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# On the problem of minimizing the epidemic final size for SIR model by social distancing

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**Abstract:** We revisit the problem of minimizing the epidemic final size in the SIR model through social distancing. In the existing literature, this problem has been considered by imposing a priori interval structure on the time period during which interventions are enforced. We show that when considering the more general class of controls with an  $L^1$  constraint on the confinement effort to reduce the infection rate, the support of the optimal control is still a single time interval. In other words, for the problem of minimizing the epidemic final size in the SIR model through social distancing, there is no benefit in splitting interventions on several disjoint time periods. The techniques we deploy here are different than what has been proposed in the literature, and could be potentially applied to other problems.

*Keywords:* Epidemiological models, optimal control,  $L^1$  controls, infinite horizon.

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## 1. INTRODUCTION

The pandemic of Covid-19 represented in many aspects an unprecedented experiment for the societies all over the world. One aspect of this has been the use of social distancing measures implemented in various forms and periods. This aspect has revived interest in optimal control methods for the control of transmissible diseases, a subject already well-researched in the past, see e.g., Lenhart and Workman (2007); Sharomi and Malik (2017) and bibliographical references in Bliman et al. (2021).

In the same vein as several works cited and analyzed below, we focus here on the optimal control of a quite simple epidemiological model, and attempt to fully characterize the optimal policy. We consider more precisely the specific issue of optimal control of the classical SIR model through non-pharmaceutical interventions that reduce the transmission rate  $\beta$  by a proportion  $u(t)$ , which plays the role of a control variable, usually during a limited period. The corresponding controlled system is

$$\begin{cases} \dot{S} = -(1 - u(t))\beta SI \\ \dot{I} = (1 - u(t))\beta SI - \gamma I \end{cases} \quad u(t) \in U := [0, \bar{u}], \quad (1)$$

where  $S$  and  $I$  stand for the proportions of the susceptible and infected individuals in the population, respectively, and  $\bar{u}$  is a fixed number in  $(0, 1]$ . The solution  $(S, I)$  depends upon the input  $u$  and the initial values (such that  $S(0), I(0) \geq 0$ ,  $S(0) + I(0) \leq 1$ ), but for sake of simplicity, this dependence is not explicitly indicated in the sequel.

Tables 1 to 4 show the diversity of the approaches considered in the studies of optimal control for problem (1), and the obtained results. These studies differ through the choice of the cost function, of the state constraint, and of the class of admissible input functions. The cost functions

that are commonly considered are the *peak prevalence* (i.e., the maximal value of  $I(\cdot)$ , abbreviated PP below), the *epidemic final size* (i.e., the total number of individuals  $S(0) - S(+\infty)$  infected during the outbreak, abbreviated FS below), the *control effort* and the *time to reach the sub-epidemic ‘safe’ region*, or some combination of them. The constraints are usually on the peak prevalence or on the cumulative control effort. Last, the class of control inputs among which optimization is achieved may be constant or piecewise constant during the lockdown phases, possibly with values chosen in a prescribed finite set, or taken in full generality in the set  $\mathcal{U}$  of measurable functions  $u(\cdot)$  defined on  $[0, +\infty)$  taking values in  $U$  defined in (1) (in practice one may look for piece-wise continuous controls). Their support, denoted  $\text{supp}(u)$ , may be imposed as a single interval or a union of intervals of prescribed length, may be simply included in a fixed interval ( $\text{supp}(u) \subset [0, \Delta]$ , for a given  $\Delta > 0$ ), or initially unspecified.

We are concerned in the present paper by the issue of reduction of the final size (see Table 1). Bliman et al. (2021) and Ketcheson (2021) have shown that the problem obtained when imposing  $\text{supp}(u)$  to be included in an interval  $[0, T]$  has a bang-bang solution consisting in at most three phases. In a first phase (possibly of zero duration) no action is taken until a certain optimal initiation time, which in any case is located before reaching the prevalence peak. Then the control is activated at its strongest value  $\bar{u}$  until reaching the time horizon  $\Delta$ . Last,  $u \equiv 0$  after the end of the permitted confinement interval. The fact that the optimal intervention may have to start after the beginning of the latter, that is at a positive time, suggests that a more efficient use of prescribed confinement duration may exist. This is the subject of Bliman and Duprez (2020), in which intervals of the type  $[t, t + \Delta]$  were considered,

where  $t \geq 0$  is here a decision variable. This corresponds to the following problem

$$\mathcal{P}_S : \sup_{t \geq 0} \sup_{u(\cdot) \in \mathcal{U}} S(+\infty) \text{ subject to } \text{supp}(u) \subset [t, t + \Delta].$$

Not quite surprisingly, it is demonstrated therein that there exists an optimal initiation time  $t^*$ , and that the optimal cost is obtained by taking  $u \equiv \bar{u}$  on the whole interval  $[t^*, t^* + \Delta]$ . Therefore, if confinement is allowed only for a given amount of time  $\Delta$ , in order to reduce the epidemic final size, it may be more rational to wait for some time, and then exhaust completely the lockdown ability.

