



HAL
open science

Simultaneous stabilization of quadratic input nonlinear systems

Abderrahman Iggidr, Mohamed Oumoun

► **To cite this version:**

Abderrahman Iggidr, Mohamed Oumoun. Simultaneous stabilization of quadratic input nonlinear systems. 2024. hal-04038270v2

HAL Id: hal-04038270

<https://inria.hal.science/hal-04038270v2>

Preprint submitted on 7 Oct 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Simultaneous stabilization of quadratic input nonlinear systems

A. Iggidr^a and M. Oumoun^b

^a Université de Lorraine, CNRS, Inria, IECL, F57000 Metz, France. Email: abderrahman.iggidr@inria.fr

^b Université Cadi Ayyad, École Nationale des Sciences Appliquées, Av. Abdelkrim Khattabi, B.P. 575, Marrakech, Morocco. Email: m.oumoun@uca.ac.ma

ABSTRACT

In this paper simultaneous stabilization of a collection of quadratic input nonlinear systems is considered. First, with the help of the technique of control Lyapunov functions (CLFs), a necessary condition, then sufficient conditions for the existence of simultaneously stabilizing state feedback controller are derived. Moreover, we present a constructive method for designing continuous simultaneously stabilizing controller when the provided sufficient conditions are satisfied. Second, by using the sliding mode technique, this constructive method has been extended to establish a discontinuous control that simultaneously stabilizes two levitation systems.

Keywords. Quadratic nonlinear systems, simultaneous stabilization, control Lyapunov function, state feedback.

2000 Mathematics Subject Classification: 93D15.

1. Introduction

The simultaneous stabilization problem arises frequently in practice, due to plant uncertainty, plant variation, failure modes or plants with various modes of operation. This problem is concerned with determining a single controller which simultaneously stabilizes a finite collection of systems (Wu, 2005). Current applications of simultaneous stabilization include many real systems, such as Hamiltonian systems (Cai, 2015; Wang et al., 2007; Wei & Wang, 2014), power converters (Chaudhuri et al., 2006), chemical reaction systems (Zhu & Yang, 2016, 2017), and descriptor systems (Chen et al., 2012) to name a few.

Therefore, the investigation of simultaneous stabilization problems is quite relevant from a practical point of view, and many important results have been obtained for linear systems: see (Galindo, 2012; Guan et al., 2011; Kohan-Sedgh et al., 2016; Li et al., 2016; Miller & Chen, 2002; Yu & Chi, 2014) and references therein. For nonlinear systems, we note that it has been pointed out in (Wu, 2010) that the simultaneous stabilization is more difficult to solve. Hence, there are fewer results in the literature related to this problem, see, for examples, (Iggidr & Oumoun, 2020; Wu, 2010, 2005; Zhang et al., 2011) for nonlinear deterministic systems case, and (Iggidr & Oumoun, 2019; Iggidr et al., 2020) for nonlinear stochastic systems case. Recently, the problem of simultaneous stabilization has been addressed for nonlinear systems with delays in (Dastaviz & Binazadeh, 2019, 2020; Dastaviz et al., 2021).

Nowadays, quadratic input nonlinear systems represents an important class of nonlinear systems which can model many practical applications, such as in magnetic problems (Moulay & Perruquetti, 2005) and oscillation problems of the electromagnetic oscillator (Sane & Bernstein, 2002). Therefore, it is interesting to investigate such systems.

To the best of our knowledge, there are even fewer results on simultaneous stabilization of quadratic nonlinear systems because of the difficulties involved in synthesis of these systems. Following the stream of research on the simultaneous stabilization of nonlinear systems and motivated by the approach in (Iggidr & Oumoun, 2020; Wu, 2010, 2005; Zhang et al., 2011), in the context of quadratic input nonlinear systems, we study the problem of designing constructively a simultaneously asymptotically stabilizing control for these systems. Based on the control Lyapunov functions approach (CLF), we first give a necessary condition for the existence of simultaneously asymptotically stabilizing feedback. After that, we present sufficient conditions for the existence of simultaneously stabilizing feedback. Furthermore, we give an explicit formula for constructing the simultaneously stabilizing controller. It is shown that, under the small control property (SCP) assumption, the obtained feedback is continuous everywhere in the state space.

