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# Isolated singularities, inverse systems and the punctual Hilbert scheme

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*To the memory of Agnes Szanto*

ABSTRACT. In this paper, we present an effective study of the punctual Hilbert scheme. First, we recall algorithms to compute the inverse system of an isolated singular point. These inverse systems are points of the punctual Hilbert scheme, that we define as a subvariety of a Grassmannian variety, providing explicit defining equations. We localise our study to the algebraic variety  $\text{Hilb}_B$  of inverse systems, which admits a given dual (monomial) basis  $B$ . Building on the algorithm to compute inverse systems and exploiting combinatorial properties of the monomial basis  $B$ , we derive a minimal set of equations defining  $\text{Hilb}_B$ . Using this effective presentation, we prove new properties of transversality of  $\text{Hilb}_B$ , give new dimension formula for its irreducible components and prove that they are birational.

## 1. Introduction

Solving a system of equations  $f_1 = 0, \dots, f_s = 0$  with multivariate polynomials  $f_i \in \mathbb{K}[x_1, \dots, x_n]$  is a task which appears in many contexts and applications. One of the most popular method used to find a solution of such non-linear equations is Newton method (or variants like Gauss-Newton method). However, the methods fail when the solution is not a simple isolated point.

To address this problem and solve more efficiently non-linear equations with isolated singular solutions, one can analyse the multiplicity structure and then exploit the properties of this structure to develop efficient algorithms. This leads to the conceptualisation of families of multiplicity structures of a given length as an algebraic variety, which geometric properties give structural insights into the multiplicity classes themselves.

This approach, which underlies the construction of (punctual Hilbert) schemes by A. Grothendieck [14], has opened up a very rich field of mathematical investigation. The study of Hilbert schemes is an active domain of algebraic geometry since several decades. In [6] an analysis of the punctual Hilbert scheme in two variables and of the generic staircase strata is developed. Bounds on the dimension of the punctual Hilbert scheme are provided in [7]. A review of geometric properties of the punctual Hilbert scheme can be found in [19]. The study of the punctual Hilbert schemes is further developed in [13]. The subject also progresses on the analysis of the *global* Hilbert scheme and the functorial properties. See e.g. [2] for a general overview. In a more effective perspective, works like [15] aimed to provide an explicit description of the global Hilbert scheme (in two variables). In [27], graphs of irreducible components of the Hilbert scheme associated to connections of Borel initial ideals are analyzed. In [26], a stratification of the Hilbert scheme in terms of initial ideals is further investigated. In [16], Hilbert schemes in multigraded algebra are considered. A case-study of the Hilbert

schemes of 8 points is developed in [8]. In [18], charts of the Hilbert scheme in two variables are analysed. Explicit coordinate rings and relations related to corner monomials are described. This approach is further developed in [28, 20], with the study of so-called Grobner strata. In [9], strata of the Hilbert scheme of points are analysed via initial ideals, in particular those associated with Borel initial ideals. This approach is further investigated in [4]. Charts of the Hilbert scheme with marked bases are studied in [3]. In [5], explicit equations of low degree in the Plücker coordinates defining general Hilbert schemes are described. In [1], the tangent space of the punctual Hilbert scheme in two variables is analysed. Bounds on its dimension are provided in terms of Young diagrams associated with monomial bases. This list, which is far from being exhaustive, shows that many works study the properties of the Hilbert scheme, via explicit descriptions coming from the algebraic calculus on ideals.

Despite these numerous developments, a complete understanding of the geometry of Hilbert schemes is not yet achieved and many questions remain open.

In this paper, we approach the study of the punctual Hilbert scheme from an algorithmic point of view. We first present algorithms, which allow to compute the inverse system or multiplicity structure associated to an isolated point. The main steps of the algorithm are the solution of linear systems, which are built by formal integration. We define the punctual Hilbert scheme as a subvariety of a Grassmannian variety and provide explicit equations defining it. Then we localised our study to the varieties  $\text{Hilb}_B$  of inverse systems, which admits a given dual (monomial) basis  $B$ . Building on the algorithm to compute the inverse system, we derive a set of integration and commutation relations, which define  $\text{Hilb}_B$ . Exploiting combinatorial properties of the monomial basis  $B$ , we provide a minimal set of defining equations. Using this effective presentation of  $\text{Hilb}_B$ , we prove new properties of transversality of  $\text{Hilb}_B$ , give new dimension formula for its irreducible components and prove that they are birational.

## 2. The inverse system of a singular point

In this section, we provide the notation and basic material needed for the analysis of the punctual Hilbert scheme. Without loss of generality, we will assume that the singular point that we analyse is the origin  $\mathbf{0} = (0, \dots, 0)$ .

**2.1. Preliminaries.** Let  $\mathbb{K}$  be an algebraic closed field (not necessarily of characteristic 0). Throughout the paper we use the notation  $\mathbf{e}_1, \dots, \mathbf{e}_n$  for the standard basis of  $\mathbb{K}^n$  or for a *canonical* basis of a vector space  $V$  of dimension  $n$ .

The algebra of polynomials in the variables  $\mathbf{x} = (x_1, \dots, x_n)$  with coefficients in  $\mathbb{K}$  is denoted  $\mathbb{K}[\mathbf{x}]$ . The degree of a polynomial  $p = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{x}^\alpha \in \mathbb{K}[\mathbf{x}]$  with  $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  is  $\deg(p) = \max_{|\alpha| \text{ s.t. } p_\alpha \neq 0} |\alpha|$  where  $|\alpha| = |(\alpha_1, \dots, \alpha_n)| = \sum_i \alpha_i$ . Its valuation is  $\text{val}(p) = \min_{\alpha \text{ s.t. } p_\alpha \neq 0} \{|\alpha|\}$ . For  $t \in \mathbb{N}$ , the space of polynomials of degree  $\leq t$  is  $\mathbb{K}[\mathbf{x}]_t$ . Let  $s_{n,t} = \binom{n+t}{n} = \dim \mathbb{K}[\mathbf{x}]_t$  be the number of monomials of degree  $\leq t$  in  $n$  variables.

For a set  $B \subset \mathbb{K}[\mathbf{x}]$ ,  $B_t = B \cap \mathbb{K}[\mathbf{x}]_t$  and  $B_{[t]}$  is the subset of elements of exactly degree  $t$  in  $B$ . For  $B \subset \mathbb{K}[\mathbf{x}]$  let  $B^+ = B \cup x_1 B \cup \cdots \cup x_n B$  be the *prolongation* of  $B$  and let  $\partial B = B^+ \setminus B$  be the *border* of  $B$ . We say that  $B$  is closed by division if for any  $b \in B$  of the form  $b = x_j b'$  for some  $1 \leq j \leq n$ , we have  $b' \in B$ . In particular, if  $B$  is a set of monomials closed by division, it must contains 1. In the following, we will identify a monomial  $\mathbf{x}^\beta$  with its exponent  $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ .

For a  $\mathbf{m}$ -primary ideal of  $Q$  (with  $\sqrt{Q} = \mathbf{m}$ ),  $\mathcal{A} = \mathbb{K}[\mathbf{x}]/Q$  is a local Artinian algebra. Its dimension  $\delta$  over  $\mathbb{K}$  is the *multiplicity* of  $\zeta$ .

The dual  $\mathbb{K}[\mathbf{x}]^* = \text{Hom}_{\mathbb{K}}(\mathbb{K}[\mathbf{x}], \mathbb{K})$  is the space of linear functionals  $\lambda : p \in \mathbb{K}[\mathbf{x}] \mapsto \lambda(p) \in \mathbb{K}$  on  $\mathbb{K}[\mathbf{x}]$ . An element  $\lambda \in \mathbb{K}[\mathbf{x}]^*$  can be represented by an element of the ring of formal power series  $\mathbb{K}[[\mathbf{y}]] := \mathbb{K}[[y_1, \dots, y_n]]$  as

$$\lambda = \sum_{\alpha \in \mathbb{N}^n} \lambda(\mathbf{x}^\alpha) \mathbf{y}^\alpha$$

where  $\mathbf{y}^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$  is the basis dual to the monomial basis  $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$  of  $\mathbb{K}[\mathbf{x}]$ . Notice that  $1 \in \mathbb{K}[\mathbf{y}]$  represents the evaluation  $p \in \mathbb{K}[\mathbf{x}] \mapsto p(\zeta)$  at  $\zeta \in \mathbb{K}^n$ . This defines a pairing between  $\mathbb{K}[\mathbf{x}]^*$  and  $\mathbb{K}[\mathbf{x}]$ : For  $\lambda = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \mathbf{y}^\alpha \in \mathbb{K}[\mathbf{x}]^*$  and  $p = \sum_{\beta, |\beta| \leq d} p_\beta \mathbf{x}^\beta$  where  $d = \deg(p)$ ,

$$(1) \quad \langle \lambda | p \rangle = \lambda(p) = \sum_{\alpha} \lambda_\alpha p_\alpha.$$

As  $\mathbb{K}[\mathbf{x}]$  acts on  $\mathbb{K}[\mathbf{x}]$  by multiplication, it also acts on its dual  $\mathbb{K}[\mathbf{x}]^*$ , by the transposed multiplication defined as follows. For  $p \in \mathbb{K}[\mathbf{x}]$ ,  $\lambda \in \mathbb{K}[\mathbf{x}]^*$ ,  $p \star \lambda : q \mapsto \lambda(pq)$ . We check that  $p = x_i$  acts as a divided difference on  $\mathbb{K}[[\mathbf{y}]]$  for this transposed product:

$$x_i \star \mathbf{y}^\alpha = \begin{cases} y_1^{\alpha_1} \cdots y_{i-1}^{\alpha_{i-1}} y_i^{\alpha_i-1} y_{i+1}^{\alpha_{i+1}} \cdots y_n^{\alpha_n} & \text{if } \alpha_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

We also denote this transpose product by  $x_i$  or *divided difference*  $\partial_i : \lambda \in \mathbb{K}[\mathbf{x}]^* \mapsto \partial_i(\lambda) := x_i \star \lambda$ . For this product  $x_i$  acts as the *inverse* of  $y_i$ . This explains why, in the (old) literature, the elements in  $\mathbb{K}[\mathbf{x}]^*$  generate so called *inverse systems*. In characteristic 0, using a scaled dual basis, one can see the transpose product as a derivation [25].

