of computability to countable sets only. Computability over non countable sets is an unexplored world, except for one set: that of real numbers. As real numbers might be useful in physics, we have to say a few words about the computability of real functions.

To know a real number  $x$  is not to be able to answer the query “give me all the digits of  $x$ ”, but to be able to answer the query “give me the  $n^{th}$  first digits of  $x$ ” for all  $n$ , or equivalently to the query “give me a rational approximation of  $x$  with an error less than  $\varepsilon$ ” for all  $\varepsilon$  [29, 33].

To know a real function  $f$  is to be able to answer all the queries of the form “give me a rational approximation of  $f(x)$  with an error less than  $\varepsilon$ ” provided we are answered all queries of the form “give me a rational approximation of  $x$  with an error less than  $\eta$ ”.

Thus a function  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$  is computable if there exists a computable function  $F$  from  $\mathbb{Q} \times (\mathbb{Q}^+ \setminus \{0\})$  to  $\mathbb{Q} \times (\mathbb{Q}^+ \setminus \{0\} \cup \{+\infty\})$  such that for all  $x$  in  $\mathbb{R}$ ,  $q$  and  $r$  in  $\mathbb{Q}$ ,  $\eta$  and  $\varepsilon$  in  $\mathbb{Q}^+ \setminus \{0\}$

$$(F(q, \eta) = (r, \varepsilon) \text{ and } |x - q| \leq \eta) \Rightarrow |f(x) - r| \leq \varepsilon$$

and if  $(q_n)_n, (\eta_n)_n,$  are sequences such that for all  $n$ ,  $|x - q_n| \leq \eta_n$  and the sequence  $\eta$  goes to 0 at infinity, then, calling  $(r_n, \varepsilon_n) = F(q_n, \eta_n)$ , the sequence  $\varepsilon$  also goes to 0 at infinity.

### 2.3 Non-determinism

Computability theory is a theory of computable functions. However, when describing a machine we spoke, not about functions, but about relations. This is because several sequences of measured values  $b$  may correspond to the same sequence of parameters  $a$ . This non-determinism may have different sources, it may either come from the absolute non-determinism of nature, or because some variables are kept hidden: for instance, in the experiment above, if we choose the resistance  $R$  and measure the intensity  $I$ , then  $I$  does not depend functionally on  $R$ .

We know that a non deterministic algorithm from a set  $A$  to a set  $B$  may always be considered as a deterministic algorithm from the set  $A$  to the powerset  $\mathcal{P}(B)$ : a relation  $R$  may always be considered as a function mapping  $a$  to the set  $R_a = \{b \in B \mid a R b\}$ . If we represent the set  $R_a$  by an algorithm computing its partial characteristic function, then a relation  $R$  may be called computable when there exists an algorithm mapping  $a$  to an algorithm mapping  $b$  to 1 if and only if  $a R b$ . Using the fact that a function taking values in a functional space is equivalent to a function of several arguments, the relation  $R$  is represented by its own partial characteristic function. Thus, a relation is computable if and only if it is semi-decidable.

We could also represent the set  $R_a$  by an enumeration function and this would lead to define computable relations as effectively enumerable relations. On natural numbers, and on any set on which equality is decidable, these two notions coincide. This is not the case on real numbers and only the second notion makes sense in this case [21].

## 3 Gandy's proof

As previously noticed, some statements are too complex to be taken as axioms or principles and must rather be derived from more basic ones. This is probably the case with the physical Church-Turing thesis. And indeed Gandy [26] has shown how to derive this thesis from three more basic hypotheses about nature.

### 3.1 Gandy's hypotheses

Gandy's hypotheses are



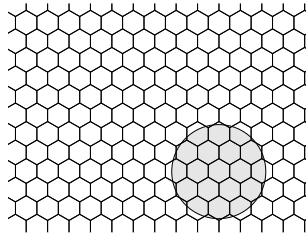
- the homogeneity of space and time,
- the boundedness of the velocity of propagation of information,
- the boundedness of the density of information.

The boundedness of the velocity of propagation of information can be expressed as the fact that the state of a system in one place can only affect the state of a system in another only after a delay, proportional to their distance. This principle has been part of the principles of physics, at least since special relativity.

