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RESEARCH ARTICLE

Control Lyapunov function method for robust stabilization of multistable affine nonlinear systems

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Abstract

In this paper, we study the problem of robust stabilization of affine nonlinear multistable systems in the presence of exogenous disturbances. The results are based on the theory of input-to-state stability (ISS) and integral input-to-state stability (iISS) for systems with multiple invariant sets. The notions of ISS and iISS control Lyapunov functions (CLFs) and the small control property are extended within the multistability framework. Such properties are also complemented by the concept of a weak iISS CLF and corresponding small control property. It is verified that the universal control formula can be applied to yield the ISS (iISS) property for the closed-loop system. The efficiency of the extended CLF framework in the multistable sense is illustrated for a Duffing system and in application to a noise-induced transition in a semiconductor-gas-discharge gap system.

KEYWORDS:

Multistability, Control Lyapunov Functions, (integral) Input-to-state stability

1 | INTRODUCTION

The multistability phenomenon is present in almost all natural sciences (ranging, for example, from mechanics and electronics^{1,2} to biology^{3,4,5} and neurosciences⁶), and it arises when it is necessary to analyze the global behavior of complex nonlinear dynamics, taking into account all its possible final states and motions. Therefore, analyzing convergence and robustness properties of multistable systems has become increasingly attractive from the systems and control theory perspective. Different kinds of multistable systems are studied in the literature: bistable systems (with at least two stable equilibria)^{7,8}, almost globally stable systems (with only one attracting invariant set)⁹, and nonlinear systems with generic invariant sets^{10,11,12,13,14,15,16,17}. Due to the nontrivial relationships between different regions that compose its state space and their complex intertwined boundaries, multistable systems are extremely sensitive to initial conditions and perturbations, making their stability analysis and control design extremely challenging.

In¹⁸, a global asymptotic stability notion was proposed for multistable systems with the respective necessary and sufficient Lyapunov characterization, having as the object of investigation all compact invariant solutions of the system (including locally stable and unstable ones, which can be disconnected). The global nature of stability and the strict properties of Lyapunov functions have also been proven useful in robustness analysis with respect to external disturbances. To this end, in¹⁹, it was shown that the most natural way to solve this problem is to relax the Lyapunov stability requirement by relatively mild additional assumptions on the possibility of decomposition of invariant sets. This insight has led to a new line of research, starting from the characterizations of input-to-state stability (ISS) for a class of multistable systems in terms of usual Lyapunov dissipation inequalities, generalizing the classical ISS theory^{20,21}. In its turn, integral input-to-state stability (iISS) property^{22,23,24,25,19,26} is weaker than ISS, and it was extended in²⁷ for systems with multiple invariant sets.

Once ISS and iISS characterizations in the multistable sense are available, the problem of designing robust stabilizing control laws in this framework naturally arises. In this setting, such a problem consists of finding state feedback control laws that make the closed-loop system ISS or iISS with respect to a family of finite disjoint compact invariant sets in the presence of external disturbances. The classical approach's most generic results are formulated within the control Lyapunov function (CLF) theory^{28,29,30,31,32}. In²⁸ and²⁹, it was shown that the existence of a CLF is necessary and sufficient for stabilizing an equilibrium and that it leads to an explicit formulation for a stabilizing control law. Similar results were proven in³³ for the ISS (iISS) case resulting in appropriated universal CLF formulation, which is equivalent to the existence of a feedback rendering the closed-loop system with ISS (iISS) properties. In this work, we are interested in the robust stabilization of multistable affine nonlinear systems in the presence of disturbance inputs. Our approach is based on the theory developed in^{19,27}, and it aims to find conditions of CLF existence in the context of systems with multiple invariant sets (compact and maybe disconnected) and to show how a control formula from²⁹ based on CLF can be explicitly developed for robust stabilization in ISS (iISS) multistable sense.

The outline of this work is as follows. The main used definitions are given in Section 2. The problem statement is formulated in Section 3, while the main results are presented in Section 4. Section 5 contains examples of the application of the developed CLF approach. Final remarks and discussion are summarized in Section 6.

Notation

- Denote the set of real numbers by \mathbb{R} , and $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$. The set of natural numbers is denoted by \mathbb{N} .
- For $u : \mathbb{R}_+ \rightarrow \mathbf{U} \subseteq \mathbb{R}^m$, define its L_∞ -norm as $\|u\|_\infty = \text{ess sup}_{t \geq 0} |u(t)|$, where $|\cdot|$ is the respective Euclidean norm. The set of all such functions $u : \mathbb{R}_+ \rightarrow \mathbf{U}$ with the property $\|u\|_\infty < +\infty$ is denoted by $\mathcal{L}_\infty(\mathbf{U})$.
- We denote by \mathcal{M} a n -dimensional connected and orientable Riemannian manifold without a boundary. For $x \in \mathcal{M}$, its tangent space is denoted by $T_x \mathcal{M} \subseteq \mathbb{R}^n$. The distance from $x \in \mathcal{M}$ to a set $\mathcal{W} \subset \mathcal{M}$ is given by:

$$|x|_{\mathcal{W}} = \inf_{a \in \mathcal{W}} \delta(x, a),$$

where $\delta(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in \mathcal{M} , with the convention that for a point $x_{\text{or}} \in \mathcal{M}$, selected as the origin on \mathcal{M} , $|x| = |x|_{\{x_{\text{or}}\}}$ can be considered as a norm of a vector $x \in \mathcal{M}$ (with a minor ambiguity we use the same symbol as for the Euclidean norm).

- The Lie derivative of a continuously differentiable function $V : \mathcal{M} \rightarrow \mathbb{R}_+$ along a vector field $f : \mathcal{M} \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ is defined by:

$$DV(x)f(x, v) = \frac{\partial V(x)}{\partial x} f(x, v)$$

for any $x \in \mathcal{M}$ and $v \in \mathbb{R}^k$.

- Definitions of functions belonging to classes \mathcal{K} and \mathcal{K}_∞ can be found in²⁵.

2 | PRELIMINARIES

2.1 | The multistability framework

Consider nonlinear dynamical systems of the following form:

$$\dot{x}(t) = f(x(t), v(t)), \quad \forall t \in \mathbb{R}_+, \quad (1)$$

$$y(t) = h(x(t)), \quad (2)$$

evolving on a n -dimensional connected and orientable Riemannian manifold \mathcal{M} without boundary, and let $f : \mathcal{M} \times \mathbb{R}^k \rightarrow T_x \mathcal{M}$ be a locally Lipschitz continuous function on \mathcal{M} mapping the state vector $x(t) \in \mathcal{M}$ and the input vector $v(t) \in \mathbb{R}^k$ to the tangent space $T_x \mathcal{M} \subseteq \mathbb{R}^n$; $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^p$ is the corresponding output vector, $h : \mathcal{M} \rightarrow \mathcal{Y}$ is a continuous function.

For any $x_0 \in \mathcal{M}$ and $v \in \mathcal{L}_\infty(\mathbb{R}^k)$ we denote by $x(t, x_0, v)$ the uniquely defined solution of (1) at time $t \in \mathbb{R}_+$ such that $x(0, x_0, v) = x_0$. The short notation $x(t)$ is used when the initial conditions x_0 and the input v are clear.

A set $\mathcal{W} \subset \mathcal{M}$ is called (forward) *invariant* if the solutions of the unperturbed version of (1):

$$\dot{x}(t) = f(x(t), 0), \quad t \in \mathbb{R}_+, \quad (3)$$

do not leave \mathcal{W} for all initial conditions $x_0 \in \mathcal{W}$, i.e., if $x_0 \in \mathcal{W}$ implies that $x(t, x_0, 0) \in \mathcal{W}$ for all $t \in \mathbb{R}_+$.

Definition 1.¹⁴ A point \underline{x} is called an α -*limit point* of (3) if there exists a sequence $(t_k)_{k \in \mathbb{N}}$ of time instants, with $t_k \rightarrow -\infty$ as $k \rightarrow +\infty$, for which the following holds:

$$x(t_k, x_0, 0) \rightarrow \underline{x}, \quad k \rightarrow +\infty. \quad (4)$$

The set of all such points is called α -limit of $x(t, x_0, 0)$, and it is denoted by $\alpha(x_0)$.