Considering the reverse operation, we use the following integral notation to denote the linear operator of  $\mathbb{K}[[\mathbf{y}]]$  defined for  $\alpha \in \mathbb{N}^n$  on the dual basis  $(\mathbf{y}^\alpha)$  by

$$(2) \quad \oint_k \mathbf{y}^\alpha := \begin{cases} y_1^{\alpha_1} \cdots y_{k-1}^{\alpha_{k-1}} y_k^{\alpha_k+1} & \text{if } \alpha_{k+1} = \cdots = \alpha_n = 0 \\ 0 & \text{otherwise} \end{cases}$$

**2.2. The orthogonal to an ideal.** For an ideal  $I \subset \mathbb{K}[\mathbf{x}]$ , let  $I^\perp = \{\lambda \in \mathbb{K}[[\mathbf{y}]] \mid \forall p \in I, \langle \lambda | p \rangle = 0\}$ . For a vector space  $\mathcal{L} \subset \mathbb{K}[[\mathbf{y}]]$ , we denote  $\mathcal{L}^\perp = \{p \in \mathbb{K}[\mathbf{x}] \mid \forall \lambda \in \mathcal{L}, \langle \lambda | p \rangle = 0\}$ .

**Definition 2.1.** The vector space  $I^\perp \subset \mathbb{K}[[\mathbf{y}]]$  is called the *inverse system* of  $I$ .

The vector space  $I^\perp$  is naturally identified with the dual space of  $\mathbb{K}[\mathbf{x}]/I$ . We check that  $\mathcal{L} = I^\perp$  is a vector subspace of  $\mathbb{K}[[\mathbf{y}]]$ , which is closed by divided difference  $\partial_i$  for  $i = 1, \dots, n$ . Conversely,  $\mathcal{L} \subset \mathbb{K}[[\mathbf{y}]]$  is closed by divided difference iff  $Q = \mathcal{L}^\perp$  is an ideal.

The following classical property says that inverse systems of  $\mathfrak{m}$ -primary ideals are vector spaces of polynomials  $\subset \mathbb{K}[\mathbf{y}]$ :

**Lemma 2.2.** *If  $Q$  is a  $\mathfrak{m}$ -primary isolated component of  $I$ , then  $Q^\perp = I^\perp \cap \mathbb{K}[\mathbf{y}]$ . Its dimension is  $\delta(I) = \dim \mathcal{A}$ .*

Therefore, we can obtain  $Q^\perp$ , as the space of all polynomials  $\in \mathbb{K}[\mathbf{y}]$  which are in  $I^\perp$ , without knowing the primary decomposition of  $I$ .

Let us denote this set  $\mathcal{L} = I^\perp \cap \mathbb{K}[\mathbf{y}]$ . It is a vector space stable under the divided differences  $\partial_i$ . Its dimension is the dimension of  $Q^\perp$  or  $\mathbb{K}[\mathbf{x}]/Q$ , that is the *multiplicity* of  $\mathbf{0}$ , denoted  $\delta(I)$ , or simply  $\delta$  if  $I$  is clear from the context.

**Example 2.3.** Let  $Q = (x_1^2, x_2^2) \subset \mathbb{K}[x_1, x_2]$  defining  $\zeta = (0, 0) \in \mathbb{K}^2$  as an isolated point of multiplicity 4. Then its inverse system is  $Q^\perp = \text{span}_{\mathbb{K}}(1, y_1, y_2, y_1 y_2)$ .

For an element  $\lambda = \sum_{\beta} \lambda_{\beta} \mathbf{y}^{\beta} \in \mathbb{K}[\mathbf{y}]$  we define the degree or *order*  $\omega(\lambda)$  to be the maximal  $|\beta|$  s.t.  $\mathbf{y}^{\beta}$  appears in  $\lambda$  with non-zero coefficient  $\lambda_{\beta}$ .

For  $t \in \mathbb{N}$ , let  $\mathcal{L}_t$  be the elements of  $\mathcal{L}$  of order  $\leq t$ . We directly check that  $\mathcal{L}_t = (Q + \mathfrak{m}^{t+1})^\perp$ . As  $\mathcal{L}$  is of dimension  $\delta$ , there exists a smallest  $t \geq 0$  s.t.  $\mathcal{L}_{t+1} = \mathcal{L}_t$ . Let us call this smallest  $t$ , the *nil-index* of  $\mathcal{L}$  and denote it by  $\omega(I)$ , or simply by  $\omega$ .

**Remark 2.4.** As  $\mathcal{L}$  is stable by the divided differences  $\partial_i$ , we easily check that  $\omega < \delta$  so that  $\mathcal{L}_{\delta-1} = \mathcal{L}$ .

Let  $B = \{\mathbf{x}^{\beta_1}, \dots, \mathbf{x}^{\beta_{\delta}}\}$  be a basis of  $\mathbb{K}[\mathbf{x}]/Q$ . We can identify the elements of  $\mathbb{K}[\mathbf{x}]/Q$  with the elements of the vector space  $\text{span}_{\mathbb{K}}(B)$  spanned by  $B$ .

We define the normal form  $N(p)$  of a polynomial  $p$  in  $\mathbb{K}[\mathbf{x}]$  as the projection of  $p$  on  $\text{span}_{\mathbb{K}}(B)$  along  $Q$ , that is, the unique element  $b$  of  $\text{span}_{\mathbb{K}}(B)$  such that  $p - b \in Q$ . We easily verify that if  $B = \{\mathbf{x}^{\beta_1}, \dots, \mathbf{x}^{\beta_\delta}\}$ , then  $\forall p \in \mathbb{K}[\mathbf{x}]$

$$N(p) = \sum_{i=1}^{\delta} \lambda_i(p) \mathbf{x}^{\beta_i}$$

where  $\lambda_i \in Q^\perp$  is a linear functional, which vanishes on  $Q$ . We have the following property:

**Lemma 2.5.** *The coefficients  $\Lambda = \{\lambda_1, \dots, \lambda_\delta\}$  of the normal form  $N$  modulo  $Q$  in the basis  $B$  of  $\mathbb{K}[\mathbf{x}]/Q$  form a basis of  $Q^\perp$  dual to  $B$ .*

**Example 2.6.** Let  $Q = (x_1^2, x_2^2) \subset \mathbb{K}[x_1, x_2]$  defining  $\zeta = (0, 0) \in \mathbb{K}^2$  with multiplicity 4. Let  $B = \{1, x_1, x_2, x_1x_2\}$ . It is a basis of  $\mathbb{K}[x_1, x_2]/Q$  and for any  $p = \sum_{\alpha} p_{\alpha} \mathbf{x}^{\alpha}$ , we have

$$N(p) = p_{0,0} + p_{1,0}x_1 + p_{0,1}x_2 + p_{1,1}x_1x_2 = \langle 1 | p \rangle + \langle y_1 | p \rangle x_1 + \langle y_2 | p \rangle x_2 + \langle y_1y_2 | p \rangle x_1x_2$$

and the coefficients of the normal form in the basis  $B = \{1, x_1, x_2, x_1x_2\}$  are the linear functionals  $\{1, y_1, y_2, y_1y_2\}$ , which form a basis of  $Q^\perp$  dual to  $B$ .

**2.3. Computing inverse systems.** We describe now how the inverse system of an isolated point  $\zeta$  can be computed from the input polynomials  $\mathbf{f} = \{f_1, \dots, f_s\} \subset \mathbb{K}[\mathbf{x}]$ . The way a basis of the inverse system is computed relies on structural properties, that will be useful to analyse the geometry of the Punctual Hilbert Scheme.

**Macaulay's dialytic method.** The first "algorithm" was proposed by F.S. Macaulay [21]. It is an iterative method, which degree by degree computes a basis of the kernel of matrices built from monomial multiples of  $\mathbf{f}$ .

As  $f_i(\mathbf{0}) = 0$ , the valuation  $\text{val}(f_i)$  is greater than 1 for  $i = 1, \dots, s$ . We denote by  $\mathbf{m} = (x_1, \dots, x_n)$  the maximal ideal defining  $\mathbf{0}$ . For  $t \in \mathbb{N}$  let  $\mathbf{m}^{[t]}$  be the set of monomials of degree  $t$ . Then the ideal  $\langle \mathbf{f} \rangle$  is the vector space generated by  $\mathbf{f} \cdot \mathbf{m}^{[t]}$  for  $t \in \mathbb{N}$ . Notice that the polynomials  $\mathbf{f} \cdot \mathbf{m}^{[t]}$  are of valuation  $\geq t + 1$ . Let  $\mathbf{h}_t = \{\mathbf{f}, \mathbf{f} \cdot \mathbf{m}^{[1]}, \dots, \mathbf{f} \cdot \mathbf{m}^{[t-1]}\} \subset I$  and  $N_t = |\mathbf{h}_t|$ . The elements  $p \in I$  of valuation  $\leq t$  are in the space spanned by  $\mathbf{h}_t$ . We consider the linear map defined as follows:

$$\begin{aligned} H_t : \mathbb{K}[\mathbf{y}]_t &\rightarrow \mathbb{K}^{s_t} \\ \lambda &\mapsto ((\lambda | h))_{h \in \mathbf{h}_t} \end{aligned}$$

By Lemma 2.2, the elements of the inverse systems  $\mathcal{L}_t$  of degree  $\leq t$  are the elements of  $\mathbb{K}[\mathbf{y}]_t$  orthogonal to  $I$ . Thus they are the elements of  $\mathbb{K}[\mathbf{y}]_t$  orthogonal to the elements of  $I$  of valuation  $\leq t$ . In other words,  $\mathcal{L}_t = \ker H_t$ .

We easily check that  $\ker H_0 = \text{span}_{\mathbb{K}}(1) \subset \mathbb{K}[\mathbf{y}]$ . The kernel of  $H_t$  can be computed from the kernel  $H_{t-1}$  since  $H_{t-1}$  is a submap of  $H_t$  and its matrix in monomial basis is of the form

$$\begin{bmatrix} H_{t-1} & H'_t \\ \mathbf{0} & N_t \end{bmatrix}.$$

Thus  $\ker H_{t-1} \hookrightarrow \ker H_t$  and new basis elements of  $\ker H_t$  are obtained from the basis of  $\ker H_{t-1}$  by computing elements  $\varphi \in \ker N_t$  and  $\lambda \in \mathbb{K}[\mathbf{y}]_{t-1}$  such that  $H_t(\lambda) + H'_t(\varphi) = 0$ . This leads to Algorithm 1.

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**Algorithm 1** INVERSE SYSTEM BY THE DIALYTIC METHOD

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Starting with  $\mathcal{D}_0 = \{1\}$  and  $t = 0$

–  $t := t + 1$

– Compute  $H_t$ , a basis of  $\ker N_t$  and the new basis elements in  $\mathcal{L}_t$ .

until  $\ker N_t = \{0\}$ .

