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On the problem of minimizing the epidemic final size for SIR model by social distancing

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Abstract: We revisit the problem of minimizing the epidemic final size in the SIR model through social distancing. In the existing literature, this problem has been considered by imposing a priori interval structure on the time period during which interventions are enforced. We show that when considering the more general class of controls with an $L^1$ constraint on the confinement effort to reduce the infection rate, the support of the optimal control is still a single time interval. In other words, for the problem of minimizing the epidemic final size in the SIR model through social distancing, there is no benefit in splitting interventions on several disjoint time periods. The techniques we deploy here are different than what has been proposed in the literature, and could be potentially applied to other problems.

Keywords: Epidemiological models, optimal control, $L^1$ controls, infinite horizon.

1. INTRODUCTION

The pandemic of Covid-19 represented in many aspects an unprecedented experiment for the societies all over the world. One aspect of this has been the use of social distancing measures implemented in various forms and periods. This aspect has revived interest in optimal control methods for the control of transmissible diseases, a subject already well-researched in the past, see e.g., Lenhart and Workman (2007); Sharomi and Malik (2017) and bibliographical references in Bliman et al. (2021).

In the same vein as several works cited and analyzed below, we focus here on the optimal control of a quite simple epidemiological model, and attempt to fully characterize the optimal policy. We consider more precisely the specific issue of optimal control of the classical SIR model through non-pharmaceutical interventions that reduce the transmission rate $\beta$ by a proportion $u(t)$, which plays the role of a control variable, usually during a limited period. The corresponding controlled system is

$$\begin{cases}
\dot{S} = -(1-u(t))\beta SI \\
\dot{I} = (1-u(t))\beta SI - \gamma I
\end{cases} \quad u(t) \in U := [0, \bar{u}], \quad (1)
$$

where $S$ and $I$ stand for the proportions of the susceptible and infected individuals in the population, respectively, and $\bar{u}$ is a fixed number in $(0, 1]$. The solution $(S, I)$ depends upon the input $u$ and the initial values (such that $S(0), I(0) \geq 0, S(0) + I(0) \leq 1$), but for sake of simplicity, this dependence is not explicitly indicated in the sequel.

Tables 1 to 4 show the diversity of the approaches considered in the studies of optimal control for problem (1), and the obtained results. These studies differ through the choice of the cost function, of the state constraint, and of the class of admissible input functions. The cost functions that are commonly considered are the peak prevalence (i.e., the maximal value of $I(\cdot)$, abbreviated $PP$ below), the epidemic final size (i.e., the total number of individuals $S(0) + S(+\infty)$ infected during the outbreak, abbreviated $FS$ below), the control effort and the time to reach the sub-epidemic ‘safe’ region, or some combination of them. The constraints are usually on the peak prevalence or on the cumulative control effort. Last, the class of control inputs among which optimization is achieved may be constant or piecewise constant during the lockdown phases, possibly with values chosen in a prescribed finite set, or taken in full generality in the set $U$ of measurable functions $u(\cdot)$ defined on $[0, +\infty)$ taking values in $U$ defined in (1) (in practice one may look for piece-wise continuous controls). Their support, denoted $\text{supp}(u)$, may be imposed as a single interval or a union of intervals of prescribed length, may be simply included in a fixed interval $(\text{supp}(u) \subset [0, \Delta])$, for a given $\Delta > 0$, or initially unspecified.

We are concerned in the present paper by the issue of reduction of the final size (see Table 1). Bliman et al. (2021) and Ketcheson (2021) have shown that the problem obtained when imposing $\text{supp}(u)$ to be included in an interval $[0, T]$ has a bang-bang solution consisting in at most three phases. In a first phase (possibly of zero duration) no action is taken until a certain optimal initiation time, which in any case is located before reaching the prevalence peak. Then the control is activated at its strongest value $\bar{u}$ until reaching the time horizon $\Delta$. Last, $u \equiv 0$ after the end of the permitted confinement interval. The fact that the optimal intervention may have to start after the beginning of the latter, that is at a positive time, suggests that a more efficient use of prescribed confinement duration may exist. This is the subject of Bliman and Duprez (2020), in which intervals of the type $[t, t + \Delta]$ were considered,
There exists an optimal initiation time $t^*$, and that the optimal cost is obtained by taking $u = \bar{u}$ on the whole interval $[t^*, t^* + \Delta]$. Therefore, if confinement is allowed only for a given amount of time $\Delta$, in order to reduce the epidemic final size, it may be more rational to wait for some time, and then exhaust completely the lockdown ability.