In the end, based on the sliding mode technique, the design procedure developed in this paper is also used to construct a discontinuous control that simultaneously stabilizes two Levitation systems. Note that magnetic levitation systems have practical importance in many engineering systems such as in high-speed Japanese “Maglev” and German “Transrapid” passenger trains, frictionless bearings ... to name a few.

2. Problem formulation and preliminaries

2.1. Problem formulation

Consider a collection of m quadratic input nonlinear systems described as follows:

$$S_i : \dot{x} = f_i(x) + ug_i(x) + u^2h_i(x), \quad i \in \{1, \dots, m\} = I, \quad (2.1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control, $f_i(\cdot)$, $g_i(\cdot)$ and $h_i(\cdot)$ are continuous functions with $f_i(0) = 0$.

The objective of this paper is to find a continuous function $k(\cdot)$ such that the state feedback controller $u = k(x)$ globally asymptotically stabilizes the collection of systems in (2.1) simultaneously, that is, the origin is a globally asymptotically stable equilibrium for the resulting closed-loop systems

$$S_i : \dot{x} = f_i(x) + k(x)g_i(x) + k(x)^2h_i(x), \quad i \in I. \quad (2.2)$$

2.2. Preliminaries

We first give some notations and basic definitions concerning control Lyapunov functions, then we recall some results on the stabilization of quadratic input nonlinear systems.

For each $i = 1, 2, \dots, m$, let V_i be a C^1 positive definite and proper function defined on \mathbb{R}^n . Its derivative, denoted by \dot{V}_i , along the trajectories of system S_i can be written as

$$\dot{V}_i(x) = a_i(x)u^2 + b_i(x)u + c_i(x), \quad (2.3)$$

with, $a_i = \langle h_i, \nabla V_i \rangle$, $b_i = \langle g_i, \nabla V_i \rangle$, $c_i = \langle f_i, \nabla V_i \rangle$.

In the sequel, we will denote by $\Delta_i(x), i \in I$, the discriminant of the right-hand side of (2.3), viewed as a polynomial in u , at every $x \in \mathbb{R}^n$, i.e., $\Delta_i(x) = b_i^2(x) - 4a_i(x)c_i(x)$. Moreover, when $\Delta_i(x) \geq 0$, we denote by $\lambda_{1i}(x)$ and $\lambda_{2i}(x)$ the two roots of the equation $a_i(x)\lambda^2 + b_i(x)\lambda + c_i(x) = 0$ given by

$$\lambda_{1i}(x) = \frac{-b_i(x) - \sqrt{\Delta_i(x)}}{2a_i(x)},$$

$$\lambda_{2i}(x) = \frac{-b_i(x) + \sqrt{\Delta_i(x)}}{2a_i(x)}.$$

We start by recalling the following definitions.

Definition 2.1. (*Maniar et al. , 2017; Moulay & Perruquetti, 2005*) For $i \in I$. The system S_i is almost stabilizable, if there exists a feedback control law $u : \mathcal{V} \rightarrow \mathbb{R}$ continuous on $\mathcal{V} \setminus \{0\}$, such that

- $u(0)=0$;
- the origin is an asymptotically stable equilibrium of the closed-loop system deduced from S_i and $u = u(x)$.

Moreover, if u is continuous on \mathcal{V} , we say that the system S_i is stabilizable.

The system S_i is globally stabilizable if $\mathcal{V} = \mathbb{R}^n$.

Definition 2.2. (*Maniar et al. , 2017; Moulay & Perruquetti, 2005*) For $i \in I$. A C^1 positive definite and proper function V_i , defined on \mathbb{R}^n is a control Lyapunov function (CLF) for system S_i if for any $x \in \mathbb{R}^n \setminus \{0\}$

$$\inf_{u \in \mathbb{R}} \dot{V}_i(x) = \inf_{u \in \mathbb{R}} (a_i(x)u^2 + b_i(x)u + c_i(x)) < 0.$$

A CLF V_i for System S_i is said to satisfy the small control property (SCP), if for each $\varepsilon > 0$ there is $\delta > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some u with $\|u\| < \varepsilon$ such that $\dot{V}_i(x) = a_i(x)u^2 + b_i(x)u + c_i(x) < 0$.

