



HAL
open science

Techniques de quadratisation pour les équations d'onde linéaires : modélisation et simulations

Juliette Chabassier

► **To cite this version:**

Juliette Chabassier. Techniques de quadratisation pour les équations d'onde linéaires : modélisation et simulations. Journées Ondes du Sud Ouest, Mar 2023, Toulouse, France. hal-04039870

HAL Id: hal-04039870

<https://inria.hal.science/hal-04039870v1>

Submitted on 21 Mar 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



JO-SO
2023

Journées Ondes du Sud-Ouest
14 - 16 mars 2023 - Toulouse

Techniques de quadratisation pour les équations d'onde linéaires : modélisation et simulations

Juliette Chabassier - Equipe Makutu - INRIA

Introduction

Soit $B : U = H(B) \subset P \rightarrow D$ un opérateur linéaire continu.

Introduction

Soit $B : U = H(B) \subset P \rightarrow D$ un opérateur linéaire continu.

Chercher $u \in U = H(B) \subset P$ tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

Introduction

Soit $B : U = H(B) \subset P \rightarrow D$ un opérateur linéaire continu.

Chercher $u \in U = H(B) \subset P$ tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

En prenant $\tilde{p} = \dot{u}$, on obtient le bilan de puissance :

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

Introduction

Soit $B : U = H(B) \subset P \rightarrow D$ un opérateur linéaire continu.

Chercher $u \in U = H(B) \subset P$ tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

En prenant $\tilde{p} = \dot{u}$, on obtient le bilan de puissance :

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

Introduction

Soit $B : U = H(B) \subset P \rightarrow D$ un opérateur linéaire continu.

Chercher $u \in U = H(B) \subset P$ tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

En prenant $\tilde{p} = \dot{u}$, on obtient le bilan de puissance :

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

Introduction

Soit $B : U = H(B) \subset P \rightarrow D$ un opérateur linéaire continu.

Chercher $u \in U = H(B) \subset P$ tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

En prenant $\tilde{p} = \dot{u}$, on obtient le bilan de puissance :

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

I. Discrétisation et analyse de (1)

Introduction

Soit $B : U = H(B) \subset P \rightarrow D$ un opérateur linéaire continu.

Chercher $u \in U = H(B) \subset P$ tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

En prenant $\tilde{p} = \dot{u}$, on obtient le bilan de puissance :

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

I. Discrétisation et analyse de (1)

II. On pose $v(x, t) = Bu(x, t) \Rightarrow$ Formulation Mixte

Introduction

Soit $B : U = H(B) \subset P \rightarrow D$ un opérateur linéaire continu.

Chercher $u \in U = H(B) \subset P$ tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

En prenant $\tilde{p} = \dot{u}$, on obtient le bilan de puissance :

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

I. Discrétisation et analyse de (1)

II. On pose $v(x, t) = Bu(x, t)$ \Rightarrow Formulation Mixte

III. On pose $v(t) = \sqrt{(Bu, Bu)_D + 2\alpha}$ \Rightarrow Scalar Auxiliary Variable

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h \in U_h \subset U$ tel que

$$(\ddot{u}_h, \tilde{p}_h)_P + (B u_h, B \tilde{p}_h)_D = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (2)$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h \in U_h \subset U$ tel que

$$(\ddot{u}_h, \tilde{p}_h)_P + (B u_h, B \tilde{p}_h)_D = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (2)$$

$$M_h \ddot{U}_h + K_h U_h = M_h F_h$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h \in U_h \subset U$ tel que

$$(\ddot{u}_h, \tilde{p}_h)_P + (Bu_h, B\tilde{p}_h)_D = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (2)$$

$$M_h \ddot{U}_h + K_h U_h = M_h F_h$$

Projection elliptique $P_h : U \rightarrow U_h$ telle que $\forall u \in U$

$$(B(P_h u - u), Bu_h)_P + c_a(P_h u - u, u_h)_P = 0, \quad \forall u_h \in U_h$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h \in U_h \subset U$ tel que

$$(\ddot{u}_h, \tilde{p}_h)_P + (Bu_h, B\tilde{p}_h)_D = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (2)$$

$$M_h \ddot{U}_h + K_h U_h = M_h F_h$$

Projection elliptique $P_h : U \rightarrow U_h$ telle que $\forall u \in U$

$$(B(P_h u - u), Bu_h)_P + c_a(P_h u - u, u_h)_P = 0, \quad \forall u_h \in U_h$$

Lemme de Céa

$$\|u - P_h u\|_U \leq C \inf_{u_h \in U_h} \|u - u_h\|_U$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h^n \in U_h \subset U$ tel que

$$(\delta^2 u_h^n, \tilde{p}_h)_P + (B\{u_h\}_\theta^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (3)$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h^n \in U_h \subset U$ tel que

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + (B\{u_h\}_\theta^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (3)$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h^n \in U_h \subset U$ tel que

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + (B\{u_h\}_\theta^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (3)$$

$$M_h \delta^2 U_h^n + K_h \{U_h\}_\theta^n = M_h F_h^n$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h^n \in U_h \subset U$ tel que

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + (B\{u_h\}_\theta^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (3)$$

$$M_h \delta^2 U_h^n + K_h \{U_h\}_\theta^n = M_h F_h^n$$

$$M_h + \Delta t^2 \theta K_h$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h^n \in U_h \subset U$ tel que

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + (B\{u_h\}_\theta^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (3)$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (B\delta^2 u_h^n, B\tilde{p}_h)_D + (B\mu^2 u_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h^n \in U_h \subset U$ tel que

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + (B\{u_h\}_\theta^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (3)$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (B\delta^2 u_h^n, B\tilde{p}_h)_D + (B\mu^2 u_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$m_\theta(\delta^2 u_h^n, \tilde{p}_h) + (B\mu^2 u_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$m_\theta(u, v) = (u, v)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (Bu, Bv)_D$$

I. Discrétisation et Analyse Ordre 2

I.1 Discrétisation variationnelle et θ -schema

Chercher $u_h^n \in U_h \subset U$ tel que

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$\mu^2 u_h^n = \frac{u_h^{n+1} + 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + (B\{u_h\}_\theta^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \quad (3)$$

$$(\delta^2 u_h^n, \tilde{p}_h)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (B\delta^2 u_h^n, B\tilde{p}_h)_D + (B\mu^2 u_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$m_\theta(\delta^2 u_h^n, \tilde{p}_h) + (B\mu^2 u_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$m_\theta(u, v) = (u, v)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (Bu, Bv)_D$$

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$m_\theta(\delta^2 \mathbf{u}_h^n, \tilde{p}_h) + (B\mu^2 \mathbf{u}_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$m_\theta(u, v) = (u, v)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (Bu, Bv)_D$$

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$m_\theta(\delta^2 \mathbf{u}_h^n, \tilde{p}_h) + (B\mu^2 \mathbf{u}_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

Bilan de puissance discret

$$\tilde{p}_h = \mu \delta \mathbf{u}_h^n$$

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta \mathbf{u}_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} m_\theta(\delta \mathbf{u}_h^{n+\frac{1}{2}}, \delta \mathbf{u}_h^{n+\frac{1}{2}}) + \frac{1}{2} (B\mu \mathbf{u}_h^{n+\frac{1}{2}}, B\mu \mathbf{u}_h^{n+\frac{1}{2}})_D$$

$$m_\theta(u, v) = (u, v)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (Bu, Bv)_D$$

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$m_\theta(\delta^2 \mathbf{u}_h^n, \tilde{p}_h) + (B\mu^2 \mathbf{u}_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

Bilan de puissance discret

$$\tilde{p}_h = \mu \delta \mathbf{u}_h^n$$

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta \mathbf{u}_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} m_\theta(\delta \mathbf{u}_h^{n+\frac{1}{2}}, \delta \mathbf{u}_h^{n+\frac{1}{2}}) + \frac{1}{2} (B\mu \mathbf{u}_h^{n+\frac{1}{2}}, B\mu \mathbf{u}_h^{n+\frac{1}{2}})_D$$

$$m_\theta(u, v) = (u, v)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (Bu, Bv)_D$$

- L'énergie est positive si m_θ est positive, cad

- ▶ si $\theta \geq \frac{1}{4}$, ou

- ▶ si $\Delta t^2 \sup_{v_h \in U_h} \frac{(Bv_h, Bv_h)_D}{(v_h, v_h)_P} \leq \frac{4}{1 - 4\theta}$

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$m_\theta(\delta^2 u_h^n, \tilde{p}_h) + (B\mu^2 u_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} m_\theta(\delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}}) + \frac{1}{2} (B\mu u_h^{n+\frac{1}{2}}, B\mu u_h^{n+\frac{1}{2}})_D$$

$$m_\theta(u, v) = (u, v)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (Bu, Bv)_D$$

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$m_\theta(\delta^2 u_h^n, \tilde{p}_h) + (B\mu^2 u_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} m_\theta(\delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}}) + \frac{1}{2} (B\mu u_h^{n+\frac{1}{2}}, B\mu u_h^{n+\frac{1}{2}})_D$$

$$m_\theta(u, v) = (u, v)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (Bu, Bv)_D$$

- Éléments de preuve pour la stabilité et la convergence espace / temps

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$m_\theta(\delta^2 u_h^n, \tilde{p}_h) + (B\mu^2 u_h^n, B\tilde{p}_h)_D = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} m_\theta(\delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}}) + \frac{1}{2} (B\mu u_h^{n+\frac{1}{2}}, B\mu u_h^{n+\frac{1}{2}})_D$$

$$m_\theta(u, v) = (u, v)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (Bu, Bv)_D$$

- Éléments de preuve pour la stabilité et la convergence espace / temps

Chercher $u_h^n \in U_h \subset U$ tel que $P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$

$$P_p(x) = 1 \quad P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1 \qquad P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