Definition 2.¹⁴ A point \bar{x} is called an ω -*limit point* of (3) if there exists a sequence $(t_k)_{k \in \mathbb{N}}$ of time instants, with $t_k \rightarrow +\infty$ as $k \rightarrow +\infty$, for which the following holds:

$$x(t_k, x_0, 0) \rightarrow \bar{x}, \quad k \rightarrow +\infty. \quad (5)$$

The set of all such points is called ω -limit of $x(t, x_0, 0)$, and it is denoted by $\omega(x_0)$.

2.2 | Decomposition of a compact invariant set \mathcal{W}

It is useful to decompose the invariant set \mathcal{W} and explicitly determine the existence of solutions traveling between different components of its decomposition, to characterize the evolution of $x(t, x_0, 0)$ on \mathcal{M} .

Definition 3.³⁴ A *decomposition* of \mathcal{W} is a finite, disjoint family of compact invariant sets $\{\mathcal{W}_1, \dots, \mathcal{W}_k\}$ such that

$$\mathcal{W} = \bigsqcup_{i=1}^k \mathcal{W}_i.$$

For an invariant set \mathcal{W} , its attracting and repulsing subsets can be defined, respectively, as follows:

$$\begin{aligned} \mathfrak{A}(\mathcal{W}) &= \{x_0 \in \mathcal{M} : |x(t, x_0; 0)|_{\mathcal{W}} \rightarrow 0 \text{ as } t \rightarrow +\infty\}, \\ \mathfrak{R}(\mathcal{W}) &= \{x_0 \in \mathcal{M} : |x(t, x_0; 0)|_{\mathcal{W}} \rightarrow 0 \text{ as } t \rightarrow -\infty\}. \end{aligned}$$

We can define a relation between two invariant sets $\mathcal{D}_1 \subset \mathcal{M}$ and $\mathcal{D}_2 \subset \mathcal{M}$ by $\mathcal{D}_1 < \mathcal{D}_2$ if $\mathfrak{A}(\mathcal{D}_1) \cap \mathfrak{R}(\mathcal{D}_2) \neq \emptyset$. This relation implies that there is a solution connecting set \mathcal{D}_2 with set \mathcal{D}_1 .

Definition 4.³⁴ Let $\{\mathcal{W}_1, \dots, \mathcal{W}_k\}$ be a decomposition of \mathcal{W} , then

- 1) An r -*cycle* ($r \geq 2$) is an ordered r -tuple of distinct indices i_1, \dots, i_r such that $\mathcal{W}_{i_1} < \dots < \mathcal{W}_{i_r} < \mathcal{W}_{i_1}$.
- 2) A 1-*cycle* is an index i such that $(\mathfrak{R}(\mathcal{W}_i) \cap \mathfrak{A}(\mathcal{W}_i)) \setminus \mathcal{W}_i \neq \emptyset$.
- 3) A *filtration ordering* is a numbering of the \mathcal{W}_i so that $\mathcal{W}_i < \mathcal{W}_j \Rightarrow i \leq j$.

Remark 1. The existence of an r -cycle with $r \geq 2$ results in a *heteroclinic trajectory* for (3)¹⁴. Also, the existence of a 1-cycle results in a *homoclinic trajectory*¹⁴. It was recognized in¹⁹ that if one wishes to have a strict Lyapunov function (with a proper negative definite derivative) in the multistable scenario, the presence of cycles has to be ruled out, and further, we need a strict Lyapunov function for robust stability analysis.

For the sequel, we will assume that \mathcal{W} satisfies the following hypothesis:

Assumption 1. ¹⁹ The compact invariant set \mathcal{W} , containing all α - and ω -limit sets of the unperturbed system (3), admits a finite decomposition without cycles: $\mathcal{W} = \bigsqcup_{i=1}^r \mathcal{W}_i$ for some non-empty disjoint compact sets \mathcal{W}_i and $r \in \mathbb{N}$, $r \neq 0$, which form a filtration ordering of \mathcal{W} .

2.3 | Robust stability notions with respect to a compact invariant set \mathcal{W}

Below we list several stability properties for (1) and (2) with respect to a compact invariant set \mathcal{W} . Most of these properties are direct extensions of the classical ISS and iISS notions introduced in ^{20,21,23,24}.

Definition 5. ^{19,27} The system (1) has the *practical asymptotic gain (pAG)* property if there exist $\eta \in \mathcal{K}_\infty$ and $q \geq 0$ such that for all $x_0 \in \mathcal{M}$ and all $v \in \mathcal{L}_\infty(\mathbb{R}^k)$, the solutions are defined for all $t \geq 0$ and the following holds:

$$\limsup_{t \rightarrow +\infty} |x(t, x_0, v)|_{\mathcal{W}} \leq \eta(\|v\|_\infty) + q. \quad (6)$$

If $q = 0$, we say the *asymptotic gain (AG)* property holds. Moreover, if (6) is satisfied for $q = 0$ for the system (3) only, then we say that (1) has the *zero-global attraction (0-GATT)* property with respect to \mathcal{W} .

Definition 6. ¹⁹ The system (1) has the *limit property (LIM)* with respect to \mathcal{W} if there exists $\mu \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathcal{M}$ and all $v \in \mathcal{L}_\infty(\mathbb{R}^k)$, the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t \geq 0} |x(t, x_0, v)|_{\mathcal{W}} \leq \mu(\|v\|_\infty).$$

Definition 7. ¹⁹ The system (1) has the *practical global stability (pGS)* property with respect to \mathcal{W} if there exists $\beta \in \mathcal{K}_\infty$ and $c \geq 0$ such that for all $x_0 \in \mathcal{M}$ and all $v \in \mathcal{L}_\infty(\mathbb{R}^k)$, the solutions are defined for all $t \geq 0$ and the following holds:

$$|x(t, x_0, v)|_{\mathcal{W}} \leq \beta(\max\{|x_0|_{\mathcal{W}} + c, \|v\|_\infty\}).$$

Definition 8. ^{19,27} A continuously differentiable function $V : \mathcal{M} \rightarrow \mathbb{R}_+$ is a *practical ISS-Lyapunov function* for (1) if there exist $\alpha_1, \alpha_2, \alpha_3, \gamma \in \mathcal{K}_\infty$ and $q, c \geq 0$ such that for all $x \in \mathcal{M}$:

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}}) + c, \quad (7)$$

and the following dissipation inequality holds for all $(x, v) \in \mathcal{M} \times \mathbb{R}^k$:

$$DV(x)f(x, v) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|v|) + q. \quad (8)$$

If (8) holds for $q = 0$, then V is an *ISS-Lyapunov function*. If (8) holds for $q = 0$ and $\alpha_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a positive definite function, then V is an *iISS-Lyapunov function*.

The existence of α_2 and c follows, without any additional hypothesis, by standard continuity arguments.

Definition 9. ²⁷ The system (1) has the *uniform bounded-energy bounded-state (UBEBS)* property if there exist $\alpha, \gamma, \sigma \in \mathcal{K}_\infty$ and $c \geq 0$ such that the following estimate holds for all $t \geq 0$, all $x_0 \in \mathcal{M}$ and all $v \in \mathcal{L}_\infty(\mathbb{R}^k)$:

$$\alpha(|x(t, x_0, v)|_{\mathcal{W}}) \leq \gamma(|x_0|_{\mathcal{W}}) + \int_0^t \sigma(|v(\tau)|) d\tau + c.$$

Definition 10. ²⁷ The system (1), (2) has the *smooth dissipativity* property if there exist a C^1 function $V : \mathcal{M} \rightarrow \mathbb{R}_+$, $\alpha_1, \alpha_2, \sigma \in \mathcal{K}_\infty$, a continuous positive definite function α_4 , and a continuous output map $h : \mathcal{M} \rightarrow \mathbb{R}^p$ with

$$|x|_{\mathcal{W}} = 0 \Rightarrow h(x) = 0, \quad \forall x \in \mathcal{M}$$

such that (7) is satisfied for all $x \in \mathcal{M}$ and the following dissipation inequality holds for all $(x, v) \in \mathcal{M} \times \mathbb{R}^k$:

$$DV(x)f(x, v) \leq -\alpha_4(|h(x)|) + \sigma(|v|). \quad (9)$$

Definition 11. ²⁷ The system (1), (2) has the *weak zero-detectability* property if:

$$h(x(t, x_0, 0)) = 0, \quad \forall t \geq 0$$

implies $|x(t, x_0, 0)|_{\mathcal{W}} \rightarrow 0$ as $t \rightarrow +\infty$.