---

The iteration stops at a degree  $t$  which is the null-index  $\omega$  of the singular point  $\zeta = \mathbf{0}$ . This method has been used and improved in recent works. See e.g. [10, 11]. However, its complexity is in  $\mathcal{O}(N_\omega s_{n,\omega}^2)$  where  $N_\omega = |\mathbf{h}_\omega|$  and  $s_{n,\omega} = \dim \mathbb{K}[\mathbf{y}]_\omega$ .

**Integration method.** Another approach to compute a basis of the inverse system  $\mathcal{L}$  exploits the stability of  $\mathcal{L}$  by divided difference. In particular,  $\partial_k \mathcal{L}_t \subset \mathcal{L}_{t-1}$  for  $k = 1, \dots, n$ . Therefore, a basis of  $\mathcal{L}_t$  can be computed from a basis of  $\mathcal{L}_{t-1}$  by *integration*, which consists in “inverting” the actions of the divided differences  $\partial_k$ .

The method computes a so-called *graded primal-dual pair*  $(B, \Lambda)$  with  $B = \{\mathbf{x}^{\beta_1}, \dots, \mathbf{x}^{\beta_\delta}\}$ ,  $|\beta_1| \leq \dots \leq |\beta_\delta|$ ,  $\Lambda = \{\lambda_{\beta_1}, \dots, \lambda_{\beta_\delta}\}$ ,  $\omega(\lambda_{\beta_i}) \leq |\beta_i|$  and

$$(3) \quad \langle \lambda_{\beta_i} | \mathbf{x}^{\beta_j} \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

It is based on the following characterisation of spaces stable by divided differences:

**THEOREM 2.7** ([24, 22, 23]). *Let  $B = \{\mathbf{x}^{\beta_1}, \dots, \mathbf{x}^{\beta_\delta}\} \subset \mathbb{K}[\mathbf{x}]$  with  $|\beta_1| \leq \dots \leq |\beta_\delta|$ . Let  $\mathcal{L} = \langle \lambda_{\beta_1}, \dots, \lambda_{\beta_\delta} \rangle \subset \mathbb{K}[\mathbf{y}]$  with  $\omega(\lambda_{\beta_i}) \leq |\beta_i|$  and s.t.*

*Then  $\mathcal{L} = \langle \lambda_1, \dots, \lambda_r \rangle$  is closed by divided difference with*

$$(4) \quad \partial_k \lambda_{\beta_i} = \sum_{|\beta_j| < |\beta_i|} \mu_{\beta_i, \beta_j + \mathbf{e}_k} \lambda_{\beta_j}.$$

iff for  $1 \leq i \leq \delta$  we have the integration relations

$$(5) \quad \lambda_{\beta_i} = \sum_{|\beta_j| < |\beta_i|} \sum_{k=1}^n \mu_{\beta_i, \beta_j + \mathbf{e}_k} \oint_k \lambda_{\beta_j}$$

and for all  $i, s = 1, \dots, r$ ,  $|\beta_s| < |\beta_i|$ ,  $k \neq l \in \{1, \dots, n\}$  we have the commutation relations

$$(6) \quad \sum_{j: |\beta_h| < |\beta_j| < |\beta_i|} \mu_{\beta_i, \beta_j + \mathbf{e}_k} \mu_{\beta_j, \beta_h + \mathbf{e}_l} - \mu_{\beta_i, \beta_j + \mathbf{e}_l} \mu_{\beta_j, \beta_h + \mathbf{e}_k} = 0.$$

Furthermore,  $(B, \Lambda)$  is a primal-dual basis pair iff they satisfy (5), (6) and orthogonality relations

$$(7) \quad \mu_{\beta_i, \beta_j + \mathbf{e}_k} = \begin{cases} 1 & \text{for } \beta_i = \beta_j + \mathbf{e}_k \\ 0 & \text{for } \beta_j + \mathbf{e}_k = \beta_l \text{ for some } 1 \leq l \leq \delta, \beta_i \neq \beta_j + \mathbf{e}_k, \end{cases}$$

Moreover  $\Lambda$  is a basis of the inverse system of  $\mathbf{f} = (f_1, \dots, f_s)$  at  $\zeta = \mathbf{0}$  iff

$$(8) \quad \langle \lambda_{\beta_i} | f_j \rangle = 0 \quad \text{for } 1 \leq i \leq \delta, 1 \leq j \leq s.$$

This leads to Algorithm 2 to compute a graded primal-dual basis pair  $(B, \Lambda)$  for the inverse system  $\mathcal{L}$  of  $\mathbf{f}$  at  $\zeta = \mathbf{0}$ .

The monomials of  $B_{[t]}$  computed at step  $t$  are of degree  $t$ . They can be obtained by triangularization of the system (c) so that  $(B_{[t]}, \Lambda_{[t]})$  is a graded primal-dual basis pair. By construction, the algorithm yields a graded primal-dual basis pair  $(B, \Lambda)$  of  $\mathcal{L}$ . It can moreover compute a basis  $B$ , which is closed by derivation. Its complexity is in  $\mathcal{O}(\delta^3)$ . For more details on the algorithm, we refer to [24, 22, 23].

**Example 2.8.** We consider the polynomials  $f_1 = x_1 - x_2 + x_1^2$ ,  $f_2 = x_1 - x_2 + x_2^2 \in \mathbb{Q}[x_1, x_2]$ , for which  $\zeta = (0, 0)$  is an isolated root of multiplicity 3 as we will see by applying the integration algorithm:

- We start with  $B_0 = \{1\}$ ,  $\Lambda_0 = \{1\}$ ;
- At step (b) we have  $\lambda = \nu_{1,0} y_1 + \nu_{0,1} y_2$ ;
- Solving the system (c) yields  $\nu_{1,0} = \nu_{0,1} = 1$  and  $B_{[1]} = \{x_1\}$ ,  $\Lambda_{[1]} = \{y_1 + y_2\}$ ;
- At the iteration for  $t = 2$ , we have  $\lambda = \nu_{1,0} y_1 + \nu_{0,1} y_2 + \nu_{2,0} y_1^2 + \nu_{1,1} (y_1 y_2 + y_2^2)$ .
- In step (c), we obtain  $B_{[2]} = \{x_1^2\}$ ,  $\Lambda_{[2]} = \{y_1^2 + y_1 y_2 + y_2^2\}$ ;
- At  $t = 3$ , there is no solution and the iteration stops.

**Algorithm 2** INVERSE SYSTEM BY INTEGRATION

Starting with  $B_0 = \{1\} \subset \mathbb{K}[\mathbf{x}]$ ,  $\Lambda_0 = \{1\} \subset \mathbb{K}[\mathbf{y}]$ ,  $t = 0$

(a)  $t := t + 1$

(b) For unknown coefficients  $\boldsymbol{\nu} = (\nu_{\beta_j + \mathbf{e}_k})$ , let

$$\lambda = \sum_{|\beta_j| < t} \sum_{k=1}^n \nu_{\beta_j + \mathbf{e}_k} \oint_k \lambda_{\beta_j}$$

(c) Compute  $B_{[t]} = \{\mathbf{x}^{\beta_{\delta_{t-1}+1}}, \dots, \mathbf{x}^{\beta_{\delta_t}}\}$  and a basis  $\Lambda_{[t]} = \{\lambda_{\beta_{\delta_{t-1}+1}}, \dots, \lambda_{\beta_{\delta_t}}\}$  of the solutions of the linear system in  $\boldsymbol{\nu}$ :

$$\sum_{j: |\beta_h| < |\beta_j| < t} \nu_{\beta_j + \mathbf{e}_k} \mu_{\beta_j, \beta_h + \mathbf{e}_l} - \nu_{\beta_j + \mathbf{e}_l} \mu_{\beta_j, \beta_h + \mathbf{e}_k} = 0 \quad \text{for } 1 \leq h \leq \delta_{t-1}, 1 \leq k < l \leq n,$$

$$\langle \lambda | f_j \rangle = 0 \quad \text{for } 1 \leq j \leq s,$$

$$\langle \lambda | \mathbf{x}^{\beta_h} \rangle = 0 \quad \text{for } 1 \leq h \leq \delta_{t-1}$$

$$\text{such that } \langle \lambda_{\beta_j} | \mathbf{x}^{\beta_i} \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \delta_{t-1} + 1 \leq i, j \leq \delta_t.$$

(d)  $B_t := B_{t-1} \cup B_{[t]}$ ,  $\Lambda_t := \Lambda_{t-1} \cup \Lambda_{[t]}$ ,  $\mu_{\beta_i, \beta_j + \mathbf{e}_k} = \langle \lambda_{\beta_i} | \mathbf{x}^{\beta_j + \mathbf{e}_k} \rangle$  for  $\delta_{t-1} + 1 \leq i \leq \delta_t$ ,  $1 \leq j \leq \delta_{t-1}$ ,  $1 \leq k \leq n$ .

until  $B_{[t]} = \{\}$ .

We obtain the primal-dual basis pair  $B = \{1, x_1, x_1^2\}$ ,  $\Lambda = \{1, y_1 + y_2, y_1^2 + y_1 y_2 + y_2^2\}$ . This shows that  $\zeta = (0, 0)$  is an isolated point of multiplicity  $3 = |B| = |\Lambda|$ . We verify that  $\Lambda$  is stable by the divided differences  $\partial_k$  for  $k = 1, 2$ .

### 3. The punctual Hilbert scheme $\text{Hilb}_\delta$

In this section, we define the punctual Hilbert scheme, affine charts and describe their defining equations.

**3.1. The projective variety  $\text{Hilb}_\delta$ .** For a vector space  $V$ , let  $\mathcal{G}_\delta(V)$  be the projective variety of the  $\delta$  dimensional linear subspaces of  $V$ , also known as the *Grassmannian* of  $\delta$ -spaces of  $V$ . The points in  $\mathcal{G}_\delta(V)$  are the projective points of  $\mathbb{P}(\wedge^\delta V)$  of the form  $\mathbf{v} = v_1 \wedge \dots \wedge v_\delta$  for  $v_i \in V$ .

Fixing a basis  $\mathbf{e}_1, \dots, \mathbf{e}_s$  of  $V$ , the Plücker coordinates of  $\mathbf{v} = v_1 \wedge \dots \wedge v_\delta \in \wedge^\delta V$  are the coefficients of  $\Delta_{i_1, \dots, i_\delta}(\mathbf{v})$  of  $\mathbf{v} = \sum_{i_1 < \dots < i_\delta} \Delta_{i_1, \dots, i_\delta}(\mathbf{v}) \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_\delta}$ .