As far as possible, our objective is to explicitly design a common state feedback law u , which is continuous on $\mathbb{R}^n \setminus \{0\}$, and is such that the resulting closed-loop systems (2.2) are globally asymptotically stable. To be more precise, if V_i is a known collection of CLFs for the systems $S_i, i \in I$, we shall construct a feedback law u such that for each $i \in I$, V_i is a Lyapunov function for the closed-loop system $\dot{x} = f_i(x) + u(x)g_i(x) + u^2(x)h_i(x)$. This means to construct a continuous feedback such that, for $x \neq 0, i \in I$,

$$\dot{V}_i(x) = a_i(x)u^2(x) + b_i(x)u(x) + c_i(x) < 0. \quad (2.4)$$

Remark 2.1. (I) Since V_i is a CLF for system S_i , $i \in I$, it is easy to see that for $x \neq 0$, if $a_i(x) > 0$, then $\Delta_i(x) > 0$.

(II) Inequality (2.4) shows that in the setting of simultaneous quadratic input nonlinear systems, a continuous feedback u simultaneously stabilizes the collection of systems in (2.1) has to verify for each $i \in I$,

- (1) $u(x) \in]\lambda_{1i}(x), \lambda_{2i}(x)[$, if $a_i(x) > 0$,
- (2) $u(x) < -c_i(x)/b_i(x)$, if $a_i(x) = 0$ and $b_i(x) > 0$,
- (3) $u(x) > -c_i(x)/b_i(x)$, if $a_i(x) = 0$ and $b_i(x) < 0$,
- (4) $u(x) \notin [\lambda_{2i}(x), \lambda_{1i}(x)]$, if $a_i(x) < 0$ and $\Delta_i(x) \geq 0$.

In what follows, we recall some results (from (Maniar et al. , 2017)) that will be used in this paper.

Lemma 2.1. (Maniar et al. , 2017) Assume that V_i is a CLF for system S_i , $i \in I$. Then the following assertions hold:

- (1) the inequality $\Delta_i(x) < 0$ implies $a_i(x) < 0$;
- (2) $\Delta_i(x) = 0$, $a_i(x) \neq 0$ implies $a_i(x) < 0$;
- (3) $\Delta_i(x) = 0$, $a_i(x) = 0$ implies $b_i(x) = 0$ and $c_i(x) < 0$.

Lemma 2.2. (Maniar et al. , 2017) Assume that V_i is a CLF for System S_i , $i \in I$. Then for any $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that $a_i(x_0) = 0$, one has:

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{1i}(x) = -\frac{c_i(x_0)}{b_i(x_0)}, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{2i}(x) = +\infty \text{ if } b_i(x_0) < 0. \quad (2.5)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0}} \lambda_{1i}(x) = -\frac{c_i(x_0)}{b_i(x_0)}, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0}} \lambda_{2i}(x) = -\infty \text{ if } b_i(x_0) < 0. \quad (2.6)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{1i}(x) = -\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{2i}(x) = +\infty \text{ if } b_i(x_0) = 0. \quad (2.7)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0 \\ b_i(x) < 0}} \lambda_{1i}(x) = -\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0 \\ b_i(x) < 0}} \lambda_{2i}(x) = -\infty \text{ if } b_i(x_0) = 0. \quad (2.8)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0 \\ b_i(x) > 0}} \lambda_{1i}(x) = +\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0 \\ b_i(x) > 0}} \lambda_{2i}(x) = +\infty \text{ if } b_i(x_0) = 0. \quad (2.9)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{1i}(x) = -\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{2i}(x) = -\frac{c_i(x_0)}{b_i(x_0)} \text{ if } b_i(x_0) > 0. \quad (2.10)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0}} \lambda_{1i}(x) = +\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0}} \lambda_{2i}(x) = -\frac{c_i(x_0)}{b_i(x_0)} \text{ if } b_i(x_0) > 0. \quad (2.11)$$

Assumption 2.1. Let $i \in I$. For every $x \in \mathbb{R}^n \setminus \{0\}$, $(a_i(x) < 0 \text{ and } \Delta_i(x) \geq 0) \implies b_i(x) \neq 0$.

