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# Techniques de quadratisation pour les équations d'onde linéaires : modélisation et simulations

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**JO-SO**  
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**Journées Ondes du Sud-Ouest**  
14 - 16 mars 2023 - Toulouse

# **Techniques de quadratisation pour les équations d'onde linéaires : modélisation et simulations**

**Juliette Chabassier - Equipe Makutu - INRIA**

# Introduction

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III. On pose  $v(t) = \sqrt{(Bu, Bu)_D + 2\alpha}$   $\Rightarrow$  Scalar Auxiliary Variable

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## I.1 Discrétisation variationnelle et $\theta$ -schema

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Lemme de Céa

$$\|u - P_h u\|_U \leq C \inf_{u_h \in U_h} \|u - u_h\|_U$$

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$$\tilde{p}_h = \mu \delta \mathbf{u}_h^n$$

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- L'énergie est positive si  $m_\theta$  est positive, cad

- ▶ si  $\theta \geq \frac{1}{4}$ , ou

- ▶ si  $\Delta t^2 \sup_{v_h \in U_h} \frac{(Bv_h, Bv_h)_D}{(v_h, v_h)_P} \leq \frac{4}{1 - 4\theta}$

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- Éléments de preuve pour la stabilité et la convergence espace / temps

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

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Chercher  $u_h^n \in U_h \subset U$  tel que  $P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$

$$P_p(x) = 1 \quad P_k(x) = 1 + \left( \theta - \frac{1}{4} \right) x$$

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## I.2 Analyse

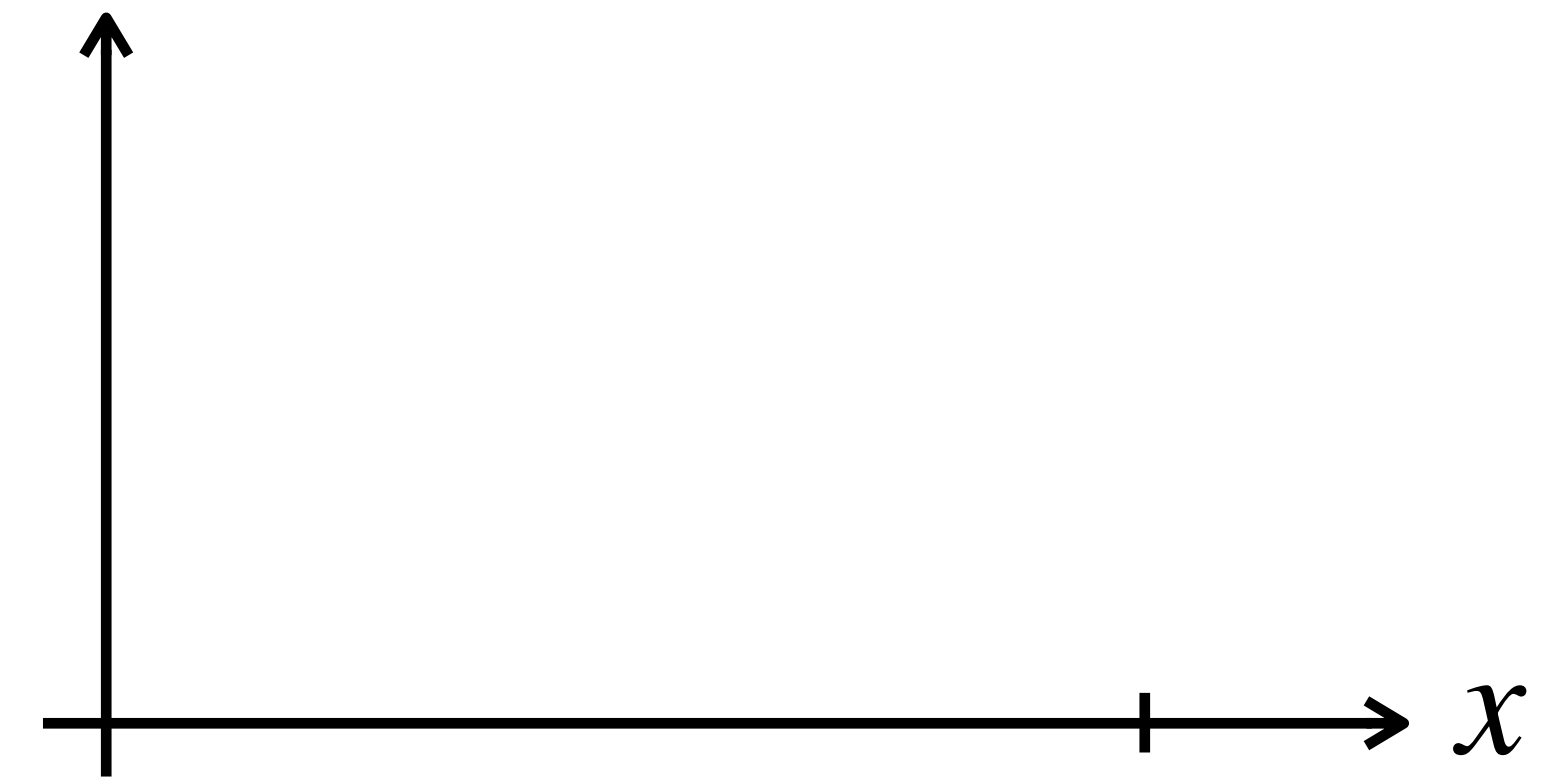
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# I. Discrétisation et Analyse Ordre 2

## I.2 Analyse

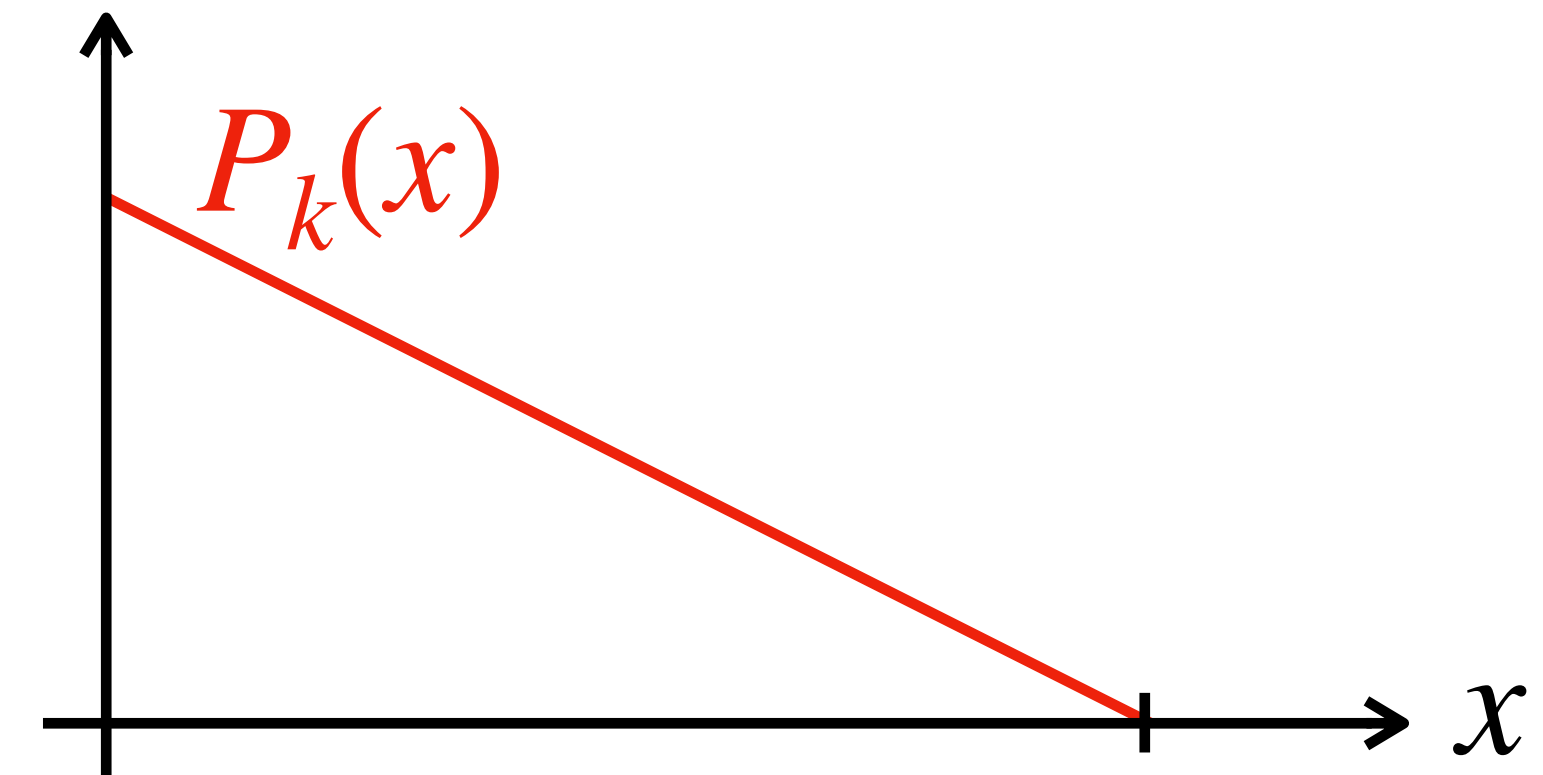
$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1$$

$$P_k(x) = 1 + \left( \theta - \frac{1}{4} \right) x$$



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# I. Discrétisation et Analyse Ordre 2