However, this does not answer the following more involved issue: would not it be even more efficient to split the confinement period and consider more general control phases, keeping a total treatment time at most equal to $\Delta$? Writing $\text{meas} \sup \Delta$ writing $\text{phases}$, keeping a total treatment time at most equal to $\Delta$? Writing $\text{meas} \sup \Delta$ for the cost function for $\mathcal{R}_S$ is $S(T)$, different from $S(\infty)$ for problems $\mathcal{P}_S$ and $\mathcal{Q}_S$. However, a common behavior will be recovered for $T \rightarrow +\infty$.

Now, the $L^1$ constraint on the control variable, $\|u(\cdot)\|_1 \leq K$, can be easily tackled by augmenting the state of the system

$$\begin{cases}
\dot{S} = -(1 - u)\beta SI \\
\dot{I} = (1 - u)\beta SI - \gamma I \\
\dot{C} = u
\end{cases}$$

with $C(0) = 0$, so that the integral constraint (2) is transformed into the terminal constraint $C(T) \leq K$. Then, the optimal control problem $\mathcal{R}_S$ is formulated as a Mayer problem with a target. The (extended) dynamics being linear with respect to the control, existence of optimal controls follows from the classical Filippov Theorem, see for instance (Lee and Markus, 1967, Theorem 4, p. 259).

### 3. STUDY OF PROBLEM $\mathcal{R}_S$

This section is devoted to the study of Problem $\mathcal{R}_S$, and culminates in the statement of Proposition 4. We begin by writing the necessary optimality conditions of the Pontryagin Maximum Principle for the latter.

The Hamiltonian writes

$$H = (p_1 - p_S)(1 - u)\beta SI - p_1\gamma I + p_C u$$

and the adjoint equations

$$\begin{aligned}
\dot{p}_S &= -\partial_S H = (p_S - p_1)(1 - u)\beta I, \\
\dot{p}_I &= -\partial_I H = (p_S - p_1)(1 - u)\beta S + p_1\gamma, \\
\dot{p}_C &= -\partial_C H = 0,
\end{aligned}$$

along with the transversality conditions

$$\begin{aligned}
p_S(T) &= 1, \\
p_I(T) &= 0, \\
p_C(T) &\leq 0 \text{ (} < 0 \text{ if the constraint is saturated).}
\end{aligned}$$

An optimal control $u(\cdot)$ is a maximizer of the Hamiltonian for almost any $t$, which allows to claim that one has for any optimum control $u$

$$\begin{aligned}
\phi(t) > 0 &\Rightarrow u(t) = \bar{u} \text{ a.e. } t, \\
\phi(t) < 0 &\Rightarrow u(t) = 0 \text{ a.e. } t,
\end{aligned}$$

and the (switching function) $\phi$ defined by

$$\phi := (p_S - p_1)\beta SI + p_C$$

has the property that $H = \phi \dot{\phi} + (p_S - p_1)\beta SI - p_1\gamma I$. Moreover, for any optimum solution, there exists a number $\bar{H}$ such that $H(t) = \bar{H}$ a.e. $t \in [0, T]$.