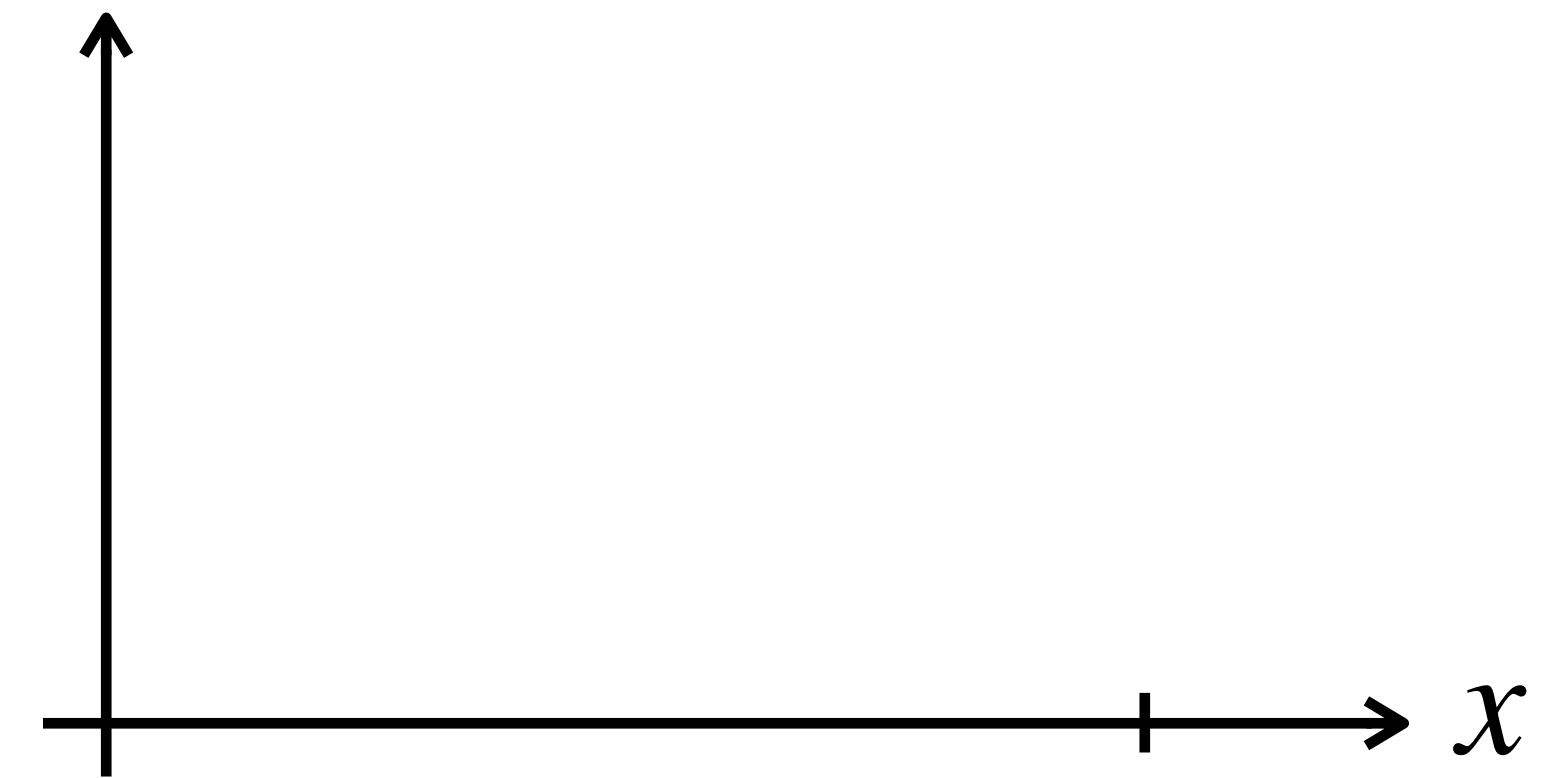
$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1$$

$$P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$



Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

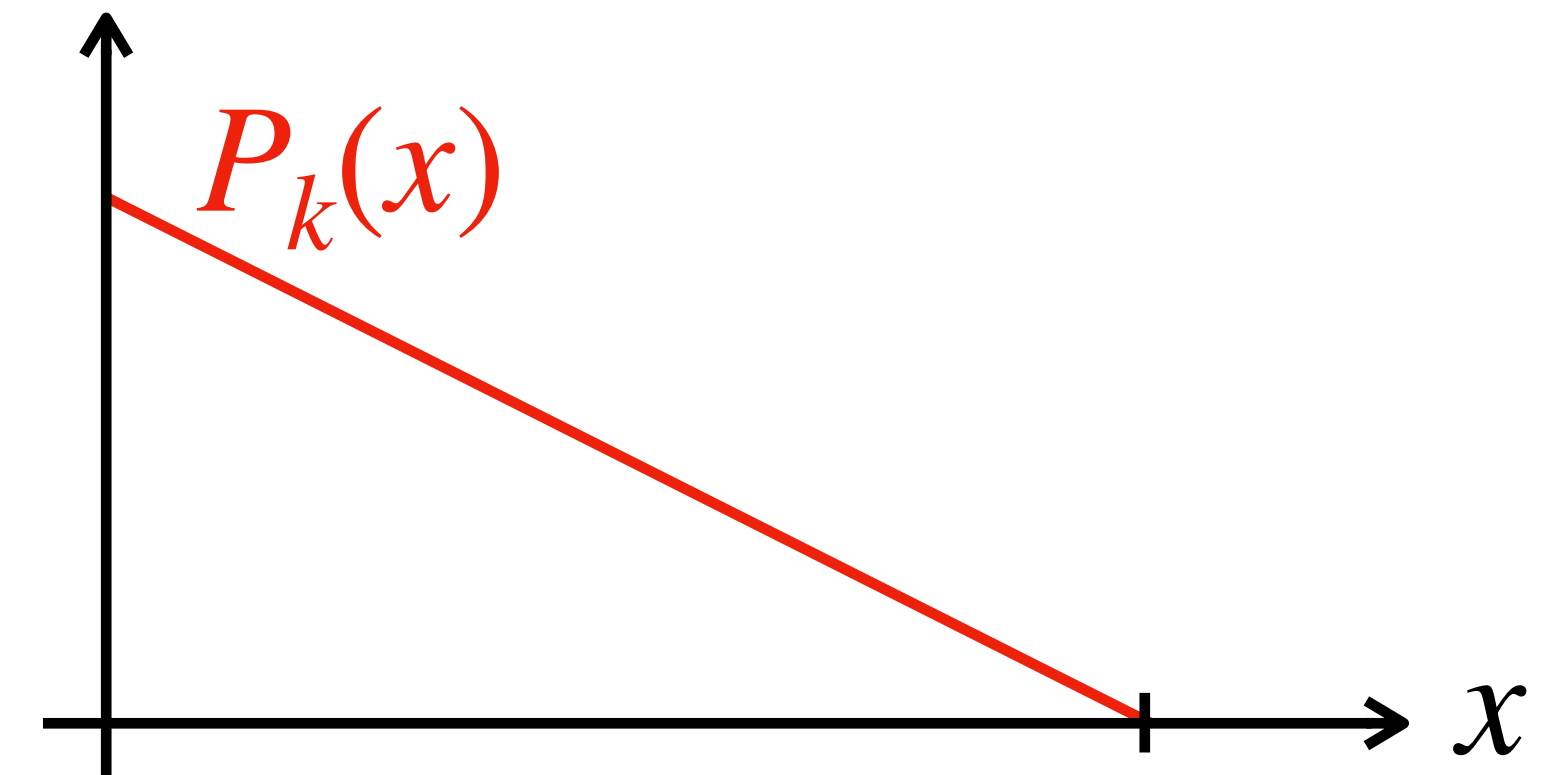
$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1$$

$$P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$



Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

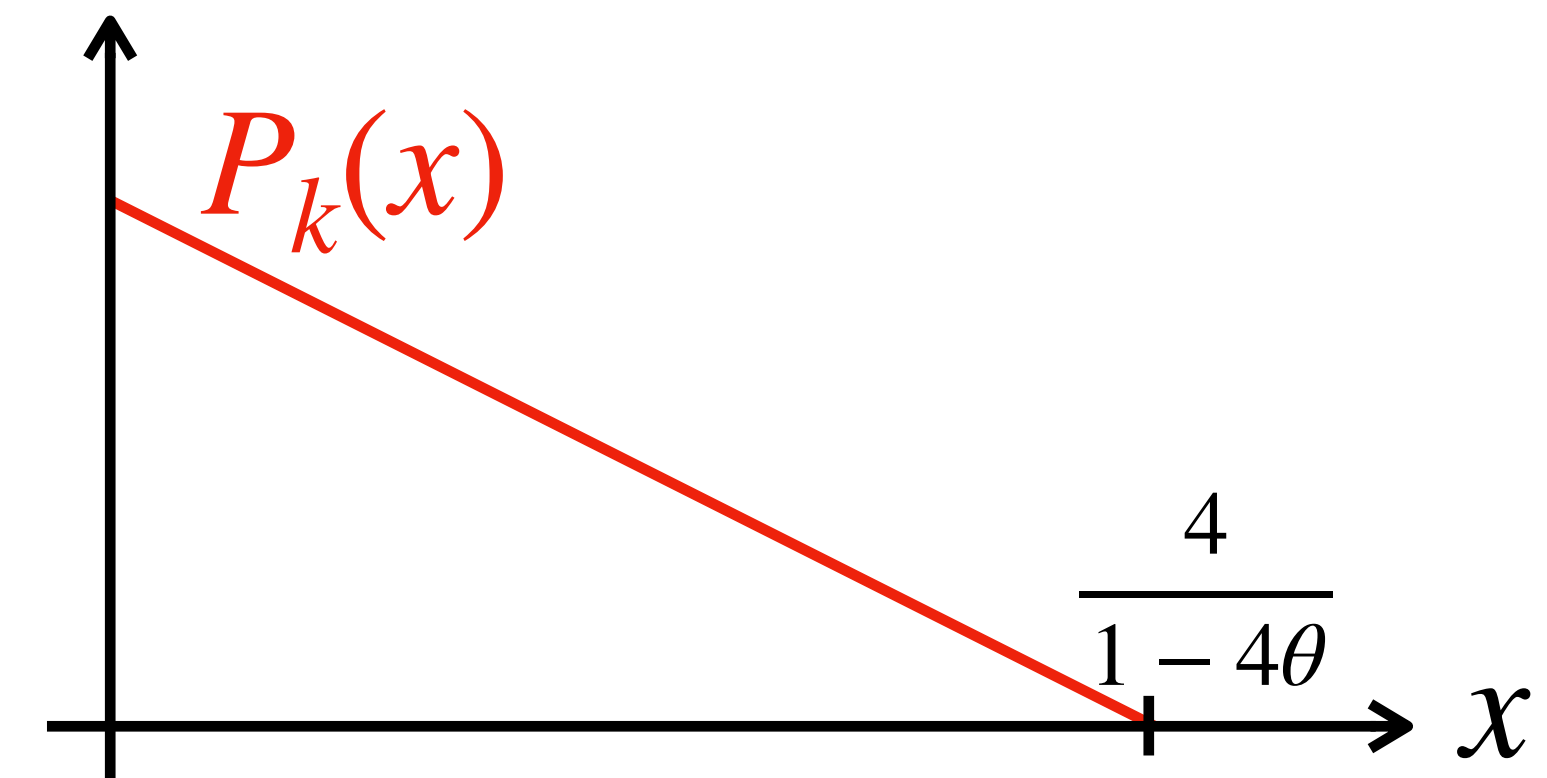
$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1$$