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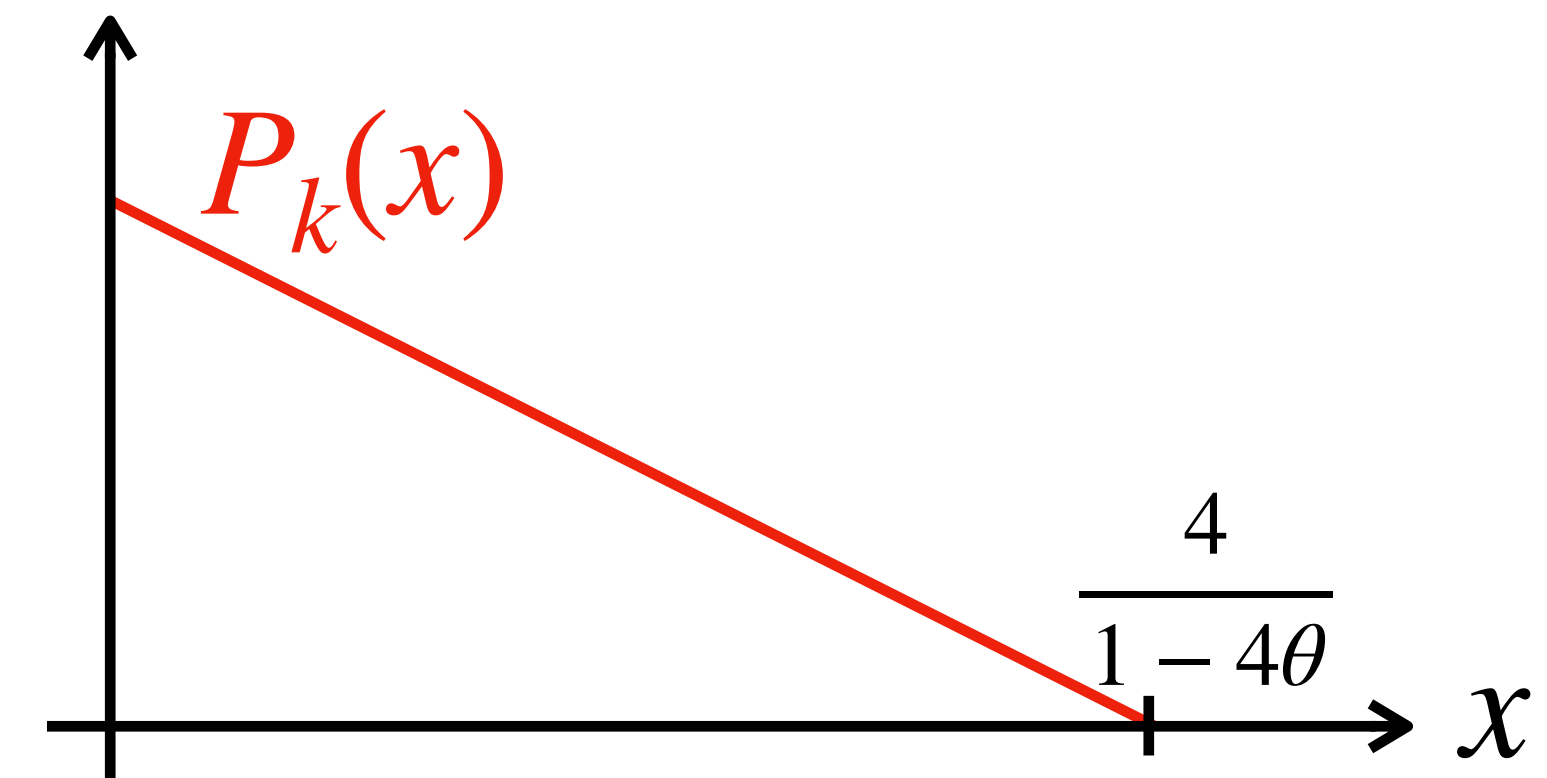
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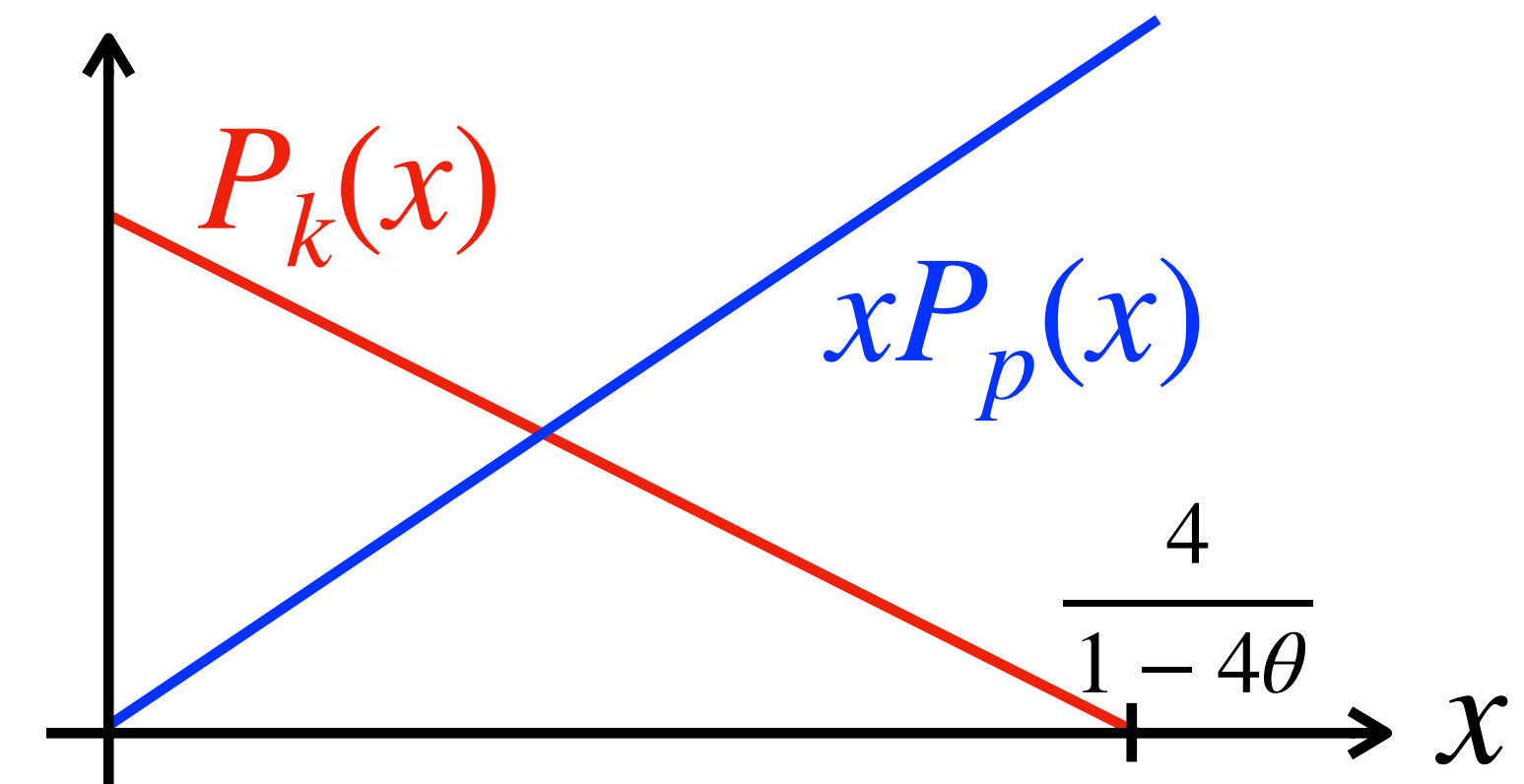
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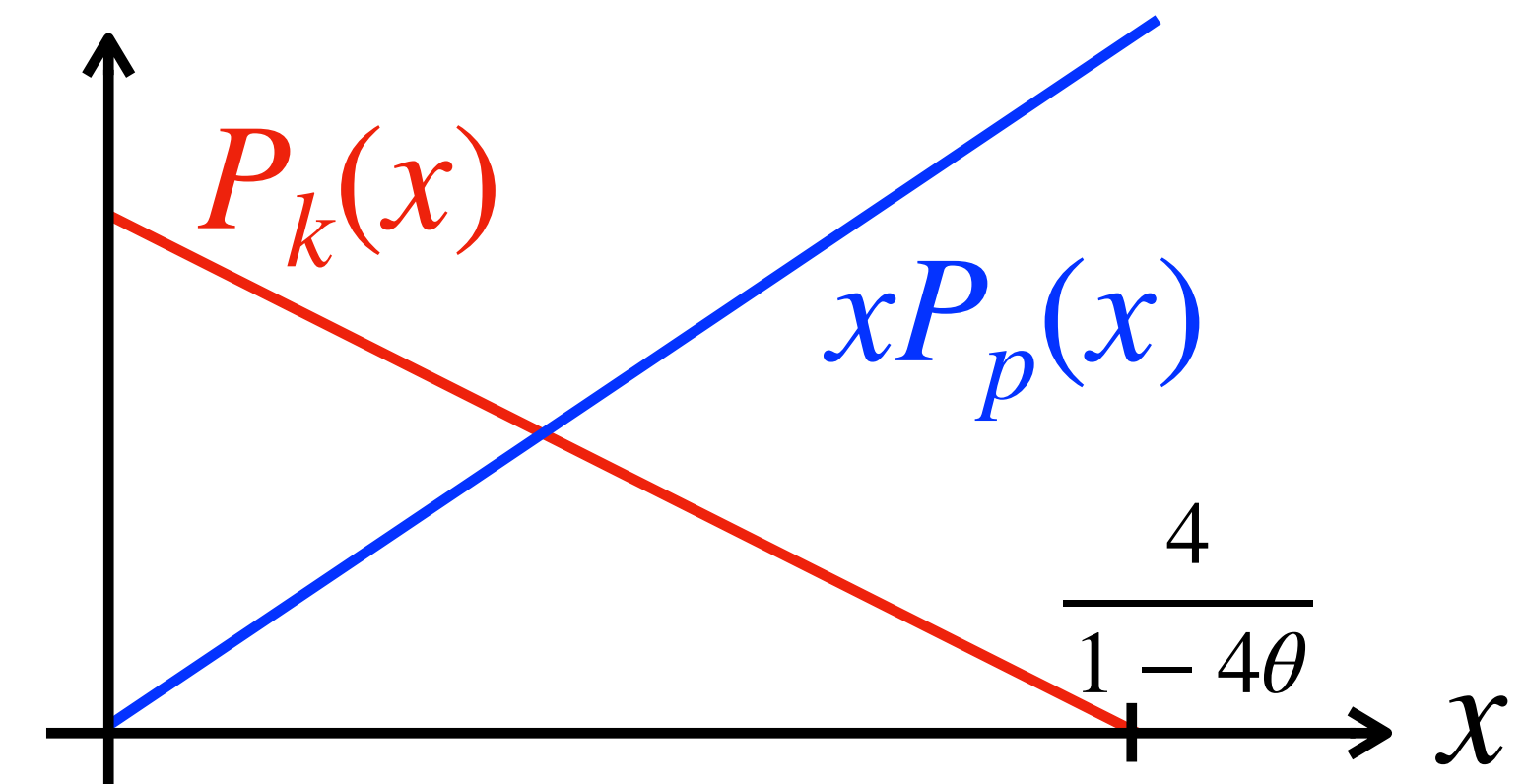
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$$P_k(x) = 1 + \left( \theta - \frac{1}{4} \right) x$$