The principal results connecting these properties are given by:

Theorem 1. ¹⁹ Consider a nonlinear system (1) and let Assumption 1 hold. Then the following properties are equivalent:

1. The system enjoys the pAG or AG property.
2. The system admits an ISS Lyapunov function.
3. The system admits an ISS Lyapunov function constant on \mathcal{W} .
4. The system admits a practical ISS Lyapunov function.
5. The system enjoys the LIM property and the pGS.

A system (1) satisfying these properties is called *ISS in the multistable sense* from the input v with respect to the set \mathcal{W} .

Theorem 2. ²⁷ Consider a nonlinear system (1) and let Assumption 1 hold. Then the following properties are equivalent:

1. 0-GATT and UBEBs properties.
2. Existence of an iISS Lyapunov function constant on \mathcal{W} .
3. Existence of an iISS Lyapunov function.
4. Existence of an output function that makes the system smoothly dissipative and weakly zero-detectable.

A system (1) satisfying these properties is called *iISS in the multistable sense* from the input v with respect to the set \mathcal{W} .

3 | PROBLEM STATEMENT

In this work, we deal with a class of nonlinear dynamical systems that are affine in the control of the following form:

$$\dot{x}(t) = f(x(t), v(t)) + G(x(t))u(t), \quad t \in \mathbb{R}_+, \quad (10)$$

$$y(t) = h(x(t)), \quad (11)$$

where $x(t) \in \mathcal{M}$ is the state vector; $v(t) \in \mathbb{R}^k$ is a disturbance with $v \in \mathcal{L}_\infty(\mathbb{R}^k)$; $u(t) \in \mathbf{U} \subseteq \mathbb{R}^m$ is the input vector and $u \in \mathcal{U} = \mathcal{L}_\infty(\mathbf{U})$, where \mathbf{U} defines the set of admissible values for the control; $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^p$ is the corresponding output vector. For this system, $f : \mathcal{M} \times \mathbb{R}^k \rightarrow \mathbb{R}^n$ and the columns of the matrix function $G : \mathcal{M} \rightarrow \mathbb{R}^{n \times m}$ are assumed to be locally Lipschitz continuous on \mathcal{M} . We also assume that $f(0, 0) = h(0) = 0$ (without losing generality let $x_{\text{or}} = 0 \in \mathcal{M}$).

Systems in the form (10), (11) are said to be *ISS (iISS) stabilizable* if there exists a control law $u = K(x)$, where $K : \mathcal{M} \rightarrow \mathbf{U}$, rendering the closed-loop system with the ISS (iISS) property from $v \in \mathbb{R}^k$ with respect to a given set $\mathcal{W} \subset \mathcal{M}$. **By assuming the restrictions given in Assumption 1 on \mathcal{W} the problem studied in this work can be formally determined as follows:**

Problem 1. For the model (10), (11) under Assumption 1, find the conditions of its ISS (iISS) stabilizability by u from v with respect to \mathcal{W} .

Special attention has to be paid to the continuity of the control map K .

4 | MAIN RESULTS

As in the case of ISS/iISS stabilization with respect to a compact set^{23,29}, we search for existence conditions of a stabilizing feedback using the CLF concept. To present our main result, we introduce a suitable notion of ISS (iISS) CLF in a multistability framework.

Definition 12. A *practical ISS CLF* for the system (10) and control $u \in \mathcal{U}$, is a twice continuously differentiable function $V : \mathcal{M} \rightarrow \mathbb{R}_+$ satisfying:

1. There exist \mathcal{K}_∞ functions α_1, α_2 and $c \geq 0$ such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}}) + c \quad (12)$$

for all $x \in \mathcal{M}$, and $\frac{\partial V(x)}{\partial x} = 0$ for any $x \in \mathcal{W}$.

2. There exist \mathcal{K}_∞ functions χ, α_3 and $q \geq 0$ such that for all $x \in \mathcal{M}$ and all $v \in \mathbb{R}^k$:

$$\inf_{u \in \mathbf{U}} \{a(x, v) + b(x)u\} \leq -\alpha_3(|x|_{\mathcal{W}}) + \chi(|v|) + q, \quad (13)$$

where $a(x, v) = DV(x)f(x, v)$, $b(x) = DV(x)G(x)$.

If (13) holds for $q = 0$, then V is an *ISS CLF*. In addition, V is an *iISS CLF* if both items are satisfied for $q = 0$ and for a continuous positive definite function $\alpha_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

Observe that the inequality (13) can be rewritten as follows:

$$\inf_{u \in \mathbf{U}} \{a(x, v) - \tilde{\chi}(|v|) - q + b(x)u\} \leq -\alpha_3(|x|_{\mathcal{W}}),$$

where $\tilde{\chi} \in \mathcal{K}_\infty$ satisfies $\tilde{\chi}(s) \geq \chi(s)$ for all $s \in \mathbb{R}_+$ and such that there exists a function

$$\psi(x) = \sup_{v \in \mathbb{R}^k} \{a(x, v) - \tilde{\chi}(|v|) - q\}, \quad \forall x \in \mathcal{M}$$

(supremum over all disturbances v), e.g., take $\tilde{\chi}(s) = \max\{\chi(s), \sup_{|x|_{\mathcal{W}} \leq s, |v| \leq s} |a(x, v)|\}$ for any $s \in \mathbb{R}_+$ ²³, then with such a choice $\psi(x) = \sup_{|v| \leq |x|_{\mathcal{W}}} \{a(x, v) - \tilde{\chi}(|v|) - q\}$. By construction $\psi(x) \leq 0$ for all $x \in \mathcal{W}$. Since a is at least Lipschitz continuous in x , then the function ψ admits the same property. In addition, for all $x \in \mathcal{M}$

$$\inf_{u \in \mathbf{U}} \{\psi(x) + b(x)u\} \leq -\alpha_3(|x|_{\mathcal{W}}).$$

Finally, define

$$\Psi(x) = \psi(x) + \frac{1}{2}\alpha_3(|x|_{\mathcal{W}}),$$

then for all $x \in \mathcal{M}$ and all $v \in \mathbb{R}^k$:

$$\Psi(x) \geq \frac{1}{2}\alpha_3(|x|_{\mathcal{W}}) + a(x, v) - \tilde{\chi}(|v|) - q,$$

$$\inf_{u \in \mathbf{U}} \{\Psi(x) + b(x)u\} \leq -\frac{1}{2}\alpha_3(|x|_{\mathcal{W}}).$$

Conclude that now the condition (13) can be replaced with another one:

$$b(x) = 0 \Rightarrow \Psi(x) \leq -\frac{1}{2}\alpha_3(|x|_{\mathcal{W}}) \quad (14)$$

for all $x \in \mathcal{M}$, and α_3 is in the class \mathcal{K}_∞ or a positive definite function for the cases of ISS or iISS, respectively. Therefore, the condition (14) formulates the main restriction to be checked for an ISS (iISS) CLF in the system (10), (11).

Definition 13. A continuously differentiable function $V : \mathcal{M} \rightarrow \mathbb{R}_+$ possesses the *small control property* (SCP) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x|_{\mathcal{W}} < \delta$ there exists some control $u \in \mathbf{U}$ with $|u| < \varepsilon$ such that

$$\Psi(x) + b(x)u < 0. \quad (15)$$

Note that (15) can be reformulated as an existence condition for each $\varepsilon > 0$ of $\delta > 0$ such that $\frac{\Psi(x)}{|b(x)|} < \varepsilon$ whenever $0 < |x|_{\mathcal{W}} < \delta$, then to check (15) one can verify the continuity of a scalar function $\frac{\Psi(x)}{|b(x)|}$ in a neighborhood of \mathcal{W} with $\Psi(x) > 0$.