The boundedness of the density of information can be expressed as the fact that a physical system of finite diameter has a finite state space. This principle is a more original idea. We may see it as an abstract formulation of the quantization idea: the fact that a variable cannot take any real number as a value, but only one of a discrete set. A similar principle has been stated more explicitly by Bekenstein [8].

### 3.2 Gandy's proof

Then, to prove the physical Church-Turing thesis from these hypotheses, we just need to partition the space into an infinite number of identical cells of finite diameter. Because information has a bounded density, the state space of each cell is finite. Because space is homogeneous, this state space is the same for all cells. At the origin of time all the cells except a finite number are in a particular *quiescent* state. Like space, time can be discretized. Because the velocity of propagation of information is bounded and space and time are homogeneous, the state of a cell at a given time step is function of the state of a finite number of neighbors cells at the previous time step.



This function of a finite number of variables varying in a finite set is computable. Hence the state of each cell at each time step can be computed from the initial state.

In other words, using Gandy's hypotheses, a tiling of space and time allows to describe the evolution of any system as a cellular automata [35]. Of course, the choice of the tiling should not affect the evolution of the system, as the tiling is not a property of the system itself, but of our description of the system.

### 3.3 Criticizing Gandy's hypotheses

Gandy's hypotheses can be, and have been [18], criticized. For instance, it is well-known that, in Newtonian mechanics, gravity is instantaneous and thus information can be propagated with an infinite velocity. Also, the position of a body between two points  $A$  and  $B$ , the distance  $AB$  taken as a unit, can be any real number between 0 and 1 and thus can contain an infinite quantity of information: any infinite sequence in a finite alphabet can be encoded as the digits of such a number, in an appropriate base. Yet, these properties of Newtonian mechanics seem to be rather weaknesses of this theory than properties that would allow to communicate instantaneously or to store an infinite amount of information in a finite space: Gandy's hypotheses have not been refuted experimentally, for instance by the construction of an instantaneous computer network or by the construction of a hard drive with an unbounded capacity.

More importantly, even if Gandy's hypotheses must be refined, Gandy's proof shows that the physical Church-Turing thesis is a consequence of some hypotheses about nature, that do not refer to notions such as those of language or computability.

## 4 The physical Church-Turing thesis and the Galileo thesis

### 4.1 The Galileo thesis

**The effectiveness of mathematics in natural sciences** The thesis that mathematics are effective in the natural sciences has been formulated by Galileo in 1623: "Philosophy is written in this vast book, which continuously lies open before our eyes (I mean the Universe). But it cannot be understood unless you have first learned the language and recognize the characters in which it is written. It is written in the language of mathematics" [25].

Galileo formulated this thesis, but did not give any explanation why it held. And long after Galileo, the lack of such an explanation was noticed by Einstein according to whom "The eternal mystery of the world is its comprehensibility" [24] or Wigner according to whom "The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it" [34].

**Insufficient explanations** Several explanations of this unreasonable effectiveness of mathematics in the natural sciences have been attempted:

1. God is a mathematician and He wrote the vast book in the language of mathematics.
2. The mathematical concepts are built by abstracting from empirical objects.
3. Scientists select only those phenomena that can be mathematically described.
4. Scientists approximate the phenomena they study, until they can be mathematically described.

5. Our brain is part of nature, hence our concepts are natural objects, thus they are of the same kind as the objects they describe.

Each of these explanation is insufficient. The first reduces the problem to that of why God is a mathematician, which seems even harder to explain. The second is partial: if some mathematical concepts are built by abstracting from natural objects, the concept of ellipse, for instance, has not been built by abstracting from the trajectory of the planets, as it has been introduced some two thousands years before. The third leaves intact the problem of why so many — if not all — phenomena can be described in the language of mathematics. The fourth leaves intact the problem of why phenomena can be approximately — if not accurately — described in the language of mathematics. The fifth presupposes that we understand better a phenomenon from the inside than from the outside, which is not the case in general.

**Perhaps several kinds of effectiveness** The effectiveness of mathematics in the natural sciences may be of different kinds. And instead of looking for a global explanation of all kinds of effectiveness, we should perhaps look for more local explanations.