$$P_k(x) \geq C_k \quad xP_p(x) \geq C_p$$

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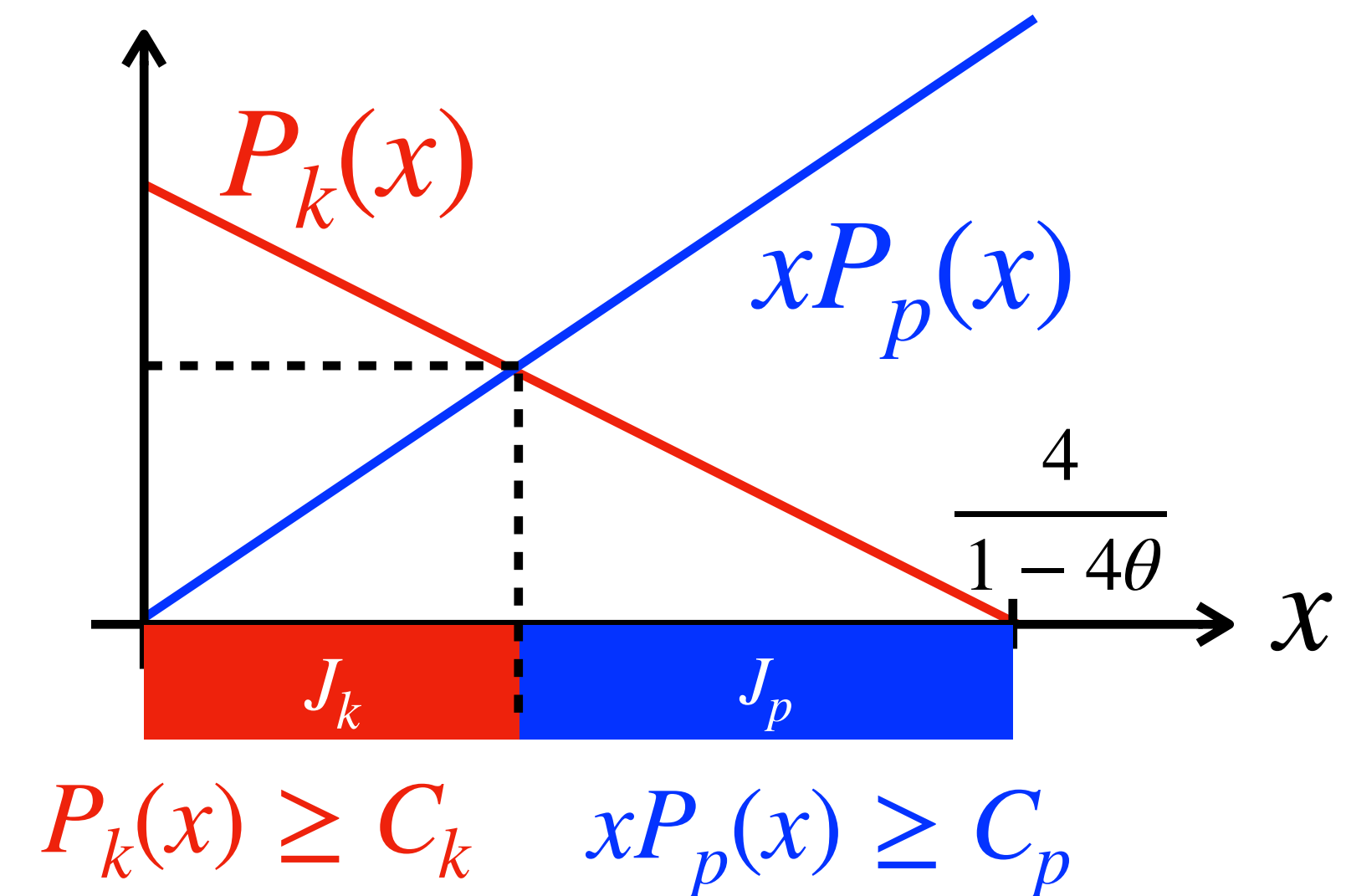
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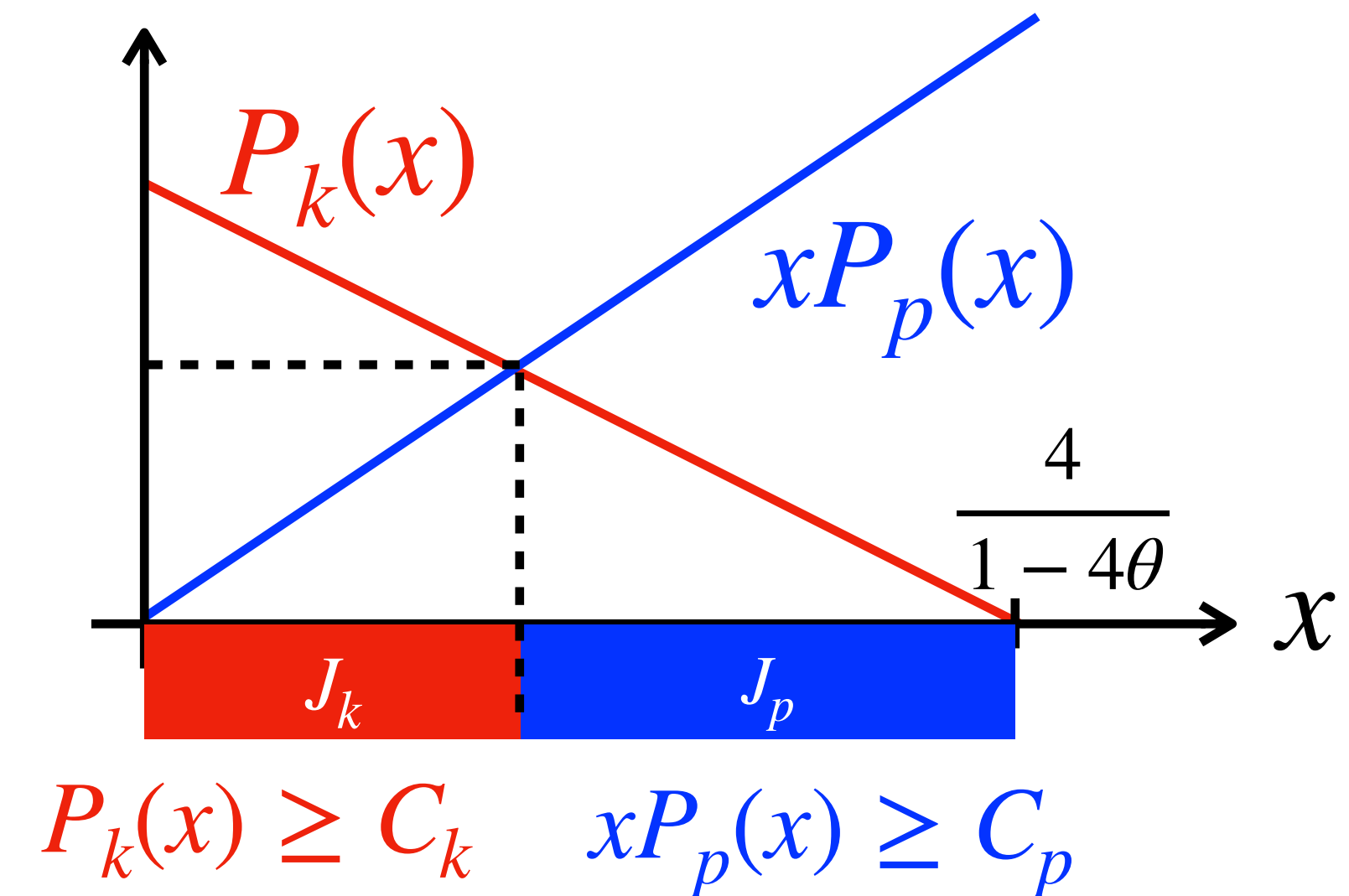
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$$\Pi_p u_h = \sum_{\substack{\Delta t^2 \lambda \in J_p \\ \lambda \in \text{Sp}(A_h)}} (w_h^\lambda, u_h) w_h^\lambda$$