Let $\mathbf{U} = \mathbb{R}^m$, then following the CLF framework from^{29,33}, we will use the “universal” control formula, which in our case takes the form for all $x \in \mathcal{M}$:

$$u = K(x) = \begin{cases} \kappa(\Psi(x), |b(x)|^2)b(x)^\top & \text{if } x \notin \mathcal{W} \\ 0 & \text{if } x \in \mathcal{W} \end{cases}, \quad (16)$$

where $\kappa : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is defined by

$$\kappa(s, r) = -\frac{s + \sqrt{s^2 + r^2}}{r}. \quad (17)$$

The chosen form of the function κ in (17) corresponds to $\mathbf{U} = \mathbb{R}^m$, then with mild modifications and by applying the formulas given in³⁰, for example, similar results can be obtained for other classes of \mathbf{U} (with bounded controls). According with the above developments we propose the following theorem:

Theorem 3. Let Assumption 1 be satisfied. If the function $V : \mathcal{M} \rightarrow \mathbb{R}_+$ is an ISS (iISS) CLF, then the feedback law (16) is continuous in $\mathcal{M} \setminus \mathcal{W}$ and it provides for (10) the ISS (iISS) property from v with respect to \mathcal{W} . If such a CLF satisfies the SCP (given in Definition 13), then the control (16) is continuous on \mathcal{M} .

Proof. First of all note that the control (16) is continuous on $\mathcal{M} \setminus \mathcal{W}$ by the standard argumentation²⁹. Indeed, it was shown in the proof of Theorem 1 of²⁹ that the function $\kappa(s, r)$ is continuous on $S = \{(s, r) \in \mathbb{R}^2 : r > 0 \text{ or } s < 0\}$, and $(\Psi(x), |b(x)|^2) \in S$ due to (14) and the definition of the control (16).

Next, consider (10) and assume that V is an ISS CLF. With (16) and since $\tilde{\chi}(|v|) + q - \frac{1}{2}\alpha_3(|x|_{\mathcal{W}}) \geq a(x, v) - \Psi(x)$ by construction, computing the derivative of V along the solution of the closed-loop system, $\dot{x} = f(x, v) + G(x)K(x)$, we obtain for all $x \notin \mathcal{W}$ (denote a short hand $\dot{V} = DV(x)[f(x, v) + G(x)K(x)]$):

$$\begin{aligned} \dot{V} &= a(x, v) + b(x)K(x) \\ &= a(x, v) + \kappa(\Psi(x), |b(x)|^2)|b(x)|^2 \\ &= a(x, v) - \Psi(x) - \sqrt{\Psi^2(x) + |b(x)|^4} \\ &\leq \tilde{\chi}(|v|) + q - \frac{1}{2}\alpha_3(|x|_{\mathcal{W}}) - \sqrt{\Psi^2(x) + |b(x)|^4} \\ &\leq -\frac{1}{2}\alpha_3(|x|_{\mathcal{W}}) + \tilde{\chi}(|v|) + q, \end{aligned} \quad (18)$$

which shows that V is an ISS Lyapunov function for the closed-loop system. Therefore, according to Theorem 1 the closed-loop system is ISS from v with respect to \mathcal{W} as needed. The same analysis can be done by considering V as an iISS CLF for (18). In this case α_3 is just a positive definite function and the closed-loop system is iISS by means of the existence of an iISS Lyapunov function (Theorem 2).

Now, we will show that if V satisfies the SCP, then u is also continuous at \mathcal{W} . Since $u = 0$ whenever $b(x) = 0$ or $x \in \mathcal{W}$, we can assume that $b(x) \neq 0$ and $x \notin \mathcal{W}$ in what follows. Let us analyze the amplitude of the control law (16) rewriting it as

$$|u| = |\kappa(\Psi(x), |b(x)|^2)||b(x)| = \frac{|\Psi(x) + \sqrt{\Psi(x)^2 + |b(x)|^4}|}{|b(x)|}. \quad (19)$$

The gradient of V is continuous and zero on the set \mathcal{W} , then for any $\varepsilon > 0$ there is $\delta > 0$ such that $|b(x)| \leq \varepsilon$ when $|x|_{\mathcal{W}} \leq \delta$. Similarly for $\Psi(x) = \psi(x) + \frac{1}{2}\alpha_3(|x|_{\mathcal{W}})$, where the function ψ is locally Lipschitz continuous, from SCP of V (Definition 13) we know that for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\Psi(x) < |b(x)|\varepsilon$$

with $|x|_{\mathcal{W}} \leq \delta$, hence, the positive value of Ψ is upper bounded in the vicinity $|x|_{\mathcal{W}} \leq \delta$ by ε^2 .

Now let us analyze (19) for $\Psi(x) \leq 0$ and any $|x|_{\mathcal{W}} \leq \delta$:

$$|u| \leq \frac{|\Psi(x) + |\Psi(x)| + |b(x)|^2|}{|b(x)|} = |b(x)| \leq \varepsilon.$$

On the other hand, in the case when $\Psi(x) > 0$, we obtain for $|x|_{\mathcal{W}} \leq \delta$:

$$\begin{aligned} |u| &\leq \frac{|\Psi(x)| + \sqrt{\Psi(x)^2 + |b(x)|^4}}{|b(x)|} \\ &\leq 2\frac{|\Psi(x)|}{|b(x)|} + |b(x)| \leq 3\varepsilon, \end{aligned}$$

and since ε is arbitrary, that implies the continuity of the control (16) on the set \mathcal{W} also. \square

Note that $V(x) = 0$ for $x \in \mathcal{A}$ from some subset $\mathcal{A} \subset \mathcal{W}$ (where V reaches its minimum; and it has other kinds of extremum for the decomposition components in $\mathcal{W} \setminus \mathcal{A}$), then the control (16) provides almost global or local attraction of the set \mathcal{A} in the case $v = 0$. If V and f, G are smooth, then the control (16) also inherits this property out of the set \mathcal{W} by construction, and as it has been proven above it is continuous everywhere. However, there is no problem with the uniqueness of solutions on the set \mathcal{W} for the closed-loop system since this set contains invariant solutions.

Let us formulate a direct consequence of our main result:

Lemma 1. Let Assumption 1 be satisfied. Then there exists a continuous control $u = K(x)$ that provides for (10) the ISS (iISS) property from v with respect to \mathcal{W} if and only if the system admits an ISS (iISS) CLF with SCP.

Proof. The proof that the existence of a CLF with SCP implies the corresponding stability property under a continuous control (16) is presented in Theorem 3. Conversely, assume that there is a continuous feedback u that renders the closed-loop system ISS (iISS) in v with respect to \mathcal{W} , then by Theorems 1 or 2 there is a corresponding Lyapunov function (constant on \mathcal{W}), which can serve as a searched CLF (in³³ it was shown how to deal with continuity of the closed-loop system on \mathcal{W}). \square

The notion of CLF can be extended for the iISS case by means of *relaxed* concepts of the output smooth dissipativity and the weak zero-detectability (definitions 10 and 11, respectively), whose existence also is equivalent to iISS (Theorem 2). In such a case we can introduce a weak iISS CLF as follows:

Definition 14. A *weak iISS CLF* for the system (10), (11) with

$$\mathcal{W} \subseteq \mathcal{Z} = \{x \in \mathcal{M} : h(x) = 0\}$$

and control $u \in \mathcal{U}$ is a twice continuously differentiable function $V : \mathcal{M} \rightarrow \mathbb{R}_+$ satisfying:

1. There exist \mathcal{K}_∞ functions α_1, α_2 , and χ , a continuous positive definite function $\alpha_4 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that (12) is satisfied for all $x \in \mathcal{M}$, $b(x) = 0$ for any $x \in \mathcal{Z}$, and the following dissipation inequality holds for all $x \in \mathcal{M}$ and all $v \in \mathbb{R}^k$:

$$\inf_{u \in \mathcal{U}} \{a(x, v) + b(x)u\} \leq -\alpha_4(|h(x)|) + \chi(|v|). \quad (20)$$