A straightforward computation gives

$K := \Delta \bar{u}$.
Also, at any point of continuity of $u$, 

\[ \dot{\phi} = -\beta \gamma S I p_S. \tag{5} \]

Also, at any point of continuity of $u$, 

\[ \frac{1}{\beta \gamma} \phi = (S I + S I) p_S + S I \dot{p}_S \]

\[ = (-1 - u) \beta S I^2 + (1 - u) \beta S I - \gamma S I) \]

\[ = (1 - u) \beta S I(-I + S) + \frac{1}{\beta} \dot{\phi} + (1 - u) I(\phi - p_C) \]

\[ = \frac{1}{\gamma} - (u - I) \gamma I \phi + \frac{1}{\beta} \dot{\phi} + (1 - u) I(\phi - p_C), \]

so that, using (5), 

\[ \ddot{\phi} = - (\phi - p_C)(1 - u) \beta \gamma I + \phi \dot{\phi}(1 - u)(S - I) - \gamma. \tag{6} \]

\textbf{Lemma 1.} For $K < T \bar{u}$, one has $p_C < 0$. \hfill \Box

\textbf{Proof.} Assume by contradiction that $p_C = 0$. One has

\[ \phi(T) = (p_S(T) - p_I(T)) \beta S(T) I(T) = \beta S(T) I(T) > 0. \]

By continuity, $\phi(t)$ is positive for $t \in [T - \eta, T]$ for some $\eta > 0$, and then any optimal control verifies $u(t) = \bar{u}$ for a.e. $t \in [T - \eta, T]$ for some $\eta > 0$.

If $\bar{u} = 1$, one obtains from the adjoint equations $p_S = \dot{p}_I = 0$, so that $p_S = 1$ and $p_I = 0$ on this time interval, and this constant solution is propagated on the whole interval $[0, T]$. Any optimal control has then to satisfy $u(t) = \bar{u}$ for a.e. $t \in [0, T]$.

If $\bar{u} < 1$, as $p_I$ is continuous with $p_I(T) = 0$, one gets from the conservation of the Hamiltonian along an optimal solution the identity

\[ H = -(1 - u(t)) \phi(t) - p_I(t) \gamma I(t) = H(T) \quad \text{a.e. } t \in [0, T]. \]

On the other hand, using the assumption that $p_C = 0$, 

\[ H(T) = -p_S(T)(1 - \bar{u}) \beta S(T) I(T) + p_C \bar{u} \]

\[ = -(1 - \bar{u}) \beta S(T) I(T) < 0, \]

and thus:

\[ H = -(1 - u(t)) \phi(t) - p_I(t) \gamma I(t) < 0 \quad \text{a.e. } t \in [0, T]. \tag{7} \]

Consider now the set 

\[ E := \{ t \in [0, T] ; \phi(t) \leq 0 \}. \]

If $E$ is non empty, let $t_c = \sup E < T$. By continuity of $\phi$, one has $\phi(t_c) = 0$, and thus $p_I(t_c) = p_S(t_c) > 0$ using inequality (7). In view of (5), this implies $\dot{\phi}(t_c) < 0$ which contradicts the definition of $t_c$. We conclude that $E$ has to be empty and that any optimal control has also to satisfy $u(t) = \bar{u}$ for a.e. $t \in [0, T]$. 

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Input set</th>
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<tbody>
<tr>
<td>$\supp(u) \subseteq [0, \Delta]$</td>
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<tr>
<th>Constraint</th>
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<tbody>
<tr>
<td>$\supp(u) \subseteq [I, T + \Delta, t \geq 0]$</td>
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<td>$\supp(u) = [0, \Delta]$</td>
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<tr>
<td>$\supp(u) \subseteq I$, $\meas \supp(u) = \Delta, I \equiv \max I(t)$ on $\supp(u)$</td>
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<td>$\meas \supp(u) = \Delta, u</td>
<td>_{\supp(u)}$ constant</td>
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Table 2. Contributions with peak prevalence as cost function.

Table 3. Contributions taking the $L^1$-norm of $u$ as cost function.

Table 4. Contribution taking as cost function the time to the ‘safe zone’, i.e., the zone in which absence of control cannot yield violating of the constraint.
We have shown that, in the case $\bar{u} = 1$ as well as in the case $\bar{u} < 1$, $pc = 0$ implies that $u(t) = \bar{u}$ almost everywhere on $[0, T]$. One then obtains
\[
\int_0^T u(t)dt = T\bar{u} > K,
\]
which violates the constraint (2). Therefore, one has $pc < 0$, which demonstrates Lemma 1. □

As the objective is to consider the criterion for large $T$, we assume in the sequel that the condition $T\bar{u} > K$ is verified, so that an optimal control saturates constraint (2).