$$P_k(x) = 1 + \left( \theta - \frac{1}{4} \right) x$$

$$\Pi_k u_h = \sum_{\substack{\Delta t^2 \lambda \in J_k \\ \lambda \in \text{Sp}(A_h)}} (w_h^\lambda, u_h) w_h^\lambda$$



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$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

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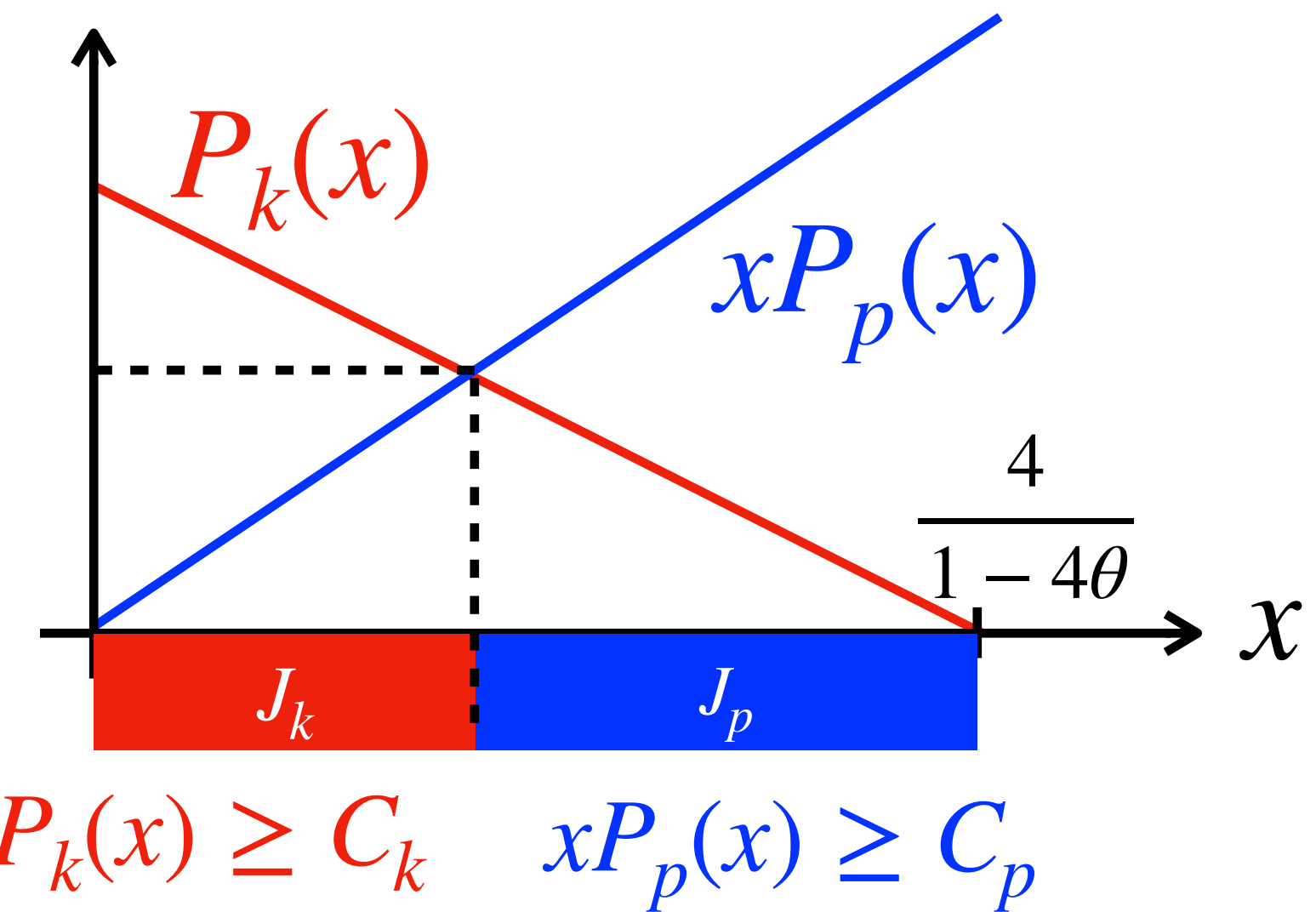
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$$\|\Pi_p \tilde{u}_h\|_P^2 \leq C_p^{-1} (\Delta t^2 A_h P_p(\Delta t^2 A_h) \tilde{u}_h, \tilde{u}_h)_P$$

$$\|\Pi_k \tilde{u}_h\|_P^2 \leq C_k^{-1} (P_k(\Delta t^2 A_h) \tilde{u}_h, \tilde{u}_h)_P$$



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$$\|\delta \mu u_h^n\|_P \leq 2\gamma\mu(\sqrt{2\mathcal{E}_h})^n$$

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$$\|\delta \mu u_h^n\|_P \leq 2\gamma \mu (\sqrt{2\mathcal{E}_h})^n \quad \Rightarrow \quad \sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma \Delta t \sum_{\ell=1}^n \|f\|_P$$

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$$\|u_h^{n+1}\|_P \leq \|\Pi_k u_h^n\|_P + \|\Pi_p u_h^n\|_P + \Delta t \gamma \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \Rightarrow \|u_h^{n+1}\|_P \leq \sqrt{2} \|u_h^1\|_P + 2\sqrt{2} \Delta t \gamma \sum_{\ell=1}^n \sqrt{\mathcal{E}_h^{\ell+\frac{1}{2}}}$$

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# II. Formulation Mixte

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## II.1 Formulation primale/duale

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Chercher  $u \in U = H(B) \subset P$  tel que  $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

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On pose  $v(x, t) = Bu(x, t) :$

$$\begin{cases} (\ddot{u}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} = B\dot{u} & \text{in } D \end{cases}$$

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On pose  $p = \dot{u} : \begin{cases} (\dot{p}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D = (Bp, \tilde{v})_D, & \forall \tilde{v} \in D \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$

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On pose  $\mathbf{v}(x, t) = B\mathbf{u}(x, t) :$  
$$\begin{cases} (\ddot{\mathbf{u}}, \tilde{p})_P + (\mathbf{v}, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{\mathbf{v}} = B\dot{\mathbf{u}} & \text{in } D \end{cases}$$

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$$\begin{cases} (\dot{\mathbf{p}}, \tilde{p})_P + (\mathbf{v}, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{\mathbf{v}}, \tilde{v})_D = (B\mathbf{p}, \tilde{v})_D, & \forall \tilde{v} \in D \\ (\dot{\mathbf{u}}, \tilde{u})_P = (\mathbf{p}, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$



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On pose  $b(v, p) = (v, Bp)_D, \quad \forall v \in D, \forall p \in U$ . On cherche  $p \in U$  et  $v \in D$  tq

$$\begin{cases} (\dot{p}, \tilde{p})_P + b(v, \tilde{p}) = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D - b(\tilde{v}, p) = 0, & \forall \tilde{v} \in D \end{cases}$$

