

The physical Church thesis as an explanation of the Galileo thesis

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Abstract

The Galileo thesis is a consequence of the physical Church thesis.

1 The Galileo thesis

1.1 The effectiveness of mathematics in natural sciences

The thesis that mathematics are effective in the natural sciences has been formulated by Galileo in 1623: “Philosophy is written in this vast book, which continuously lies open before our eyes (I mean the Universe). But it cannot be understood unless you have first learned the language and recognize the characters in which it is written. It is written in the language of mathematics” [15].

Galileo formulated this thesis, but did not give any explanation why it held. And long after Galileo, the lack of such an explanation was noticed by Albert Einstein according to whom “The eternal mystery of the world is its comprehensibility” [12] or Eugene Wigner according to whom “The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it” [19].

1.2 Insufficient explanations

Several explanations of this unreasonable effectiveness of mathematics in the natural sciences have been attempted:

1. God is a mathematician and He wrote the vast book in the language of mathematics.
2. The mathematical concepts are built by abstracting from empirical objects.
3. Scientists select only those phenomena that can be mathematically described.

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4. Scientists approximate the phenomena they study, until they can be mathematically described.
5. Our brain is part of nature, hence our concepts are natural objects, thus they are of the same kind as the objects they describe.

Each of these explanation is insufficient. The first reduces the problem to that of why God is a mathematician, which seems even harder to explain. The second is partial: if some mathematical concepts are built by abstracting from natural objects, the concept of ellipse, for instance, has not been built by abstracting from the trajectory of the planets, as it has been introduced some two thousands years before. The third leaves intact the problem of why so many — if not all — phenomena can be described in the language of mathematics. The fourth leaves intact the problem of why phenomena can be approximately — if not accurately — described in the language of mathematics. The fifth presupposes that we understand better a phenomenon from the inside than from the outside, which is not the case in general.

1.3 Perhaps several kinds of effectiveness

The effectiveness of mathematics in the natural sciences may be of different kinds. And instead of looking for a global explanation of all kinds of effectiveness, we should perhaps look for more local explanations.

For instance, the atomic masses of the chemical elements have a regular structure, as they are the integer multiples of some unit. When this regularity was discovered, there were three exceptions to this rule, because no elements of atomic mass 45, 68, and 70 were known. Yet, as some chemists trusted the structure of the natural numbers better than their observations, they predicted the existence of these three elements, that were later discovered. This is a striking example of the effectiveness of the structure of natural numbers in chemistry.

But, this striking regularity is easily explained by the fact that the mass of the atoms is mostly due to the mass of protons and neutrons that constitute their nucleus and that each nucleus contains an whole number of such particles.

Yet, this explanation sheds light on the effectiveness of the structure of natural numbers to describe the atomic masses of the chemical elements, but it does not shed light on the effectiveness of mathematics in the natural sciences in general, for instance, it does not shed light on the effectiveness of the quadratic functions to describe the free fall.

Thus, in this note, we shall focus on a particular instance of the general thesis that mathematics are effective in in the natural sciences: the fact that physically realized relations can be expressed by a proposition of the language of mathematics.

1.4 Physically realized relations

Let us imagine an experiment where one prepares a physical system by choosing some parameters and measures others. Let us call $a = \langle a_1, \dots, a_n \rangle$ the value of the chosen parameters and $b = \langle b_1, \dots, b_p \rangle$ that of the measured ones. This experiment, *i.e.* this physical system together with the protocol defining the chosen parameters and the measured ones, defines a relation: $a R b$ if b is a possible result for the measures when the chosen parameters are a . We say that these relations are *physically realized*.

For instance, if one applies an electrical tension U to a conductor of resistance R and measures the current I passing through this conductor, then $a = \langle U, R \rangle$ and $b = I$ and the realized relation is that relating $\langle U, R \rangle$ and I when $U = RI$. The relation between $\langle U, R \rangle$ and I can thus be expressed by a proposition of the language of mathematics. In the same way, the relation between the time and the position of a body freely falling in vacuum can be described by the proposition of the language of mathematics $x = \frac{1}{2}gt^2$. Among the uncountable number of relations between numbers, only a countable number can be defined by a proposition of the language of mathematics, such as $U = RI$ or $x = \frac{1}{2}gt^2$ and all the physically realized relations seem to be in this small set.

As Galileo stressed the role of the language of mathematics, we can call *the Galileo thesis* the statement that all physically realized relation can be expressed by a proposition of the language of mathematics.

Instead of attempting to explain the general thesis that mathematics are effective in the natural sciences, we shall restrict our investigation to attempt to explain this unreasonable effectiveness of the propositions of the language of mathematics to express physically realized relations.

2 The physical Church thesis

2.1 The physical Church thesis

The physical Church thesis expresses that if we are able to construct a computing machine, *i.e.* a physical system together with an interaction protocol defining a way to communicate some information to the system by choosing some parameters a and to extract some information from the system, by measuring others b , then the relation between a and b is a computable relation.