$$P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$



Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

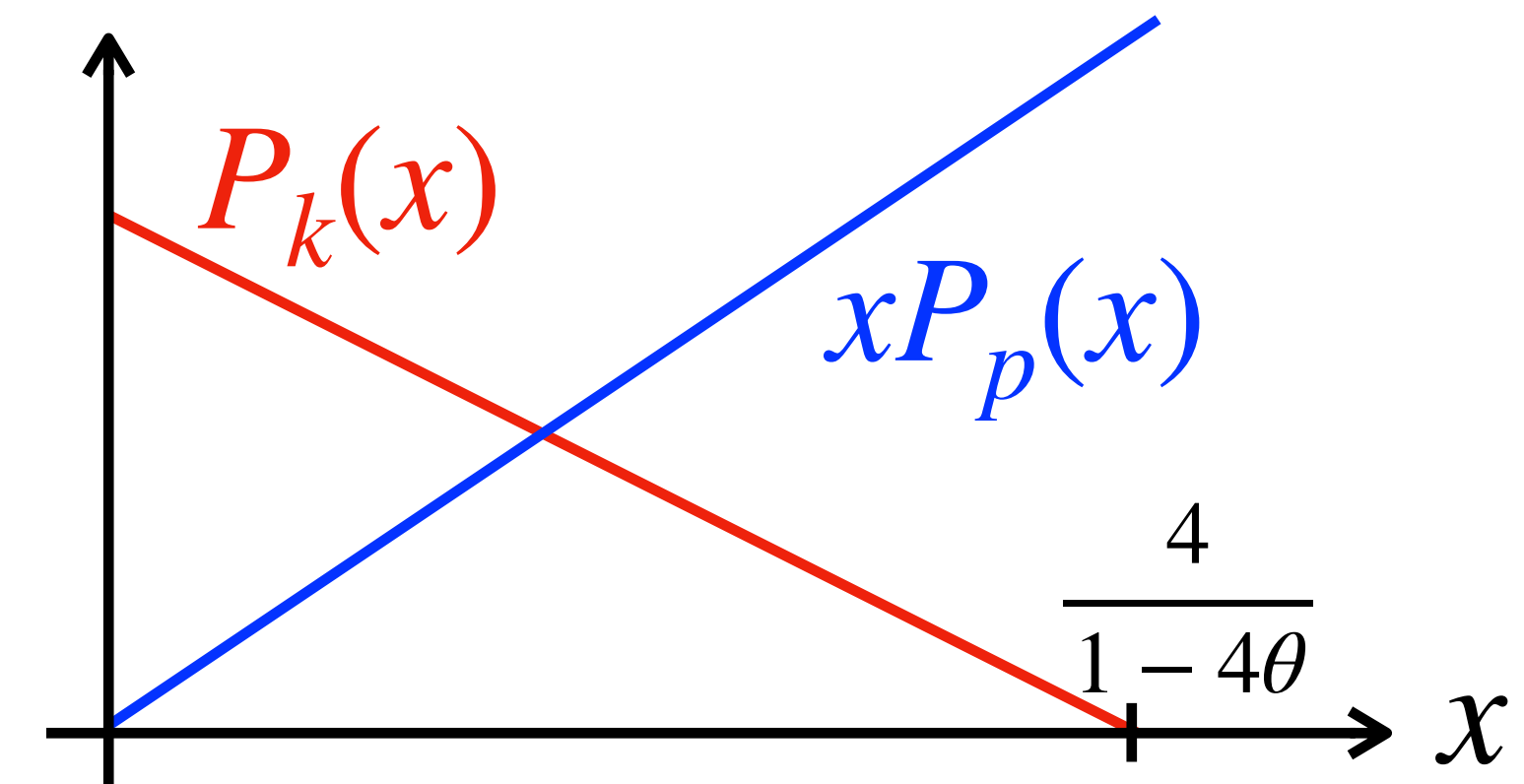
$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1$$

$$P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$



Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

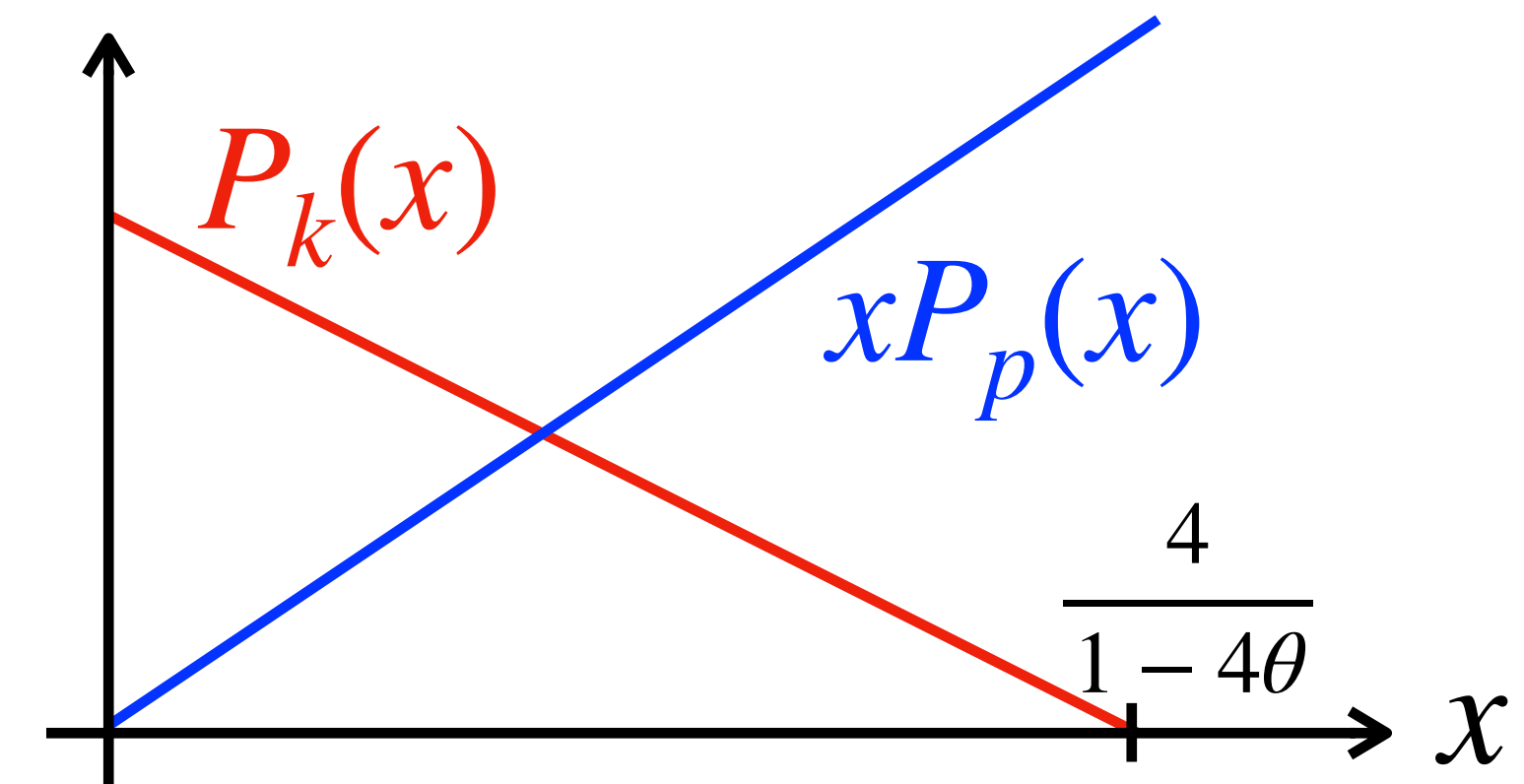
$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1$$

$$P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$



$$P_k(x) \geq C_k \quad xP_p(x) \geq C_p$$

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

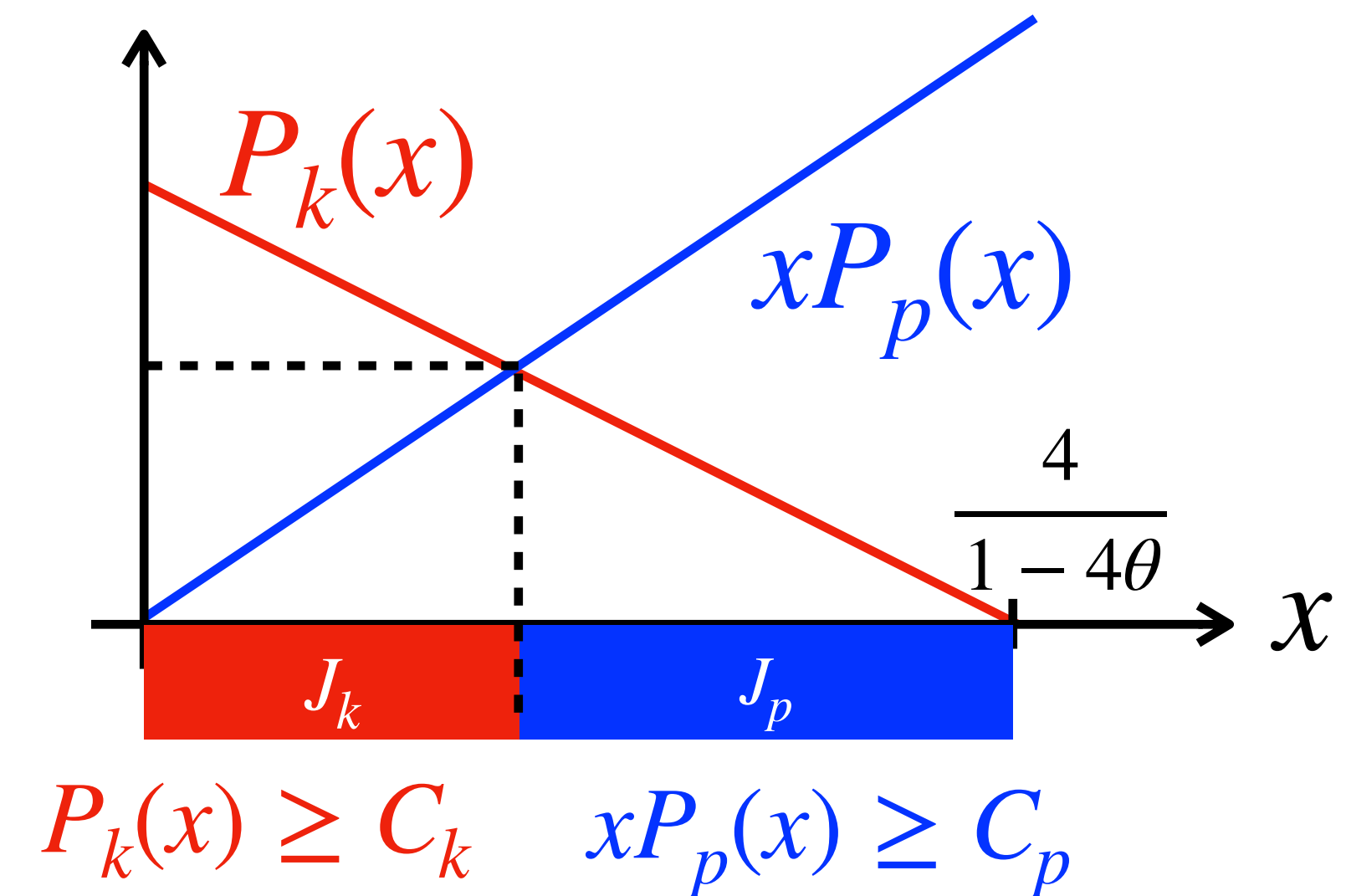
$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1$$