Lemma 2. For $\bar{u} < 1$, an optimal solution has no singular arc. For $\bar{u} = 1$, a singular arc is given by $u = \bar{u}$. □

Proof. If $\phi$ is null on a non-empty open time interval $J$, one should have $\hat{\phi}(t) = 0$ for $t \in J$ and $\hat{\phi}(t) < 0$ for almost every (a.e.) $t \in J$. From expression (6), one obtains
\[
\hat{\phi}(t) = pc(1 - u(t))\beta\gamma I(t) = 0 \text{ a.e. } t \in J.
\]
If $\bar{u} < 1$, one obtains a contradiction, due to the fact that $pc < 0$, see Lemma 1: there is no singular arc. If $\bar{u} = 1$, $u = 1$ is optimal on $J$. □

Lemma 3. For any optimal solution, the set $F := \{t \in [0, T]; \phi(t) \geq 0\}$ is connected. □

Proof. By contradiction, assume that there exists $t_1 < t_2$ in $(0, T)$ such that $\phi(t_1) = \phi(t_2) = 0$ with $\phi(t) < 0$ for $t \in J := (t_1, t_2)$. Then the optimal solution is given by $u = 0$ on $J$, see (3). The switching function $\phi(t)$ being $C^1$, there exists $\tau_1 < \tau_2$ in $J$ as close as desired respectively to $t_1$ and $t_2$, and such that $\phi(\tau_1) < 0$ and $\phi(\tau_2) > 0$, which, due to (5), amounts to have $ps(\tau_1) > 0$ and $ps(\tau_2) < 0$.

For the constant control $u = 0$, we get from expression (4) the property
\[
\phi(t) = \hat{ps}(t)S(t) + pc, \quad t \in J. \tag{8}
\]
As $pc < 0$ (by Lemma 1) and $\phi(t_2) = 0$, one has $\hat{ps}(t_2) > 0$ for $t_2$ chosen close enough to $t_2$. We deduce, from this and the fact that $ps(\tau_1) > 0 > ps(\tau_2)$, that $ps$ has a (negative) minimum in $J$, reached at a certain $t \in (\tau_1, \tau_2)$.

At $t$, one has $\hat{ps}(t) = 0$ and thus $ps(t) = \hat{p}_s(\hat{t})$ from the adjoint equations. From the conservation of the Hamiltonian, one gets
\[
H = -p_I(t)\gamma I(t) + pcu(t)
= -p_I(t)\gamma I(t) = -(1 - u(t))\beta S(T)I(T) + pcu(T),
\]
so that $H < 0$, which implies $p_I(t) = ps(t) > 0$. From (8) and (5), one now obtains
\[
0 > -\beta S(t)I(t)ps(t) = \hat{\phi}(t) = \hat{ps}(t)S(t) + \hat{p}_s(t)\hat{S}(t),
\]
for $t \in J$. Because $\hat{ps}(t) = 0$, we deduce that $\hat{ps}(t) < 0$, which contradicts $\hat{t}$ as a local minimizer of $ps$.

In conclusion, there exists no $t_1 < t_2$ in $(0, T)$ such that $\phi(t_1) = \phi(t_2) = 0$ with $\phi(t) < 0$ for $t \in J := (t_1, t_2)$. In short, the set $F$ defined in the statement is connected. □

We then obtain the following characterization of the optimal solutions for the problem $\mathcal{R}_S$.

Proposition 4. For any positive initial condition and time horizon $T > \Delta$, any optimal control $u(\cdot)$ for problem $\mathcal{R}_S$ is such that
\[
u(t) = \begin{cases} \bar{u} & \text{if } t \in [t^*_c, t^*_c + \Delta], \\ 0 & \text{otherwise} \end{cases} \quad \text{a.e. } t \in [0, T] \tag{9}
\]
for some $t^*_c \in [0, T - \Delta]$.