Theorem 2.2. (Maniar et al., 2017) For $i \in I$, assume V_i is a CLF for System S_i , and Assumption 2.1 holds. Then, S_i is almost stabilizable with the feedback

$$k_i(x) = \begin{cases} v_i(x), & \text{if } \Delta_i(x) > 0, \\ 0, & \text{if } \Delta_i(x) \leq 0, \end{cases} \quad (2.12)$$

where

$$v_i(x) = \begin{cases} \min\left(\lambda_{2i}(x) - \frac{\sqrt{\Delta_i(x)}}{2(1+a_i^2(x))}, 0\right) & \text{if } a_i(x) \neq 0 \text{ and } b_i(x) > 0, \\ \max\left(\lambda_{1i}(x) + \frac{\sqrt{\Delta_i(x)}}{2(1+a_i^2(x))}, 0\right) & \text{if } a_i(x) \neq 0 \text{ and } b_i(x) < 0, \\ \min\left(-\frac{c_i(x)}{b_i(x)} - \frac{b_i(x)}{2}, 0\right) & \text{if } a_i(x) = 0 \text{ and } b_i(x) > 0, \\ \max\left(-\frac{c_i(x)}{b_i(x)} - \frac{b_i(x)}{2}, 0\right) & \text{if } a_i(x) = 0 \text{ and } b_i(x) < 0, \\ 0 & \text{if } a_i(x) > 0 \text{ and } b_i(x) = 0, \end{cases}$$

In addition, if V_i satisfies the small control property, then S_i is stabilizable.

Hereafter we give a result that shows how to verify the small control property (SCP).

Lemma 2.3. For $i \in I$, assume V_i is a CLF for System S_i . Then V_i satisfy the SCP if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in B(0, \delta) \setminus \{0\}$ and $\Delta_i(x) \geq 0$, we have

$$\begin{aligned} \lambda_{1i}(x) < \varepsilon \text{ and } -\varepsilon < \lambda_{2i}(x) & \text{ if } a_i(x) > 0, \\ \lambda_{2i}(x) < \lambda_{1i}(x) < \varepsilon & \text{ if } a_i(x) < 0 \text{ and } b_i(x) < 0, \\ -\varepsilon < \lambda_{2i}(x) < \lambda_{1i}(x) & \text{ if } a_i(x) < 0 \text{ and } b_i(x) > 0, \\ -\lambda_{2i}(x) = \lambda_{1i}(x) < \varepsilon & \text{ if } a_i(x) < 0 \text{ and } b_i(x) = 0, \\ -\frac{c_i(x)}{b_i(x)} < \varepsilon & \text{ if } a_i(x) = 0 \text{ and } b_i(x) < 0, \\ -\varepsilon < -\frac{c_i(x)}{b_i(x)} & \text{ if } a_i(x) = 0 \text{ and } b_i(x) > 0. \end{aligned}$$

Proof. The proof is an immediate consequence of the definition of the SCP and the study of the sign of the polynomial in u : $a_i(x)u^2 + b_i(x)u + c_i(x)$, according to the sign of $a_i(x)$, x in a neighborhood of the origin.