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## II.1 Formulation primale/duale

Chercher  $u \in U = H(B) \subset P$  tel que  $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

On pose  $v(x, t) = Bu(x, t) : \begin{cases} (\ddot{u}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} = B\dot{u} & \text{in } D \end{cases}$

On pose  $p = \dot{u} : \begin{cases} (\dot{p}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D = (Bp, \tilde{v})_D, & \forall \tilde{v} \in D \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$

On pose  $b(v, p) = (v, Bp)_D, \quad \forall v \in D, \forall p \in U$ . On cherche  $p \in U$  et  $v \in D$  tq

$$\begin{cases} (\dot{p}, \tilde{p})_P + b(v, \tilde{p}) = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D - b(\tilde{v}, p) = 0, & \forall \tilde{v} \in D \end{cases} \quad (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, \quad \forall \tilde{u} \in P$$

# II. Formulation Mixte

## II.1 Formulation primale/duale/primale !

On cherche  $p \in U$  et  $v \in D$  tq

$$\begin{cases} (\dot{p}, \tilde{p})_P + b(v, \tilde{p}) = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D - b(\tilde{v}, p) = 0, & \forall \tilde{v} \in D \end{cases}$$

$$(\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, \quad \forall \tilde{u} \in P$$

# II. Formulation Mixte

## II.1 Formulation primale/duale/primale !

On cherche  $p \in U$  et  $v \in D$  tq

$$\begin{cases} (\dot{p}, \tilde{p})_P + b(v, \tilde{p}) = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D - b(\tilde{v}, p) = 0, & \forall \tilde{v} \in D \end{cases} \quad (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, \quad \forall \tilde{u} \in P$$

En prenant  $\tilde{p} = p$  et  $\tilde{v} = v$ ,  $\frac{d\mathcal{E}(p, v)}{dt} = (f, p)_P$  où  $\mathcal{E}(p, v) = \frac{1}{2}(p, p)_P + \frac{1}{2}(v, v)_D$

En utilisant que  $p = \dot{u}$  et  $v = Bu$ , on retrouve le bilan de puissance initial.

# II. Formulation Mixte

## II.2 Discrétisation de la formulation primale-duale(-primale)

# II. Formulation Mixte

## II.2 Discrétisation de la formulation primale-duale(-primale)

Soient  $U_h \subset U$  et  $D_h \subset D$ . On cherche  $p_h \in U_h$  et  $v_h \in D_h$  tq

$$(\dot{p}_h, \tilde{p}_h)_P + b(v_h, \tilde{p}_h) = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\dot{v}_h, \tilde{v}_h)_D - b(\tilde{v}_h, p_h) = 0, \quad \forall \tilde{v}_h \in D_h$$

# II. Formulation Mixte

## II.2 Discrétisation de la formulation primale-duale(-primale)

Soient  $U_h \subset U$  et  $D_h \subset D$ . On cherche  $p_h \in U_h$  et  $v_h \in D_h$  tq

$$(\dot{p}_h, \tilde{p}_h)_P + b(v_h, \tilde{p}_h) = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\dot{v}_h, \tilde{v}_h)_D - b(\tilde{v}_h, p_h) = 0, \quad \forall \tilde{v}_h \in D_h$$

On suppose qu' $\exists$  opérateur linéaire  $\Pi_h : U \times D \rightarrow U_h \times D_h$  tel que  $\forall v \in D, \forall p \in U,$

$$\begin{cases} (p - \Pi_h p, \tilde{p}_h)_P + b(v - \Pi_h v, \tilde{p}_h) = 0, & \forall \tilde{p}_h \in U_h \\ (v - \Pi_h v, \tilde{v}_h)_D - b(\tilde{v}_h, p - \Pi_h p) = 0, & \forall \tilde{v}_h \in D_h \end{cases}$$

# II. Formulation Mixte

## II.2 Discrétisation de la formulation primale-duale(-primale)

Soient  $U_h \subset U$  et  $D_h \subset D$ . On cherche  $p_h \in U_h$  et  $v_h \in D_h$  tq

$$(\dot{p}_h, \tilde{p}_h)_P + b(v_h, \tilde{p}_h) = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\dot{v}_h, \tilde{v}_h)_D - b(\tilde{v}_h, p_h) = 0, \quad \forall \tilde{v}_h \in D_h$$

On suppose qu' $\exists$  opérateur linéaire  $\Pi_h : U \times D \rightarrow U_h \times D_h$  tel que  $\forall v \in D, \forall p \in U,$

$$\begin{cases} (p - \Pi_h p, \tilde{p}_h)_P + b(v - \Pi_h v, \tilde{p}_h) = 0, & \forall \tilde{p}_h \in U_h \\ (v - \Pi_h v, \tilde{v}_h)_D - b(\tilde{v}_h, p - \Pi_h p) = 0, & \forall \tilde{v}_h \in D_h \end{cases}$$

Tel que  $\|p - \Pi_h p\|_U + \|v - \Pi_h v\|_D \xrightarrow{h \rightarrow 0} 0$



# II. Formulation Mixte

## II.2 Discrétisation de la formulation primale-duale(-primale)

Soient  $U_h \subset U$  et  $D_h \subset D$ . On cherche  $p_h \in U_h$  et  $v_h \in D_h$  tq

$$(\dot{p}_h, \tilde{p}_h)_P + b(v_h, \tilde{p}_h) = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\dot{v}_h, \tilde{v}_h)_D - b(\tilde{v}_h, p_h) = 0, \quad \forall \tilde{v}_h \in D_h$$

On suppose qu' $\exists$  opérateur linéaire  $\Pi_h : U \times D \rightarrow U_h \times D_h$  tel que  $\forall v \in D, \forall p \in U,$

$$\begin{cases} (p - \Pi_h p, \tilde{p}_h)_P + b(v - \Pi_h v, \tilde{p}_h) = 0, & \forall \tilde{p}_h \in U_h \\ (v - \Pi_h v, \tilde{v}_h)_D - b(\tilde{v}_h, p - \Pi_h p) = 0, & \forall \tilde{v}_h \in D_h \end{cases}$$

Tel que  $\|p - \Pi_h p\|_U + \|v - \Pi_h v\|_D \xrightarrow{h \rightarrow 0} 0$

Soit  $B_h : U_h \rightarrow D_h$  l'opérateur défini par  $b(v_h, p_h) = (v_h, B_h p_h)_D, \forall v_h \in D_h, \forall p_h \in U_h$

# II. Formulation Mixte

## II.3 Schéma de Crank–Nicolson et analyse

# II. Formulation Mixte

## II.3 Schéma de Crank–Nicolson et analyse

On discrétise  $p_h$  et  $v_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  :

# II. Formulation Mixte

## II.3 Schéma de Crank–Nicolson et analyse

On discrétise  $p_h$  et  $v_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  :

$$(\delta p_h^n, \tilde{p}_h)_P + b(\mu v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\delta v_h^n, \tilde{v}_h)_D - b(\tilde{v}_h, \mu p_h^n) = 0, \quad \forall \tilde{v}_h \in D_h$$

# II. Formulation Mixte

## II.3 Schéma de Crank–Nicolson et analyse

On discrétise  $p_h$  et  $v_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  :

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$$(\delta v_h^n, \tilde{v}_h)_D - b(\tilde{v}_h, \mu p_h^n) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}})_D$$

# II. Formulation Mixte

## II.3 Schéma de Crank–Nicolson et analyse

On discrétise  $p_h$  et  $v_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  :

$$(\delta p_h^n, \tilde{p}_h)_P + b(\mu v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\delta v_h^n, \tilde{v}_h)_D - b(\tilde{v}_h, \mu p_h^n) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}})_D$$

- L'énergie est toujours positive

$$\|p_h^{n+\frac{1}{2}}\|_P \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \text{et} \quad \|v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}}$$

# II. Formulation Mixte

## II.3 Schéma de Crank–Nicolson et analyse

On discrétise  $p_h$  et  $v_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  :

$$\begin{aligned} (\delta p_h^n, \tilde{p}_h)_P + b(\mu v_h^n, \tilde{p}_h) &= (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \\ (\delta v_h^n, \tilde{v}_h)_D - b(\tilde{v}_h, \mu p_h^n) &= 0, \quad \forall \tilde{v}_h \in D_h \end{aligned} \quad (\delta u_h^{n+\frac{1}{2}}, \tilde{u})_P = (p_h^{n+\frac{1}{2}}, \tilde{u})_P, \quad \forall \tilde{u} \in U_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}})_D$$

- L'énergie est toujours positive

$$\|p_h^{n+\frac{1}{2}}\|_P \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \text{et} \quad \|v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}}$$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse



# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

On discrétise  $p_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  et  $v_h$  sur la grille  $t^n$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

On discrétise  $p_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  et  $v_h$  sur la grille  $t^n$

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left( \delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

On discrétise  $p_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  et  $v_h$  sur la grille  $t^n$