However, this does not answer the following more involved issue: would not it be even more efficient to split the confinement period and consider more general control phases, keeping a total treatment time at most equal to  $\Delta$ ? Writing  $\text{meas}$  the set measure, this yields the problem

$$\mathcal{Q}_S : \sup_{u(\cdot) \in \mathcal{U}} S(+\infty) \text{ subject to } \text{meas } \text{supp}(u) \leq \Delta.$$

Our goal in this paper is to study Problem  $\mathcal{Q}_S$ . More precisely, we will establish that *Problem  $\mathcal{Q}_S$  has the same optimal solution as Problem  $\mathcal{P}_S$* . In other words, there is no gain in attempting to split the control period: the solution of Problem  $\mathcal{Q}_S$  is also obtained uniquely, by taking  $u \equiv \bar{u}$  on the interval  $[t^*, t^* + \Delta]$ , for some optimal time  $t^*$  that may be computed. Notice that this result is by no means a straightforward consequence of the previous ones, and necessitates a quite different approach.

The paper is organized as follows. An auxiliary optimal control problem on finite time horizon  $T$  (problem  $\mathcal{R}_S$ ) is formulated in Section 2. Compared to  $\mathcal{Q}_S$ , it consists of two changes: first, the cost  $S(+\infty)$  is replaced by  $S(T)$  for  $T \gg \Delta$ ; second, the constraint  $\text{meas } \text{supp}(u) \leq \Delta$  is replaced by the alternative integral condition  $\|u(\cdot)\|_1 \leq \Delta \bar{u}$ . This problem is studied in Section 3, where the form of the optimal solution is provided (Proposition 4). Problem  $\mathcal{R}_S$  on an infinite time horizon (i.e., for  $T \rightarrow +\infty$ ) is then studied in Section 4. This finally allows in Section 5 to establish the equivalence between Problems  $\mathcal{P}_S$  and  $\mathcal{Q}_S$  (Theorem 5). Last, Section 6 provides concluding remarks.

## 2. FORMULATION OF THE PROBLEM

The rationale adopted in this paper in order to prove the main claim is as follows. We first introduce an auxiliary problem on a finite horizon (problem  $\mathcal{R}_S$  below), in which a restriction is introduced on the maximal value of  $\|u\|_1$ , instead of a constraint on  $\text{supp}(u)$ . The idea of the argument then consists in showing, for large enough time horizons  $T$ , on the one hand that the optimal solution of this problem has a structure independent of the time horizon, and on the other hand that the optimal control has a connected support.

Let us first introduce the auxiliary problem. Posit

$$K := \Delta \bar{u}.$$

Given a positive initial condition  $(S(0), I(0)) = (S_0, I_0)$  and a time horizon  $T > 0$ , we consider the following optimal control problem:

$$\mathcal{R}_S : \sup_{u(\cdot) \in \mathcal{U}} S(T) \text{ subject to } \|u(\cdot)\|_1 \leq K.$$

The number  $K$  represents the maximal amount of effort on reducing the incidence that is allowed without fixing any a priori time window of action on the interval  $[0, T]$ . What is expected is that, for large values of  $T$  (larger than  $\Delta$ ), the constraint on  $\|u\|_1$  will be active, while the optimal solution will conserve the same structure, whatever the value of  $T$ . Notice also that the cost function for  $\mathcal{R}_S$  is  $S(T)$ , different from  $S(+\infty)$  for problems  $\mathcal{P}_S$  and  $\mathcal{Q}_S$ . However, a common behavior will be recovered for  $T \rightarrow +\infty$ .

Now, the  $L^1$  constraint on the control variable,

$$\|u(\cdot)\|_1 \leq K, \quad (2)$$

can be easily tackled by augmenting the state of the system

$$\begin{cases} \dot{S} = -(1-u)\beta SI \\ \dot{I} = (1-u)\beta SI - \gamma I \\ \dot{C} = u \end{cases}$$

with  $C(0) = 0$ , so that the integral constraint (2) is transformed into the terminal constraint  $C(T) \leq K$ . Then, the optimal control Problem  $\mathcal{R}_S$  is formulated as a Mayer problem with a target. The (extended) dynamics being linear with respect to the control, existence of optimal controls follows from the classical Filippov Theorem, see for instance (Lee and Markus, 1967, Theorem 4, p. 259).