To define the punctual Hilbert scheme  $\text{Hilb}_\delta$ , we will consider the family of linear spaces  $\mathcal{L}$ , which are the orthogonal of an ideal  $Q$  defining a point  $\zeta$  with multiplicity  $\delta$ . By Lemma 2.2 and Remark 2.4,  $\mathcal{L}$  is a linear subspace of  $\mathbb{K}[\mathbf{y}]_{\delta-1}$ , that is a point of the Grassmannian  $\mathcal{G}_\delta(\mathbb{K}[\mathbf{y}]_{\delta-1})$ .

A natural basis of  $\mathbb{K}[\mathbf{y}]_{\delta-1}$  is the dual monomial basis  $(\mathbf{y}^\alpha)_{|\alpha| < \delta}$ . The Plücker coordinates of an element  $\mathcal{L} \in \mathcal{G}_\delta(\mathbb{K}[\mathbf{y}]_{\delta-1})$  for this canonical basis are denoted  $\Delta_{\alpha_1, \dots, \alpha_\delta}(\mathcal{L})$  where  $\alpha_i \in \mathbb{N}^n$ ,  $|\alpha_i| < \delta$ . If  $\Lambda = \{\lambda_1, \dots, \lambda_\delta\}$  is a basis of  $\mathcal{L}$  in  $\mathbb{K}[\mathbf{y}]_{\delta-1}$  with  $\lambda_i = \sum_{|\alpha| < \delta} \mu_{i, \alpha} \mathbf{y}^\alpha$ , its coefficient matrix in the monomial basis is  $[\mu_{i, \alpha}]_{1 \leq i \leq \delta, |\alpha| < \delta}$ . The Plücker coordinates of  $\mathcal{L} = \text{span}_{\mathbb{K}}(\Lambda)$  are, up to a scalar,

$$\Delta_{\alpha_1, \dots, \alpha_\delta}(\mathcal{L}) = \det [\mu_{i, \alpha_j}]_{1 \leq i, j \leq \delta} = \det [\langle \lambda_i | \mathbf{x}^{\alpha_j} \rangle]_{1 \leq i, j \leq \delta}.$$

**Definition 3.1.** Let  $\text{Hilb}_\delta \subset \mathcal{G}_\delta(\mathbb{K}[\mathbf{y}]_{\delta-1})$  be the set of linear spaces  $\mathcal{L}$  of dimension  $\delta$  in  $\mathbb{K}[\mathbf{y}]_{\delta-1}$  which are stable by the divided differences  $\partial_i$  with respect to the variables  $\mathbf{y}$  (i.e.  $\partial_i \mathcal{L} \subset \mathcal{L}$  for  $i = 1, \dots, n$ ). We called  $\text{Hilb}_\delta$  the *punctual Hilbert scheme* of points of multiplicity  $\delta$ .

If  $\mathcal{L} \subset \mathbb{K}[\mathbf{y}]$  is stable by the divided differences  $\partial_i$ , then by duality  $Q = \mathcal{L}^\perp := \{p \in \mathbb{K}[\mathbf{x}] \mid \forall \lambda \in \mathcal{L}, \langle \lambda | p \rangle = 0\}$  is a vector space of  $\mathbb{K}[\mathbf{x}]$  stable by multiplication by  $x_i$ , i.e. an ideal of  $\mathbb{K}[\mathbf{x}]$ .

Consequently,  $\mathcal{L} \in \text{Hilb}_\delta$  iff  $\mathcal{L}^\perp = Q$  is the  $(x_1, \dots, x_n)$ -primary ideal such that  $\dim \mathbb{K}[\mathbf{x}]/Q = \delta$  (see [23]).

To describe equations defining  $\text{Hilb}_\delta$ , we recall that  $\partial_k$  ( $1 \leq k \leq n$ ) is a linear operator in  $\mathbb{K}[\mathbf{y}]_{\delta-1}$  sending  $\mathbf{y}^\alpha \in \mathbb{K}[\mathbf{y}]_\delta$  either to  $\mathbf{y}^{\alpha - \mathbf{e}_k}$  if  $\alpha_k > 0$  or to 0. We recall also that the proper intersection of the linear spaces  $\mathcal{L}$  and  $\mathbf{v} = \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_\sigma$ ,  $\mathbf{v}_i \in \mathbb{K}[\mathbf{y}]_{\delta-1}$  respectively of dimension  $\delta$  and  $\sigma = s_{n, \delta-1} - \delta + 1$  is given by the intersection product ([12])

$$\mathcal{L} \cdot \mathbf{v} = \sum_{|\alpha| < t} I_\alpha(\Delta(\mathcal{L}), \mathbf{v}_1, \dots, \mathbf{v}_\sigma) \mathbf{x}^\alpha \in \mathbb{K}[\mathbf{y}]_{\delta-1}$$

where  $I_\alpha(\Delta(\mathcal{L}), \mathbf{v}_1, \dots, \mathbf{v}_\sigma)$  is a multilinear function of  $\Delta(\mathcal{L}), \mathbf{v}_1, \dots, \mathbf{v}_\sigma$  for  $|\alpha| < \delta$ .

**Proposition 3.2.** *The punctual Hilbert scheme  $\text{Hilb}_\delta$  is the projective variety of linear spaces  $\mathcal{L} \in \mathcal{G}_\delta(\mathbb{K}[\mathbf{y}]_{\delta-1})$  such that*

$$(9) \quad \partial_k(\mathcal{L} \cdot \mathbf{y}^{\alpha_1} \wedge \dots \wedge \mathbf{y}^{\alpha_\sigma}) \wedge \mathcal{L} = 0 \text{ for } |\alpha_i| < \delta, \quad 1 \leq k \leq n.$$

*These are quadratic equations in the Plücker coordinates  $\Delta(\mathcal{L})$  of  $\mathcal{L}$ .*

PROOF. As the set of  $\mathcal{L} \cdot \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_\sigma$  for  $\mathbf{v}_i \in \mathbb{K}[\mathbf{y}]_{\delta-1}$  is the whole linear space  $\mathcal{L}$ ,  $\mathcal{L}$  is stable by  $\partial_k$  for  $k = 1, \dots, b$  iff  $\partial_k(\mathcal{L} \cdot \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_\sigma) \in \mathcal{L}$ , that is,

$$\partial_k(\mathcal{L} \cdot \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_\sigma) \wedge \mathcal{L} = 0$$

for all  $\mathbf{v}_i \in \mathbb{K}[\mathbf{y}]_{\delta-1}$ ,  $1 \leq k \leq n$ . Since these are multilinear relations in  $\mathbf{v}_1, \dots, \mathbf{v}_\sigma$ , they are satisfied iff they vanish when  $\mathbf{v}_i$  is replaced by any element of the canonical basis  $(\mathbf{y}^\alpha)_{|\alpha| < \delta}$ . This yields the relations (9).  $\square$

We are going to analyse locally the punctual Hilbert scheme, fixing a basis  $B$  of the quotient  $\mathbb{K}[\mathbf{x}]/Q$  where  $Q = \mathcal{L}^\perp$ :

**Definition 3.3.** For  $B \subset \mathbb{K}[\mathbf{x}]$  such that  $|B| = \delta$ , let  $\text{Hilb}_B = \{\mathcal{L} \in \text{Hilb}_\delta \mid \Delta_B(\mathcal{L}) \neq 0\}$  be a (possibly empty) chart of  $\text{Hilb}_\delta$ .

We verify that if  $\Delta_B(\mathcal{L}) \neq 0$  where  $B = \{\mathbf{x}^{\beta_j}\}_{j=1}^\delta$  then  $\mathcal{L}$  admits a basis  $\Lambda = \{\lambda_1, \dots, \lambda_\delta\}$  such that  $[\langle \lambda_i \mid \mathbf{x}^{\beta_j} \rangle]_{1 \leq i, j \leq \delta} = \text{Id}$ , that is,  $(B, \Lambda)$  is a primal-dual basis pair. Moreover,  $\Delta_B(\mathcal{L}) \neq 0$  iff  $B$  is a basis of  $\mathbb{K}[\mathbf{x}]/Q$  where  $Q = \mathcal{L}^\perp$ .

It is clear that  $\cup_{|B|=\delta} \text{Hilb}_B$  is open covering of  $\text{Hilb}_\delta$ . According to Algorithm 2, every  $\mathcal{L} \in \text{Hilb}_\delta$  admits a graded primal-dual basis pair  $(B, \Lambda)$  such that  $\Delta_B(\mathcal{L}) \neq 0$ . Moreover, we can assume that  $B$  is closed by division.

Therefore we can restrict the covering to charts  $\text{Hilb}_B$  such that  $(B, \Lambda)$  is a graded primal-dual basis pair and  $B$  is closed by division.

**3.2. The affine chart  $\text{Hilb}_B$ .** Let  $B = \{\mathbf{x}^{\beta_i}\}_{i=1}^\delta \subset (\mathbf{x}^\alpha)_{|\alpha| < \delta}$  closed by division with  $|B| = \delta$ ,  $\mathbf{x}^{\beta_1} = 1$ ,  $0 = |\beta_1| \leq |\beta_2| \leq \dots \leq |\beta_\delta|$ .

We assume that there exists  $\Lambda = \{\lambda_i\}_{i=1}^\delta \subset \mathbb{K}[\mathbf{y}]_{\delta-1}$  such that  $(B, \Lambda)$  is a graded primal-dual basis pair. As  $[\langle \lambda_i \mid \mathbf{x}^{\beta_j} \rangle]_{1 \leq i, j \leq \delta} = \text{Id}$ , the coefficient matrix of the dual basis  $\Lambda$  in the monomial basis of  $\mathbb{K}[\mathbf{y}]_{\delta-1}$  is of the form

$$(10) \quad \begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_\delta \end{array} \begin{bmatrix} \beta_1 & \cdots & \beta_\delta & \gamma_1 & \cdots & \gamma_l \\ 1 & & \mathbf{0} & \mu_{1, \gamma_1} & \cdots & \mu_{1, \gamma_c} \\ & \ddots & & \vdots & & \vdots \\ \mathbf{0} & & 1 & \mu_{\delta, \gamma_1} & \cdots & \mu_{\delta, \gamma_c} \end{bmatrix}$$

where  $C = \{\mathbf{x}^{\gamma_1}, \dots, \mathbf{x}^{\gamma_c}\}$  is the set of monomials of  $\mathbb{K}[\mathbf{y}]_{\delta-1}$ , which are not in  $B$ . For  $d \in \mathbb{N}$ , we denote  $C_d = C \cap \mathbb{K}[\mathbf{y}]_d$ . As  $\mathbf{x}^{\beta_1} = 1$ , we have  $\lambda_1 = 1$  and  $\mu_{\beta_1, \gamma_j} = 0$  for  $j = 1, \dots, c$ . Since  $(B, \Lambda)$  is graded, we have

$$\lambda_i = \mathbf{y}^{\beta_i} + \sum_{\gamma \in C_{|\beta_i|}} \mu_{\beta_i, \gamma} \mathbf{y}^\gamma$$



and  $\mu_{\beta_i, \alpha} = 0$  for  $\alpha \in C \setminus C_{|\beta_i|}$ . Let  $\boldsymbol{\mu} = \{\mu_{\beta_i, \gamma}, 1 \leq i \leq \delta, \gamma \in C_{|\beta_i|}\}$  be the set of non-constant parameters or variables, that we need to describe  $\Lambda$ . Let  $N_B = |\boldsymbol{\mu}|$  is size, which is less than  $N_\delta = (\delta-1) \times (s_{n, \delta-1} - \delta)$  where  $s_{n, \delta-1} = \dim \mathbb{K}[\mathbf{x}]_{\delta-1}$ . We define  $\boldsymbol{\mu}' = \{\mu_{\beta_i, \gamma_j} \in \boldsymbol{\mu} \mid \mathbf{x}^{\gamma_j} \in \partial B\}$ . Let  $N'_B = |\boldsymbol{\mu}'|$  be the number of variables of  $\boldsymbol{\mu}'$ .