For instance, the atomic masses of the chemical elements have a regular structure, as they are the integer multiples of some unit. When this regularity was discovered, there were three exceptions to this rule, because no elements of atomic mass 45, 68, and 70 were known. Yet, as some chemists trusted the structure of the natural numbers better than their observations, they predicted the existence of these three elements, that were later discovered. This is a striking example of the effectiveness of the structure of natural numbers in chemistry.

But, this striking regularity is easily explained by the fact that the mass of the atoms is mostly due to the mass of protons and neutrons that constitute their nucleus and that each nucleus contains a whole number of such particles.

Yet, this explanation sheds light on the effectiveness of the structure of natural numbers to describe the atomic masses of the chemical elements, but it does not shed light on the effectiveness of mathematics in the natural sciences in general, for instance, it does not shed light on the effectiveness of the quadratic functions to describe the free fall in vacuum.

Thus, we shall focus on a particular instance of the general thesis that mathematics are effective in the natural sciences: the fact that physically realized relations can be expressed by a proposition of the language of mathematics.

**Physically realized relations** Let us imagine an experiment where one prepares a physical system by choosing some parameters  $a = \langle a_1, \dots, a_n \rangle$  and measures some values  $b = \langle b_1, \dots, b_p \rangle$ . Such an experiment defines a relation: the relation  $R$  such that  $a R b$  if  $b$  is a possible result for the measurements when the chosen parameters are  $a$ . We say that this relation  $R$  is *realized* by this experiment. A relation that is realized by an experiment is said to be *physically realized*.

For instance, if one applies an electrical tension  $U$  to a conductor of resistance  $R$  and measures the current  $I$  passing through this conductor, then  $a = \langle U, R \rangle$  and  $b = I$  and the realized relation is that relating  $\langle U, R \rangle$  and  $I$  when  $U = RI$ . In the same way, if one lets a body freely falling in vacuum for some time  $t$  and measures the distance crossed by this body, the realized relation is that relating  $t$  and  $x$  when  $x = \frac{1}{2}gt^2$ .

The relation between  $\langle U, R \rangle$  and  $I$  can be expressed by a proposition in the language of mathematics  $U = RI$ . In the same way, the relation between the time and the position of a body freely falling in vacuum can be described by a proposition in the language of mathematics  $x = \frac{1}{2}gt^2$ . Among the uncountable number of relations between numbers, only a countable number can be defined by a proposition in the language of mathematics, such as  $U = RI$  or  $x = \frac{1}{2}gt^2$  and all the physically realized relations seem to be in this small set.

As Galileo stressed the *rôle* of the *language* of mathematics, we can call *the Galileo thesis* the thesis that all physically realized relation can be expressed by a proposition in the language of mathematics.

Instead of attempting to explain the general thesis that mathematics are effective in the natural sciences, we shall restrict our investigation to attempt to explain this unreasonable effectiveness of the propositions of the language of mathematics to express physically realized relations.

## 4.2 The physical Church-Turing thesis implies the Galileo thesis

Our definition of the notions of a machine and of an experiment are identical, both a machine and an experiment is a physical system prepared by choosing some parameters  $a = \langle a_1, \dots, a_n \rangle$  and on which some values  $b = \langle b_1, \dots, b_p \rangle$  are measured. Thus the notion of a relation realized by a machine and of a physically realized relation are identical.

The physical Church-Turing thesis states that physically realized relations are computable and the Galileo thesis states that they can be expressed by a proposition in the language of mathematics.

Once we have identified that the Galileo thesis and the physical Church-Turing thesis are theses about the same set of relations, we may remark that the physical Church-Turing thesis implies the Galileo thesis [22]. Indeed, as any program expressing a computable relation is a mathematical expression, all computable relations can be defined by a proposition in the language of mathematics. In fact, computable relations can even be expressed by a proposition in a very small fragment of mathematics: the language of Peano arithmetic. Thus, if the physical Church-Turing thesis holds, then all physically realized relations are computable, hence they can be expressed by a proposition of the language of mathematics, *i.e.* the Galileo thesis holds.

We have seen that

- Gandy’s hypotheses imply the physical Church-Turing thesis,
- and the physical Church-Turing thesis implies the Galileo thesis.