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$$\left( \delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+1}, v_h^n)_D$$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

On discrétise  $p_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  et  $v_h$  sur la grille  $t^n$

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$$\left( \delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+1}, v_h^n)_D$$

$$\begin{aligned} v_h^{n+1} &= \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \\ v_h^n &= \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \end{aligned}$$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

On discrétise  $p_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  et  $v_h$  sur la grille  $t^n$

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left( \delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$\begin{aligned} v_h^{n+1} &= \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \\ v_h^n &= \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \end{aligned}$$

$$\tilde{m}(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P - \frac{\Delta t^2}{4} (B_h p_h, B_h \tilde{p}_h)_D$$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

On discrétise  $p_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  et  $v_h$  sur la grille  $t^n$

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left( \delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$\begin{aligned} v_h^{n+1} &= \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \\ v_h^n &= \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \end{aligned}$$

$$\tilde{m}(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P - \frac{\Delta t^2}{4} (B_h p_h, B_h \tilde{p}_h)_D$$

- L'énergie est positive si  $\tilde{m}$  est positive, cad si  $\Delta t^2 \sup_{p_h \in U_h} \frac{(B_h p_h, B_h p_h)_D}{(p_h, p_h)_P} \leq 4$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

On discrétise  $p_h$  sur la grille  $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$  et  $v_h$  sur la grille  $t^n$

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left( \delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$(\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, \quad \forall \tilde{u}_h \in U_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$v_h^{n+1} = \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$v_h^n = \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$\tilde{m}(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P - \frac{\Delta t^2}{4} (B_h p_h, B_h \tilde{p}_h)_D$$

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# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse



# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^n \|f^k\|_P$$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^n \|f^k\|_P$$

Mais on n'a pas de contrôle sur  $p_h^n$  tout entier, seulement sur sa projection  $\Pi_k p_h^n$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^n \|f^k\|_P$$

Mais on n'a pas de contrôle sur  $p_h^n$  tout entier, seulement sur sa projection  $\Pi_k p_h^n$

On définit :  $u_h^{n+1} = u_h^n + \Delta t p_h^{n+\frac{1}{2}}$ ,  $u_h^0$  tel que  $B_h u_h^0 = v_h^0$

# II. Formulation Mixte

## II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^n \|f^k\|_P$$

Mais on n'a pas de contrôle sur  $p_h^n$  tout entier, seulement sur sa projection  $\Pi_k p_h^n$

On définit :  $u_h^{n+1} = u_h^n + \Delta t p_h^{n+\frac{1}{2}}$ ,  $u_h^0$  tel que  $B_h u_h^0 = v_h^0$

$$\|u_h^{n+1}\|_P \leq \sqrt{2}\|u_h^0\| + \gamma\Delta t \sum_{k=0}^n \sqrt{2\mathcal{E}_h^{k+\frac{1}{2}}}$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

Chercher  $\boldsymbol{u} \in U$  tel que  $(\ddot{\boldsymbol{u}}, \tilde{p})_P + (B\boldsymbol{u}, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{\boldsymbol{u}}, \dot{\boldsymbol{u}})_P + \frac{1}{2} (B\boldsymbol{u}, B\boldsymbol{u})_D \right] = (f, \dot{\boldsymbol{u}})_P$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

Chercher  $\boldsymbol{u} \in U$  tel que  $(\ddot{\boldsymbol{u}}, \tilde{p})_P + (B\boldsymbol{u}, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{\boldsymbol{u}}, \dot{\boldsymbol{u}})_P + \frac{1}{2} (B\boldsymbol{u}, B\boldsymbol{u})_D \right] = (f, \dot{\boldsymbol{u}})_P$$

$$v(t) = \sqrt{(B\boldsymbol{u}, B\boldsymbol{u})_D + 2\alpha} \in \mathbb{R} \Rightarrow$$



# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

Chercher  $u \in U$  tel que  $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

Chercher  $u \in U$  tel que  $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

$$b[u, \tilde{u}] = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

# IV. Scalar Auxiliary Variable

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Chercher  $u \in U$  tel que  $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

$$b[u, \tilde{u}] = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

$$\begin{cases} (\dot{u}, \tilde{p})_P + v b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, \dot{u}] = 0 \end{cases}$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

Chercher  $u \in U$  tel que  $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

$$b[u, \tilde{u}] = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

$$\begin{cases} (\ddot{u}, \tilde{p})_P + v b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, \dot{u}] = 0 \end{cases}$$

$$\dot{u} = p \in U$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

Chercher  $\mathbf{u} \in U$  tel que  $(\ddot{\mathbf{u}}, \tilde{\mathbf{p}})_P + (B\mathbf{u}, B\tilde{\mathbf{p}})_D = (f, \tilde{\mathbf{p}})_P, \quad \forall \tilde{\mathbf{p}} \in U$

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{\mathbf{u}}, \dot{\mathbf{u}})_P + \frac{1}{2} (B\mathbf{u}, B\mathbf{u})_D \right] = (f, \dot{\mathbf{u}})_P$$

$$\mathbf{v}(t) = \sqrt{(B\mathbf{u}, B\mathbf{u})_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{\mathbf{v}} = \left( \frac{B\mathbf{u}}{\sqrt{\|B\mathbf{u}\|_D^2 + 2\alpha}}, B\dot{\mathbf{u}} \right)_D$$

$$b[\mathbf{u}, \tilde{\mathbf{u}}] = \left( \frac{B\mathbf{u}}{\sqrt{\|B\mathbf{u}\|_D^2 + 2\alpha}}, B\tilde{\mathbf{u}} \right)_D$$

$$\begin{cases} (\dot{\mathbf{p}}, \tilde{\mathbf{p}})_P + \mathbf{v} b[\mathbf{u}, \tilde{\mathbf{p}}] = (f, \tilde{\mathbf{p}})_P, & \forall \tilde{\mathbf{p}} \in U \\ \dot{\mathbf{v}} - b[\mathbf{u}, \mathbf{p}] = 0 \\ (\dot{\mathbf{u}}, \tilde{\mathbf{u}})_P = (\mathbf{p}, \tilde{\mathbf{u}})_P, & \forall \tilde{\mathbf{u}} \in P \end{cases}$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

$$\begin{cases} (\dot{p}, \tilde{p})_P + \nu b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

$$\begin{cases} (\dot{p}, \tilde{p})_P + \nu b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

Prenons  $\tilde{p} = p$  et multiplions la deuxième équation par  $\nu$ , on obtient :

$$\frac{d\mathcal{E}}{dt} = (f, p)_P \quad \mathcal{E} = \frac{1}{2}(p, p)_P + \frac{1}{2}\nu^2$$



# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

$$\begin{cases} (\dot{p}, \tilde{p})_P + \nu b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D = F_\alpha[u] (Bu, B\tilde{u})_D$$

Prenons  $\tilde{p} = p$  et multiplions la deuxième équation par  $\nu$ , on obtient :

$$\frac{d\mathcal{E}}{dt} = (f, p)_P \quad \mathcal{E} = \frac{1}{2} (p, p)_P + \frac{1}{2} \nu^2$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

$$\begin{cases} (\dot{\boldsymbol{p}}, \tilde{\boldsymbol{p}})_P + \nu b[\boldsymbol{u}, \tilde{\boldsymbol{p}}] = (f, \tilde{\boldsymbol{p}})_P, & \forall \tilde{\boldsymbol{p}} \in U \\ \dot{\boldsymbol{v}} - b[\boldsymbol{u}, \boldsymbol{p}] = 0 \\ (\dot{\boldsymbol{u}}, \tilde{\boldsymbol{u}})_P = (\boldsymbol{p}, \tilde{\boldsymbol{u}})_P, & \forall \tilde{\boldsymbol{u}} \in P \end{cases}$$

$$b[\boldsymbol{u}, \tilde{\boldsymbol{u}}] = \left( \frac{B\boldsymbol{u}}{\sqrt{\|B\boldsymbol{u}\|_D^2 + 2\alpha}}, B\tilde{\boldsymbol{u}} \right)_D = F_\alpha[\boldsymbol{u}] (B\boldsymbol{u}, B\tilde{\boldsymbol{u}})_D$$

$$F_\alpha[\boldsymbol{u}] = \frac{1}{\sqrt{\|B\boldsymbol{u}\|_D^2 + 2\alpha}}$$

Prenons  $\tilde{\boldsymbol{p}} = \boldsymbol{p}$  et multiplions la deuxième équation par  $\nu$ , on obtient :

$$\frac{d\mathcal{E}}{dt} = (f, \boldsymbol{p})_P \quad \mathcal{E} = \frac{1}{2} (\boldsymbol{p}, \boldsymbol{p})_P + \frac{1}{2} \nu^2$$

# IV. Scalar Auxiliary Variable

## IV.1 Formulation primale/scalaire/primale ?

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \quad \forall u, \tilde{u} \in U^2$$

$$\begin{cases} (\dot{p}, \tilde{p})_P + v b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases} \quad \begin{cases} (\dot{p}, \tilde{p})_P + F_\alpha[u] v a(u, \tilde{p}) = (f, \tilde{p})_P \\ \dot{v} - F_\alpha[u] a(u, p) = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left( \frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D = F_\alpha[u] (Bu, B\tilde{u})_D$$

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

Prenons  $\tilde{p} = p$  et multiplions la deuxième équation par  $v$ , on obtient :

$$\frac{d\mathcal{E}}{dt} = (f, p)_P \quad \mathcal{E} = \frac{1}{2} (p, p)_P + \frac{1}{2} v^2$$

# IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

## IV.2 Discretisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \forall u, \tilde{u} \in U^2$$

# IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

## IV.2 Discretisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \forall u, \tilde{u} \in U^2$$

Chaolong Jiang and Wenjun Cai and Yushun Wang. « A Linearly Implicit and Local Energy-Preserving Scheme for the Sine-Gordon Equation Based on the Invariant Energy Quadraticization Approach », Journal of Scientific Computing vol 80(3), 2019.