By properties of the determinant, the Plücker coordinates of  $\mathcal{L}$  are such that

$$(11) \quad \mu_{\beta_i, \alpha} = \frac{\Delta_{\beta_1, \dots, \beta_{i-1}, \alpha, \beta_{i+1}, \dots, \beta_\delta}}{\Delta_{\beta_1, \dots, \beta_\delta}} \quad i = 1, \dots, \delta, \quad |\alpha| < \delta.$$

By Equation (11), we can replace the homogeneous Plücker coordinates

$$\Delta_B^{i+\gamma_j}(\mathcal{L}) := \Delta_{\beta_1, \dots, \beta_{i-1}, \gamma_j, \beta_{i+1}, \dots, \beta_\delta}(\mathcal{L})$$

of  $\mathcal{L} \in \text{Hilb}_\delta$  by the affine coordinates  $\boldsymbol{\mu} = (\mu_{\beta_i, \gamma_j})_{2 \leq i \leq \delta, 1 \leq \gamma_j \leq l}$ . As the remaining Plücker coordinates  $\Delta_{\alpha_1, \dots, \alpha_\delta} = \det([\mu_{\beta_i, \alpha_j}])$  are polynomial functions of  $\boldsymbol{\mu}$ , we will associate the coordinate ring  $\mathbb{K}[\boldsymbol{\mu}]$  to the chart  $\text{Hilb}_B$ .

By Equation (4), the matrices of multiplication by  $x_k$  or divided difference

$$\begin{aligned} \partial_k : \mathcal{L} &\rightarrow \mathcal{L} \\ \lambda &\mapsto x_k \star \lambda = \partial_k \lambda \end{aligned}$$

in the basis  $\Lambda$  of  $\mathcal{L}$  are

$$(12) \quad M_{\mu, k}^T = \begin{bmatrix} 0 & \mu_{\beta_2, \mathbf{e}_k} & \mu_{\beta_3, \mathbf{e}_k} & \cdots & \mu_{\beta_\delta, \mathbf{e}_k} \\ 0 & 0 & \mu_{\beta_3, \beta_2 + \mathbf{e}_k} & \cdots & \mu_{\beta_\delta, \beta_2 + \mathbf{e}_k} \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{\beta_\delta, \beta_{\delta-1} + \mathbf{e}_k} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

for  $1 \leq k \leq n$ . We refer to this operator

$$\begin{aligned} M_{\mu, k}^T : \text{span}_{\mathbb{K}[\boldsymbol{\mu}]}(\Lambda) &\rightarrow \text{span}_{\mathbb{K}[\boldsymbol{\mu}]}(\Lambda) \\ \lambda_i &\mapsto \sum_{j: |\beta_j| < |\beta_i|} \mu_{\beta_i, \beta_j + \mathbf{e}_k} \lambda_j \end{aligned}$$

as the *parametric operator of divided difference*  $\partial_k$ . The transpose operator

$$\begin{aligned} M_{\mu, k} : \text{span}_{\mathbb{K}[\boldsymbol{\mu}]}(B) &\rightarrow \text{span}_{\mathbb{K}[\boldsymbol{\mu}]}(B) \\ \mathbf{x}^{\beta_i} &\mapsto \sum_{j: |\beta_i| < |\beta_j|} \mu_{\beta_j, \beta_i + \mathbf{e}_k} \mathbf{x}^{\beta_j} \end{aligned}$$

is the *parametric operator of multiplication* by  $x_k$  in  $B$ . If  $\mathcal{L} \in \text{Hilb}_B$ , then (c) is satisfied and the matrices  $\mathbf{M}_\boldsymbol{\mu} = (M_{\mu, 1}, \dots, M_{\mu, n})$  commute.

The commutation relations (6) correspond to the coefficients

$$(13) \quad \langle (M_{\mu, l}^T M_{\mu, k}^T - M_{\mu, k}^T M_{\mu, l}^T)(\lambda_i) \mid \mathbf{x}^{\beta_j} \rangle = \langle \lambda_{\beta_j} \mid (M_{\mu, k} M_{\mu, l} - M_{\mu, l} M_{\mu, k})(\mathbf{x}^{\beta_h}) \rangle.$$

that is, the entries of the matrix  $M_{\mu, k} M_{\mu, l} - M_{\mu, l} M_{\mu, k}$ . We are going to analyse in more details, which entries are not zero, by exploiting the structure of these multiplication operators.

Let  $B_t = B \cap \mathbb{K}[\mathbf{x}]_t$ ,  $B_{[t]} = B_t \setminus B_{t-1}$  the set of elements of  $B$  of degree  $t$  and  $h_t = |B_{[t]}|$ . We define  $\omega_t = |(\partial B)_t|$ .

**Definition 3.4.** For  $1 \leq k, l \leq n$  and  $\mathbf{x}^\beta \in B$ , we define the *commutator polynomial*

$$\mathcal{C}_\beta^{k, l}(\mathbf{x}, \boldsymbol{\mu}) := (M_{\mu, k} M_{\mu, l} - M_{\mu, l} M_{\mu, k})(\mathbf{x}^\beta) = \sum_{\varsigma \in B: |\varsigma| \geq |\beta| + 2} \mathcal{C}_{\beta, \varsigma}^{k, l}(\boldsymbol{\mu}) \mathbf{x}^\varsigma.$$

The non-zero coefficients  $\mathcal{C}_{\beta, \varsigma}^{k, l}(\boldsymbol{\mu})$  are called *commutator relations*. We say that  $\mathbf{x}^\beta$  is a *commutator monomial* with respect to  $k, l$   $\mathcal{C}_{\beta, \varsigma}^{k, l}(\boldsymbol{\mu}) \neq 0$ .

As the matrices  $M_{\mu,k}$  are upper triangular and nilpotent, we have  $\mathcal{C}_{\beta}^{k,l}(\mathbf{x}) = 0$  for  $|\varsigma| \leq |\beta| + 1$ . We notice that  $\mathcal{C}_{\beta,\varsigma}^{k,l} \in \mathbb{K}[\boldsymbol{\mu}']$ . We verify from (13) that

$$\mathcal{C}_{\beta,\varsigma}^{k,l}(\boldsymbol{\mu}) = \sum_{\theta \in B: |\beta| < |\theta| < |\varsigma|} \mu_{\varsigma, \theta + \mathbf{e}_k} \mu_{\theta, \beta + \mathbf{e}_l} - \mu_{\varsigma, \theta + \mathbf{e}_l} \mu_{\theta, \beta + \mathbf{e}_k}.$$

**Lemma 3.5.** *Assume that  $B$  is closed by division. If  $|\varsigma| \leq |\beta| + 1$  or  $\{\mathbf{x}^\beta, \mathbf{x}^{\beta + \mathbf{e}_k}, \mathbf{x}^{\beta + \mathbf{e}_l}, \mathbf{x}^{\beta + \mathbf{e}_k + \mathbf{e}_l}\} \cap \partial B \geq 2$ , then*

$$\mathcal{C}_{\beta,\varsigma}^{k,l}(\boldsymbol{\mu}) = 0.$$

PROOF. For  $\mathbf{x}^\beta \in B$ , we have

$$M_{\mu,i}(\mathbf{x}^\beta) = \sum_{i=1}^{\delta} \mu_{\beta_i, \beta + \mathbf{e}_k} \mathbf{x}^{\beta_i}.$$

By the orthogonality relations (7),  $M_{\mu,i}(\mathbf{x}^\beta) = \mathbf{x}^{\beta + \mathbf{e}_k}$  if  $\mathbf{x}^{\beta + \mathbf{e}_k} \in B$ . If  $\mathbf{x}^\beta \in B$  is not a border commutator monomial, then

- either  $\mathbf{x}^{\beta + \mathbf{e}_k + \mathbf{e}_l}, \mathbf{x}^{\beta + \mathbf{e}_k}, \mathbf{x}^{\beta + \mathbf{e}_l} \in B$ ,
- or  $\mathbf{x}^{\beta + \mathbf{e}_k + \mathbf{e}_l} \notin B$  and  $\mathbf{x}^{\beta + \mathbf{e}_k}, \mathbf{x}^{\beta + \mathbf{e}_l} \in B$ ,

since  $B$  is closed by division. In the first case,

$$\mathcal{C}_{\beta}^{k,l}(\mathbf{x}) = M_{\mu,k}(\mathbf{x}^{\beta + \mathbf{e}_l}) - M_{\mu,l}(\mathbf{x}^{\beta + \mathbf{e}_k}) = \mathbf{x}^{\beta + \mathbf{e}_l + \mathbf{e}_k} - \mathbf{x}^{\beta + \mathbf{e}_k + \mathbf{e}_l} = 0.$$

In the second case,

$$\mathcal{C}_{\beta}^{k,l}(\mathbf{x}) = M_{\mu,k}(\mathbf{x}^{\beta + \mathbf{e}_l}) - M_{\mu,l}(\mathbf{x}^{\beta + \mathbf{e}_k}) = \sum_{\theta} \mu_{\theta, \beta + \mathbf{e}_l + \mathbf{e}_k} \mathbf{x}^{\theta} - \sum_{\theta} \mu_{\theta, \beta + \mathbf{e}_k + \mathbf{e}_l} \mathbf{x}^{\theta} = 0.$$

This proves that  $\mathcal{C}_{\beta,\varsigma}^{k,l}(\boldsymbol{\mu}) = 0$  when  $\mathbf{x}^\beta \in B$  is not a commutator monomial.