## 3. STUDY OF PROBLEM $\mathcal{R}_S$

This section is devoted to the study of Problem  $\mathcal{R}_S$ , and culminates in the statement of Proposition 4. We begin by writing the necessary optimality conditions of the Pontryagin Maximum Principle for the latter.

The Hamiltonian writes

$$H = (p_I - p_S)(1-u)\beta SI - p_I \gamma I + p_C u$$

and the adjoint equations

$$\begin{aligned} \dot{p}_S &= -\partial_S H = (p_S - p_I)(1-u)\beta I, \\ \dot{p}_I &= -\partial_I H = (p_S - p_I)(1-u)\beta S + p_I \gamma, \\ \dot{p}_C &= -\partial_C H = 0, \end{aligned}$$

along with the transversality conditions

$$\begin{aligned} p_S(T) &= 1, \\ p_I(T) &= 0, \\ p_C(T) &\leq 0 \text{ (} < 0 \text{ if the constraint is saturated)}. \end{aligned}$$

An optimal control  $u(\cdot)$  is a maximizer of the Hamiltonian for almost any  $t$ , which allows to claim that one has for any optimum control  $u$

$$\begin{aligned} \phi(t) > 0 &\Rightarrow u(t) = \bar{u} \text{ a.e. } t, \\ \phi(t) < 0 &\Rightarrow u(t) = 0 \text{ a.e. } t, \end{aligned} \quad (3)$$

and the (switching function)  $\phi$  defined by

$$\phi := (p_S - p_I)\beta SI + p_C \quad (4)$$

has the property that  $H = u\phi + (p_I - p_S)\beta SI - p_I \gamma I$ . Moreover, for any optimal solution, there exists a number  $\bar{H}$  such that  $H(t) = \bar{H}$  a.e.  $t \in [0, T]$ .

A straightforward computation gives

Table 1. Contributions with final size as cost function. Ketcheson (2021) uses a more general, FS + state/control term, cost.

	Constraint	Input set
Bliman et al. (2021)	—	$\text{supp}(u) \subset [0, \Delta]$
Bliman and Duprez (2020)	—	$\text{supp}(u) \subset [t, t + \Delta], t \geq 0$
Di Lauro et al. (2021)	—	$\text{supp}(u) = [0, \Delta]$
González et al. (2021)	—	$\text{supp}(u) = [0, \Delta], u _{\text{supp}(u)}$ constant
Ketcheson (2021)	—	$\text{supp}(u) \subset [0, \Delta]$
Sereno et al. (2022)	PP	$\text{meas supp}(u) = \Delta, I \equiv \max I(t)$ on $\text{supp}(u)$

Table 2. Contributions with peak prevalence as cost function.

	Constraint	Input set
Di Lauro et al. (2021)	—	$\text{supp}(u) = [0, \Delta]$
Greene and Sontag (2022)	—	$\text{meas supp}(u) = \Delta, u _{\text{supp}(u)}$ constant
Morris et al. (2021)	—	$\text{supp}(u) \subset I, \text{meas } I = \Delta$
Sontag (2021)	—	$\text{supp}(u) = \cup_{k=1}^K I_k, \text{meas } I_k = \Delta_k, u _{\text{supp}(u)}$ constant
Molina and Rapaport (2022)	$\ u\ _1$	—

Table 3. Contributions taking the  $L^1$ -norm of  $u$  as cost function.

	Cost	Constraint	Input set
Avram et al. (2022)	$\ u\ _{L^1}$	PP	$\text{supp}(u) \subset [0, \Delta]$
Miclo et al. (2022)	$\ u\ _{L^1}$	PP	$\text{supp}(u) \subset [0, \Delta]$

Table 4. Contribution taking as cost function the time to the ‘safe zone’, i.e., the zone in which absence of control cannot yield violating of the constraint.