$$P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$



Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

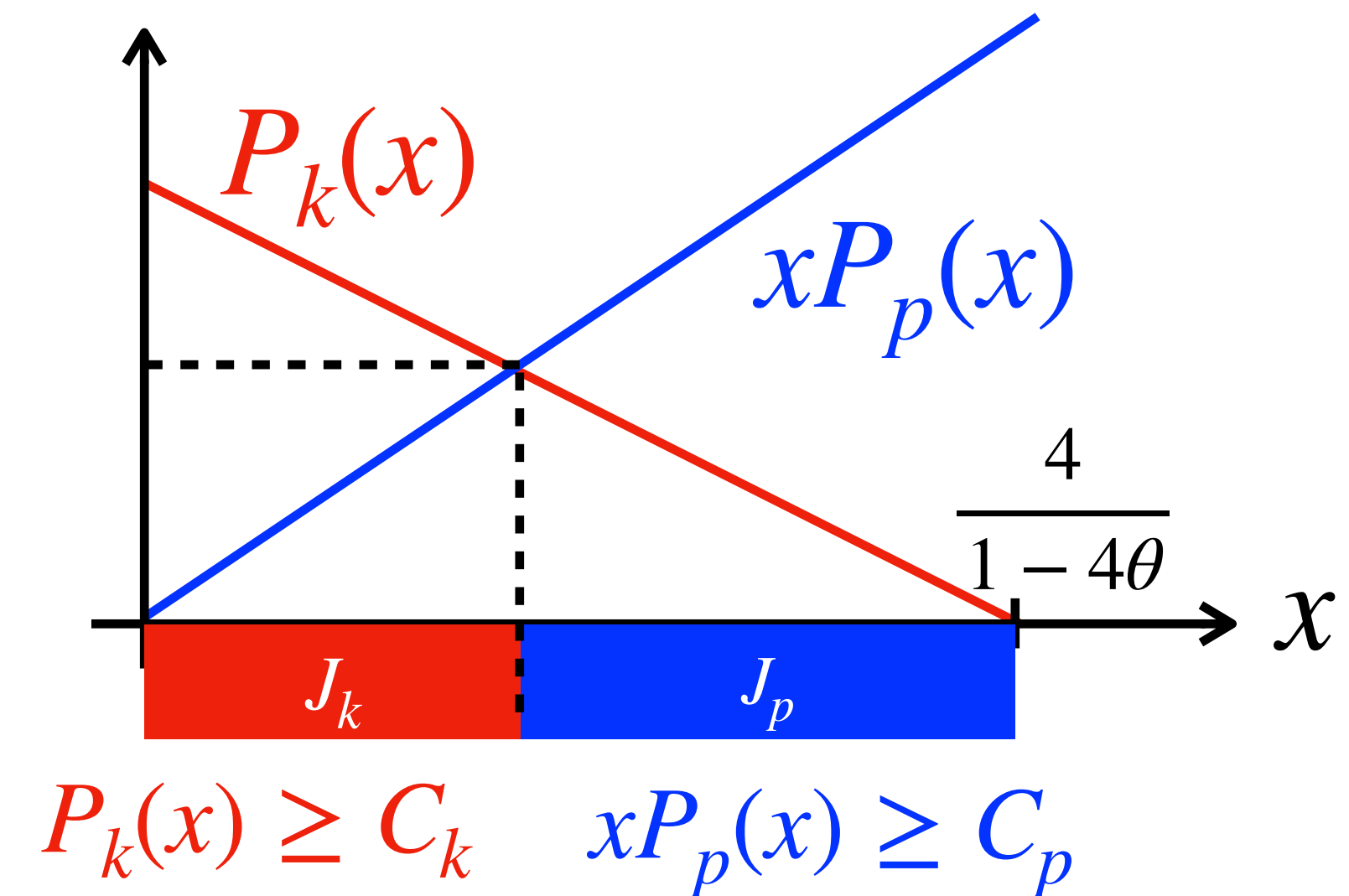
$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

$$P_p(x) = 1$$

$$\Pi_p u_h = \sum_{\substack{\Delta t^2 \lambda \in J_p \\ \lambda \in \text{Sp}(A_h)}} (w_h^\lambda, u_h) w_h^\lambda$$

$$P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$

$$\Pi_k u_h = \sum_{\substack{\Delta t^2 \lambda \in J_k \\ \lambda \in \text{Sp}(A_h)}} (w_h^\lambda, u_h) w_h^\lambda$$



Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

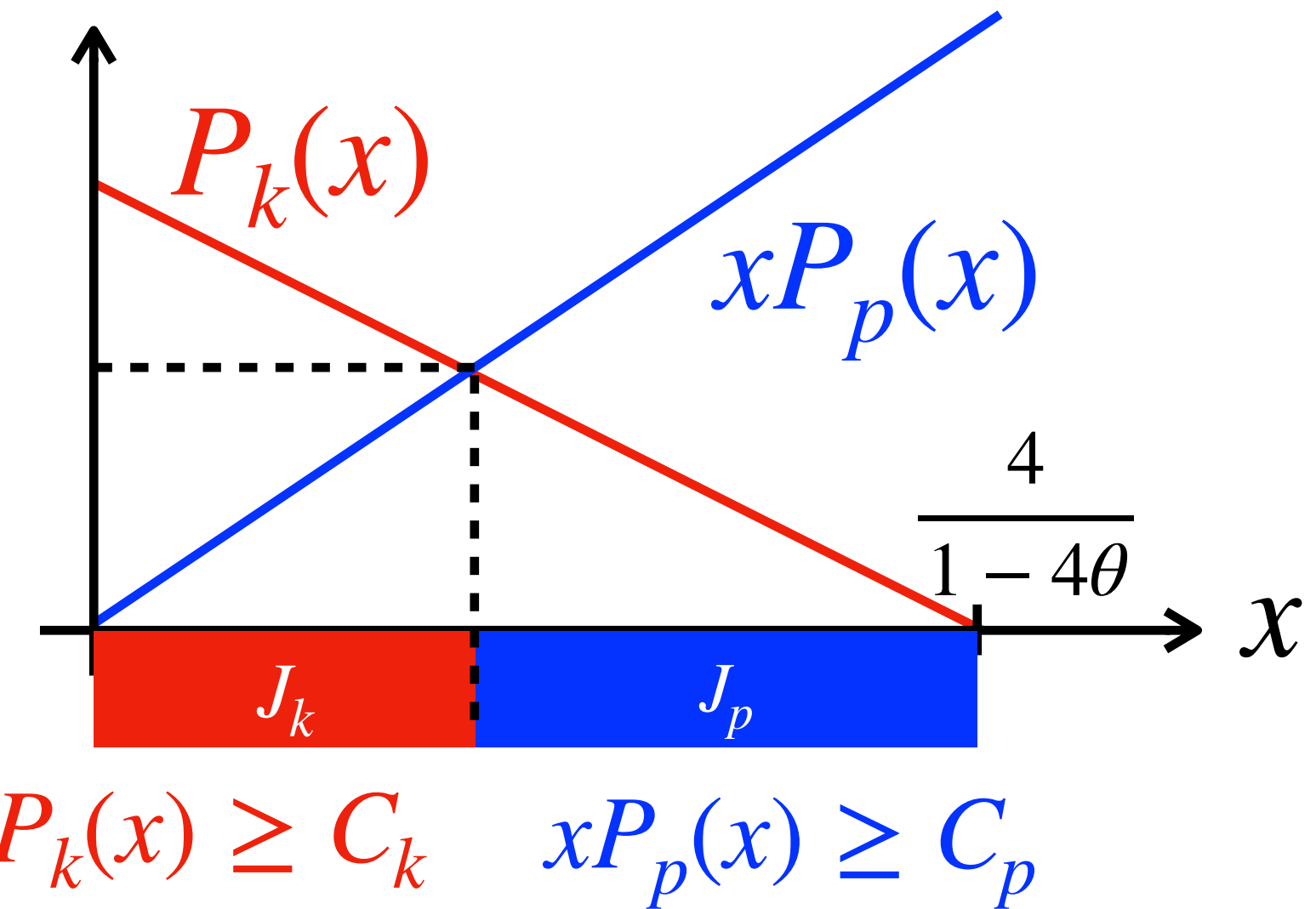
$$P_p(x) = 1 \qquad P_k(x) = 1 + \left(\theta - \frac{1}{4} \right) x$$

$$\Pi_p u_h = \sum_{\substack{\Delta t^2 \lambda \in J_p \\ \lambda \in \text{Sp}(A_h)}} (w_h^\lambda, u_h) w_h^\lambda$$

$$\Pi_k u_h = \sum_{\substack{\Delta t^2 \lambda \in J_k \\ \lambda \in \text{Sp}(A_h)}} (w_h^\lambda, u_h) w_h^\lambda$$

$$\|\Pi_p \tilde{u}_h\|_P^2 \leq C_p^{-1} (\Delta t^2 A_h P_p(\Delta t^2 A_h) \tilde{u}_h, \tilde{u}_h)_P$$

$$\|\Pi_k \tilde{u}_h\|_P^2 \leq C_k^{-1} (P_k(\Delta t^2 A_h) \tilde{u}_h, \tilde{u}_h)_P$$



Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

- Éléments de preuve pour la stabilité et la convergence espace / temps

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

- Éléments de preuve pour la stabilité et la convergence espace / temps

$$\|\delta \mu u_h^n\|_P \leq 2\gamma\mu(\sqrt{2\mathcal{E}_h})^n$$

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

- Éléments de preuve pour la stabilité et la convergence espace / temps

$$\|\delta \mu u_h^n\|_P \leq 2\gamma \mu (\sqrt{2\mathcal{E}_h})^n \quad \Rightarrow \quad \sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma \Delta t \sum_{\ell=1}^n \|f\|_P$$

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

- Éléments de preuve pour la stabilité et la convergence espace / temps

$$\|\delta \mu u_h^n\|_P \leq 2\gamma \mu (\sqrt{2\mathcal{E}_h})^n \quad \Rightarrow \quad \sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma \Delta t \sum_{\ell=1}^n \|f\|_P$$

$$\|u_h^{n+1}\|_P \leq \|\Pi_k u_h^n\|_P + \|\Pi_p u_h^n\|_P + \Delta t \gamma \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}}$$

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

I. Discrétisation et Analyse Ordre 2

I.2 Analyse

$$P_k(\Delta t^2 A_h) \delta^2 u_h^n + A_h P_p(\Delta t^2 A_h) \mu^2 u_h^n = f_h^n$$

Bilan de puissance discret

$$\frac{\mathcal{E}_h^{n+\frac{1}{2}} - \mathcal{E}_h^{n-\frac{1}{2}}}{\Delta t} = (f, \mu \delta u_h^n)_P, \quad \text{où } \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (P_k(\Delta t^2 A_h) \delta u_h^{n+\frac{1}{2}}, \delta u_h^{n+\frac{1}{2}})_P + \frac{1}{2} (A_h P_p(\Delta t^2 A_h) \mu u_h^{n+\frac{1}{2}}, \mu u_h^{n+\frac{1}{2}})_P$$

- Éléments de preuve pour la stabilité et la convergence espace / temps

$$\|\delta \mu u_h^n\|_P \leq 2\gamma \mu (\sqrt{2\mathcal{E}_h})^n \quad \Rightarrow \quad \sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma \Delta t \sum_{\ell=1}^n \|f\|_P$$

$$\|u_h^{n+1}\|_P \leq \|\Pi_k u_h^n\|_P + \|\Pi_p u_h^n\|_P + \Delta t \gamma \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \Rightarrow \quad \|u_h^{n+1}\|_P \leq \sqrt{2} \|u_h^1\|_P + 2\sqrt{2} \Delta t \gamma \sum_{\ell=1}^n \sqrt{\mathcal{E}_h^{\ell+\frac{1}{2}}}$$

Chabassier, J. and Imperiale, S. (2017). Space/time convergence analysis of a class of conservative schemes for linear wave equations. *Comptes Rendus Mathématiques*, 355(3):282–289.

Chabassier, J. and Imperiale, S. (2021). Construction and convergence analysis of conservative second order local time discretisation for linear wave equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 55(4):1507–1543.