When  $|\varsigma| \leq |\beta| + 1$ ,  $\mathcal{C}_{\beta,\varsigma}^{k,l}(\boldsymbol{\mu}) = 0$  by construction, since the matrices  $M_{\mu,k}$  are upper triangular and nilpotent.  $\square$

We can now provide a minimal set of equations in  $\mathbb{K}[\boldsymbol{\mu}]$  defining  $\text{Hilb}_B$ .

**THEOREM 3.6.** *The affine chart  $\text{Hilb}_B$  is defined by the equations:*

- (i)  $\mu_{\beta,\alpha} - \sum_{k=1}^n \sum_{\theta \in B: |\theta| < |\beta|} \sum_{\gamma \notin B: |\gamma| \leq |\theta|} \mu_{\beta, \theta + \mathbf{e}_k} \mu_{\theta, \gamma} \langle \oint_k \mathbf{y}^\gamma | \mathbf{x}^\alpha \rangle = 0$  (integration)  
for  $\beta \in B$ ,  $|\alpha| \leq |\beta|$  with  $\alpha \notin B^+$ .
- (c)  $\sum_{\theta \in B: |\beta| < |\theta| < |\varsigma|} \mu_{\varsigma, \theta + \mathbf{e}_k} \mu_{\theta, \beta + \mathbf{e}_l} - \mu_{\varsigma, \theta + \mathbf{e}_l} \mu_{\theta, \beta + \mathbf{e}_k} = 0$  (commutation)  
for  $\beta \in B$  commutator monomial with respect to  $k \neq l \in \{1, \dots, n\}$   
and  $|\beta| + 2 \leq |\varsigma|$ .

in the variables  $\mu_{\beta_i, \gamma_j} = \frac{\Delta_B^{i \leftarrow \gamma_j}}{\Delta_B}$  where  $(\Delta_A)_{|A|=\delta, A \subset \mathcal{M}_{\delta-1}}$  are the Plücker coordinates in  $\mathcal{G}_\delta(\mathbb{K}[\mathbf{y}]_{\delta-1})$ , with  $\mu_{\beta_i, \beta_j} = \delta_{i,j}$  for  $1 \leq i, j \leq \delta$  and  $\mu_{\beta, \gamma} = 0$  if  $|\beta| < |\gamma|$ .

PROOF. By Theorem 2.7,  $\mathcal{L} \in \text{Hilb}_B$  iff we have, for  $1 \leq i \leq \delta$ , the integration relations

$$\lambda_i := \sum_{|\alpha| < \delta} \mu_{\beta_i, \alpha} \mathbf{y}^\alpha = \sum_{|\beta_j| < |\beta_i|} \sum_{k=1}^n \mu_{\beta_i, \beta_j + \mathbf{e}_k} \oint_k \lambda_{\beta_j}$$

and the commutation relations (6). By Lemma 3.5, it is enough to consider the relations (c), associated to commutator monomials. We deduce from the integration relations that

$$\mu_{\beta_i, \alpha} = \langle \lambda_{\beta_i} | \mathbf{x}^\alpha \rangle = \sum_{|\beta_j| < |\beta_i|} \sum_{k=1}^n \mu_{\beta_i, \beta_j + \mathbf{e}_k} \langle \oint_k \lambda_{\beta_j} | \mathbf{x}^\alpha \rangle,$$

which gives the relations (i) by linearity. To conclude the proof, let us show that if  $\alpha \in B^+$ , then the relation (i) is identically 0. If  $\alpha = \mathbf{0}$ , then  $\mu_{\beta_i, \mathbf{0}} = 0$  and  $\langle \oint_k \lambda_{\beta_j} | 1 \rangle = 0$ , which shows that the relation (i) is identically 0.

Let  $\mathbf{x}^\alpha = x_m \mathbf{x}^\beta \in B^+$  with  $|\alpha| > 0$ ,  $\mathbf{x}^\beta \in B$  and where  $m$  is the maximal index of the variable such that  $\mathbf{x}^\alpha = x_m \mathbf{x}^\beta$  with  $\mathbf{x}^\beta \in B$ . Then the integration relation (i) for  $\alpha = \beta + \mathbf{e}_m$  writes as

$$\begin{aligned} \mu_{\beta_i, \alpha} &= \sum_{|\beta_j| < |\beta_i|} \sum_{k=1}^n \mu_{\beta_i, \beta_j + \mathbf{e}_k} \langle \oint_k \lambda_{\beta_j} | \mathbf{x}^\alpha \rangle \\ &= \mu_{\beta_i, \alpha} - \sum_{|\beta_j| < |\beta_i|} \sum_{k=1}^n \mu_{\beta_i, \beta_j + \mathbf{e}_k} \langle \partial_m \oint_k \lambda_{\beta_j} | \mathbf{x}^\beta \rangle \\ &= \mu_{\beta_i, \alpha} - \sum_{|\beta_j| < |\beta_i|} \mu_{\beta_i, \beta_j + \mathbf{e}_m} \langle \lambda_{\beta_j} | \mathbf{x}^\beta \rangle \\ &= \mu_{\beta_i, \alpha} - \mu_{\beta_i, \beta + \mathbf{e}_m} = 0 \end{aligned}$$

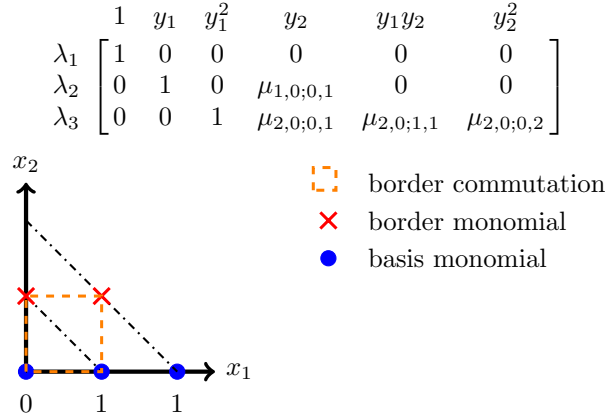
by the orthogonality relations (7) and since  $\alpha = \beta + \mathbf{e}_m$ . This shows that the relations (i) are not needed if  $\alpha \in B^+$ , which concludes the proof.  $\square$

**Definition 3.7.** Let  $\mathcal{I}_B$  be the ideal of  $\mathbb{K}[\boldsymbol{\mu}]$  generated by the equations (i), (c).

By Theorem 3.6,  $\text{Hilb}_B = \mathcal{V}(\mathcal{I}_B)$ . We can remark that the equations (i) and (c) are filtered by degree. We define  $B_t = B \cap \mathbb{K}[\mathbf{x}]_t$  as the set of elements in  $B$  of degree  $\leq t$ . Let  $\delta_t = |B_t|$ . Let  $\boldsymbol{\mu}_t = \{\mu_{\beta, \alpha} \in \boldsymbol{\mu} \text{ s.t. } |\beta| \leq t\}$  and let  $\mathcal{I}_{B,t}$  be the ideal of  $\mathbb{K}[\boldsymbol{\mu}_t]$  generated by the relations (i) with  $|\beta| \leq t$  and the relations (c) with  $|\varsigma| \leq t$ . Then by restriction, we deduce from Theorem 3.6 the following:

**Corollary 3.8.** For  $t \leq \delta$ , the ideal  $\mathcal{I}_{B,t} \subset \mathbb{K}[\boldsymbol{\mu}_t]$  defines  $\text{Hilb}_{B_t} \subset \text{Hilb}_{\delta_t}$ .

**Example 3.9.** We consider  $B = \{1, x_1, x_1^2\} \subset \mathbb{K}[x_1, x_2]$  with  $\delta = 3$ . We have  $(\partial B)_2 = \{x_2, x_1 x_2\}$ .



The only monomial of degree  $\leq 2 = \delta - 1$ , which is not in  $(\partial B)_2$  is  $x_2^2$  and the corresponding integration relation (i) is

$$\mu_{2,0;0,2} - \mu_{2,0;1,1} \mu_{1,0;0,1} = 0.$$

The commutation relation (c) for the commutator monomial 1 is

$$\mu_{2,0;1,1} - \mu_{1,0;0,1} = 0.$$

These relations define  $\text{Hilb}_B$  as an irreducible algebraic variety parametrized by  $\mu_{1,0;0,1}, \mu_{2,0;1,1}$  of dimension 2.

#### 4. The geometry of $\text{Hilb}_B$

In this section, we discuss some properties of  $\text{Hilb}_B$ , exploiting the effective description of Section 3.2.

**4.1. Transversality of  $\text{Hilb}_B$ .** Having equations defining  $\text{Hilb}_B$  at hand, a natural question is whether  $\mathcal{L} \in \text{Hilb}_B$  is the transversal intersection of  $\text{Hilb}_B$  with the Grassmannian of  $\delta$ -linear spaces, which are orthogonal to  $Q = \mathcal{L}^\perp$ . To analyse this property, we study the ideal defining this intersection in  $\mathbb{K}[\boldsymbol{\mu}]$ . For that purpose, we define the *parametric normal form*:

$$\begin{aligned} N : \mathbb{K}[\mathbf{x}, \boldsymbol{\mu}] &\rightarrow \text{span}_{\mathbb{K}[\boldsymbol{\mu}]}(B) \\ p(\mathbf{x}, \boldsymbol{\mu}) &\mapsto \sum_{j=1}^{\delta} \langle \lambda_j | p \rangle \mathbf{x}^{\beta_j} \end{aligned}$$

We verify that for  $|\alpha| < \delta$ ,

$$N(\mathbf{x}^\alpha) = \sum_{j=1}^{\delta} \mu_{\beta_j, \alpha} \mathbf{x}^{\beta_j} = \sum_{j=1}^{\delta} \frac{\Delta_{B^{j \leftarrow \alpha}}}{\Delta_B} \mathbf{x}^{\beta_j}.$$

For  $\alpha \in \mathbb{N}^n$ , let  $\mathbf{M}_{\boldsymbol{\mu}}^\alpha = M_{\mu,1}^{\alpha_1} \cdots M_{\mu,n}^{\alpha_n}$  and for  $p = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{x}^\alpha \in \mathbb{K}[\mathbf{x}]$ , let

$$p(\mathbf{M}_{\boldsymbol{\mu}}) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha \mathbf{M}_{\boldsymbol{\mu}}^\alpha : \text{span}_{\mathbb{K}[\boldsymbol{\mu}]}(B) \rightarrow \text{span}_{\mathbb{K}[\boldsymbol{\mu}]}(B)$$

be the operator of multiplication by  $p$ . Then as  $\beta_1 = \mathbf{0}$ ,  $p(\mathbf{M}_{\boldsymbol{\mu}})(1)$  is a well defined element of  $\text{span}_{\mathbb{K}[\boldsymbol{\mu}]}(B)$ .