	Constraint	Input set
Angulo et al. (2021)	PP	—

$$\begin{aligned}
\dot{\phi} &= (\dot{p}_S - \dot{p}_I)\beta SI + (p_S - p_I)\beta(\dot{S}I + S\dot{I}) + \dot{p}_C \\
&= ((p_S - p_I)(1 - u)\beta(I - S) - p_I\gamma)\beta SI \\
&\quad + (p_S - p_I)\beta(-(1 - u)\beta SI^2 + (1 - u)\beta S^2 I - \gamma SI) \\
&= (p_S - p_I)(1 - u)(I - S - I + S)\beta^2 SI \\
&\quad + \beta\gamma SI(-p_I - p_S + p_I),
\end{aligned}$$

and thus

$$\dot{\phi} = -\beta\gamma SI p_S. \quad (5)$$

Also, at any point of continuity of  $u$ ,

$$\begin{aligned}
-\frac{1}{\beta\gamma}\ddot{\phi} &= (\dot{S}I + S\dot{I})p_S + SI\dot{p}_S \\
&= (-(1 - u)\beta SI^2 + (1 - u)\beta S^2 I - \gamma SI)p_S \\
&\quad + SI(p_S - p_I)(1 - u)\beta I \\
&= (1 - u)\beta SI(-I + S)p_S + \frac{1}{\beta}\dot{\phi} + (1 - u)I(\phi - p_C) \\
&= -\frac{1}{\gamma}(1 - u)(S - I)\dot{\phi} + \frac{1}{\beta}\dot{\phi} + (1 - u)I(\phi - p_C),
\end{aligned}$$

so that, using (5),

$$\ddot{\phi} = -(\phi - p_C)(1 - u)\beta\gamma I + \dot{\phi}(\beta(1 - u)(S - I) - \gamma). \quad (6)$$

*Lemma 1.* For  $K < T\bar{u}$ , one has  $p_C < 0$ .  $\square$

**Proof.** Assume by contradiction that  $p_C = 0$ . One has

$$\phi(T) = (p_S(T) - p_I(T))\beta S(T)I(T) = \beta S(T)I(T) > 0.$$

By continuity,  $\phi(t)$  is positive for  $t \in [T - \eta, T]$  for some  $\eta > 0$ , and then any optimal control verifies  $u(t) = \bar{u}$  for a.e.  $t \in [T - \eta, T]$  for some  $\eta > 0$ .

If  $\bar{u} = 1$ , one obtains from the adjoint equations  $\dot{p}_S = \dot{p}_I = 0$ , so that  $p_S = 1$  and  $p_I = 0$  on this time interval, and this constant solution is propagated on the whole interval  $[0, T]$ . Any optimal control has then to satisfy  $u(t) = \bar{u}$  for a.e.  $t \in [0, T]$ .

If  $\bar{u} < 1$ , as  $p_I$  is continuous with  $p_I(T) = 0$ , one gets from the conservation of the Hamiltonian along an optimal solution the identity

$$H = -(1 - u(t))\phi(t) - p_I(t)\gamma I(t) = H(T) \quad \text{a.e. } t \in [0, T].$$

On the other hand, using the assumption that  $p_C = 0$ ,

$$\begin{aligned}
H(T) &= -p_S(T)(1 - \bar{u})\beta S(T)I(T) + p_C\bar{u} \\
&= -(1 - \bar{u})\beta S(T)I(T) < 0,
\end{aligned}$$

and thus:

$$H = -(1 - u(t))\phi(t) - p_I(t)\gamma I(t) < 0 \quad \text{a.e. } t \in [0, T]. \quad (7)$$

Consider now the set

$$E := \{t \in [0, T]; \phi(t) \leq 0\}.$$

If  $E$  is non empty, let  $t_c = \sup E < T$ . By continuity of  $\phi$ , one has  $\phi(t_c) = 0$ , and thus  $p_I(t_c) = p_S(t_c) > 0$  using inequality (7). In view of (5), this implies  $\dot{\phi}(t_c) < 0$  which contradicts the definition of  $t_c$ . We conclude that  $E$  has to be empty and that any optimal control has also to satisfy  $u(t) = \bar{u}$  for a.e.  $t \in [0, T]$ .

We have shown that, in the case  $\bar{u} = 1$  as well as in the case  $\bar{u} < 1$ ,  $p_C = 0$  implies that  $u(t) = \bar{u}$  almost everywhere on  $[0, T]$ . One then obtains

$$\int_0^T u(t) dt = T\bar{u} > K,$$

which violates the constraint (2). Therefore, one has  $p_C < 0$ , which demonstrates Lemma 1.  $\blacksquare$

As the objective is to consider the criterion for large  $T$ , we assume in the sequel that the condition  $T\bar{u} > K$  is verified, so that an optimal control saturates constraint (2).