Thus, we can deduce that Gandy's hypotheses imply the Galileo thesis and attempt to prove this directly.

Yet, from a historical point of view, it is important to notice the *rôle* of computability theory and of the physical Church-Turing thesis in connecting Gandy's hypotheses to the Galileo thesis.

### 4.3 An algorithmic description of the laws of nature

A side effect of this explanation of the Galileo thesis is that the laws of nature can be described not only in the language of mathematics, but also in an algorithmic language.

Instead of expressing the law of free fall in vacuum by the proposition  $x = \frac{1}{2}gt^2$ , we could express it by the algorithm `fun t -> g * t * t / 2`, leading to a second Galilean revolution in the language of natural sciences. In particular, as long as differential equations have computable solutions [29, 12, 15, 16] the language of differential equations can be seen as an algorithmic language.

Yet, this algorithmic description of the laws of nature may have a broader scope than the description of the laws of nature with differential equations. For instance, the transformation of a messenger RNA string to a protein is easily expressed by an algorithm, while it cannot be expressed by a differential equation.

### 4.4 A property of nature or of the theory?

An objection to Galileo's formulation of the Galileo thesis "The Universe [...] is [...] a book [...] written in the language of mathematics" is that it confuses nature with our description of nature: nature itself is neither a book written in the language of mathematics, nor a book written in another language: it is not a book at all. Only our description of nature is a book. Thus, we could imagine that our description of nature is written in the language of mathematics because we have chosen to write it this way, nature having nothing to do with our decision [36].

Yet, nature and our description of nature are not independent: our description must have an experimental adequation of some sort with nature. And as we shall see, when constructing a theory, the scientists have very little freedom when "choosing" the set of physically realized relations.

Let us consider first a particular case where all the realized relations are functional. Then, we show that if two theories differ on the set of realized relations, one of the theories can be, at least in principle, experimentally refuted. Indeed, if the set of the realized relations differ, then there exists a relation  $R$  that is realized according to one theory  $\mathcal{T}$  but not according to the other theory  $\mathcal{T}'$ . Let  $E$  be the experiment realizing  $R$  according to the theory  $\mathcal{T}$  and  $R'$  be the relation realized by this experiment according to the theory  $\mathcal{T}'$ . As  $R$  is not realized according to  $\mathcal{T}'$ , the relations  $R$  and  $R'$  are different. Thus, there exists  $a$ ,  $b$ , and  $b'$ , such that  $b \neq b'$ ,  $a R b$ , and  $a R' b'$ . Then, if we perform the experiment  $E$

with the parameters  $a$ , the measures will either give the result  $b$  and refute  $\mathcal{T}'$ , or  $b'$  and refute  $\mathcal{T}$ , or an other value and refute both theories.

When the realized relations need not be functional, we have a weaker result: either a theory can be refuted, or it predicts, among others, a result that never occurs, whatever the number of times the experiment is repeated. Again, if the set of the realized relations differ, then there exists an experiment that realizes a relation  $R$  according to one theory and a relation  $R'$ ,  $R' \neq R$ , according to the other. Thus, there exists an  $a$ , such that the set  $R_a$  of the  $b$  such that  $a R b$  and the set  $R'_a$  of the  $b$  such of  $b$  such that  $a R' b$  differ. As these sets are different, they are not both equal to  $R_a \cap R'_a$ . Then, if we repeat the experiment with the parameters  $a$ , either the measures give one result that is not in  $R_a \cap R'_a$  and at least one of the theories is refuted, or the measures always give results in  $R_a \cap R'_a$  and at least one theory predicts a result that never occurs.

## 5 The physical Church-Turing thesis and the quantum theory

### 5.1 The bounded density of information in the quantum case

The hypothesis of a bounded density of information seems to be inspired by the idea of quantization of the state space, but is in contradiction with quantum theory. Indeed in quantum theory even a system with two distinct states  $\mathbf{u}$  and  $\mathbf{v}$  has an infinite number of states  $\alpha\mathbf{u} + \beta\mathbf{v}$ , for  $|\alpha|^2 + |\beta|^2 = 1$ . Thus, in quantum theory, this hypothesis cannot be formulated as the fact that the state space of a cell is finite.