**Lemma 4.1.** *For any  $p \in \mathbb{K}[\mathbf{x}; \boldsymbol{\mu}]$ , we have*

$$(14) \quad N(p) - p(\mathbf{M}_{\boldsymbol{\mu}})(1) \equiv 0 \pmod{\mathcal{I}_B}$$

**PROOF.** We proof by induction on  $|\alpha|$  that  $N(\mathbf{x}^\alpha) - \mathbf{M}_{\boldsymbol{\mu}}^\alpha(1) \pmod{\mathcal{J}}$  for  $\alpha \in \mathbb{N}^n$ . The property is true for  $|\alpha| = 0$  since  $N(1) = 1$ . Assume it is true for  $|\alpha'| = t - 1$  and let  $\alpha = \alpha' + \mathbf{e}_j$  for some  $1 \leq j \leq n$ . Then

$$\begin{aligned} N(\mathbf{x}^\alpha) &= \sum_{j=1}^{\delta} \langle \lambda_j | \mathbf{x}^\alpha \rangle \mathbf{x}^{\beta_j} = \sum_{j=1}^{\delta} \langle \partial_j \lambda_j | \mathbf{x}^{\alpha'} \rangle \mathbf{x}^{\beta_j} \\ &= M_{\mu,j} \left( \sum_{j=1}^{\delta} \langle \lambda_j | \mathbf{x}^{\alpha'} \rangle \mathbf{x}^{\beta_j} \right) = M_{\mu,j} N(\mathbf{x}^{\alpha'}) \\ &= M_{\mu,j} \mathbf{M}_{\boldsymbol{\mu}}^{\alpha'}(1) \pmod{\mathcal{I}_B} \quad (\text{by induction hypothesis}) \\ &= \mathbf{M}_{\boldsymbol{\mu}}^\alpha(1) \pmod{\mathcal{I}_B} \quad (\text{by commutation of } M_{\mu,i} \text{ modulo } \mathcal{I}_B) \end{aligned}$$

This proves by induction that the coefficient  $\mu_{\beta_j, \alpha}$  of  $\mathbf{x}^{\beta_j}$  is the coefficient  $\langle \mathbf{y}^{\beta_j} | \mathbf{M}_{\boldsymbol{\mu}}^\alpha(1) \rangle$  of  $\mathbf{x}^{\beta_j}$  in the the image of 1 by the multiplication operator  $\mathbf{M}_{\boldsymbol{\mu}}^\alpha$ , modulo  $\mathcal{I}_B$ .  $\square$

We now prove that the equations (i) and (c) combined with the relations (8) define a simple point, which is the inverse system associated to the isolated singular point  $\mathbf{0}$ . The proof follows the same lines as in [17].

For  $p \in \mathbb{K}[\mathbf{x}]$ , let  $p^{(\perp)} = \{\mathcal{L} \in \mathcal{G}(\mathbb{K}[\mathbf{y}]_{\delta-1}) \mid \forall \lambda \in \mathcal{L}, \langle \lambda | p \rangle = 0\}$  be the Grassmannian of  $\delta$ -linear spaces, which elements are orthogonal to  $p$ . For  $P \subset \mathbb{K}[\mathbf{x}]$ , let  $P^{(\perp)} = \bigcap_{p \in P} p^{(\perp)}$ .

**THEOREM 4.2.** *Let  $\mathcal{L} \in \text{Hilb}_B$ ,  $Q = \mathcal{L}^\perp$  and  $\mathbf{f} = \{f_1, \dots, f_s\} \subset \mathbb{K}[\mathbf{x}]$  such that  $Q$  is an isolated  $\mathbf{m}$ -primary component of the ideal  $I = \langle \mathbf{f} \rangle$ . Then,  $\mathbf{f}^{(\perp)}$  intersects transversally  $\text{Hilb}_B$  at  $\mathcal{L}$ .*

PROOF. Let  $\mathcal{L} \in \text{Hilb}_B$  with Plücker coordinates  $\boldsymbol{\nu} = (\nu_{\beta_i, \gamma}) \in \mathbb{K}^N$  satisfying equations (i) and (c), which generate the ideal  $\mathcal{I}_B(\boldsymbol{\mu}) \subset \mathbb{K}[\boldsymbol{\mu}]$ . Let  $I_{\mathbf{f}} = \langle \langle \lambda_i | f_j \rangle \rangle$  be the ideal of  $\mathbb{K}[\boldsymbol{\mu}]$  defining  $\mathbf{f}^{(\perp)}$ . Let  $\mathcal{J} = \mathcal{I}_B + I_{\mathbf{f}} \subset \mathbb{K}[\boldsymbol{\mu}]$ . By construction, we have

$$\mathcal{L} \in \mathcal{V}(\mathcal{J}) \text{ or } \mathcal{J} \subset \mathbf{m}_{\boldsymbol{\nu}},$$

where  $\mathbf{m}_{\boldsymbol{\nu}} := (\mu_{\beta_i, \gamma} - \nu_{\beta_i, \gamma})$  is the maximal ideal defining  $\boldsymbol{\nu}$ .

Let us prove that  $\mathbb{A} = \mathbb{K}[\boldsymbol{\mu}]/\mathcal{J}$  is of dimension 1, which implies that  $\mathbf{f}^{(\perp)}$  intersects transversally  $\text{Hilb}_B$  and that  $\mathcal{L}$  is the only intersection point. We denote consider the map

$$\begin{aligned} \Phi : \mathbb{A}[\mathbf{x}] &\rightarrow \text{span}_{\mathbb{A}}(B) \\ p &\mapsto p(M_{\boldsymbol{\mu}})(1) \end{aligned}$$

where  $p(M_{\boldsymbol{\mu}})$  is the linear operator of  $\text{span}_{\mathbb{A}}(B)$  obtained by replacing  $x_i$  by  $M_{\boldsymbol{\mu}, i}$ .

Let  $K$  be its kernel. Since the matrices  $M_{\mu, i}$  are commuting modulo  $\mathcal{J}$ ,  $K$  is an ideal of  $\mathbb{A}[\boldsymbol{\mu}]$ . By Lemma 4.1, we have for all  $p \in \mathbb{K}[\mathbf{x}]$ ,

$$(15) \quad \Phi(p) \equiv \sum_i \langle \lambda_i | p \rangle \mathbf{x}^{\beta_i} \pmod{\mathcal{J}}.$$

As  $i = 1, \dots, \delta$  and for  $i = 1, \dots, s$ ,  $\langle \lambda_i | f_j \rangle = 0 \pmod{\mathcal{J}}$ , we deduce that  $\mathbf{f} \subset K$  and thus  $I = \langle \mathbf{f} \rangle \subset K$ .

Next we show that  $Q \subset K$ . Let  $q \in Q$ . As  $Q$  is the  $(x_1, \dots, x_n)$ -primary component of  $I$ , there exists  $p \in \mathbb{K}[\mathbf{x}]$  such that  $p(\mathbf{0}) \neq 0$  and  $pq \in I$ . Since the matrices  $M_{\mu, i}$  commute modulo  $\mathcal{J}$ , we have

$$\Phi(pq) = p(M_{\boldsymbol{\mu}}) \Phi(q) = 0.$$

Since  $p(\mathbf{0}) \neq 0$  and  $p(M_{\boldsymbol{\mu}}) = p(\mathbf{0})Id + N$  with  $N$  lower triangular and nilpotent,  $p(M_{\boldsymbol{\mu}})$  is invertible. We deduce that  $\Phi(q) = p(M_{\boldsymbol{\mu}})^{-1} \Phi(pq) = 0$  and  $q \in K$ , which implies that  $Q \subset K$ .

The relation (15) and the orthogonality relations (7) implies that  $\Phi(\mathbf{x}^{\beta_k}) = \mathbf{x}^{\beta_k}$  for  $k = 1, \dots, \delta$ . This shows that the restriction of  $\Phi$  on  $\text{span}_{\mathbb{A}}(B) \subset \mathbb{A}[\mathbf{x}]$  is the identity.

We can now prove that  $\mathbf{m}_{\boldsymbol{\nu}} \subset \mathcal{J}$ . Let substitute the parameters  $\boldsymbol{\mu}$  by the values  $\boldsymbol{\nu} \in \mathbb{K}^N$  in  $M_{\nu, i}$ . We get the multiplication by  $x_i$  in the basis  $B$  of  $\mathbb{K}[\mathbf{x}]/Q$ . Thus, for any  $b \in B$  and  $i = 1, \dots, n$ , we have  $x_i b = M_{\nu, i}(b) + q$  with  $q \in Q$ . As  $Q \subset K$  and  $\Phi|_{\text{span}_{\mathbb{A}}(B)} = \text{Id}$ , we deduce that for  $\beta \in B$ ,

$$\Phi(x_i \mathbf{x}^{\beta}) = M_{\nu, i} \Phi(\mathbf{x}^{\beta}) = M_{\nu, i}(\mathbf{x}^{\beta}) = \Phi(M_{\nu, i}(\mathbf{x}^{\beta})) = M_{\nu, i}(\mathbf{x}^{\beta}).$$

This shows that for  $\beta \in B$  and  $|\alpha| < \delta$ ,  $\mu_{\beta, \alpha} - \nu_{\beta, \alpha} \in \mathcal{J}$  and thus that  $\mathbf{m}_{\boldsymbol{\nu}} = \mathcal{J}$ , which concludes the proofs.  $\square$

This property of transversal intersection is the basis for Newton-type methods developed in [17, 23] to numerically compute the inverse system and the associated multiple point.

**4.2. Irreducible components of  $\text{Hilb}_B$  and commutation rank.** For  $t \in \mathbb{N}$  and  $\varsigma \in B_{[t]}$ , let  $\boldsymbol{\mu}_{\varsigma} = \{\mu_{\gamma, \delta} \in \boldsymbol{\mu} \text{ s.t. } |\gamma| \in (\partial B)_t\}$ . Let  $\omega_t = |\boldsymbol{\mu}_{\varsigma}| = |(\partial B)_t|$ .

We verify that the commutation relations (c) are linear in the variables  $\boldsymbol{\mu}_{\varsigma}$  (and in the variables  $\boldsymbol{\mu}_{t-1} := \{\mu_{\beta, \alpha} \in \boldsymbol{\mu} \text{ s.t. } |\beta| \leq t-1, |\alpha| \leq |\beta|, \alpha \notin B\}$ ).