In the formulation of this thesis nothing is said about the nature of the machine. It may be electronic or not, digital or not, deterministic or not, inanimate or not, *etc.* This notion of machine encompasses all the physical systems, equipped with an interaction protocol. Thus, it is co-extensive with the notion of experiment, we have defined above.

Therefore, the physical Church thesis can be stated as the fact that physically realized relations are computable.

2.2 The physical Church thesis implies the Galileo thesis

Once we have identified the similarities between the Galileo thesis and the physical Church thesis by stating them as theses about the set of physically realized relations, we may remark that the physical Church thesis implies the Galileo thesis.

Indeed, as any program expressing a computable relation is a mathematical expression, all computable relations can be defined by a proposition of the language of mathematics. In fact, computable relations can even be expressed by a proposition in a very small fragment of mathematics: the language of Peano arithmetic.

Thus, if the physical Church thesis holds, then all physically realized relations are computable, hence they can be expressed by a proposition of the language of mathematics, *i.e.* the Galileo thesis holds.

2.3 Gandy's proof of the physical Church thesis

This explanation of the Galileo thesis reduces the problem of explaining the Galileo thesis to that of explaining why the physical Church thesis holds. But, such an explanation has already been attempted. For instance, Robin Gandy [14] has shown that the physical Church thesis is a consequence of three assumptions about nature:

- the homogeneity of space and time,
- the boundedness of the velocity of propagation of information,
- the boundedness of the density of information.

The boundedness of the density of information can be expressed, in physical terms, as the fact that a physical system of finite diameter has a finite state space and the boundedness of the velocity of propagation of information can be expressed as the fact that the state of a system in one place can only affect the state of a system in another after a delay, proportional to their distance.

Then, to prove the physical Church thesis from these assumptions, we just need to partition the space into an infinite number of identical cells of finite size. Because information has a bounded density, the state of each cell is an element of a finite set. Because of the homogeneity of space, this state space is the same for all cells. At the origin of time all the cells except a finite number are in a particular *quiescent* state. Like space, time can be discretized. Because the velocity of propagation of information is bounded, the state of a cell at a given time step is function of the state of a finite number of neighbors cells at the previous time step. This function of a finite number of variables varying in a finite set is obviously computable. Hence the state of each cell at each time step can be computed from the initial state.

Gandy's hypotheses can be, and have been [8], criticized. For instance, it is well-known that in Newtonian mechanics, gravity is instantaneous and thus information can be propagated with an infinite velocity. Also, the position of a

body between two points A and B — the distance AB taken as a unit — can be any real number between 0 and 1 and thus can contain an infinite quantity of information: any infinite sequence in a finite alphabet can be encoded as the digits of such a number, in an appropriate base. Yet, Gandy's hypotheses have not been refuted experimentally — for instance by the construction of an instantaneous computer network or by the construction of a hard drive with an infinite capacity.

More importantly, even if Gandy's hypotheses must be refined, Gandy's proof shows that the physical Church thesis is a consequence of some hypotheses about nature, that do not refer to notions such as those of language or computability. And, the Galileo thesis also is a consequence of these hypotheses.

If these hypotheses are true, the fact that natural phenomena can be described by propositions of the language of mathematics is a consequence of objective properties of nature, such as the fact that a system of finite size has a finite state space.

2.4 Eliminating the physical Church thesis

We have seen that

- Gandy's hypotheses imply the physical Church thesis,
- and the physical Church thesis implies the Galileo thesis.

Thus, we can deduce that Gandy's hypotheses imply the Galileo thesis and attempt to prove this directly.

Yet, from a historical point of view, it is important to notice the role of computability theory and the physical Church thesis in connecting Gandy's hypotheses to the Galileo thesis.

2.5 An algorithmic description of the laws of nature

A side effect of this explanation of the Galileo thesis is that the laws of nature can be described not only in the language of mathematics, but also in a language designed to express algorithms: a programming language.

Instead of expressing the law of free fall in vacuum by the proposition $x = \frac{1}{2}gt^2$, we could express it by the algorithm `fun t -> g * t * t / 2`, leading to a second Galilean revolution in the language of natural sciences. In particular, as long as differential equations have computable solutions [18, 5, 6, 7] the language of differential equations can be seen as a language to define algorithms: a programming language.

Yet, this algorithmic description of the laws of nature may have a broader scope than the description of the laws of nature with differential equations. For instance, the transformation of a messenger RNA string to a protein is easily expressed by an algorithm, while it cannot be expressed by a differential equation.

3 A property of nature or of the theory?

An objection to Galileo’s formulation of the Galileo thesis “The Universe [...] is [...] a book [...] written in the language of mathematics” is that it confuses nature with our description of nature: nature itself is neither a book written in the language of mathematics, nor a book written in another language: it is not a book at all. Only our description of nature is a book. Thus, we could imagine that our description of nature is written in the language of mathematics because we have chosen to write it this way, nature having nothing to do with our decision [20]. Rather than a property of nature itself, this set of relations seems to be a property of a particular theory chosen to describe nature.

