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► **To cite this version:**

| Gilles Dowek. A theory independent Curry-De Bruijn-Howard correspondence. 2012. hal-04070185

HAL Id: hal-04070185

<https://inria.hal.science/hal-04070185>

Preprint submitted on 15 Apr 2023

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A theory independent Curry-De Bruijn-Howard correspondence

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Brouwer, Heyting, and Kolmogorov have proposed to define constructive proofs as algorithms, for instance, a proof of $A \Rightarrow B$ as an algorithm taking proofs of A as input and returning proofs of B as output. Curry, De Bruijn, and Howard have developed this idea further. First, they have proposed to express these algorithms in the lambda-calculus, writing for instance $\lambda f^{A \Rightarrow A \Rightarrow B} \lambda x^A (f x x)$ for the proof of the proposition $(A \Rightarrow A \Rightarrow B) \Rightarrow A \Rightarrow B$ taking a proof f of $A \Rightarrow A \Rightarrow B$ and a proof x of A as input and returning the proof of B obtained by applying f to x twice. Then, they have remarked that, as proofs of $A \Rightarrow B$ map proofs of A to proofs of B , their type $proof(A \Rightarrow B)$ is $proof(A) \rightarrow proof(B)$. Thus the function *proof* mapping propositions to the type of their proofs is a morphism transforming the operation \Rightarrow into the operation \rightarrow . In the same way, this morphism transforms cut-reduction in proofs into beta-reduction in lambda-terms.

This expression of proofs as lambda-terms has been extensively used in proof processing systems: Automath, Nuprl, Coq, Elf, Agda, etc. Lambda-calculus is a more compact representation of proofs, than natural deduction or sequent calculus proof-trees. This representation is convenient, for instance to store proofs on a disk and to communicate them through a network.

This has lead to the development of several typed lambda-calculi: Automath, the system F, the system $F\omega$, the lambda-Pi-calculus, Martin-Löf intuitionistic type theory, the Calculus of Constructions, the Calculus of Inductive Constructions, etc. And we may wonder why so many different calculi are needed.

In some cases, the differences in the lambda-calculi reflect differences in the logic where proofs are expressed: some calculi, for instance, express constructive proofs, others classical ones. In other cases, they reflect differences in the inductive rules used to define proofs: some calculi are based on natural deduction, others on sequent calculus. But most of the times, the differences reflect differences in the theory where the proofs are expressed: arithmetic, the theory of classes—a.k.a. second-order logic—, simple type theory—a.k.a. higher-order logic—, predicative type theory, etc.

Instead of developing a customized typed lambda-calculus for each specific theory, we may attempt to design a general parametric calculus that permits to express the proofs of any theory. This way, the problem of expressing proofs in the lambda-calculus would be completely separated from that of choosing a theory.

A way to do this is to start from the lambda-Pi-calculus, that is designed to express proofs in minimal predicate logic and to define a theory in an axiomatic

way, declaring a variable, or a constant, for each axiom. This is the approach of the *Logical framework* [8]. Yet, a limit of this approach is that the beta-reduction is too weak in presence of axioms, and we need to add axiom-specific proof-reduction rules, such as the rules of Gödel system T for the induction axiom, to emulate cut-reduction in specific theories.

We have proposed in [5] a different approach, where a theory is expressed, not with axioms, but with rewrite rules, as in Deduction modulo [6, 7]. This has led to the *lambda-Pi-calculus modulo*, and its implementation, the system *Dedukti* [2].

Although it is just a proof-checker, *Dedukti* is a universal proof-checker [3]. By choosing appropriate rewrite rules, the lambda-Pi-calculus modulo can be parametrized to express proofs of any theory that can be expressed in Deduction modulo, such as arithmetic, the theory of classes, simple type theory, some versions of set theory, etc. By choosing appropriate rewrite rules, the lambda-Pi-calculus can also emulate the system F, the system $F\omega$, the Calculus of Constructions [5], the Calculus of Inductive Constructions [4], etc. This has led to the development of systems to translate proofs from the system Coq to *Dedukti* [4] and from the system HOL to *Dedukti* [1].

This universal proof-checker opens new research directions that still remain to be investigated. First, what happens if we prove the proposition $A \Rightarrow B$ in a theory \mathcal{T}_1 and the proposition A in a theory \mathcal{T}_2 ? Is there a theory in which we can deduce B ? Of course, if the theories \mathcal{T}_1 and \mathcal{T}_2 are incompatible—such as set theory with the axiom of choice and set theory with the negation of the axiom of choice—, it makes no sense to deduce B anywhere. But, there are also cases where one of the rewrite systems expressing \mathcal{T}_1 and \mathcal{T}_2 in the lambda-Pi-calculus modulo is a subset of the other, or where the union of these two systems defines a consistent theory, or where propositions and proofs of one theory may be translated into the other, and in all these cases, it makes sense to deduce B from the proofs of $A \Rightarrow B$ and A , even if these proofs have been developed in different theories and different systems.

More generally, although most proof processing systems are based on strong theories—simple type theory, the Calculus of Inductive Constructions, etc.—we know that many proofs developed in these systems use only a small part of this strength. Making explicit the axioms or rewrite rules defining these theories permits to identify which axiom, or which rule, is used in which proof, in a similar way as we, more or less, know which part of informal mathematics depends on the axiom of choice and which part does not.

Such an analysis may be a first step towards the development of libraries of proofs, where proofs would not be classified in function of the system in which they have been developed, but in function of the axioms and rules they use, i.e. to a true interoperability between proof systems.

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