Proof. From Lemma 2, any optimal control takes only the values 0 and $\bar{u}$. From Lemma 3, it has at most one arc with $u = \bar{u}$; and with Lemma 1, that the constraint is saturated, therefore, any optimal control has the structure (9). □

Uniqueness of the optimal switching time $t^*_c$ has not been demonstrated. It will be proved in the next section that any sequence of maximizers $t^*_c(T)$ converges to a unique switching time when $T$ tends to $+\infty$.

4. PROBLEM $\mathcal{R}_S$ OVER AN INFINITE HORIZON

We investigate now the optimal solution of the limiting problem $\mathcal{R}_S$ when $T \to +\infty$.

For any $T < +\infty$, the structure of the optimal control for problem $\mathcal{R}_S$ being known (‘0-$\bar{u}$-0’), one may characterize the corresponding optimal value of $S(T)$ as follows. Define
\[
S_h := \frac{1}{\beta}, \quad F(S) := S - S_h \log(S). \tag{10}
\]
One may recognize in $S_h$ the herd immunity threshold, that is the proportion of susceptible below which the number of infected starts decreasing.

On the two time intervals $[0, t^*_c)$ and $(t^*_c + \Delta, T]$, the optimal control $u$ is identically equal to 0 and we can write the invariant properties
\[
F(S(t^*_c)) + I(t^*_c) = F(S(0)) = I(0) =: V_0, \tag{11}
\]
\[
F(S(T)) + I(T) = F(S(t^*_c + \Delta)) + I(t^*_c + \Delta). \tag{12}
\]
On the time interval $(t^*_c, t^*_c + \Delta)$, the optimal control $u$ is constant equal to $\bar{u}$. We distinguish two cases.

(1) $\bar{u} = 1$. Then one has
\[
S(t^*_c + K) = S(t^*_c), \quad I(t^*_c + K) = e^{-\gamma K}I(t^*_c). \tag{13}
\]
(2) $\bar{u} < 1$. We can write the invariance property
\[
S(t^*_c + \Delta) + I(t^*_c + \Delta) - \frac{S_h}{1 - \bar{u}} \log(S(t^*_c + \Delta)) = S(t^*_c) + I(t^*_c) - \frac{S_h}{1 - \bar{u}} \log(S(t^*_c)). \tag{14}
\]
Let us combine expressions (11), (13) or (14), and (12). We obtain (with $V_0$ defined in (11)):

(1) for $\bar{u} = 1$,
\[
F(S(T)) = V_0 - (1 - e^{-\gamma K})I(t^*_c) - I(T), \tag{15}
\]
(2) for $\bar{u} < 1$,
\[
F(S(T)) = V_0 - \frac{\bar{u}}{1 - \bar{u}} S_h (\log(S(t^*_c)) - \log(S(t^*_c + \Delta))) - I(T). \tag{16}
\]

Then, we posit
\[ \Gamma_T(t_c) := \begin{cases} 
(1 - e^{-\gamma K}) I(t_c) + I(T), & \bar{u} = 1 \\
\frac{-\bar{u}}{1 - \bar{u}} \left( \log(S(t_c)) - \log(S(t_c + \Delta)) \right) + I(T), & \bar{u} < 1 
\end{cases} \]

along the solution for the bang-bang control `0-\bar{u}-0' with commutations at \( t_c \) and \( t_c + \Delta \). Then from (15) and (16),

\[ F(S(T)) = V_0 - \Gamma_T(t_c). \]  

Let us first show that for \( T \) large enough, one has necessarily \( S(T) < S_h \) for any admissible control. By contradiction, if one had \( S(t) \geq S_h \) for any \( t > 0 \), then one would have

\[ \frac{d}{dt} \log I = \beta S(1 - u(t)) - \gamma \geq -\gamma u(t), \]

so that \( I(t) \geq I(0) e^{-\gamma K} > 0, t > 0 \). But then,

\[ \dot{S} + \dot{I} = -\gamma I \Rightarrow S(t) + I(t) = S(0) + I(0) - \gamma \int_0^t I(\tau) d\tau, \]

and \( S + I \) would be negative for large \( t \), a contradiction.

Note that the function \( F \) defined in (10) is decreasing for \( S < S_h \). We deduce from (17) that for \( T \) large enough, the optimal commutation time \( t^*_c \), which gives the maximal value \( S(T) \), has to maximize the function \( \Gamma_T \).