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# IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

## IV.2 Discretisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \quad \forall u, \tilde{u} \in U^2$$

$$\left\{ \begin{array}{l} (\delta p_h^n, \tilde{p}_h)_P + F_\alpha[u_h^n] \mu v_h^n a(u_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \\ \delta v_h^n - F_\alpha[u_h^n] a(u_h^n, \mu p_h^n) = 0 \\ (\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, \quad \forall \tilde{u}_h \in U_h \end{array} \right.$$

Chaolong Jiang and Wenjun Cai and Yushun Wang. « A Linearly Implicit and Local Energy-Preserving Scheme for the Sine-Gordon Equation Based on the Invariant Energy Quadraticization Approach », Journal of Scientific Computing vol 80(3), 2019.

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# IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

## IV.2 Discrétisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \quad \forall u, \tilde{u} \in U^2$$

$$\left\{ \begin{array}{l} (\delta p_h^n, \tilde{p}_h)_P + F_\alpha[u_h^n] \mu v_h^n a(u_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \\ \delta v_h^n - F_\alpha[u_h^n] a(u_h^n, \mu p_h^n) = 0 \\ (\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, \quad \forall \tilde{u}_h \in U_h \end{array} \right.$$

Prenons  $\tilde{p}_h = \mu p_h^n$ , multiplions la deuxième équation par  $\mu v_h^n$ . On a :

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# IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

## IV.2 Discrétisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \quad \forall u, \tilde{u} \in U^2$$

$$\begin{cases} (\delta p_h^n, \tilde{p}_h)_P + F_\alpha[u_h^n] \mu v_h^n a(u_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, & \forall \tilde{p}_h \in U_h \\ \delta v_h^n - F_\alpha[u_h^n] a(u_h^n, \mu p_h^n) = 0 \\ (\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, & \forall \tilde{u}_h \in U_h \end{cases}$$

Prenons  $\tilde{p}_h = \mu p_h^n$ , multiplions la deuxième équation par  $\mu v_h^n$ . On a :

$$\delta \mathcal{E}_h^n = (f^n, \mu p_h^{n+\frac{1}{2}})_P \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+\frac{1}{2}})^2$$

Chaolong Jiang and Wenjun Cai and Yushun Wang. « A Linearly Implicit and Local Energy-Preserving Scheme for the Sine-Gordon Equation Based on the Invariant Energy Quadraticization Approach », Journal of Scientific Computing vol 80(3), 2019.

Michele Ducceschi and Stefan Bilbao. « Simulation of the geometrically exact nonlinear string via energy quadratisation », Journal of Sound and Vibration vol 534, 2022.



# IV. Scalar Auxiliary Variable

## IV.2 Discretisation

# IV. Scalar Auxiliary Variable

## IV.2 Discrétisation

Connaissant  $U_h^n$ ,  $P_h^{n-\frac{1}{2}}$ ,  $v_h^{n-\frac{1}{2}}$ , Trouver  $P_h^{n+\frac{1}{2}}$  et  $v_h^{n+\frac{1}{2}}$  puis  $U_h^{n+1}$  tels que :

# IV. Scalar Auxiliary Variable

## IV.2 Discrétisation

Connaissant  $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$ , Trouver  $P_h^{n+\frac{1}{2}}$  et  $v_h^{n+\frac{1}{2}}$  puis  $U_h^{n+1}$  tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha [U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha [U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases}$$

# IV. Scalar Auxiliary Variable

## IV.2 Discrétisation

Connaissant  $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$ , Trouver  $P_h^{n+\frac{1}{2}}$  et  $v_h^{n+\frac{1}{2}}$  puis  $U_h^{n+1}$  tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

# IV. Scalar Auxiliary Variable

## IV.2 Discrétisation

Connaissant  $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$ , Trouver  $P_h^{n+\frac{1}{2}}$  et  $v_h^{n+\frac{1}{2}}$  puis  $U_h^{n+1}$  tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

# IV. Scalar Auxiliary Variable

## IV.2 Discrétisation

Connaissant  $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$ , Trouver  $P_h^{n+\frac{1}{2}}$  et  $v_h^{n+\frac{1}{2}}$  puis  $U_h^{n+1}$  tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

Ce schéma :

# IV. Scalar Auxiliary Variable

## IV.2 Discrétisation

Connaissant  $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$ , Trouver  $P_h^{n+\frac{1}{2}}$  et  $v_h^{n+\frac{1}{2}}$  puis  $U_h^{n+1}$  tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

Ce schéma :

- est inconditionnellement stable

# IV. Scalar Auxiliary Variable

## IV.2 Discrétisation

Connaissant  $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$ , Trouver  $P_h^{n+\frac{1}{2}}$  et  $v_h^{n+\frac{1}{2}}$  puis  $U_h^{n+1}$  tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

Ce schéma :

- est inconditionnellement stable

- couple linéairement les inconnues  $P_h^{n+\frac{1}{2}} / U_h^{n+1}$  et  $v_h^{n+\frac{1}{2}}$  par une matrice  $(N+1) \times (N+1)$



# IV. Scalar Auxiliary Variable

## IV.2 Discrétisation

Connaissant  $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$ , Trouver  $P_h^{n+\frac{1}{2}}$  et  $v_h^{n+\frac{1}{2}}$  puis  $U_h^{n+1}$  tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

Ce schéma :

- est inconditionnellement stable
- couple linéairement les inconnues  $P_h^{n+\frac{1}{2}} / U_h^{n+1}$  et  $v_h^{n+\frac{1}{2}}$  par une matrice  $(N+1) \times (N+1)$
- peut **se ramener** à une simple inversion de  $M_h$  accompagnée d'une évaluation de  $K_h U_h^n + \text{prod scal}$

# IV. Scalar Auxiliary Variable

## IV.3 Simulations préliminaires

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Pour les  $\theta$ -schémas,