2. The system (10), (11) has the weak zero-detectability property for $v \equiv 0$ and $u \equiv 0$.

Following the same reasoning as above, we can rewrite (20) as:

$$\inf_{u \in \mathcal{U}} \{a(x, v) - \tilde{\chi}(|v|) + b(x)u\} \leq -\alpha_4(|h(x)|),$$

where $\tilde{\chi} \in \mathcal{K}_\infty$ is such that $\tilde{\chi}(s) \geq \chi(s)$ for all $s \in \mathbb{R}_+$, then

$$\psi_o(x) = \sup_{v \in \mathbb{R}^k} \{a(x, v) - \tilde{\chi}(|v|)\}$$

is well defined. The function $\tilde{\chi}$ is selected in a way that $\psi_o(x) \leq 0$ for all $x \in \mathcal{Z}$, and since a is at least Lipschitz continuous in x , then the function ψ_o admits the same property. In addition,

$$\inf_{u \in \mathcal{U}} \{\psi_o(x) + b(x)u\} \leq -\alpha_4(|h(x)|)$$

for all $x \in \mathcal{M}$.

Finally, by defining

$$\Psi_o(x) = \psi_o(x) + \frac{1}{2}\alpha_4(|h(x)|)$$

we have

$$\Psi_o(x) \geq \frac{1}{2}\alpha_4(|h(x)|) + a(x, v) - \tilde{\chi}(|v|),$$

therefore,

$$\inf_{u \in \mathcal{U}} \{\Psi_o(x) + b(x)u\} \leq -\frac{1}{2}\alpha_4(|h(x)|),$$

for all $x \in \mathcal{M}$ and $v \in \mathbb{R}^k$. Observe that for this case the condition (20) can be replaced by:

$$b(x) = 0 \Rightarrow \Psi_o(x) \leq -\frac{1}{2}\alpha_4(|h(x)|) \quad (21)$$

for all $x \in \mathcal{M}$ and a positive definite function α_4 .

Definition 15. A weak iISS CLF $V : \mathcal{M} \rightarrow \mathbb{R}_+$ possesses a variant of SCP if for each $\varepsilon > 0$ there is $\delta > 0$ such that there exists some control $u \in \mathbf{U}$ with $|u| < \varepsilon$ providing

$$\Psi_o(x) + b(x)u < 0 \quad (22)$$

whenever $0 < |x|_{\mathcal{Z}} < \delta$.

Hence, taking into account the weakly zero-detectability property, the condition (21) formulates the main restriction to be checked for a weak iISS CLF in the system (10), (11) and we can formulate the following theorem.

Theorem 4. Let Assumption 1 be satisfied. If the function $V : \mathcal{M} \rightarrow \mathbb{R}_+$ is a weak iISS CLF, then the feedback law

$$u = K(x) = \begin{cases} \kappa(\Psi_o(x), |b(x)|^2)b(x)^\top & \text{if } x \notin \mathcal{Z} \\ 0 & \text{if } x \in \mathcal{Z} \end{cases}, \quad (23)$$

is a continuous function in $\mathcal{M} \setminus \mathcal{Z}$ (with κ as in (17)) providing (10), (11) the iISS property from v with respect to \mathcal{W} . If such a CLF satisfies the SCP (given in Definition 15), then the control (23) is continuous on \mathcal{M} .

Proof. The proof comes from the same ideas applied in the proof of Theorem 3. Note, first, that the control (23) is continuous on $\mathcal{M} \setminus \mathcal{Z}$ by the usual argumentation²⁹. Next, consider (10), (11) and assume that V is a weak iISS CLF. By means of control (23) and since $\tilde{\chi}(|v|) - \frac{1}{2}\alpha_4(|h(x)|) \geq a(x, v) - \Psi_o(x)$ by construction, the derivative of V along the solution of the closed-loop system, $\dot{x} = f(x, v) + G(x)K(x)$, is given by

$$\begin{aligned} \dot{V} &= a(x, v) + b(x)K(x) \\ &= a(x, v) + \kappa(\Psi_o(x), |b(x)|^2)|b(x)|^2 \\ &= a(x, v) - \Psi_o(x) - \sqrt{\Psi_o^2(x) + |b(x)|^4} \\ &\leq \tilde{\chi}(|v|) - \frac{1}{2}\alpha_4(|h(x)|) - \sqrt{\Psi_o^2(x) + |b(x)|^4} \\ &\leq -\frac{1}{2}\alpha_4(|h(x)|) + \tilde{\chi}(|v|), \end{aligned}$$

which shows that V verifies the output dissipation inequality, then under the weak-zero detectability property the closed-loop system is iISS with respect to \mathcal{W} (by Theorem 2). The SCP in this case implies that for any $\varepsilon > 0$ there is $\delta > 0$ such that $|b(x)| \leq \varepsilon$ when $|x|_{\mathcal{Z}} \leq \delta$. Similarly for $\Psi_o(x) = \psi_o(x) + \frac{1}{2}\alpha_4(|h(x)|)$, where the function ψ_o is locally Lipschitz continuous and α_4 is a positive definite function of a continuous function h with $|x|_{\mathcal{W}} = 0$ implying $h(x) = 0$. Then, from the SCP (Definition 15) we know that for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\Psi_o(x) = \psi_o(x) + \frac{1}{2}\alpha_4(|h(x)|) < |b(x)|\varepsilon.$$

Therefore, the rest of the proof follows the same arguments given in Theorem 3 and we can conclude the continuity of the control (23) on the set \mathcal{Z} . \square

A difference between iISS CLF (Definition 12) and its weak formulation (Definition 14) is that the derivative of the latter, when $u = 0$ in (23) and $v = 0$, is not zero (in general case, the trajectory is in \mathcal{Z} and has to converge to \mathcal{W}), while the former with the control (16) in such a case implies that the trajectory is already at \mathcal{W} .

Remark 2. The requirement of Assumption 1 introduced in Theorems 3 and 4 can be relaxed. It is straightforward to check that the results of the theorems are preserved if the set \mathcal{W} is compact and contain a part of α - and ω -limit sets of the unperturbed system (3), admitting a finite decomposition without cycles. This feature will be used in the example of subsection 5.2.

5 | EXAMPLES

In this section two examples are presented to illustrate the efficacy and flexibility of the approach.

5.1 | Duffing oscillator

Consider the following variant of Duffing oscillator:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + u(t) + v_1(t), \\ \dot{x}_2(t) &= -x_2(t) + x_1(t)[x_1^2(t) - 1] + v_2(t),\end{aligned}$$

where $x(t) = (x_1(t) \ x_2(t))^T \in \mathcal{M} = \mathbb{R}^2$, $u(t) \in \mathbb{R}$ and $v(t) = (v_1(t) \ v_2(t))^T \in \mathbb{R}^2$ have the same sense as before. It is straightforward to check that for $u = 0$ and $v = 0$ the system has three equilibria, thus, $\mathcal{W} = \{(-1, 0), (0, 0), (1, 1)\}$. Linearization shows that the origin is a locally asymptotically stable steady-state, and the two equilibria, $(-1, 0)$ and $(1, 0)$ are unstable focuses. Select $\mathcal{A} = \{(-1, 0), (0, 1)\}$ as the subset of \mathcal{W} that will be almost globally attractive in the closed-loop system. To this end, as a CLF candidate, choose

$$V(x) = \frac{1}{4}(x_1^2 - 1)^2 + \frac{1}{2}x_2^2,$$

whose derivative for the oscillator takes the usual form $\dot{V} = a(x, v) + b(x)u$, where

$$\begin{aligned}a(x, v) &= 2x_1(x_1^2 - 1)x_2 - x_2^2 + x_1(x_1^2 - 1)v_1 + x_2v_2, \\ b(x) &= x_1(x_1^2 - 1).\end{aligned}$$

Remark 3. Note that V is a non-strict Lyapunov function for the unforced system $\dot{x}_1 = x_2$; $\dot{x}_2 = -x_2 - x_1^3 + x_1$, whose behavior is our target in the closed loop. Moreover, the Lyapunov function is equal to zero only in the stabilizing equilibria and is positive otherwise.