II. Formulation Mixte

II.1 Formulation primale/duale

II. Formulation Mixte

II.1 Formulation primale/duale

Chercher $u \in U = H(B) \subset P$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

II. Formulation Mixte

II.1 Formulation primale/duale

Chercher $u \in U = H(B) \subset P$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

On pose $v(x, t) = Bu(x, t) :$

$$\begin{cases} (\ddot{u}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} = B\dot{u} & \text{in } D \end{cases}$$

II. Formulation Mixte

II.1 Formulation primale/duale

Chercher $u \in U = H(B) \subset P$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

On pose $v(x, t) = Bu(x, t) : \begin{cases} (\ddot{u}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} = B\dot{u} & \text{in } D \end{cases}$

On pose $p = \dot{u} : \begin{cases} (\dot{p}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D = (Bp, \tilde{v})_D, & \forall \tilde{v} \in D \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$

II. Formulation Mixte

II.1 Formulation primale/duale

Chercher $\mathbf{u} \in U = H(B) \subset P$ tel que $(\ddot{\mathbf{u}}, \tilde{p})_P + (B\mathbf{u}, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

On pose $\mathbf{v}(x, t) = B\mathbf{u}(x, t) :$
$$\begin{cases} (\ddot{\mathbf{u}}, \tilde{p})_P + (\mathbf{v}, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{\mathbf{v}} = B\dot{\mathbf{u}} & \text{in } D \end{cases}$$

On pose $\mathbf{p} = \dot{\mathbf{u}} :$
$$\begin{cases} (\dot{\mathbf{p}}, \tilde{p})_P + (\mathbf{v}, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{\mathbf{v}}, \tilde{v})_D = (B\mathbf{p}, \tilde{v})_D, & \forall \tilde{v} \in D \\ (\dot{\mathbf{u}}, \tilde{u})_P = (\mathbf{p}, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

II. Formulation Mixte

II.1 Formulation primale/duale

Chercher $u \in U = H(B) \subset P$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

On pose $v(x, t) = Bu(x, t) : \begin{cases} (\ddot{u}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} = B\dot{u} & \text{in } D \end{cases}$

On pose $p = \dot{u} : \begin{cases} (\dot{p}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D = (Bp, \tilde{v})_D, & \forall \tilde{v} \in D \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$

On pose $b(v, p) = (v, Bp)_D, \quad \forall v \in D, \forall p \in U$. On cherche $p \in U$ et $v \in D$ tq

$$\begin{cases} (\dot{p}, \tilde{p})_P + b(v, \tilde{p}) = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D - b(\tilde{v}, p) = 0, & \forall \tilde{v} \in D \end{cases}$$

II. Formulation Mixte

II.1 Formulation primale/duale

Chercher $u \in U = H(B) \subset P$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

On pose $v(x, t) = Bu(x, t) : \begin{cases} (\ddot{u}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} = B\dot{u} & \text{in } D \end{cases}$

On pose $p = \dot{u} : \begin{cases} (\dot{p}, \tilde{p})_P + (v, B\tilde{p})_D = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D = (Bp, \tilde{v})_D, & \forall \tilde{v} \in D \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$

On pose $b(v, p) = (v, Bp)_D, \quad \forall v \in D, \forall p \in U$. On cherche $p \in U$ et $v \in D$ tq

$$\begin{cases} (\dot{p}, \tilde{p})_P + b(v, \tilde{p}) = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D - b(\tilde{v}, p) = 0, & \forall \tilde{v} \in D \end{cases} \quad (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, \quad \forall \tilde{u} \in P$$

II. Formulation Mixte

II.1 Formulation primale/duale/primale !

On cherche $p \in U$ et $v \in D$ tq

$$\begin{cases} (\dot{p}, \tilde{p})_P + b(v, \tilde{p}) = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D - b(\tilde{v}, p) = 0, & \forall \tilde{v} \in D \end{cases}$$

$$(\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, \quad \forall \tilde{u} \in P$$

II. Formulation Mixte

II.1 Formulation primale/duale/primale !

On cherche $p \in U$ et $v \in D$ tq

$$\begin{cases} (\dot{p}, \tilde{p})_P + b(v, \tilde{p}) = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_D - b(\tilde{v}, p) = 0, & \forall \tilde{v} \in D \end{cases} \quad (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, \quad \forall \tilde{u} \in P$$

En prenant $\tilde{p} = p$ et $\tilde{v} = v$, $\frac{d\mathcal{E}(p, v)}{dt} = (f, p)_P$ où $\mathcal{E}(p, v) = \frac{1}{2}(p, p)_P + \frac{1}{2}(v, v)_D$

En utilisant que $p = \dot{u}$ et $v = Bu$, on retrouve le bilan de puissance initial.

II. Formulation Mixte

II.2 Discrétisation de la formulation primale-duale(-primale)

II. Formulation Mixte

II.2 Discrétisation de la formulation primale-duale(-primale)

Soient $U_h \subset U$ et $D_h \subset D$. On cherche $p_h \in U_h$ et $v_h \in D_h$ tq

$$(\dot{p}_h, \tilde{p}_h)_P + b(v_h, \tilde{p}_h) = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\dot{v}_h, \tilde{v}_h)_D - b(\tilde{v}_h, p_h) = 0, \quad \forall \tilde{v}_h \in D_h$$

II. Formulation Mixte

II.2 Discrétisation de la formulation primale-duale(-primale)

Soient $U_h \subset U$ et $D_h \subset D$. On cherche $p_h \in U_h$ et $v_h \in D_h$ tq

$$(\dot{p}_h, \tilde{p}_h)_P + b(v_h, \tilde{p}_h) = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\dot{v}_h, \tilde{v}_h)_D - b(\tilde{v}_h, p_h) = 0, \quad \forall \tilde{v}_h \in D_h$$

On suppose qu' \exists opérateur linéaire $\Pi_h : U \times D \rightarrow U_h \times D_h$ tel que $\forall v \in D, \forall p \in U,$

$$\begin{cases} (p - \Pi_h p, \tilde{p}_h)_P + b(v - \Pi_h v, \tilde{p}_h) = 0, & \forall \tilde{p}_h \in U_h \\ (v - \Pi_h v, \tilde{v}_h)_D - b(\tilde{v}_h, p - \Pi_h p) = 0, & \forall \tilde{v}_h \in D_h \end{cases}$$

II. Formulation Mixte

II.2 Discrétisation de la formulation primale-duale(-primale)

Soient $U_h \subset U$ et $D_h \subset D$. On cherche $p_h \in U_h$ et $v_h \in D_h$ tq

$$(\dot{p}_h, \tilde{p}_h)_P + b(v_h, \tilde{p}_h) = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\dot{v}_h, \tilde{v}_h)_D - b(\tilde{v}_h, p_h) = 0, \quad \forall \tilde{v}_h \in D_h$$

On suppose qu' \exists opérateur linéaire $\Pi_h : U \times D \rightarrow U_h \times D_h$ tel que $\forall v \in D, \forall p \in U,$

$$\begin{cases} (p - \Pi_h p, \tilde{p}_h)_P + b(v - \Pi_h v, \tilde{p}_h) = 0, & \forall \tilde{p}_h \in U_h \\ (v - \Pi_h v, \tilde{v}_h)_D - b(\tilde{v}_h, p - \Pi_h p) = 0, & \forall \tilde{v}_h \in D_h \end{cases}$$

Tel que $\|p - \Pi_h p\|_U + \|v - \Pi_h v\|_D \xrightarrow{h \rightarrow 0} 0$

II. Formulation Mixte

II.2 Discrétisation de la formulation primale-duale(-primale)

Soient $U_h \subset U$ et $D_h \subset D$. On cherche $p_h \in U_h$ et $v_h \in D_h$ tq

$$(\dot{p}_h, \tilde{p}_h)_P + b(v_h, \tilde{p}_h) = (f, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\dot{v}_h, \tilde{v}_h)_D - b(\tilde{v}_h, p_h) = 0, \quad \forall \tilde{v}_h \in D_h$$

On suppose qu' \exists opérateur linéaire $\Pi_h : U \times D \rightarrow U_h \times D_h$ tel que $\forall v \in D, \forall p \in U,$

$$\begin{cases} (p - \Pi_h p, \tilde{p}_h)_P + b(v - \Pi_h v, \tilde{p}_h) = 0, & \forall \tilde{p}_h \in U_h \\ (v - \Pi_h v, \tilde{v}_h)_D - b(\tilde{v}_h, p - \Pi_h p) = 0, & \forall \tilde{v}_h \in D_h \end{cases}$$

Tel que $\|p - \Pi_h p\|_U + \|v - \Pi_h v\|_D \xrightarrow{h \rightarrow 0} 0$

Soit $B_h : U_h \rightarrow D_h$ l'opérateur défini par $b(v_h, p_h) = (v_h, B_h p_h)_D, \forall v_h \in D_h, \forall p_h \in U_h$

II. Formulation Mixte

II.3 Schéma de Crank–Nicolson et analyse

II. Formulation Mixte

II.3 Schéma de Crank–Nicolson et analyse

On discrétise p_h et v_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$:

II. Formulation Mixte

II.3 Schéma de Crank–Nicolson et analyse

On discrétise p_h et v_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$:

$$(\delta p_h^n, \tilde{p}_h)_P + b(\mu v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\delta v_h^n, \tilde{v}_h)_D - b(\tilde{v}_h, \mu p_h^n) = 0, \quad \forall \tilde{v}_h \in D_h$$

II. Formulation Mixte

II.3 Schéma de Crank–Nicolson et analyse

On discrétise p_h et v_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$:

$$(\delta p_h^n, \tilde{p}_h)_P + b(\mu v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\delta v_h^n, \tilde{v}_h)_D - b(\tilde{v}_h, \mu p_h^n) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}})_D$$

II. Formulation Mixte

II.3 Schéma de Crank–Nicolson et analyse

On discrétise p_h et v_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$:

$$(\delta p_h^n, \tilde{p}_h)_P + b(\mu v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$(\delta v_h^n, \tilde{v}_h)_D - b(\tilde{v}_h, \mu p_h^n) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}})_D$$

- L'énergie est toujours positive

$$\|p_h^{n+\frac{1}{2}}\|_P \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \text{et} \quad \|v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}}$$

II. Formulation Mixte

II.3 Schéma de Crank–Nicolson et analyse

On discrétise p_h et v_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$:

$$\begin{aligned} (\delta p_h^n, \tilde{p}_h)_P + b(\mu v_h^n, \tilde{p}_h) &= (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \\ (\delta v_h^n, \tilde{v}_h)_D - b(\tilde{v}_h, \mu p_h^n) &= 0, \quad \forall \tilde{v}_h \in D_h \end{aligned} \quad (\delta u_h^{n+\frac{1}{2}}, \tilde{u})_P = (p_h^{n+\frac{1}{2}}, \tilde{u})_P, \quad \forall \tilde{u} \in U_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+\frac{1}{2}}, v_h^{n+\frac{1}{2}})_D$$

- L'énergie est toujours positive

$$\|p_h^{n+\frac{1}{2}}\|_P \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \text{et} \quad \|v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}}$$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+1}, v_h^n)_D$$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+1}, v_h^n)_D$$

$$\begin{aligned} v_h^{n+1} &= \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \\ v_h^n &= \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \end{aligned}$$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$v_h^{n+1} = \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$v_h^n = \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$\tilde{m}(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P - \frac{\Delta t^2}{4} (B_h p_h, B_h \tilde{p}_h)_D$$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$\begin{aligned} v_h^{n+1} &= \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \\ v_h^n &= \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}} \end{aligned}$$

$$\tilde{m}(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P - \frac{\Delta t^2}{4} (B_h p_h, B_h \tilde{p}_h)_D$$