*Lemma 2.* For  $\bar{u} < 1$ , an optimal solution has no singular arc. For  $\bar{u} = 1$ , a singular arc is given by  $u = \bar{u} = 1$ .  $\square$

**Proof.** If  $\phi$  is null on a non-empty open time interval  $J$ , one should have  $\dot{\phi}(t) = 0$  for  $t \in J$  and  $\ddot{\phi}(t) = 0$  for almost every (a.e.)  $t \in J$ . From expression (6), one obtains

$$\ddot{\phi}(t) = p_C(1 - u(t))\beta\gamma I(t) = 0 \quad \text{a.e. } t \in J.$$

If  $\bar{u} < 1$ , one obtains a contradiction, due to the fact that  $p_C < 0$ , see Lemma 1: there is no singular arc. If  $\bar{u} = 1$ ,  $u = 1$  is optimal on  $J$ .  $\blacksquare$

*Lemma 3.* For any optimal solution, the set  $F := \{t \in [0, T]; \phi(t) \geq 0\}$  is connected.  $\square$

**Proof.** By contradiction, assume that there exists  $t_1 < t_2$  in  $(0, T)$  such that  $\phi(t_1) = \phi(t_2) = 0$  with  $\phi(t) < 0$  for  $t \in J := (t_1, t_2)$ . Then the optimal solution is given by  $u = 0$  on  $J$ , see (3). The switching function  $\phi$  being  $C^1$ , there exists  $\tau_1 < \tau_2$  in  $J$  as close as desired respectively to  $t_1$  and  $t_2$ , and such that  $\dot{\phi}(\tau_1) < 0$  and  $\dot{\phi}(\tau_2) > 0$ , which, due to (5), amounts to have  $p_S(\tau_1) > 0$  and  $p_S(\tau_2) < 0$ .

For the constant control  $u = 0$ , we get from expression (4) the property

$$\dot{\phi}(t) = \dot{p}_S(t)S(t) + p_C, \quad t \in J. \quad (8)$$

As  $p_C < 0$  (by Lemma 1) and  $\phi(t_2) = 0$ , one has  $\dot{p}_S(\tau_2) > 0$  for  $\tau_2$  chosen close enough to  $t_2$ . We deduce, from this and the fact that  $p_S(\tau_1) > 0 > p_S(\tau_2)$ , that  $p_S$  has a (negative) minimum in  $J$ , reached at a certain  $\underline{t} \in (\tau_1, \tau_2)$ .

At  $\underline{t}$ , one has  $\dot{p}_S(\underline{t}) = 0$  and thus  $p_S(\underline{t}) = p_I(\underline{t})$  from the adjoint equations. From the conservation of the Hamiltonian, one gets

$$\begin{aligned} H &= -p_I(\underline{t})\gamma I(\underline{t}) + p_C u(\underline{t}) \\ &= -p_I(\underline{t})\gamma I(\underline{t}) = -(1 - u(T))\beta S(T)I(T) + p_C u(T), \end{aligned}$$

so that  $H < 0$ , which implies  $p_I(\underline{t}) = p_S(\underline{t}) > 0$ . From (8) and (5), one now obtains

$$0 > -\beta\gamma S(t)I(t)p_S(t) = \dot{\phi}(t) = \ddot{p}_S(t)S(t) + \dot{p}_S(t)\dot{S}(t),$$

for  $t \in J$ . Because  $\dot{p}_S(\underline{t}) = 0$ , we deduce that  $\ddot{p}_S(\underline{t}) < 0$ , which contradicts that  $\underline{t}$  is a local minimizer of  $p_S$ .

In conclusion, there exists no  $t_1 < t_2$  in  $(0, T)$  such that  $\phi(t_1) = \phi(t_2) = 0$  with  $\phi(t) < 0$  for  $t \in J := (t_1, t_2)$ . In short, the set  $F$  defined in the statement is connected.  $\blacksquare$

We then obtain the following characterization of the optimal solutions for the problem  $\mathcal{R}_S$ .

*Proposition 4.* For any positive initial condition and time horizon  $T > \Delta$ , any optimal control  $u(\cdot)$  for problem  $\mathcal{R}_S$  is such that

$$u(t) = \begin{cases} \bar{u} & \text{if } t \in [t_c^*, t_c^* + \Delta], \\ 0 & \text{otherwise} \end{cases} \quad \text{a.e. } t \in [0, T] \quad (9)$$

for some  $t_c^* \in [0, T - \Delta]$ .  $\square$

**Proof.** From Lemma 2, any optimal control takes only the values 0 and  $\bar{u}$ . From Lemma 3, it has at most one arc with  $u = \bar{u}$ ; and with Lemma 1, that the constraint is saturated. Therefore, any optimal control has the structure (9).  $\blacksquare$

Uniqueness of the optimal switching time  $t_c^*$  has not been demonstrated. It will be proved in the next section that any sequence of maximizers  $t_c^*(T)$  converges to a unique switching time when  $T$  tends to  $+\infty$ .