Select $\chi(s) = s^2$, then

$$a(x, v) - \chi(|v|) = \rho^T Q \rho - \frac{1}{4}x_2^2 + 2.3x_1^2(x_1^2 - 1)^2,$$

where

$$\rho = [x_1(x_1^2 - 1) \ x_2 \ v_1 \ v_2]^T, \quad Q = \begin{pmatrix} -2.3 & 1 & 0.5 & 0 \\ 1 & -0.75 & 0 & 0.5 \\ 0.5 & 0 & -1 & 0 \\ 0 & 0.5 & 0 & -1 \end{pmatrix}$$

is a negative definite matrix. Hence, we can take

$$\psi(x) = -\frac{1}{4}x_2^2 + 2.3x_1^2(x_1^2 - 1)^2 \geq a(x, v) - \chi(|v|).$$

Next, let

$$|x|_{\mathcal{W}} = \sqrt{x_2^2 + x_1^2(x_1^2 - 1)^2}$$

and $\alpha_3(s) = \frac{1}{4}s^2$, then we obtain

$$\Psi(x) = \psi(x) + \frac{1}{2}\alpha_3(|x|_{\mathcal{W}}),$$

and it is easy to check that

$$b(x) = 0 \Rightarrow \Psi(x) \leq -\frac{1}{2}\alpha_3(|x|_{\mathcal{W}}).$$

Consequently, V is a CLF and for any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x|_{\mathcal{W}} < \delta$ there exists some control $|u| < \varepsilon$ such that

$$\Psi(x) + b(x)u < 0,$$

then SCP holds, and the control (16) is continuous in \mathbb{R}^2 . The behavior of the controlled system is shown in Fig. 1. Note that with the applied control, the two unstable focuses become almost attractive in the closed-loop system. The effect of the noise for both, the open-loop and closed-loop systems, is also shown. Observe that the control action forces the trajectories to different attractors by starting the system from the same initial conditions. To generate the plots, the disturbance signals were chosen $v_1(t) = v_2(t) = 0.2 \sin(10t)$.

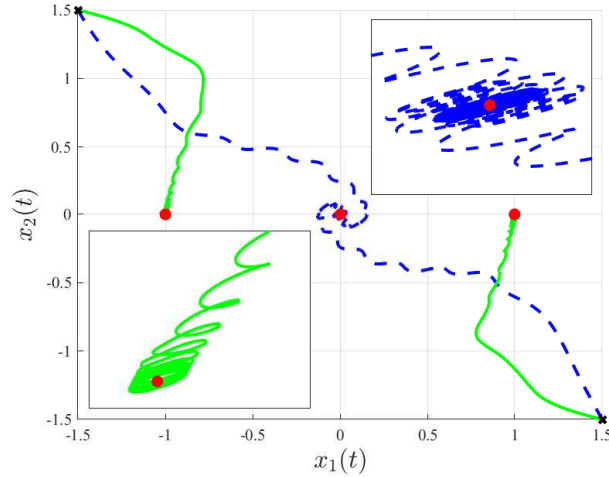


FIGURE 1 Trajectory convergence for the system with disturbance: without control (dashed line), with control (solid line)

5.2 | Noise-induced transition in a semiconductor-gas-discharge gap system

To illustrate the iISS CLF application in the multistable sense, we consider a nonlinear dynamical model of a semiconductor-gas-discharge gap affected by parametric noise resonance. Such a system is described by the following differential equations:

$$\dot{x}_1 = -\frac{x_1}{\tau} + \frac{x_1 x_2}{\tau(E_c + v)}, \quad (24)$$

$$\dot{x}_2 = -a_1 x_2 - b_1 x_1 x_2 + a_1 E_m, \quad (25)$$

where x_1 is the density of free charge carries in the gap, x_2 is the electric field strength in the discharge gap, $\tau_E = \frac{1}{a_1}$ is the characteristic time of the charging process with $a_1 > 0$, $b_1 > 0$ is a coefficient, $E_m = \frac{U_m}{d}$ is the maximal value of x_2 in the gap that can be provided by a source of constant voltage, $U_m \in \mathbb{R}$ is the voltage of the fielding source, $d > 0$ is the length of the gap in the direction of the electric current, and $\tau > 0$ defines the rate of a temporal variation of the charge carriers density when the electric field in the gap is not equal to the critical electric field strength $E_c > 0$.

The generation of free carriers in the gas-discharge gap is provided by the avalanche of gas atoms and molecules ionization. The efficiency of this process is known to fluctuate in time, which serves as a source of intrinsic noise that can initiate large amplitude oscillations in the current, which drives the system from the conductive to the dielectric state. The influence of the noise on dynamics of the system can be simulated by adding a stochastic perturbation v to the parameter E_c , which by physical principles suggests that the control action can be applied by properly varying E_m in time^{35,36}.

5.2.1 | Transformation of the system

Therefore, in this example we are interested in robust stabilization of this system, using E_m as the control, with respect to the disturbance input v in the iISS sense. In other words, we are interested in knowing under which conditions the dynamics of the system remains within an appropriate operation region. However, let us first analyze the system (24), (25) for $E_m = 0$ and $v \equiv 0$, then we have:

$$\dot{x}_1 = -\frac{x_1}{\tau} + \frac{x_1 x_2}{\tau E_c}, \quad (26)$$

$$\dot{x}_2 = -a_1 x_2 - b_1 x_1 x_2, \quad (27)$$

whose stationary solutions are given by $n_1 = (0, 0)$ and $n_2 = (-\frac{a_1}{b_1}, E_c)$. In this case, a stabilization with respect to the first equilibrium point would mean shutting the device down, while the stabilization with respect to the latter one is unfeasible since the first state component is negative.

Another way to deal with this problem is to properly transform the original system (24), (25) by means of an input signal in order to obtain a positive equilibrium point. For instance, let us consider

$$E_m = u_0 x_2 + \delta u, \quad (28)$$

where $\delta u \in \mathbb{R}$ is the new auxiliary control signal and $u_0 \in \mathbb{R}$ is a tuning parameter. Thus the system (24), (25), (28) becomes

$$\dot{x}_1 = -\frac{x_1}{\tau} + \frac{x_1 x_2}{\tau(E_c + v)}, \quad (29)$$

$$\dot{x}_2 = a_1(u_0 - 1)x_2 - b_1 x_1 x_2 + a_1 \delta u, \quad (30)$$

which means that, by choosing $u_0 > 1$, in the state space $\mathcal{M} = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ the system will have a positive equilibrium point $n = \left(\frac{a_1(u_0-1)}{b_1}, E_c\right)$ and we can use

$$V(x) = b_1 x_1 - a_1(u_0 - 1) \ln(x_1) + \frac{x_2}{\tau E_c} - \frac{\ln(x_2)}{\tau} \quad (31)$$

as an energy function. In fact (31) is a conserved quantity for the unforced version of (29), (30), *i.e.*, with $v \equiv 0$ and $\delta u \equiv 0$, which reveals the multistability character of the system by means of the existence of closed orbits around the equilibrium point n with different constant energy levels V . Therefore, a possible strategy to ensure robustness of the system consists, first, to select the level of energy V for $v \equiv 0$ guaranteeing the needed performance and, second, to design δu using the results of Section 4 in the presence of the perturbation v .

Remark 4. For this example, V is obtained by noting that the unforced, unperturbed version of the system (29), (30), has the form of a Lotka-Volterra model $\dot{x}_1 = \alpha_1 x_1 + \beta_1 x_1 x_2$; $\dot{x}_2 = \alpha_2 x_2 + \beta_2 x_1 x_2$, whose energy function has the form $V(x) = \beta_1 x_2 + \alpha_1 \ln x_2 - \beta_2 x_1 - \alpha \ln x_1$. This conserved quantity is obtained by dividing this model by $x_1 x_2$ and solving the system of equations $\dot{x}_1 = \partial V / \partial x_2$; $\dot{x}_2 = -\partial V / \partial x_1$. In such a case, the derivative of V along the trajectories of the system is equal to zero. Therefore, the same behavior can be obtained for the unforced and unperturbed version of the system (29), (30) by correctly choosing the coefficient of V .