Yet, the Universe and our description of the Universe are not independent: our description must have an experimental adequation of some sort with the Universe.

We show, in this section, that in the construction of a theory, the scientists have very little freedom when “choosing” this set of realized relations.

Let us consider first a particular case where all the realized relations are functional. Then, we show that if two theories differ on the set of realized relations, one of the theories can be, at least in principle, experimentally refuted. Indeed, if the set of the realized relations differ, then there exists a relation R that is realized according to one theory \mathcal{T} but not according to the other theory \mathcal{T}' . Let E be the experiment realizing R according to the theory \mathcal{T} and R' be the relation realized by this experiment according to the theory \mathcal{T}' . As R is not realized according to \mathcal{T}' , the relations R and R' are different. Thus, there exists a , b and b' , $b \neq b'$, such that $a R b$ and $a R' b'$. Then, if we perform the experiment E with the parameters a , the measures will either give the result b and refute \mathcal{T}' or b' and refute \mathcal{T} or an other value and refute both theories.

When the realized relations need not be functional, we have a weaker result: either a theory can be refuted, or it predicts, among others, a result that never occurs, whatever the number of times the experiment is repeated is. Again, if the set of the realized relations differ, then there exists an experiment that realizes a relation R according to one theory and a relation R' , $R' \neq R$, according to the other. Thus, there exists an a , such that the set R_a of the b such that $a R b$ and the set R'_a of the b such of b such that $a R' b$ are different. As these sets are different, they are not both equal to $R_a \cap R'_a$. Then, if we repeat the experiment with the parameters a , either the measures give one result that is not in $R_a \cap R'_a$ and at least one of the theories is refuted, or the measures always give results in $R_a \cap R'_a$ and at least one theory predicts a result that never occurs.

4 Towards a logical analysis of natural phenomena

The formulation of the Galileo thesis and the physical Church thesis as properties of the set of physically realized relations points out the importance of this

set in the natural sciences. Several other theses can be stated as a property of this set.

- The negation of the physical Church thesis, *i.e.* the existence of hyper-computations, is, of course, also a thesis about this set of relations.

A hyper-computation is an experiment that is supposed to realize a relation that is not computable. It has been argued for instance that hyper-computations exist because the quantum adiabatic theorem [17] or the properties of time-space in the neighborhood of a black hole [13, 16] allow to perform an infinite number of computation steps in a finite time. Even in classical Newtonian physics, such hyper-computations may also exist if we accept to encode a non computable set in the initial state of the system, using either the fact that the system has an infinite size [2], or that it contains arbitrarily small pieces [3], or that the position of some point is described with a real number [4].

More interesting than the refutation of the physical Church thesis is a positive characterization of the set of the relations physically realized under these hypotheses.

- Determinism and non-determinism can also be stated as theses about the set of the physically realized relations. It is too naive to state that determinism is the thesis that all physically realized relations are functional, because the functionality of these relations depends on the chosen protocol. For instance, if one applies a given electrical tension U to a conductor of resistance R and measures the current I passing through this conductor, then the realized relation contains all the pairs $\langle\langle U, R \rangle, I\rangle$ such that $U = RI$ and this relation is functional. But, if one applies an electrical tension to a conductor of resistance R and measures both the electrical tension and the current passing through this conductor, then the realized relation contains all the pairs $\langle R, \langle U, I \rangle \rangle$ such that $U = RI$ and, unlike the previous one, this relation is not functional, although no non-determinism occurs here. To define determinism, we have to take time into account and restrict to protocols where the chosen parameters are measurable parameters of the system at a given time t and the measured one are the parameters of the system at a later time t' . Then, determinism can be stated as the fact that for each physical system, there exists such a protocol, for which all realized relations are functional. Non-determinism, in contrast, is the thesis that there exist systems such that for all such protocols, there exists a non functional realized relation.
- The thesis that all physical phenomena are continuous or differentiable can also be stated as properties of the set of the physically realized relations.

In very general terms, when a set A is a subset of a set B , then the thesis that the physically realized relations are elements of the set A and that they are elements of the set B are related: the first implies the second. The fact that

the physical Church thesis implies the Galileo thesis is just a particular case of this.

For some sets A of relations, there exists a language \mathcal{L} such that the relations of the set A are those that can be expressed in the language \mathcal{L} . For instance, the polynomial relations are those that can be expressed in the language of polynomials, the computable relations are those that can be expressed in a programming language, *etc.*

In this case, the thesis that all the physically realized relations are elements of the set A can be stated as the fact that all realized relations can be expressed in the language \mathcal{L} or, in a more poetic way, as the fact that the language \mathcal{L} is (unreasonably) effective in the natural sciences. Such a thesis should be understood as a thesis about nature, not about the language.

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