#### 4. PROBLEM $\mathcal{R}_S$ OVER AN INFINITE HORIZON

We investigate now the optimal solution of the limiting problem  $\mathcal{R}_S$  when  $T \rightarrow +\infty$ .

For any  $T < +\infty$ , the structure of the optimal control for problem  $\mathcal{R}_S$  being known ('0- $\bar{u}$ -0'), one may characterize the corresponding optimal value of  $S(T)$  as follows. Define

$$S_h := \frac{\gamma}{\beta}, \quad F(S) := S - S_h \log(S). \quad (10)$$

One may recognize in  $S_h$  the *herd immunity threshold*, that is the proportion of susceptible below which the number of infected starts decreasing.

On the two time intervals  $[0, t_c^*)$  and  $(t_c^* + \Delta, T]$ , the optimal control  $u$  is identically equal to 0 and we can write the invariant properties

$$F(S(t_c^*)) + I(t_c^*) = F(S(0)) + I(0) =: V_0, \quad (11)$$

$$F(S(T)) + I(T) = F(S(t_c^* + \Delta)) + I(t_c^* + \Delta). \quad (12)$$

On the time interval  $(t_c^*, t_c^* + \Delta)$ , the optimal control  $u$  is constant equal to  $\bar{u}$ . We distinguish two cases.

(1)  $\bar{u} = 1$ . Then one has

$$S(t_c^* + K) = S(t_c^*), \quad I(t_c^* + K) = e^{-\gamma K} I(t_c^*). \quad (13)$$

(2)  $\bar{u} < 1$ . We can write the invariance property

$$\begin{aligned} S(t_c^* + \Delta) + I(t_c^* + \Delta) - \frac{S_h}{1 - \bar{u}} \log(S(t_c^* + \Delta)) \\ = S(t_c^*) + I(t_c^*) - \frac{S_h}{1 - \bar{u}} \log(S(t_c^*)). \end{aligned} \quad (14)$$

Let us combine expressions (11), (13) or (14), and (12). We obtain (with  $V_0$  defined in (11)):

(1) for  $\bar{u} = 1$ ,

$$F(S(T)) = V_0 - (1 - e^{-\gamma K})I(t_c^*) - I(T), \quad (15)$$

(2) for  $\bar{u} < 1$ ,

$$\begin{aligned} F(S(T)) = V_0 - \\ \frac{\bar{u}}{1 - \bar{u}} S_h (\log(S(t_c^*)) - \log(S(t_c^* + \Delta))) - I(T). \end{aligned} \quad (16)$$

Then, we posit

$$\Gamma_T(t_c) := \begin{cases} (1 - e^{-\gamma K})I(t_c) + I(T), & \bar{u} = 1 \\ \frac{\bar{u}}{1 - \bar{u}} (\log(S(t_c) - \log(S(t_c + \Delta))) + I(T)), & \bar{u} < 1 \end{cases}$$

along the solution for the bang-bang control ‘0- $\bar{u}$ -0’ with commutations at  $t_c$  and  $t_c + \Delta$ . Then from (15) and (16),

$$F(S(T)) = V_0 - \Gamma_T(t_c^*). \quad (17)$$

Let us first show that for  $T$  large enough, one has necessarily  $S(T) < S_h$  for any admissible control. By contradiction, if one had  $S(t) \geq S_h$  for any  $t > 0$ , then one would have

$$\frac{d}{dt} \log I = \beta S(1 - u(t)) - \gamma \geq -\gamma u(t),$$

so that  $I(t) \geq I(0)e^{-\gamma K} > 0$ ,  $t > 0$ . But then,

$$\dot{S} + \dot{I} = -\gamma I \Rightarrow S(t) + I(t) = S(0) + I(0) - \gamma \int_0^t I(\tau) d\tau,$$

and  $S + I$  would be negative for large  $t$ , a contradiction.

Note that the function  $F$  defined in (10) is decreasing for  $S < S_h$ . We deduce from (17) that for  $T$  large enough, the optimal commutation time  $t_c^*$ , which gives the maximal value  $S(T)$ , has to maximize the function  $\Gamma_T$ .