5.2.2 | Universal formula control

Introducing an auxiliary function

$$Q(x) = \frac{1}{2}(V - V_d)^2, \quad (32)$$

where $V_d \in \mathbb{R}$ is a desired level of V , which is asymptotically reached provided that $Q(x(t)) \rightarrow 0$ as $t \rightarrow +\infty$, and calculating its derivative along the trajectories of the system (29), (30) we obtain

$$\begin{aligned} \dot{Q}(x, u) &= (V(x) - V_d) a_1 \left(\frac{x_2 - E_c}{\tau E_c x_2} \right) \delta u \\ &+ (V(x) - V_d) (a_1(u_0 - 1)x_2 - b_1 x_1 x_2) \frac{v}{\tau E_c (E_c + v)}. \end{aligned} \quad (33)$$

In this case, the set of all invariant solutions of the system for $v \equiv 0$ is chosen as $\mathcal{W} = n \cup \Gamma$, where $\Gamma = \{x \in \mathcal{M} : V(x) = V_d\}$. To design δu providing iISS property of the closed loop system from the input v with respect to \mathcal{W} , let us introduce a weak iISS CLF candidate function $W(x) = \ln(1 + Q(x))$ and calculate its derivative:

$$\dot{W}(x, u) = a(x, v) + b(x) \delta u$$

where $a(x, v) = r(x) \sigma(v)$, with

$$\begin{aligned} r(x) &= \frac{(V(x) - V_d)(a_1(u_0 - 1)x_2 - b_1 x_1 x_2)}{1 + Q(x)}, \\ \sigma(v) &= \frac{v}{\tau E_c (E_c + v)}, \\ b(x) &= \frac{(V(x) - V_d) a_1 \left(\frac{x_2 - E_c}{\tau E_c x_2} \right)}{1 + Q(x)}. \end{aligned}$$

Performing simple calculations, we can observe that r is bounded by a constant $R \geq 0$. However, the boundedness of σ depends on the sign of v that cannot be equal to $-E_c$ ($\sigma \rightarrow \infty$ when $v \rightarrow -E_c$). Then, assuming $v \geq 0$, we can apply Theorem 3 to design δu . Note that for the chosen output $y = h(x) = b(x)$, the weak zero-detectability property is satisfied for $\delta u \equiv 0$ and $v \equiv 0$.

Remark 5. In³⁷, the speed-gradient method was used to design δu , and it was shown that the controlled system is iISS. Here, following the construction described in Section 4, we will show that the “universal formula” strategy also results in the iISS property for the closed-loop system.

Selecting $\tilde{x}(s) = s + R\sigma(s)$ we define ψ_o as

$$\psi_o(x) = \sup_{v \geq 0} \{r(x)\sigma(v) - R\sigma(v) - v\} \leq 0,$$

therefore, $\Psi_o(x) = \psi_o(x) + \frac{1}{2}\alpha_4(|h(x)|)$ and $b(x) = 0$ imply $\Psi_o(x) \leq -\frac{1}{2}\alpha_4(|h(x)|)$ for any positive definite function α_4 . Finally, by applying the control law (23) and choosing $\alpha_4(s) = s^2$ (a choice to guarantee SCP), we obtain

$$\begin{aligned} \dot{W}(x, u) &= r(x)\sigma(v) + b(x)K(x) \\ &= r(x)\sigma(v) + h(x)\kappa(\Psi_o(x), |h(x)|^2)h(x)^T \\ &= r(x)\sigma(v) - \sup_{v \geq 0} \{r(x)\sigma(v) - R\sigma(v) - v\} \\ &\quad - \frac{1}{2}|h(x)|^2 - \sqrt{\Psi_o^2(x) + |h(x)|^4} \\ &\leq v + R\sigma(v) - \frac{1}{2}|h(x)|^2 - \frac{\sqrt{5}}{2}|h(x)|^2 \\ &\leq -\frac{1}{2}|h(x)|^2 + R\sigma(v) + v \end{aligned}$$

which demonstrates that conditions of Definition 14 are verified. To show that the weak iISS CLF V satisfies the SCP property, we analyze the amplitude of the control law (23):

$$|\delta u| = \frac{|\Psi_o(x) + \sqrt{\Psi_o^2(x) + |b(x)|^4}|}{|b(x)|} \leq 2|h(x)|.$$

As $h(x) = b(x)$ is a continuous output map $|h(x)| \leq \varepsilon$ whenever $0 < |x|_z < \delta$ for $\varepsilon > 0$ and $\delta > 0$ (in fact in this case we can choose $\varepsilon = \delta$) and $\delta u = K(x) < \varepsilon$.

To illustrate the behavior of the closed-loop system, we choose the parameters $a_1 = 20s^{-1}$, $b = 0.4cm^3/s$, $\tau = 5 \times 10^{-3}$, and $E_c = 5V/cm$. To simulate the perturbation, a uniformly distributed stochastic noise with amplitude equal to 0.025 is used, which represents 0.5% of the critical electric field strength E_c . These values are re-scaled from the real physical parameters of a real device, which can be found in^{36,35}, to facilitate the result's simulation and analysis.

Recovering the control goal, we are interested in robust stabilization of the system near to an appropriate operation region, which in this case means near to the minimum energy level of V achieved when $x = n = (50, 5)$. To exemplify, we choose $V_d = 0.001$ and the control input applied to the original system is given by $Em = K(x) + u_0x_2$ where $u_0 = 2$. Fig. 2 shows the phase portrait for the system with and without disturbance. As we can see, for the former case the state trajectories evolve inside a small region around the desired one given by the latter case. By means of Figs. 3 and 4, we can see the behavior of the open-loop system and the closed-loop system for the same disturbance signal and conclude that for the controlled version, the state oscillations are much smaller, which means that the system remains inside the desired operation region.

6 | CONCLUSION

In this work, we studied a subclass of nonlinear multistable dynamical systems affine in the control input. Our main goal was to establish conditions for the existence of a feedback control rendering the ISS (iISS) property with respect to disturbances to a multistable system. In this direction, we have appropriately extended the notions of ISS (iISS) CLF and SCP within the multistability framework. It has also been shown that such properties can be extended to the iISS case using a continuous output map that makes the system smooth dissipative and weak zero detectable, giving rise to a weak iISS CLF notion and corresponding SCP. It is verified that an ISS (iISS) CLF or a weak iISS CLF satisfying the SCP implies the existence of a feedback control law that can be explicitly designed by the universal formula strategy. To illustrate the ISS (iISS) CLF application, such a feedback

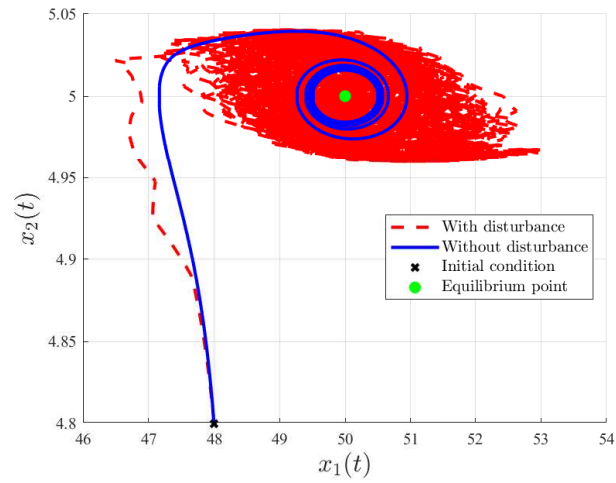


FIGURE 2 Phase portrait for the controlled system (24), (25) without and with disturbance input