We exploit this property to describe the tangent space of  $\text{Hilb}_B$  at regular points. For  $\varsigma \in B_{[t]}$ , let  $\mathcal{C}_{\varsigma} = \{C_{\beta, \varsigma}^{k, l} \neq 0\}$  be the set of non-zero commutator relations for  $\varsigma$ . Let  $H_{\varsigma} \in \mathbb{K}[\boldsymbol{\mu}_{t-1}]^{|\mathcal{C}_{\varsigma}| \times \omega_t}$  and  $\overline{H}_{\varsigma} \in \mathbb{K}[\boldsymbol{\mu}_{t-1}]^{|\mathcal{C}_{\varsigma}|}$  be coefficient matrices of the commutator relations such that

$$H_{\varsigma} \cdot [\boldsymbol{\mu}_{\varsigma}] + \overline{H}_{\varsigma} = [C_{\beta, \varsigma}]_{\beta \in \mathcal{C}_{\varsigma}}$$

For  $\mathcal{L} \in \text{Hilb}_B$ , the commutator relations are satisfied iff

$$\text{rank } H_{\varsigma}(\mathcal{L}_{t-1}) = \text{rank } [H_{\varsigma}(\mathcal{L}_{t-1}), \overline{H}_{\varsigma}(\mathcal{L}_{t-1})].$$

**Definition 4.3** (Commutation rank). The commutation rank of  $\mathcal{L} \in \text{Hilb}_B$  at  $\varsigma \in B_{[t]}$  is  $\text{cr}_{\varsigma}(\mathcal{L}_{t-1}) := \text{rank } H_{\varsigma}(\mathcal{L}_{t-1}) = \text{rank } [H_{\varsigma}(\mathcal{L}_{t-1}), \overline{H}_{\varsigma}(\mathcal{L}_{t-1})]$ .

In the case where the commutation relations have a solution, we have

$$(16) \quad \dim\{\boldsymbol{\mu}_\varsigma \in \mathbb{K}^{\omega_t} \text{ s.t. } H_\varsigma(\mathcal{L}_{t-1}) \cdot [\boldsymbol{\mu}_\varsigma] + \overline{H}_\varsigma(\mathcal{L}_{t-1}) = 0\} = \omega_t - \text{cr}_\varsigma(\mathcal{L}_{t-1}).$$

For an irreducible component  $\mathcal{V}$  of  $\text{Hilb}_B$ , the commutation rank  $\text{cr}_\varsigma(\mathcal{L}_{t-1})$  with  $\mathcal{L} \in \mathcal{V}$  is generically maximal (by semi-continuity of the rank function). We say that  $\mathcal{L} \in \mathcal{V}$  is *regular* if for all  $1 \leq t < \deg(B)$  and all  $\varsigma \in B_{[t]}$ ,  $\text{cr}_\varsigma(\mathcal{L}_{t-1})$  is the generic commutation rank at  $\varsigma$  on  $\mathcal{V}$ .

**THEOREM 4.4.** *For an irreducible component  $\mathcal{V}$  of  $\text{Hilb}_B$  and a regular point  $\mathcal{L} \in \mathcal{V}$ , we have*

$$\dim \mathcal{V} = \sum_{t \leq \deg(B)} (|B_{[t]}| |(\partial B)_t| - \sum_{\varsigma \in B_{[t]}} \text{cr}_\varsigma(\mathcal{L}_{t-1}))$$

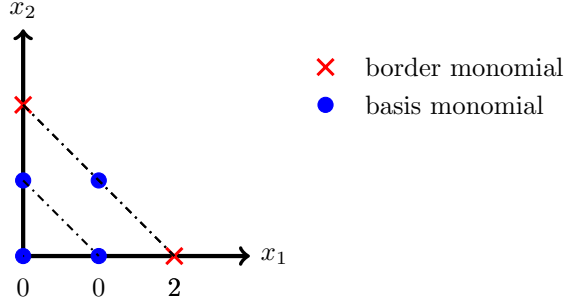
**PROOF.** For  $t \in \mathbb{N}$ , let  $\pi_t$  be the projection from  $\mathbb{K}[\mathbf{y}]_\delta$  to  $\mathbb{K}[\mathbf{y}]_t$  obtained by truncation in degree  $\leq t$ . We denote by  $\mathcal{V}_t := \pi_t(\mathcal{V})$  the induced projection of  $\mathcal{V}$ . Then  $\mathcal{V}_t$  is irreducible and projects on  $\mathcal{V}_{t-1}$ . Let  $\boldsymbol{\mu}' = \cup_{\varsigma \in B} \boldsymbol{\mu}_\varsigma$  be the set of border variables. By the integration relations (i) which express the coefficients which are not in  $\boldsymbol{\mu}'$  as polynomials in the variables  $\boldsymbol{\mu}'$  and by (16), we deduce that the fiber of  $\pi_{t-1}$  is generically irreducible and of dimension  $\sum_{\varsigma \in B} \omega_t - \text{cr}_\varsigma(\mathcal{L}_{t-1})$  for a regular point  $\mathcal{L} \in \mathcal{V}$ . Summing up the dimensions of the fibers for  $t = 1, \dots, \deg(B)$ , we obtain the dimension of the irreducible component  $\mathcal{V}$ .  $\square$

**THEOREM 4.5.** *The irreducible components of  $\text{Hilb}_B$  are birational varieties.*

**PROOF.** Let  $\mathcal{V}$  be an irreducible component of  $\text{Hilb}_B$ . We prove by induction on  $t$ , that  $\mathcal{V}_t = \pi_t(\mathcal{V})$  is birationally parametrised. The property is clearly true for  $\mathcal{V}_0 = \{1\}$ . If  $\mathcal{L}_{t-1} \in \mathcal{V}_{t-1}$  is birationally parametrised then the affine solution set of the system in (16) is also birationally parametrised. Thus  $\mathcal{V}_t$  is birational. This proves by induction that  $\mathcal{V}$  is birational.  $\square$

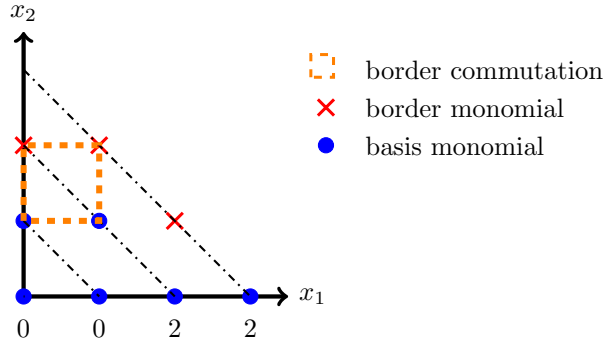
**4.3. Examples.** Let us illustrate the previous results on few simple examples.

**Example 4.6.** We consider first  $B = \{1, x_1, x_2, x_1x_2\}$  with  $\delta = 4$ .



We have  $\deg(B) = 2$ . There is no monomial of degree  $\leq 2$ , which is not in  $(\partial B)_2$ . Thus we have no integration relation (i). There is no commutator relation (c). Thus  $\mathcal{I}_B = \langle 0 \rangle \subset \mathbb{K}[\mu_{1,1;2,0}, \mu_{1,1;0,2}]$  and  $\text{Hilb}_B$  is irreducible of dimension 2.

**Example 4.7.** We consider now  $B = \{1, x_1, x_2, x_1^2, x_1x_2, x_1^3\}$  with  $\delta = 6$ .



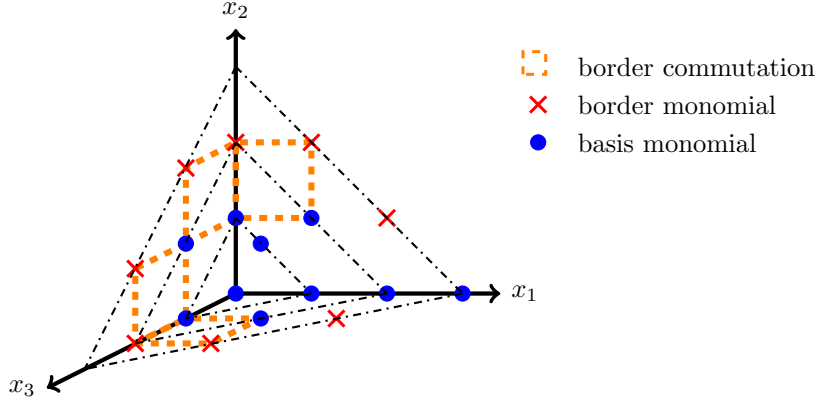
We have  $\deg(B) = 3$ . There is one integration relation (i) for the monomial  $x_2^3$ , which is not in  $(\partial B)_3$ . It expresses  $\mu_{3,0;0,3}$  as a polynomial function of the variables  $\mu'$ . There is one non-zero commutation relation (c):

$$\mu_{3,0;1,2} - \mu_{3,0;2,1}\mu_{1,1;0,2} - \mu_{2,0;0,2} = 0$$

which is irreducible and expresses  $\mu_{3,0;1,2}$  in terms of the other variables of  $\mu'$ . Thus  $\text{Hilb}_B$  is irreducible and birational. By Theorem 4.4, we have

$$\dim \text{Hilb}_B = 2 \times 2 + (1 \times 3 - 1) = 4.$$

**Example 4.8.** We take  $B = \{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1^3, x_1x_2x_3\}$  with  $\delta = 10$ .



We have  $\deg(B) = 3$ . There are 2 integration relations (i) for the monomials  $x_2^3, x_3^2$ . We have 4 commutation relations (c) corresponding to the commutator monomial  $x_2$  with respect to  $(x_1, x_2)$  and  $(x_2, x_3)$ ,  $x_3$  with respect to  $(x_1, x_3)$  and  $(x_2, x_3)$ . An explicit computation shows that they yield polynomial expressions of  $\mu_{\varsigma;1,2;0}, \mu_{\varsigma;0,2,1}, \mu_{\varsigma;0,1,2}, \mu_{\varsigma;1,0,2}$  for  $\varsigma \in \{(3, 0, 0), (1, 1, 1)\} = B_{[3]}$  as polynomial functions of the other variables. Thus  $\text{Hilb}_B$  is irreducible and birational and the generic commutation rank  $\text{cr}_{\varsigma}$  is 4 for  $\varsigma \in B_{[3]}$ . By Theorem 4.4, we have

$$\dim \text{Hilb}_B = 3 \times 4 + (2 \times 8 - 4 - 4) = 20.$$

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