Yet, this does not mean that the amount of possible outcomes of a measurement of some value associated to this system, is itself infinite and the bounded density of information principle can be formulated as the fact that each such measurement may only yield a finite number of possible outcomes.

This requirement amounts to the fact that the state space of a region of space of finite diameter is a finite-dimensional vector space. This constitutes a good alternative to the fact that the amount of possible outcomes, when measuring the system, is finite, as this formulation does not involve the notion of a measurement.

In the classical case, the finiteness of the state space  $S$  is used to deduce that any function from  $S$  to  $S$  is computable. When  $S$  is a finite-dimensional vector space, it is not the case that all functions from  $S$  to  $S$  are computable, but it is the case that all linear maps are computable, as the application of a linear map to a vector boils down to matrix multiplication, *i.e.* to addition and multiplication of scalars. Fortunately, quantum evolutions are always linear maps, hence they are computable. In short, linearity tames infinities, when infinite sets are restricted to finite-dimensional vector spaces.

To define the state space of a non necessarily bounded region of space, we first decompose this region into a finite or infinite number of cells of finite diameter. The state space of each cell is a finite-dimensional vector space. Let

$\mathbf{e}_1, \dots, \mathbf{e}_n$  be a base of this set. The state space of the whole system is the tensor product of all these finite-dimensional spaces. Its vectors are linear combinations of tensor products of elements of  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . As all the cells but a finite number of them are in a quiescent state, we can even restrict to finite tensor products.

We get this way a Fock-like space. This space is infinite-dimensional, hence it is not finitely generated with the operations  $+$  and  $\cdot$ , but as it has a base formed with vectors that are tensor products of elements of  $\mathbf{e}_1, \dots, \mathbf{e}_n$ : it is finitely generated with the operations  $+$ ,  $\cdot$  and  $\otimes$ . In particular computability is stable on such a space [3].

Thus, in the quantum case, the hypothesis of a finite density of information does not rule out infinite-dimensional vector spaces, but it explains where this infinity comes from: from the fact, that in an unbounded physical space, we can assemble an unbounded number of cells.

A difficult question is that of the impact of this hypothesis of a boundedness of information on the choice of the field of scalars we use. If we use the field  $\mathbb{C}$ , then a scalar contains an infinite quantity of information, for instance any infinite sequence of 0 and 1 may be encoded as the digits of a scalar. Thus, the linear combination  $\alpha\mathbf{u} + \beta\mathbf{v}$ , also contains an infinite quantity of information. Of course, this information cannot be retrieved in a single measurement, but it can, in principle, be probabilistically retrieved, by repeated measurements on similarly prepared systems.

On the opposite, we might consider that when superposing two base states  $\alpha\mathbf{u} + \beta\mathbf{v}$ , only a finite number of linear combinations are possible.

A intermediate solution is to assume that the state space of each cell is a finite-dimensional vector space over a finite-degree extension of the field of rationals. Since we are in discrete-time discrete-space quantum theory, such a restriction has little consequences: we have all the scalars that can be generated by a universal set of quantum gates for instance [13], see also [2] for a more in-depth discussion. Nevertheless, in the continuous picture, this kind of assumptions are not without consequences, and these are currently being investigated [17, 10].

## 5.2 The bounded velocity of information in the quantum case

Like the hypothesis of a finite density of information, that of a finite velocity of propagation of information has to be defined with caution in the quantum case. For instance, according to some interpretations of the Einstein-Podolsky-Rosen paradox, it could be said to contradict quantum theory.

In the Einstein-Podolsky-Rosen paradox, however, no *accessible* information can be communicated faster than the speed of light [9]. Similarly, it can be proved that not more than one bit of accessible information can be stored within a single qubit [27]. Drawing this distinction between the description of the infinite, non-local quantum states and the information that can actually be accessed about them, hints towards the quantum version of these hypotheses.