The functions  $\Gamma_T$ ,  $T > 0$ , are continuous, and for each given  $t_c$ , the map  $T \mapsto I(T)$  is decreasing for  $T > t_c + \Delta$ .  $\{\Gamma_T(\cdot)\}_{T>0}$  is thus a decreasing family of continuous functions, converging uniformly to the continuous function

$$\Gamma_\infty(t_c) = \begin{cases} (1 - e^{-\gamma K})I(t_c), & \bar{u} = 1 \\ \frac{\bar{u}}{1 - \bar{u}} (\log(S(t_c) - \log(S(t_c + \Delta))), & \bar{u} < 1 \end{cases} \quad (18)$$

Therefore, any sequence  $t_c^*(T_n)$  of maximizers of  $\Gamma_{T_n}$  converges to a maximizer of  $\Gamma_\infty$  when  $T_n$  tends to  $+\infty$ . The function  $\Gamma_\infty$  is maximized for  $t_c^*(\infty) < +\infty$  such that

- (1) for  $\bar{u} = 1$ ,  $t_c^*(\infty)$  is uniquely defined as  $S(t_c^*(\infty)) = S_h$  if  $S(0) \geq S_h$ , and  $t_c^*(\infty) = 0$  if the herd immunity threshold has already been reached at initial time;
- (2) for  $\bar{u} < 1$ ,  $t_c^*(\infty)$  maximizes  $t_c \mapsto \log(S(t_c)) - \log(S(t_c + \Delta))$ . As this last quantity tends to 0 when  $t_c$  tends to  $+\infty$ , we conclude that the maximum is reached for  $t_c^*(\infty) < +\infty$ . Uniqueness of  $t_c(\infty)$  is not proved here due to lack of space. It will be provided in a more complete forthcoming version of the paper.

Notice that the previous considerations show that, when complete stop of the transmission is possible ( $\bar{u} = 1$ ), the optimal policy for  $\mathcal{R}_S$  over infinite horizon consists in waiting for the proportion  $S$  to reach the herd immunity value  $S_h$  (whatever the initial condition), and then to block the transmission as long as the  $L^1$  budget is not completely used, that is during a period  $\Delta = K/\bar{u} = K$ . On the contrary, when only partial reduction of the transmission rate is possible (i.e.,  $\bar{u} < 1$ ), the optimal commutation time anticipates the time when  $S$  reaches  $S_h$  (depending on the initial condition). This general behavior has been described in Bliman and Duprez (2020). It is illustrated in Fig. 1, where is shown the determination of optimal commutation time in two different cases, and in Fig. 2 where the corresponding optimal trajectories are depicted.

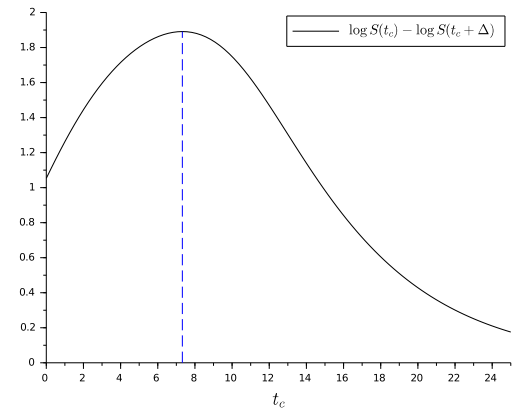
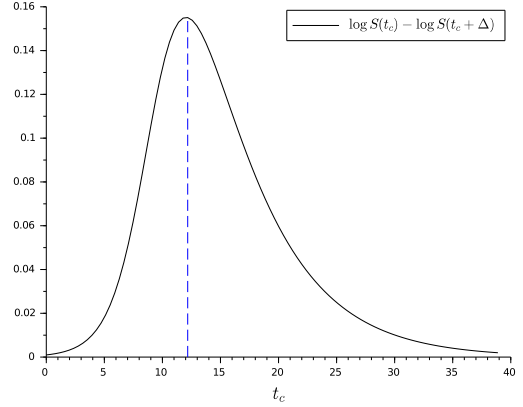


Fig. 1. Determination of the optimal commutation times as maxima (shown in dashed lines) of  $\Gamma_\infty$  defined in (18), for  $\beta = 0.8$ ,  $\gamma = 0.2$ ,  $K = 10$ ,  $I(0) = 1e - 3$ ,  $S(0) = 0.999$  and  $\bar{u} = 0.9$  (top),  $\bar{u} = 0.4$  (bottom)

## 5. EQUIVALENCE OF PROBLEMS $\mathcal{P}_S$ AND $\mathcal{Q}_S$

We fix a positive initial condition  $(S(0), I(0)) = (S_0, I_0)$  and a number  $\Delta > 0$ . By construction, the optimal value of Problem  $\mathcal{P}_S$  is not larger than the optimal cost of Problem  $\mathcal{Q}_S$ . The main result of this note is now presented. It shows that they are indeed equal, thanks to the knowledge of the optimal solution of problem  $\mathcal{R}_S$ .