- L'énergie est positive si \tilde{m} est positive, cad si $\Delta t^2 \sup_{p_h \in U_h} \frac{(B_h p_h, B_h p_h)_D}{(p_h, p_h)_P} \leq 4$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$(\delta p_h^n, \tilde{p}_h)_P + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$(\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, \quad \forall \tilde{u}_h \in U_h$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \text{avec} \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$v_h^{n+1} = \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$v_h^n = \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$\tilde{m}(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P - \frac{\Delta t^2}{4} (B_h p_h, B_h \tilde{p}_h)_D$$

- L'énergie est positive si \tilde{m} est positive, cad si $\Delta t^2 \sup_{p_h \in U_h} \frac{(B_h p_h, B_h p_h)_D}{(p_h, p_h)_P} \leq 4$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}}$$

$$\|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^n \|f^k\|_P$$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^n \|f^k\|_P$$

Mais on n'a pas de contrôle sur p_h^n tout entier, seulement sur sa projection $\Pi_k p_h^n$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^n \|f^k\|_P$$

Mais on n'a pas de contrôle sur p_h^n tout entier, seulement sur sa projection $\Pi_k p_h^n$

On définit : $u_h^{n+1} = u_h^n + \Delta t p_h^{n+\frac{1}{2}}$, u_h^0 tel que $B_h u_h^0 = v_h^0$

II. Formulation Mixte

II.4 Schéma de Störmer-Verlet et analyse

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2\mathcal{E}_h^{n+\frac{1}{2}}} \quad \|\mu p_h^n\|_P \leq 2\gamma\mu \left(\sqrt{2\mathcal{E}_h}\right)^n$$

$$\sqrt{\mathcal{E}_h^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_h^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^n \|f^k\|_P$$

Mais on n'a pas de contrôle sur p_h^n tout entier, seulement sur sa projection $\Pi_k p_h^n$

On définit : $u_h^{n+1} = u_h^n + \Delta t p_h^{n+\frac{1}{2}}$, u_h^0 tel que $B_h u_h^0 = v_h^0$

$$\|u_h^{n+1}\|_P \leq \sqrt{2}\|u_h^0\| + \gamma\Delta t \sum_{k=0}^n \sqrt{2\mathcal{E}_h^{k+\frac{1}{2}}}$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

Chercher $\boldsymbol{u} \in U$ tel que $(\ddot{\boldsymbol{u}}, \tilde{p})_P + (B\boldsymbol{u}, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{\boldsymbol{u}}, \dot{\boldsymbol{u}})_P + \frac{1}{2} (B\boldsymbol{u}, B\boldsymbol{u})_D \right] = (f, \dot{\boldsymbol{u}})_P$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

Chercher $\boldsymbol{u} \in U$ tel que $(\ddot{\boldsymbol{u}}, \tilde{p})_P + (B\boldsymbol{u}, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{\boldsymbol{u}}, \dot{\boldsymbol{u}})_P + \frac{1}{2} (B\boldsymbol{u}, B\boldsymbol{u})_D \right] = (f, \dot{\boldsymbol{u}})_P$$

$$v(t) = \sqrt{(B\boldsymbol{u}, B\boldsymbol{u})_D + 2\alpha} \in \mathbb{R} \Rightarrow$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

$$\begin{cases} (\dot{u}, \tilde{p})_P + v b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, \dot{u}] = 0 \end{cases}$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

$$\begin{cases} (\ddot{u}, \tilde{p})_P + v b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, \dot{u}] = 0 \end{cases}$$

$$\dot{u} = p \in U$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(t) = \sqrt{(Bu, Bu)_D + 2\alpha} \in \mathbb{R} \Rightarrow \dot{v} = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\dot{u} \right)_D$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

$$\begin{cases} (\dot{p}, \tilde{p})_P + v b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

$$\begin{cases} (\dot{p}, \tilde{p})_P + \nu b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

$$\begin{cases} (\dot{p}, \tilde{p})_P + \nu b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D$$

Prenons $\tilde{p} = p$ et multiplions la deuxième équation par ν , on obtient :

$$\frac{d\mathcal{E}}{dt} = (f, p)_P \quad \mathcal{E} = \frac{1}{2} (p, p)_P + \frac{1}{2} \nu^2$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

$$\begin{cases} (\dot{p}, \tilde{p})_P + \nu b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D = F_\alpha[u] (Bu, B\tilde{u})_D$$

Prenons $\tilde{p} = p$ et multiplions la deuxième équation par ν , on obtient :

$$\frac{d\mathcal{E}}{dt} = (f, p)_P \quad \mathcal{E} = \frac{1}{2} (p, p)_P + \frac{1}{2} \nu^2$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

$$\begin{cases} (\dot{p}, \tilde{p})_P + \nu b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D = F_\alpha[u] (Bu, B\tilde{u})_D$$

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

Prenons $\tilde{p} = p$ et multiplions la deuxième équation par ν , on obtient :

$$\frac{d\mathcal{E}}{dt} = (f, p)_P \quad \mathcal{E} = \frac{1}{2} (p, p)_P + \frac{1}{2} \nu^2$$

IV. Scalar Auxiliary Variable

IV.1 Formulation primale/scalaire/primale ?

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \quad \forall u, \tilde{u} \in U^2$$

$$\begin{cases} (\dot{p}, \tilde{p})_P + v b[u, \tilde{p}] = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ \dot{v} - b[u, p] = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases} \quad \begin{cases} (\dot{p}, \tilde{p})_P + F_\alpha[u] v a(u, \tilde{p}) = (f, \tilde{p})_P \\ \dot{v} - F_\alpha[u] a(u, p) = 0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P, & \forall \tilde{u} \in P \end{cases}$$

$$b[u, \tilde{u}] = \left(\frac{Bu}{\sqrt{\|Bu\|_D^2 + 2\alpha}}, B\tilde{u} \right)_D = F_\alpha[u] (Bu, B\tilde{u})_D$$

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

Prenons $\tilde{p} = p$ et multiplions la deuxième équation par v , on obtient :

$$\frac{d\mathcal{E}}{dt} = (f, p)_P \quad \mathcal{E} = \frac{1}{2} (p, p)_P + \frac{1}{2} v^2$$

IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

IV.2 Discretisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \forall u, \tilde{u} \in U^2$$

IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

IV.2 Discretisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \forall u, \tilde{u} \in U^2$$

Chaolong Jiang and Wenjun Cai and Yushun Wang. « A Linearly Implicit and Local Energy-Preserving Scheme for the Sine-Gordon Equation Based on the Invariant Energy Quadraticization Approach », Journal of Scientific Computing vol 80(3), 2019.

Michele Ducceschi and Stefan Bilbao. « Simulation of the geometrically exact nonlinear string via energy quadratisation », Journal of Sound and Vibration vol 534, 2022.

IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

IV.2 Discretisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \quad \forall u, \tilde{u} \in U^2$$

$$\left\{ \begin{array}{l} (\delta p_h^n, \tilde{p}_h)_P + F_\alpha[u_h^n] \mu v_h^n a(u_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h \\ \delta v_h^n - F_\alpha[u_h^n] a(u_h^n, \mu p_h^n) = 0 \\ (\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, \quad \forall \tilde{u}_h \in U_h \end{array} \right.$$

Chaolong Jiang and Wenjun Cai and Yushun Wang. « A Linearly Implicit and Local Energy-Preserving Scheme for the Sine-Gordon Equation Based on the Invariant Energy Quadraticization Approach », Journal of Scientific Computing vol 80(3), 2019.

Michele Ducceschi and Stefan Bilbao. « Simulation of the geometrically exact nonlinear string via energy quadratisation », Journal of Sound and Vibration vol 534, 2022.

IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

IV.2 Discrétisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \quad \forall u, \tilde{u} \in U^2$$

$$\begin{cases} (\delta p_h^n, \tilde{p}_h)_P + F_\alpha[u_h^n] \mu v_h^n a(u_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, & \forall \tilde{p}_h \in U_h \\ \delta v_h^n - F_\alpha[u_h^n] a(u_h^n, \mu p_h^n) = 0 \\ (\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, & \forall \tilde{u}_h \in U_h \end{cases}$$

Prenons $\tilde{p}_h = \mu p_h^n$, multiplions la deuxième équation par μv_h^n . On a :

Chaolong Jiang and Wenjun Cai and Yushun Wang. « A Linearly Implicit and Local Energy-Preserving Scheme for the Sine-Gordon Equation Based on the Invariant Energy Quadraticization Approach », Journal of Scientific Computing vol 80(3), 2019.

Michele Ducceschi and Stefan Bilbao. « Simulation of the geometrically exact nonlinear string via energy quadratisation », Journal of Sound and Vibration vol 534, 2022.

IV. Scalar Auxiliary Variable

$$F_\alpha[u] = \frac{1}{\sqrt{\|Bu\|_D^2 + 2\alpha}}$$

IV.2 Discrétisation

$$a(u, \tilde{u}) = (Bu, B\tilde{u})_D, \quad \forall u, \tilde{u} \in U^2$$

$$\begin{cases} (\delta p_h^n, \tilde{p}_h)_P + F_\alpha[u_h^n] \mu v_h^n a(u_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, & \forall \tilde{p}_h \in U_h \\ \delta v_h^n - F_\alpha[u_h^n] a(u_h^n, \mu p_h^n) = 0 \\ (\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, & \forall \tilde{u}_h \in U_h \end{cases}$$

Prenons $\tilde{p}_h = \mu p_h^n$, multiplions la deuxième équation par μv_h^n . On a :

$$\delta \mathcal{E}_h^n = (f^n, \mu p_h^{n+\frac{1}{2}})_P \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2} (v_h^{n+\frac{1}{2}})^2$$

Chaolong Jiang and Wenjun Cai and Yushun Wang. « A Linearly Implicit and Local Energy-Preserving Scheme for the Sine-Gordon Equation Based on the Invariant Energy Quadraticization Approach », Journal of Scientific Computing vol 80(3), 2019.

Michele Ducceschi and Stefan Bilbao. « Simulation of the geometrically exact nonlinear string via energy quadratisation », Journal of Sound and Vibration vol 534, 2022.