In the classical case we could assume that the state of a compound system was simply given by the state of each component. In the quantum setting this no longer holds, some correlation information needs to be added. In other words,

the state space of two regions is not the Cartesian product of the state space of each region, but its tensor product. Actually, if we stick to state vectors, knowing the state vector, e.g.  $(\mathbf{u} \otimes \mathbf{u}) + (\mathbf{v} \otimes \mathbf{v})$ , of the compound system, we cannot even assign a state vector to the first system. In order to do so, we need to switch to the density matrix formalism. Each state vector  $\mathbf{u}$  is then replaced by the pure density matrix  $|\mathbf{u}\rangle\mathbf{u}$  and if  $\rho$  is the density matrix of a compound system, then we can assign a density matrix to each subsystem, defined as a partial trace of  $\rho$ . The partial trace is defined by mapping  $A \times B$  to  $A$  and extending linearly to  $A \otimes B \rightarrow A$ . Still, knowing the density matrix of each subsystem is again not sufficient to reconstruct the state of the compound system.

In this setting, the most natural way to formalize the *bounded velocity of propagation of information* hypothesis is that given in [6] where it was referred to as *Causality*. Causality says that there exists a constant  $T$  such that for any region  $A$ , any point in time  $t$ , the density matrix associated to the region  $A$  at time  $t + T$ ,  $\rho(A, t + T)$  depends only on  $\rho(A', t)$ , with  $A'$  the region of radius 1 around  $A$ . Actually, this definition is a rephrase of the  $C^*$ -algebra formulation found in [31], which itself stems from quantum field theoretical approaches to enforcing causality [14]. The difficulty of this axiomatic formalization of the bounded velocity of propagation of information in the quantum case, is that it is quite non-constructive: it is no longer the case that because we know that  $\rho(A, t + T)$  is a local function  $f_A$  of  $\rho(A', t)$ , and  $\rho(B, t + T)$  is a local function  $f_B$  of  $\rho(B', t)$ , that  $\rho(A \cup B, t + T)$  can be reconstructed from  $\rho(A' \cup B', t)$  by means of these two functions.

A more constructive approach to formalizing the bounded velocity of propagation of information in the quantum case would be to, instead, state that the global evolution is *localizable* [7, 23, 32, 4], meaning that the global evolution is implementable by local mechanisms, each of them physically acceptable. Here this would say that the global evolution  $G$  is in fact quantum circuit of local gates with infinite width but finite depth. But the disadvantage of this approach is that this is a strong supposition to make. Fortunately, in [4, 5], the two approaches were shown to be equivalent. Hence we only need to assume causality and we can deduce locality.

With these two hypotheses formalized as explained, it is possible to extend Gandy's theorem to quantum theory [1].

## 6 Conclusion

The physical Church-Turing thesis explains the Galileo thesis, but also suggests an evolution of the language used to describe nature.

It can be proved from more basic principle of physics, but it also questions these principles, putting the emphasis on the principle of a bounded density of information. This principle itself questions our formulation of quantum theory, in particular the choice of a field for the scalars and the origin of the infinite dimension of the vector spaces used as state spaces.