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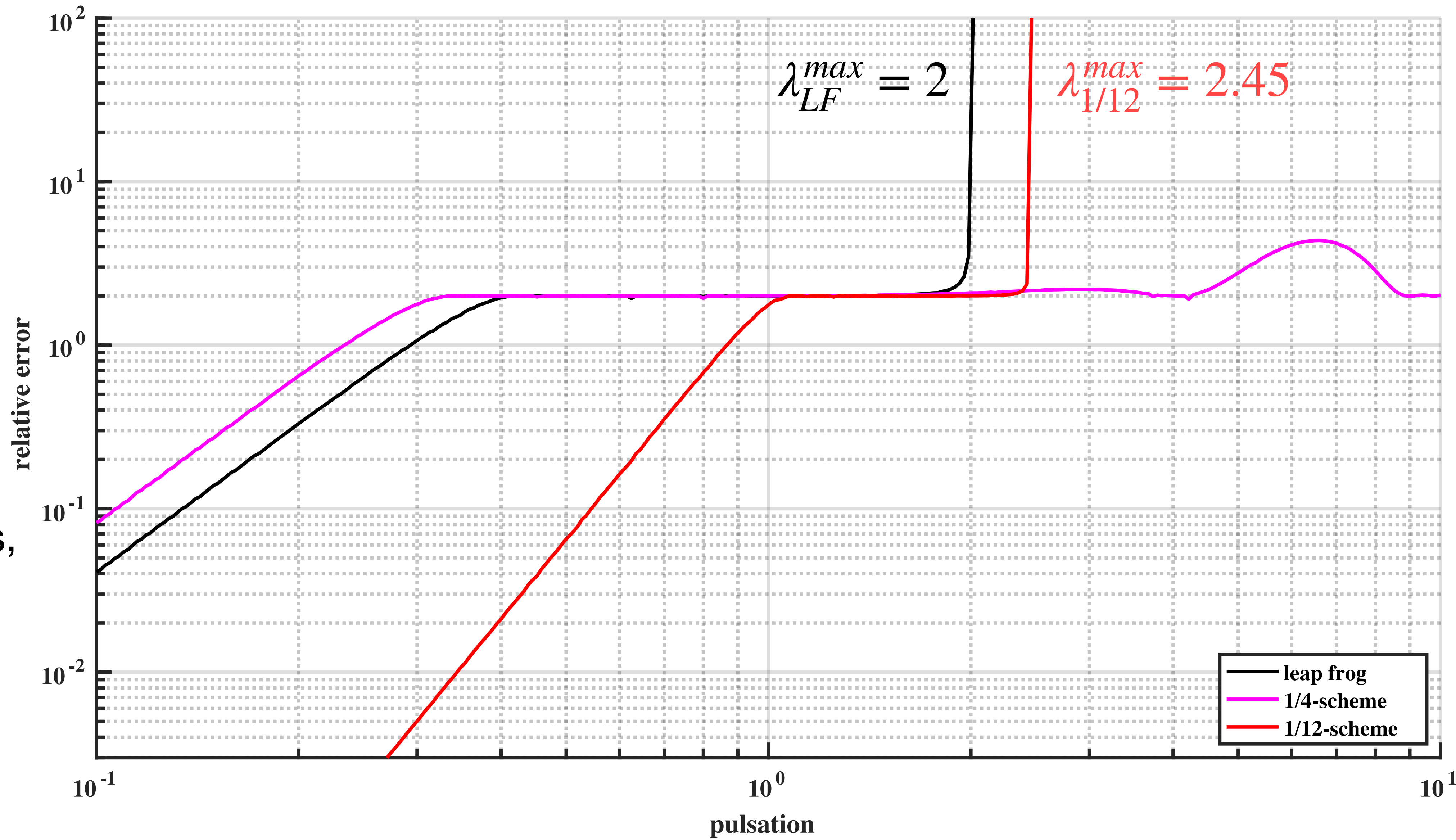
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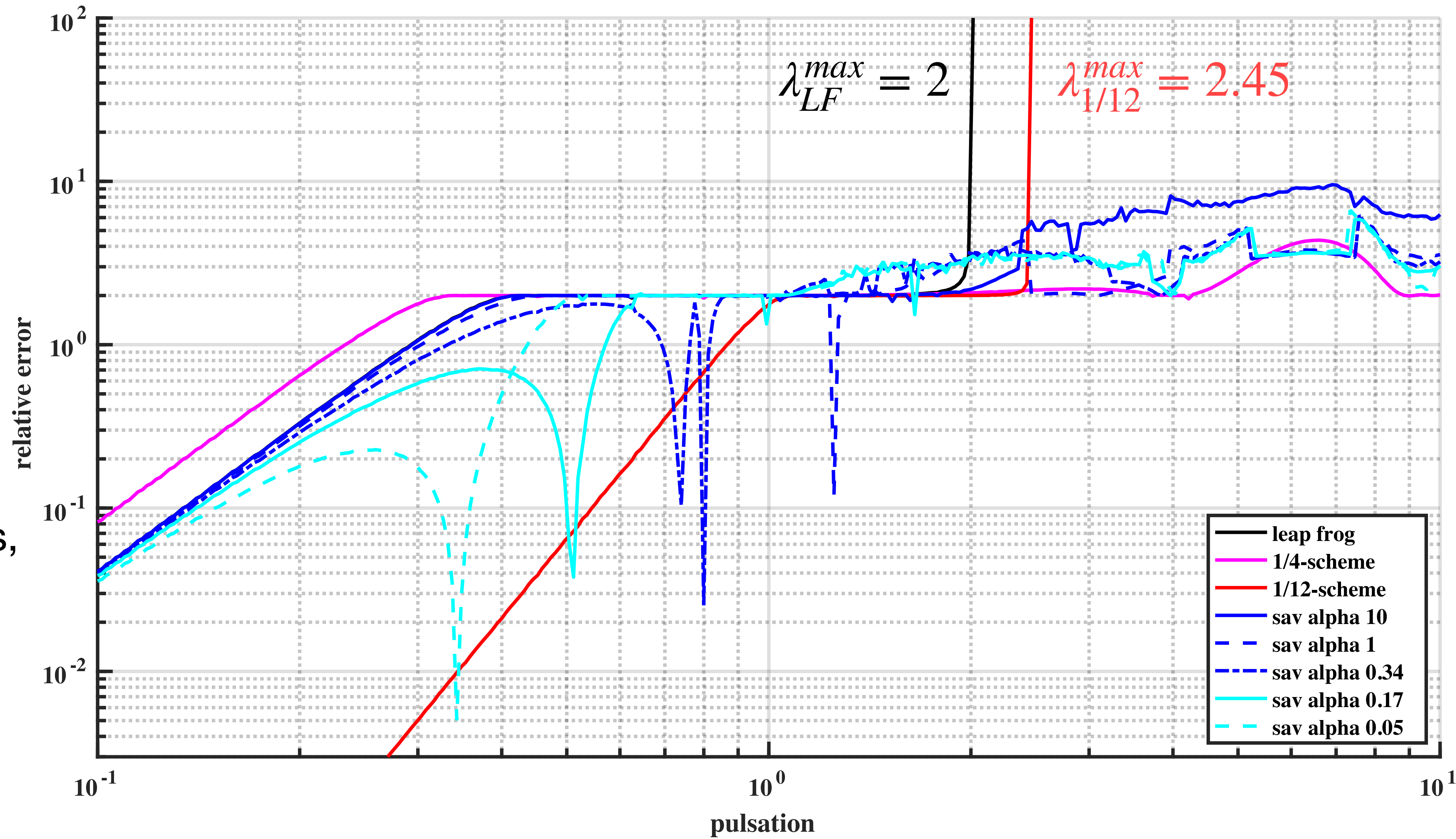
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# Conclusion - Perspectives

Chercher  $u \in U = H(B) \subset P$  tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

I. Discrétisation et analyse de (1)

II. On pose  $v(x, t) = Bu(x, t)$   $\Rightarrow$  Formulation Mixte

III. On pose  $v(t) = \sqrt{(Bu, Bu)_D + 2\alpha}$   $\Rightarrow$  Scalar Auxiliary Variable

Le nouveau schéma proposé est inconditionnellement stable et explicite.

Son analyse complète est en cours.





**JO-SO**  
*2023*

**Journées Ondes du Sud-Ouest**  
14 - 16 mars 2023 - Toulouse

**Merci de votre attention**

**Juliette Chabassier - Equipe Makutu - INRIA**

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\delta X^{n+\frac{1}{2}} = \frac{X^{n+1} - X^n}{\Delta t}, \quad \mu X^{n+\frac{1}{2}} = \frac{X^{n+1} + X^n}{2}, \quad \pi X^{n+\frac{1}{2}} = \frac{3X^n - X^{n-1}}{2}$$

$$\{X\}_\theta^n = X^n + \theta \Delta t^2 \delta^2 X = \mu^2 X^n + \left(\theta - \frac{1}{4}\right) \Delta t^2 \delta^2 X \quad (\text{A})$$

$$v_h^{n+1} = \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$v_h^n = \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

(B)

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# IV. Scalar Auxiliary Variable

## IV.3 Complexité

$$M_h \ddot{U}_h + D_h \dot{U}_h + K_h U_h = F_h$$

```
u_h(1) = u0; % initial data
u_h(2) = u1; % initial data
mu_u = (u_sav(1)+u_sav(2))*0.5;
v_h(1) = sqrt( 2 * alpha + dot ( K_h*mu_u, mu_u ) ); % initial data
Am1      = inv ( M_h + dt*0.5*D_h ); % mass matrix : easy to invert

For i = 3..N
    Kun      = Kh * u_h(i-1); % stiffness matrix * vector product
    Falphan  = 1 / sqrt( 2*alpha + dot ( Kun , u_h(i-1) ) ); % Falphan is a scalar
    Cn       = dt * 0.5 * Falphan * Kun; % vector

    RHS      = dt*0.5*F_h(n) + ( u_h(i-1)-u_h(i-2) ) / dt - v_h(i-2)*Cn; % vector
    Am1Cn    = Am1*Cn;
    mat_sherman = Am1 - dot ( Am1Cn , Am1Cn ) / ( 1 + dot ( Cn , Am1Cn ) ); % real division
    temp     = mat_sherman * RHS; % temp now contains the vector \delta\mu U^n

    u_h(i)   = u_sav(i-2) + 2*dt*temp;
    v_h(i-1) = v_h(i-2) + 2* dot ( Cn , temp ); % dot prod ( Cn , temp )
```

Complexité :  $4 \cdot O(N)$  pour les **dot**,  $O(N)$  pour l'**inv** et  $O(p \cdot N)$  pour le **produit par Kh** ( $p \approx$  ordre des FEM)

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$$m_\theta(\delta p_h^n, \tilde{p}_h) + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left( \delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

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$$m_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \theta (B_h p_h, B_h \tilde{p}_h)_D \quad \forall (p_h, \tilde{p}_h) \in U_h^2$$



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Posons  $B : U \times U \rightarrow D_0$  tel que  $B[u]\tilde{u} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\tilde{u}, \quad \forall u \in U, \tilde{u} \in U$

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$$\dot{u} = p$$

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