IV. Scalar Auxiliary Variable

IV.2 Discretisation

IV. Scalar Auxiliary Variable

IV.2 Discrétisation

Connaissant U_h^n , $P_h^{n-\frac{1}{2}}$, $v_h^{n-\frac{1}{2}}$, Trouver $P_h^{n+\frac{1}{2}}$ et $v_h^{n+\frac{1}{2}}$ puis U_h^{n+1} tels que :

IV. Scalar Auxiliary Variable

IV.2 Discrétisation

Connaissant $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$, Trouver $P_h^{n+\frac{1}{2}}$ et $v_h^{n+\frac{1}{2}}$ puis U_h^{n+1} tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha [U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha [U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases}$$

IV. Scalar Auxiliary Variable

IV.2 Discrétisation

Connaissant $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$, Trouver $P_h^{n+\frac{1}{2}}$ et $v_h^{n+\frac{1}{2}}$ puis U_h^{n+1} tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

IV. Scalar Auxiliary Variable

IV.2 Discrétisation

Connaissant $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$, Trouver $P_h^{n+\frac{1}{2}}$ et $v_h^{n+\frac{1}{2}}$ puis U_h^{n+1} tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

IV. Scalar Auxiliary Variable

IV.2 Discrétisation

Connaissant $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$, Trouver $P_h^{n+\frac{1}{2}}$ et $v_h^{n+\frac{1}{2}}$ puis U_h^{n+1} tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

Ce schéma :

IV. Scalar Auxiliary Variable

IV.2 Discrétisation

Connaissant $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$, Trouver $P_h^{n+\frac{1}{2}}$ et $v_h^{n+\frac{1}{2}}$ puis U_h^{n+1} tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

Ce schéma :

- est inconditionnellement stable

IV. Scalar Auxiliary Variable

IV.2 Discrétisation

Connaissant $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$, Trouver $P_h^{n+\frac{1}{2}}$ et $v_h^{n+\frac{1}{2}}$ puis U_h^{n+1} tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

Ce schéma :

- est inconditionnellement stable

- couple linéairement les inconnues $P_h^{n+\frac{1}{2}} / U_h^{n+1}$ et $v_h^{n+\frac{1}{2}}$ par une matrice $(N+1) \times (N+1)$

IV. Scalar Auxiliary Variable

IV.2 Discrétisation

Connaissant $U_h^n, P_h^{n-\frac{1}{2}}, v_h^{n-\frac{1}{2}}$, Trouver $P_h^{n+\frac{1}{2}}$ et $v_h^{n+\frac{1}{2}}$ puis U_h^{n+1} tels que :

$$\begin{cases} M_h \delta P_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu P_h^n = 0 \\ M_h \delta U_h^{n+\frac{1}{2}} = M_h P_h^{n+\frac{1}{2}} \end{cases} \quad \begin{cases} M_h \delta^2 U_h^n + F_\alpha[U_h^n] \mu v_h^n K_h U_h^n = F_h^n \\ \delta v_h^n - F_\alpha[U_h^n] K_h U_h^n \cdot \mu \delta U_h^n = 0 \end{cases}$$

$$\begin{pmatrix} M_h & F_\alpha[U_h^n] C_h^n \\ -F_\alpha[U_h^n] (C_h^n)^T & 1 \end{pmatrix}$$

$$F_\alpha[U_h^n] = \sqrt{K_h U_h^n \cdot U_h^n + 2\alpha}^{-1}$$

$$C_h^n = K_h U_h^n$$

Ce schéma :

- est inconditionnellement stable
- couple linéairement les inconnues $P_h^{n+\frac{1}{2}} / U_h^{n+1}$ et $v_h^{n+\frac{1}{2}}$ par une matrice $(N+1) \times (N+1)$
- peut **se ramener** à une simple inversion de M_h accompagnée d'une évaluation de $K_h U_h^n + \text{prod scal}$

IV. Scalar Auxiliary Variable

IV.3 Simulations préliminaires

IV. Scalar Auxiliary Variable

IV.3 Simulations préliminaires

$$\ddot{y} + \lambda^2 y = 0$$

IV. Scalar Auxiliary Variable

IV.3 Simulations préliminaires

$$\ddot{y} + \lambda^2 y = 0$$

$$\Delta t = 1$$

IV. Scalar Auxiliary Variable

IV.3 Simulations préliminaires

$$\ddot{y} + \lambda^2 y = 0$$

$$\Delta t = 1$$

Pour les θ -schémas,

$$\Delta t \lambda \leq \sqrt{\frac{4}{1-4\theta}}$$

IV. Scalar Auxiliary Variable

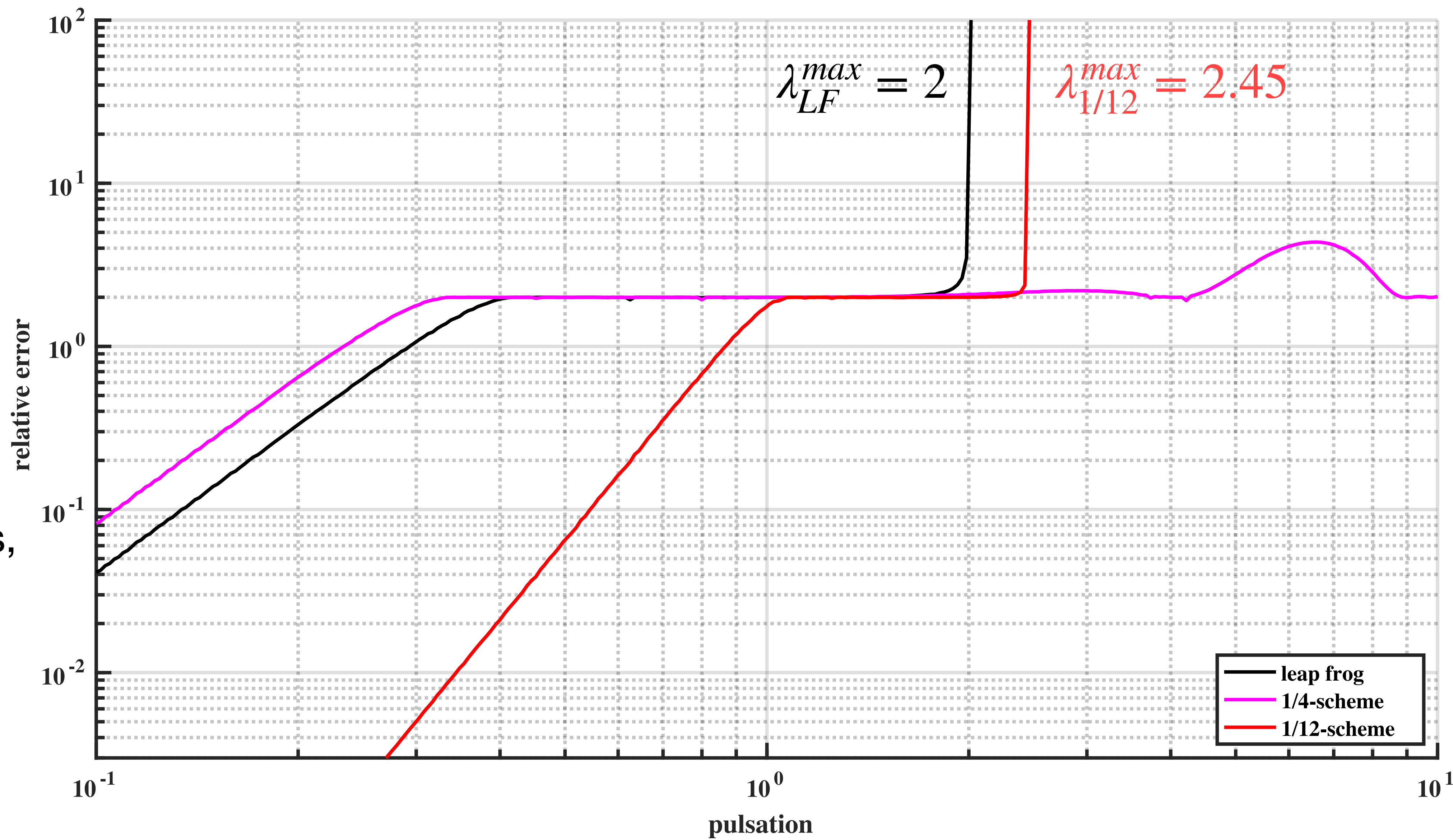
IV.3 Simulations préliminaires

$$\dot{y} + \lambda^2 y = 0$$

$$\Delta t = 1$$

Pour les θ -schémas,

$$\Delta t \lambda \leq \sqrt{\frac{4}{1-4\theta}}$$



IV. Scalar Auxiliary Variable

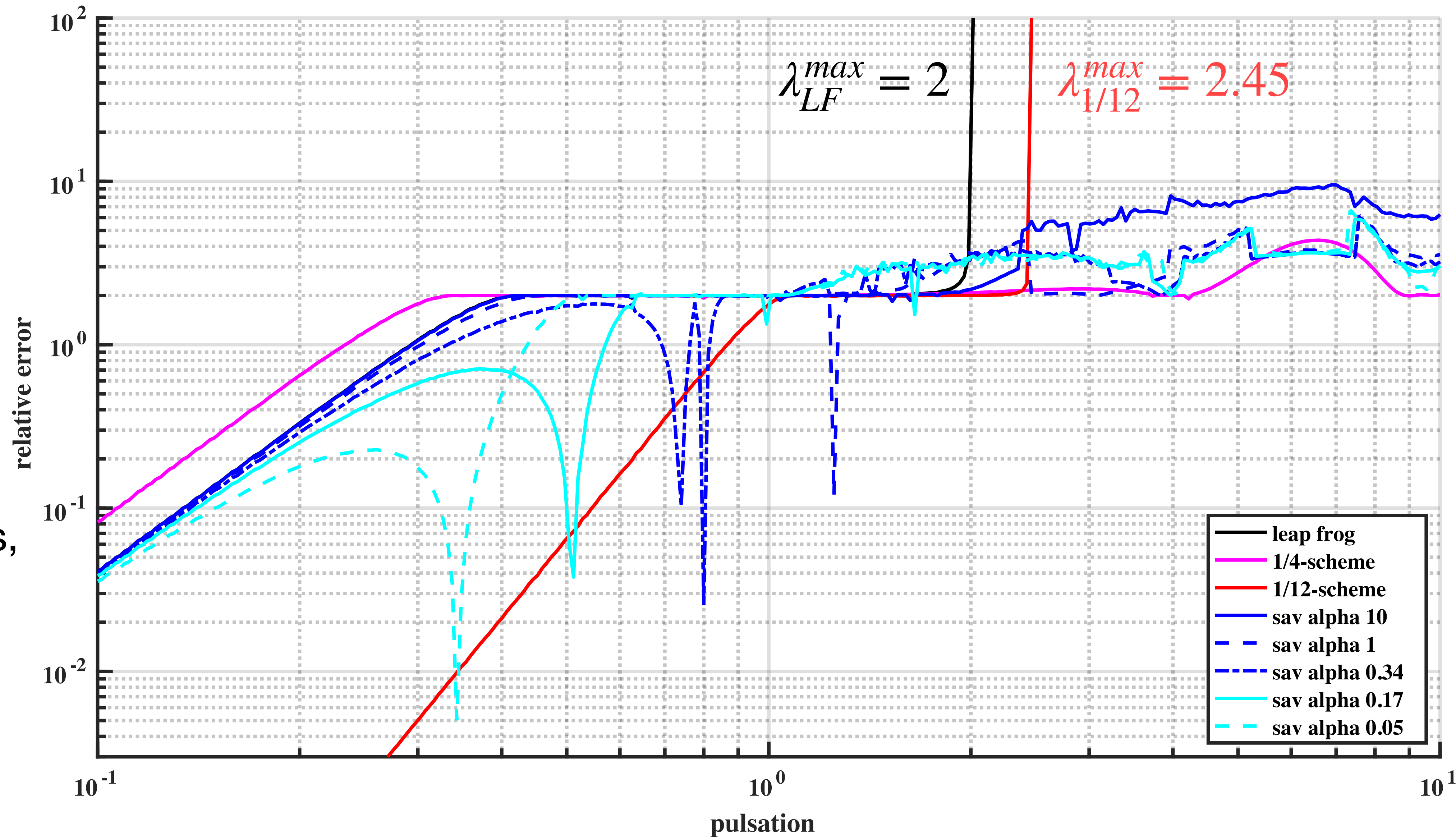
IV.3 Simulations préliminaires

$$\ddot{y} + \lambda^2 y = 0$$

$$\Delta t = 1$$

Pour les θ -schémas,

$$\Delta t \lambda \leq \sqrt{\frac{4}{1-4\theta}}$$



Conclusion - Perspectives

Chercher $u \in U = H(B) \subset P$ tel que

$$(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U \quad (1)$$

I. Discrétisation et analyse de (1)

II. On pose $v(x, t) = Bu(x, t)$ \Rightarrow Formulation Mixte

III. On pose $v(t) = \sqrt{(Bu, Bu)_D + 2\alpha}$ \Rightarrow Scalar Auxiliary Variable

Le nouveau schéma proposé est inconditionnellement stable et explicite.