The functions \( \Gamma_T, T > 0 \), are continuous, and for each given \( t_c \), the map \( T \mapsto I(T) \) is decreasing for \( T > t_c + \Delta \). \( (\Gamma_T(\cdot))_{T>0} \) is thus a decreasing family of continuous functions, converging uniformly to the continuous function

\[ \Gamma_\infty(t_c) = \begin{cases} 
(1 - e^{-\gamma K}) I(t_c), & \bar{u} = 1 \\
\frac{-\bar{u}}{1 - \bar{u}} \left( \log(S(t_c)) - \log(S(t_c + \Delta)) \right), & \bar{u} < 1 
\end{cases} \]  

Therefore, any sequence \( t^*_c(T_n) \) of maximizers of \( \Gamma_{T_n} \) converges to a maximizer of \( \Gamma_\infty \) when \( T_n \) tends to \( +\infty \). The function \( \Gamma_\infty \) is maximized for \( t^*_c(\infty) < +\infty \) such that

1. for \( \bar{u} = 1 \), \( t^*_c(\infty) \) is uniquely defined as \( S(t^*_c(\infty)) = S_h \) if \( S(0) \geq S_h \) and \( t^*_c(\infty) = 0 \) if the herd immunity threshold has already been reached at initial time;

2. for \( \bar{u} < 1 \), \( t^*_c(\infty) \) maximizes \( t_c \mapsto \log(S(t_c)) - \log(S(t_c + \Delta)) \). As this last quantity tends to 0 when \( t_c \) tends to \( +\infty \), we conclude that the maximum is reached for \( t^*_c(\infty) < +\infty \). Uniqueness of \( t^*_c(\infty) \) is not proved here due to lack of space. It will be provided in a more complete forthcoming version of the paper.

Notice that the previous considerations show that, when complete stop of the transmission is possible (\( \bar{u} = 1 \)), the optimal policy for \( \mathcal{R}_S \) over infinite horizon consists in waiting for the proportion \( S \) to reach the herd immunity value \( S_h \) (whatever the initial condition), and then to block the transmission as long as the \( L^1 \) budget is not completely used, that is during a period \( \Delta = K/\bar{u} = K \). On the contrary, when only partial reduction of the transmission rate is possible (i.e., \( \bar{u} < 1 \)), the optimal commutation time anticipates the time when \( S(\bar{u}) \) reaches \( S_h \) (depending on the initial condition). This general behavior has been described in Bliman and Duprez (2020). It is illustrated in Fig. 1, where is shown the determination of optimal commutation time in two different cases, and in Fig. 2 where the corresponding optimal trajectories are depicted.

![Fig. 1. Determination of the optimal commutation times as maxima (shown in dashed lines) of \( \Gamma_\infty \) defined in (18), for \( \beta = 0.8, \gamma = 0.2, K = 10, I(0) = 1e-3, S(0) = 0.999 \) and \( \bar{u} = 0.9 \) (top), \( \bar{u} = 0.4 \) (bottom) ]
6. CONCLUSION

In this work, we have revisited the problem of minimizing the epidemic final size in the SIR model through a term of social distancing mitigating the infection term in the evolution equations. It had been shown previously that, when the optimal solutions are searched among all functions whose support lies in an interval of length $\Delta > 0$, then the optimal solution is unique and consists in applying the strongest lockdown during a duration $\Delta$, with a starting time properly chosen. Using distinct techniques, this result has been completed here, by establishing that the same solution is indeed also optimal among the largest class of functions with (possibly non-connected) support of measure at most equal to $\Delta$. A sequence of finite horizon problems has first been considered, for which is provided the explicit solution, the $L^1$ constraint on the control being incorporated as an additional state variable. Then, the unique optimal solution of the original problem has been characterized passing the time horizon to the limit. This proves that there is indeed no benefit to expect in splitting the intervention on several disjoint time periods. We believe that this technique could be also successfully applied to other optimal control problems with $L^1$ constraint and infinite horizon. Future research will consider extensions of this result, in particular to the important case where constraints on the prevalence is added.

REFERENCES