**Theorem 5.** The Problems  $\mathcal{P}_S$  and  $\mathcal{Q}_S$  have the same optimal cost, reached by a unique optimal control  $u^*$  such that  $\text{meas supp}(u^*) = \Delta$  and  $u^* = \bar{u}$  on  $\text{supp}(u^*)$ . Moreover,  $\text{supp}(u^*)$  is an interval and  $u^*$  is the optimal control of Problem  $\mathcal{R}_S$  for  $T = +\infty$ .  $\square$

**Proof.** If  $\text{supp}(u) \in [t, t + \Delta]$ , then necessarily  $\|u(\cdot)\|_1 \leq \bar{u}T := K$ . But we proved that the optimal solution of problem  $\mathcal{R}_S$  over infinite horizon is reached for a control  $u(\cdot)$  whose support is exactly  $[t_c^*, t_c^* + \Delta]$  for some  $t_c^*$ . Therefore, the optimal value of problem  $\mathcal{P}_S$  is reached by taking  $\sup_t$  for a control  $u = \bar{u}$  on the interval  $[t, t + \Delta]$ .

If  $\text{meas supp}(u) \leq \Delta$ , then one has  $\|u(\cdot)\|_1 \leq \Delta \bar{u} = K$ . But we have shown that the solution of problem  $\mathcal{R}_S$  over infinite horizon is reached for a control  $u(\cdot)$  such that  $\text{meas supp}(u) \leq \Delta$ , indeed equal to  $\Delta$ . We conclude that Problems  $\mathcal{P}_S$  and  $\mathcal{Q}_S$  have same optimal cost.  $\blacksquare$

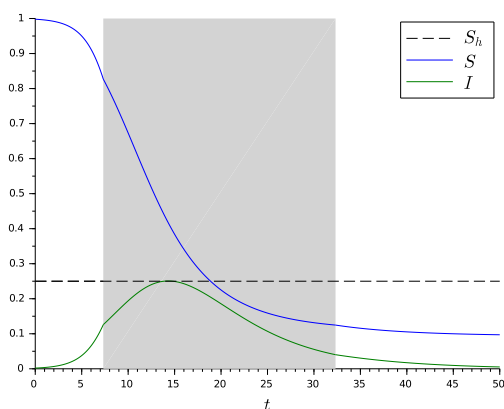
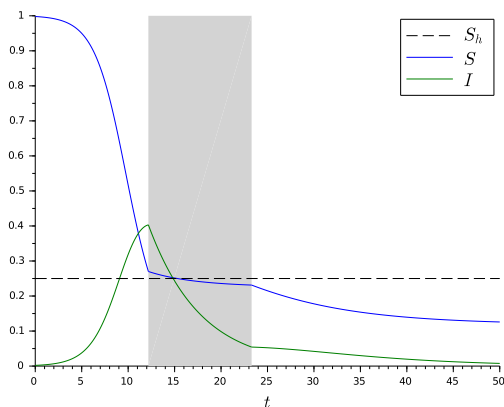


Fig. 2. Optimal solutions corresponding to the optimal commutation times in Fig. 1. On top,  $\bar{u} = 0.9$ , so that  $\Delta = 11.1$ ; on the bottom,  $\bar{u} = 0.4$  and  $\Delta = 25.0$ . Intervals with active control ( $u \equiv \bar{u}$ ) are in gray.

## 6. CONCLUSION

In this work, we have revisited the problem of minimizing the epidemic final size in the SIR model through a term of social distancing mitigating the infection term in the evolution equations. It had been shown previously that, when the optimal solutions are searched among all functions whose support lies in an interval of length  $\Delta > 0$ , then the optimal solution is unique and consists in applying the strongest lockdown during a duration  $\Delta$ , with a starting time properly chosen. Using distinct techniques, this result has been completed here, by establishing that the same solution is indeed also optimal among the largest class of functions with (possibly non-connected) support of measure at most equal to  $\Delta$ . A sequence of finite horizon problems has first been considered, for which is provided the explicit solution, the  $L^1$  constraint on the control being incorporated as an additional state variable. Then, the unique optimal solution of the original problem has been characterized passing the time horizon to the limit. This proves that there is indeed no benefit to expect in splitting the intervention on several disjoint time periods. We believe that this technique could be also successfully applied to other optimal control problems with  $L^1$  constraint and infinite horizon. Future research will consider extensions

of this result, in particular to the important case where constraints on the prevalence is added.

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