Son analyse complète est en cours.



JO-SO
2023

Journées Ondes du Sud-Ouest
14 - 16 mars 2023 - Toulouse

Merci de votre attention

Juliette Chabassier - Equipe Makutu - INRIA

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\delta X^{n+\frac{1}{2}} = \frac{X^{n+1} - X^n}{\Delta t}, \quad \mu X^{n+\frac{1}{2}} = \frac{X^{n+1} + X^n}{2}, \quad \pi X^{n+\frac{1}{2}} = \frac{3X^n - X^{n-1}}{2}$$

$$\{X\}_\theta^n = X^n + \theta \Delta t^2 \delta^2 X = \mu^2 X^n + \left(\theta - \frac{1}{4}\right) \Delta t^2 \delta^2 X \quad (\text{A})$$

$$v_h^{n+1} = \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$v_h^n = \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

(B)

$$\{u_h\}_\theta^n = \theta u_h^{n+1} + (1 - 2\theta)u_h^n + \theta u_h^{n-1}$$

$$\delta^2 u_h^n = \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

$$\delta X^{n+\frac{1}{2}} = \frac{X^{n+1} - X^n}{\Delta t}, \quad \mu X^{n+\frac{1}{2}} = \frac{X^{n+1} + X^n}{2}, \quad \pi X^{n+\frac{1}{2}} = \frac{3X^n - X^{n-1}}{2}$$

$$\{X\}_\theta^n = X^n + \theta \Delta t^2 \delta^2 X = \mu^2 X^n + \left(\theta - \frac{1}{4}\right) \Delta t^2 \delta^2 X \quad (\text{A})$$

$$v_h^{n+1} = \mu v_h^{n+\frac{1}{2}} + \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

$$v_h^n = \mu v_h^{n+\frac{1}{2}} - \frac{\Delta t}{2} \delta v_h^{n+\frac{1}{2}}$$

(B)

$$\mu^2 u_h^n = \frac{u_h^{n+1} + 2u_h^n + u_h^{n-1}}{\Delta t^2}$$

IV. Scalar Auxiliary Variable

IV.3 Complexité

$$M_h \ddot{U}_h + D_h \dot{U}_h + K_h U_h = F_h$$

```
u_h(1) = u0; % initial data
u_h(2) = u1; % initial data
mu_u = (u_sav(1)+u_sav(2))*0.5;
v_h(1) = sqrt( 2 * alpha + dot ( K_h*mu_u, mu_u ) ); % initial data
Am1      = inv ( M_h + dt*0.5*D_h ); % mass matrix : easy to invert

For i = 3..N
    Kun      = Kh * u_h(i-1); % stiffness matrix * vector product
    Falphan  = 1 / sqrt( 2*alpha + dot ( Kun , u_h(i-1) ) ); % Falphan is a scalar
    Cn       = dt * 0.5 * Falphan * Kun; % vector

    RHS      = dt*0.5*F_h(n) + ( u_h(i-1)-u_h(i-2) ) / dt - v_h(i-2)*Cn; % vector
    Am1Cn    = Am1*Cn;
    mat_sherman = Am1 - dot ( Am1Cn , Am1Cn ) / ( 1 + dot ( Cn , Am1Cn ) ); % real division
    temp     = mat_sherman * RHS; % temp now contains the vector \delta\mu U^n

    u_h(i)   = u_sav(i-2) + 2*dt*temp;
    v_h(i-1) = v_h(i-2) + 2* dot ( Cn , temp ); % dot prod ( Cn , temp )
```

Complexité : $4 \cdot O(N)$ pour les **dot**, $O(N)$ pour l'**inv** et $O(p \cdot N)$ pour le **produit par Kh** ($p \approx$ ordre des FEM)

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$m_\theta(\delta p_h^n, \tilde{p}_h) + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$m_\theta(\delta p_h^n, \tilde{p}_h) + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$m_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \theta (B_h p_h, B_h \tilde{p}_h)_D \quad \forall (p_h, \tilde{p}_h) \in U_h^2$$

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$m_\theta(\delta p_h^n, \tilde{p}_h) + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$m_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \theta (B_h p_h, B_h \tilde{p}_h)_D \quad \forall (p_h, \tilde{p}_h) \in U_h^2$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} m_\theta(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (v_h^{n+1}, v_h^n)_D$$

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$m_\theta(\delta p_h^n, \tilde{p}_h) + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$m_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \theta (B_h p_h, B_h \tilde{p}_h)_D \quad \forall (p_h, \tilde{p}_h) \in U_h^2$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}_\theta(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$m_\theta(\delta p_h^n, \tilde{p}_h) + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$m_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \theta (B_h p_h, B_h \tilde{p}_h)_D \quad \forall (p_h, \tilde{p}_h) \in U_h^2$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}_\theta(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$\tilde{m}_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (B_h p_h, B_h \tilde{p}_h)_D$$

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$m_\theta(\delta p_h^n, \tilde{p}_h) + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$m_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \theta (B_h p_h, B_h \tilde{p}_h)_D \quad \forall (p_h, \tilde{p}_h) \in U_h^2$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}_\theta(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$\tilde{m}_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (B_h p_h, B_h \tilde{p}_h)_D$$

- L'énergie est positive si \tilde{m}_θ est positive, cad si $\Delta t^2 \sup_{p_h \in U_h} \frac{(B_h p_h, B_h p_h)_D}{(p_h, p_h)_P} \leq \frac{4}{1 - 4\theta}$

II. Formulation Mixte

II.5 θ schéma sur la formulation mixte et analyse

On discrétise p_h sur la grille $t^{n+\frac{1}{2}} = (n + 1/2) \Delta t$ et v_h sur la grille t^n

$$m_\theta(\delta p_h^n, \tilde{p}_h) + b(v_h^n, \tilde{p}_h) = (f^n, \tilde{p}_h)_P, \quad \forall \tilde{p}_h \in U_h$$

$$\left(\delta v_h^{n+\frac{1}{2}}, \tilde{v}_h \right)_D - b(\tilde{v}_h, p_h^{n+\frac{1}{2}}) = 0, \quad \forall \tilde{v}_h \in D_h$$

$$(\delta u_h^{n+\frac{1}{2}}, \tilde{u}_h)_P = (p_h^{n+\frac{1}{2}}, \tilde{u}_h)_P, \quad \forall \tilde{u}_h \in U_h$$

$$m_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \theta (B_h p_h, B_h \tilde{p}_h)_D \quad \forall (p_h, \tilde{p}_h) \in U_h^2$$

$$\delta \mathcal{E}_h^n = (\delta f^n, \mu p_h^n)_P \quad \mathcal{E}_h^{n+\frac{1}{2}} = \frac{1}{2} \tilde{m}_\theta(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2} (\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D$$

$$\tilde{m}_\theta(p_h, \tilde{p}_h) = (p_h, \tilde{p}_h)_P + \Delta t^2 \left(\theta - \frac{1}{4} \right) (B_h p_h, B_h \tilde{p}_h)_D$$

- L'énergie est positive si \tilde{m}_θ est positive, cad si $\Delta t^2 \sup_{p_h \in U_h} \frac{(B_h p_h, B_h p_h)_D}{(p_h, p_h)_P} \leq \frac{4}{1 - 4\theta}$

III. Invariant Energy Quadratisation

III.1 Formulation primale/duale comprimé/primale ?

III. Invariant Energy Quadratisation

III.1 Formulation primale/duale comprimé/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

III. Invariant Energy Quadratisation

III.1 Formulation primale/duale comprimé/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(x, t) = \sqrt{|Bu(x, t)|^2 + 2\alpha} \in D_0 \Rightarrow$$

III. Invariant Energy Quadratisation

III.1 Formulation primale/duale comprimé/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2}(\dot{u}, \dot{u})_P + \frac{1}{2}(Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(x, t) = \sqrt{|Bu(x, t)|^2 + 2\alpha} \in D_0 \Rightarrow \dot{v} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\dot{u}$$

III. Invariant Energy Quadratisation

III.1 Formulation primale/duale comprimé/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2}(\dot{u}, \dot{u})_P + \frac{1}{2}(Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(x, t) = \sqrt{|Bu(x, t)|^2 + 2\alpha} \in D_0 \Rightarrow \dot{v} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\dot{u}$$

Posons $B : U \times U \rightarrow D_0$ tel que $B[u]\tilde{u} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\tilde{u}, \quad \forall u \in U, \tilde{u} \in U$

III. Invariant Energy Quadratisation

III.1 Formulation primale/duale comprimé/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(x, t) = \sqrt{|Bu(x, t)|^2 + 2\alpha} \in D_0 \Rightarrow \dot{v} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\dot{u}$$

Posons $B : U \times U \rightarrow D_0$ tel que $B[u]\tilde{u} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\tilde{u}, \quad \forall u \in U, \tilde{u} \in U$

$$\begin{cases} (\ddot{u}, \tilde{p})_P + (v, B[u]\tilde{p})_P = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_P - (B[u]\dot{u}, \tilde{v})_P = 0, & \forall \tilde{v} \in D_0 \end{cases}$$

III. Invariant Energy Quadratisation

III.1 Formulation primale/duale comprimé/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(x, t) = \sqrt{|Bu(x, t)|^2 + 2\alpha} \in D_0 \Rightarrow \dot{v} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\dot{u}$$

Posons $B : U \times U \rightarrow D_0$ tel que $B[u]\tilde{u} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\tilde{u}, \quad \forall u \in U, \tilde{u} \in U$

$$\begin{cases} (\ddot{u}, \tilde{p})_P + (v, B[u]\tilde{p})_P = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_P - (B[u]\dot{u}, \tilde{v})_P = 0, & \forall \tilde{v} \in D_0 \end{cases}$$

$$\dot{u} = p$$

III. Invariant Energy Quadratisation

III.1 Formulation primale/duale comprimé/primale ?

Chercher $u \in U$ tel que $(\ddot{u}, \tilde{p})_P + (Bu, B\tilde{p})_D = (f, \tilde{p})_P, \quad \forall \tilde{p} \in U$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{u}, \dot{u})_P + \frac{1}{2} (Bu, Bu)_D \right] = (f, \dot{u})_P$$

$$v(x, t) = \sqrt{|Bu(x, t)|^2 + 2\alpha} \in D_0 \Rightarrow \dot{v} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\dot{u}$$

Posons $B : U \times U \rightarrow D_0$ tel que $B[u]\tilde{u} = \frac{Bu}{\sqrt{|Bu|^2 + 2\alpha}} \cdot B\tilde{u}, \quad \forall u \in U, \tilde{u} \in U$

$$\begin{cases} (\dot{p}, \tilde{p})_P + (v, B[u]\tilde{p})_P = (f, \tilde{p})_P, & \forall \tilde{p} \in U \\ (\dot{v}, \tilde{v})_P - (B[u]p, \tilde{v})_P = 0, & \forall \tilde{v} \in D_0 \\ (\dot{u}, \tilde{u})_P = (p, \tilde{u})_P & \forall \tilde{u} \in P \end{cases}$$