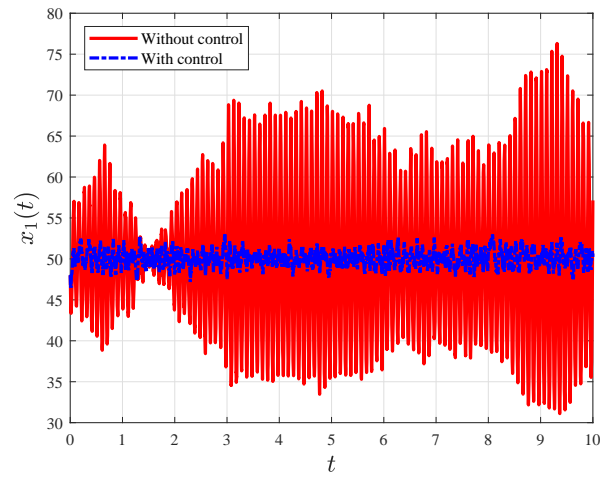


FIGURE 3 Time response of state $x_1(t)$ with and without control

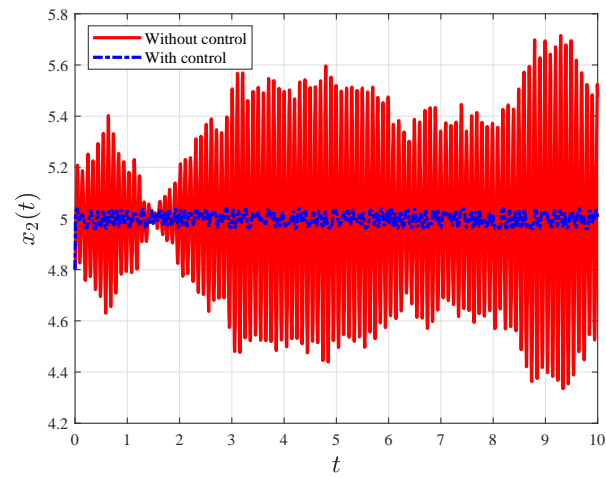


FIGURE 4 Time response of state $x_2(t)$ with and without control

control was designed to stabilize a mechanical multistable system robustly. Also, a more practical and realistic problem in the electronics field was used to exemplify the weak iISS CLF application.

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References

1. Hayashi C. *Nonlinear Oscillations in Physical Systems*. New York: McGraw-Hill Book Company . 1964.
2. Efimov D, Schiffer J, Barabanov N, Ortega R. A relaxed characterization of ISS for periodic systems with multiple invariant sets. *European Journal of Control* 2017; 37: 1–7.
3. Rouche N, Habets P, Laloy M. *Stability Theory by Liapunov's Direct Method*. Heidelberg: Springer-Verlag . 1977.
4. Laurent M, Kellershohn N. Multistability: a major means of differentiation and evolution in biological systems. *Trends Biochemical Sciences* 1999; 24: 418-422.
5. Pchelkina I, Fradkov AL. Control of oscillatory behavior of multispecies populations. *Ecological Modelling* 2012; 227: 1–6.
6. Pisarchik AN, Feudel U. Control of multistability. *Physics Reports* 2014; 540: 167–218.
7. Yakubovich VA, Leonov GA, Gelig AK. *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*. Singapore: World Scientific . 2004. [in Russian].
8. Chaves M, Eissing T, Allgower F. Bistable Biological Systems: A Characterization Through Local Compact Input-to-State Stability. *IEEE Transactions on Automatic Control* 2008; 53(Special Issue): 87–100.
9. Angeli D. An almost global notion of Input-to-state stability. *IEEE Transaction on Automatic Control* 2004; 49: 866–874.
10. Angeli D, Ferrell JE, Sontag ED. Detection of multistability, bifurcations and hysteresis in a large class of biological positive-feedback systems. *Proc. Natl. Acad. Sci. USA* 2004; 101: 1822–1827.
11. Dudkowski D, Jafari S, Kapitaniak T, Kuznetsov NV, Leonov GA, Prasad A. Hidden attractors in dynamical systems. *Physics Reports* 2016; 637: 1–50.
12. Enciso G, Sontag ED. Monotone systems under positive feedback: multistability and a reduction theorem. *Systems & Control Letters* 2005; 54: 159–168.
13. Forni P, Sepulchre R. Differential analysis of nonlinear systems: Revisiting the pendulum example. 2014: 3848–3859.
14. Guckenheimer J, Holmes PJ. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. 42 of *Applied Mathematical Sciences*. New York: Springer-Verlag . 1983.
15. Remyantsev VV, Oziraner AS. *Stability and stabilization of motion with respect to part of variables*. Moscow: Nauka . 1987. [in Russian].
16. Stan GB, Sepulchre R. Analysis of interconnected oscillators by dissipativity theory. *IEEE Trans. Automatic Control* 2007; 52: 256–270.
17. Vorotnikov VI. *Partial Stability and Control*. Boston: Birkhauser . 1998.
18. Efimov D. Global Lyapunov Analysis of Multistable Nonlinear Systems. *SIAM Journal on Control and Optimization* 2012; 50(5): 3132–3154.

19. Angeli D, Efimov D. Characterization of Input-to-State Stability for Systems with Multiple Invariant Sets. *IEEE Transactions on Automatic Control* 2015; 60: 3242–3256.
20. Sontag ED, Wang Y. On characterizations of the input-to-state stability property. *Systems & Control Letters* 1995; 24: 351–359.
21. Sontag ED, Wang Y. New Characterizations of Input-to-State Stability. *IEEE Transactions on Automatic Control* 1996; 41: 1283–1294.
22. Sontag ED. Comments on integral variants of ISS. *Systems & Control Letters* 1998; 34: 93–100.
23. Liberzon D, Sontag ED, Wang Y. On integral-input-to-state stabilization. *Proceedings of the American Control Conference* 1999; 3: 1598–1602.
24. Angeli D, Sontag ED, Wang Y. A Characterization of Integral Input-to-State Stability. *IEEE Transactions on Automatic Control* 2000; 45: 1082–1097.
25. Dashkovskiy SN, Efimov DV, Sontag ED. Input to State Stability and Allied System Properties. *Automation and Remote Control* 2011; 72: 1579–1614.
26. Forni P, Angeli D. The ISS approach to the stability and robustness properties of nonautonomous systems with decomposable invariant sets: An overview. *European Journal of Control* 2016; 30: 50–60.
27. Forni P, Angeli D. Characterization of Integral Input-to-State Stability for Systems with Multiple Invariant Sets. *IEEE Transactions on Automatic Control* 2017; 62: 3729–3743.
28. Artstein Z. Stabilization with relaxed controls. *Nonlinear Analysis, Theory, Methods and Applications* 1983; 7: 1163–1173.
29. Sontag ED. A “universal” construction of Artstein’s theorem on nonlinear stabilization. *Systems & Control Letters* 1989; 13: 117–123.
30. Lin Y, Sontag ED. Control-Lyapunov universal formulas for restricted inputs. *Control-Theory and Advanced Technology* 1995; 10: 1–22.
31. Efimov DV. Universal formula for output asymptotic stabilization. *IFAC Proceedings Volumes* 2002; 35: 239–244.
32. Efimov DV. A condition of CLF existence for affine systems. *Proceedings of the 41st IEEE Conference on Decision and Control* 2002: 1882–1887.
33. Liberzon D, Sontag ED, Wang Y. Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation. *System & Control Letters* 2002; 46: 111–127.
34. Nitecki Z, Shub M. Filtrations, Decompositions, and Explosions. *American Journal of Mathematics* 1975; 97: 1029–1047.
35. Kim JHR, Maurer H, Astrov YA, Bode M, Purwins HG. High-Speed Switch-On of a Semiconductor Gas Discharge Image Converter Using Optimal Control Methods. *Journal of Computational Physics* 2001; 170: 395–414.
36. Astrov YA, Fradkov AL, Guzenko PY. Control of a noise-induced transition in a nonlinear dynamical system. *Physical Review* 2008; 77: 026201:1–7.
37. Barroso NF, Ushirobira R, Efimov D, Fradkov AL. On robustness against disturbances of passive systems with multiple invariant sets. *International Journal of Control* 2020; 0: 1–13.