## References

1. P. Arrighi and G. Dowek. The physical Church-Turing thesis and the principles of quantum theory. To appear.
2. P. Arrighi and G. Dowek. Operational semantics for formal tensorial calculus. In *Proceedings of QPL*, volume 33, pages 21–38. Turku Centre for Computer Science General Publication, 2004. ArXiv pre-print quant-ph/0501150.
3. P. Arrighi and G. Dowek. On the completeness of quantum computation models. In *Programs, Proofs, Processes: Computability in Europe*, page 21, 2010.
4. P. Arrighi, V. Nesme, and R. Werner. Unitarity plus causality implies localizability. QIP 2010, ArXiv preprint: arXiv:0711.3975, 2007.
5. P. Arrighi, V. Nesme, and R. Werner. Unitarity plus causality implies localizability (Full version). *Journal of Computer and System Sciences*, 2010.
6. P. Arrighi, V. Nesme, and R. F. Werner. Quantum cellular automata over finite, unbounded configurations. In *Proceedings of MFCS, Lecture Notes in Computer Science*, volume 5196, pages 64–75. Springer, 2008.
7. D. Beckman, D. Gottesman, M. A. Nielsen, and J. Preskill. Causal and localizable quantum operations. *Phys. Rev. A*, 64(052309), 2001.
8. J. D. Bekenstein. Universal upper bound to entropy-to-energy ratio for bounded systems. *Phys. Rev. D*, 23:287–298, 1981.
9. J.S. Bell. On the Einstein Podolsky Rosen paradox. *Physics*, 1:195, 1964.
10. P. Benioff. New gauge fields from extension of space time parallel transport of vector spaces to the underlying number systems. Arxiv preprint arXiv:1008.3134, 2010.
11. U. Boker and N. Dershowitz. The Church-Turing thesis over arbitrary domains. *Lecture Notes in Computer Science*, 4800:199, 2008.
12. O. Bournez and M.L. Campagnolo. A survey on continuous time computations. *New computational paradigms. Changing conceptions of what is computable*, pages 383–423, 2008.
13. P. O. Boykin, T. Mor, M. Pulver, V. Roychowdhury, and F. Vatan. On universal and fault-tolerant quantum computing: A novel basis and a new constructive proof of universality for Shor’s basis. In *FOCS ’99: Proceedings of the 40th Annual Symposium on Foundations of Computer Science*, page 486, Washington, DC, USA, 1999. IEEE Computer Society.
14. D. Buchholz. Current trends in axiomatic quantum field theory. *Lect. Notes Phys.*, 558:4364, 2000.
15. P. Collins and D. S. Graça. Effective computability of solutions of ordinary differential equations — The thousand monkeys approach. In V. Brattka, R. Dillhage, T. Grubba, and A. Klutsch, editors, *5th International Conference on Computability and Complexity in Analysis*, volume 221 of *Electronic Notes Theoretical Computer Science*, pages 103–114, 2008.
16. P. Collins and D. S. Graça. Effective computability of solutions of differential inclusions — The ten thousand monkeys approach. *Journal of Universal Computer Science*, 15(6):1162–1185, 2009.
17. A. Connes. The Witt construction in characteristic one and quantization. Arxiv preprint arXiv:1009.1769, 2010.
18. B.J. Copeland and O. Shagrir. Physical Computation: How General are Gandys Principles for Mechanisms? *Minds and Machines*, 17(2):217–231, 2007.
19. N. Dershowitz and Y. Gurevich. A natural axiomatization of the computability and proof of Church’s thesis. *The Bulletin of Symbolic Logic*, 14(3), 2008.

20. D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 400(1818):97–117, 1985.
21. G. Dowek. Non deterministic computation over the real numbers. Submitted to publication.
22. G. Dowek. The physical Church thesis as an explanation of the Galileo thesis. Submitted to publication.
23. T. Eggeling, D. Schlingemann, and R.F. Werner. Semicausal operations are semilocalizable. *EPL (Europhysics Letters)*, 57:782, 2002.
24. A. Einstein. Physics and Reality. *Journal of the Franklin Institute*, 221(3):349–382, 1936.
25. G. Galilei. *Il Saggiatore*. 1623.
26. R. Gandy. Church’s thesis and principles for mechanisms. In *The Kleene Symposium*. North-Holland, 1980.
27. A.S. Holevo. Information-theoretical aspects of quantum measurement. *Problemy Peredachi Informatsii*, 9(2):31–42, 1973.
28. R. Montague. Towards a general theory of computability. *Synthese*, 12(4):429–438, 1960.
29. M.B. Pour-El and J.I. Richards. *Computability in Analysis and Physics*. Springer-Verlag, 1989.
30. M.O. Rabin. Computable algebra, general theory and theory of computable fields. *Transactions of the American Mathematical Society*, 95(2):341–360, 1960.
31. B. Schumacher and R. Werner. Reversible quantum cellular automata. ArXiv pre-print quant-ph/0405174, 2004.
32. B. Schumacher and M. D. Westmoreland. Locality and information transfer in quantum operations. *Quantum Information Processing*, 4(1):13–34, 2005.
33. K. Weihrauch. *Computable analysis: an introduction*. Springer-Verlag, 2000.
34. E. Wigner. The unreasonable effectiveness of mathematics in the natural sciences. *Communications in Pure and Applied Mathematics*, 13(1), 1960.
35. S. Wolfram. *A new Kind of Science*. Wolfram Media, 2002.
36. M. Ziegler. Physically-relativized Church-Turing hypotheses: Physical foundations of computing and complexity theory of computational physics. *Applied Mathematics and Computation*, 215(4):1431–1447, 2009.