Note that for the second assertion, we take into account the fact that $\lambda_{2i}(x) < 0$ and $|\lambda_{1i}(x)| < |\lambda_{2i}(x)|$ if $a_i(x) < 0$ and $b_i(x) < 0$. Similarly, for the third one, we use

the fact that $\lambda_{1i}(x) > 0$ and $|\lambda_{2i}(x)| < \lambda_{1i}(x)$ if $a_i(x) < 0$ and $b_i(x) > 0$. \square

Remark 2.2. Notice that if V_i , $i \in I$, is a CLF satisfying Assumption 2.1, then the set $\{x \in \mathbb{R}^n \mid a_i(x) < 0, \Delta_i(x) \geq 0, b_i(x) = 0\}$ is empty. So, in our case, to verify the SCP, it is sufficient to find functions $d_{ji} : \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, \dots, 5$, continuous in a neighborhood of the origin and $d_{ji}(0) = 0$, such that for $x \neq 0$ close enough to 0 and $\Delta_i(x) \geq 0$, one has:

$$\begin{aligned} \lambda_{1i}(x) < d_{1i}(x) < \lambda_{2i}(x) & \quad \text{if } a_i(x) > 0, \\ \lambda_{1i}(x) < d_{2i}(x) & \quad \text{if } a_i(x) < 0 \text{ and } b_i(x) < 0, \\ d_{3i}(x) < \lambda_{2i}(x) & \quad \text{if } a_i(x) < 0 \text{ and } b_i(x) > 0, \\ -\frac{c_i(x)}{b_i(x)} < d_{4i}(x) & \quad \text{if } a_i(x) = 0 \text{ and } b_i(x) < 0, \\ d_{5i}(x) < -\frac{c_i(x)}{b_i(x)} & \quad \text{if } a_i(x) = 0 \text{ and } b_i(x) > 0. \end{aligned}$$

3. Main results: Design method of simultaneously stabilizing controller

As a first step towards the construction of a simultaneously stabilizing feedback law for the collection of quadratic systems (2.1), and for easily presenting the main results, we define the following sets and functions: For each $i \in I$, suppose that V_i is a CLF for System S_i and let for each $x \in \mathbb{R}^n$,

$$\begin{aligned} I_1(x) &= \{i \in I \mid a_i(x) > 0\}, \\ I_2(x) &= \{i \in I \mid a_i(x) < 0, b_i(x) < 0, \Delta_i(x) \geq 0\}, \\ I_3(x) &= \{i \in I \mid a_i(x) < 0, b_i(x) > 0, \Delta_i(x) \geq 0\}, \\ I_4(x) &= \{i \in I \mid a_i(x) = 0, b_i(x) < 0\} \\ I_5(x) &= \{i \in I \mid a_i(x) = 0, b_i(x) > 0\}. \end{aligned}$$

$$\varphi_{1i}(x) = \begin{cases} \lambda_{1i}(x) & \text{if } i \in I_1 \cup I_2, \\ -\frac{c_i(x)}{b_i(x)} & \text{if } i \in I_4, \\ -\infty & \text{otherwise,} \end{cases}$$

$$\varphi_{2i}(x) = \begin{cases} \lambda_{2i}(x) & \text{if } i \in I_1 \cup I_3, \\ -\frac{c_i(x)}{b_i(x)} & \text{if } i \in I_5, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\psi_{1i}(x) = \begin{cases} k_i(x) & \text{if } b_i(x) < 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{2i}(x) = \begin{cases} k_i(x) & \text{if } b_i(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\varphi_1(x) = \max_{i \in I} \varphi_{1i}(x), \quad \varphi_2(x) = \min_{i \in I} \varphi_{2i}(x),$$

$$\psi_1(x) = \max_{i \in I} \psi_{1i}(x), \quad \psi_2(x) = \min_{i \in I} \psi_{2i}(x).$$

Finally, define

$$w_1(x) = \min(\psi_1(x), \varphi_2(x)) \quad \text{and} \quad w_2(x) = \max(\psi_2(x), \varphi_1(x)).$$

Remark 3.1. Due to definitions of ψ_{1i} and ψ_{2i} , it is easy to see that $\psi_{1i}(x) \geq 0$ and $\psi_{2i}(x) \leq 0$ for all $x \in \mathbb{R}^n$ and for all $i \in I$. Thus, $\psi_1(x) \geq 0$ and $\psi_2(x) \leq 0$, $\forall x \in \mathbb{R}^n$.

In preparation for the construction of a continuous simultaneously stabilizing control u , we need the following lemma.

Lemma 3.1. Consider the collection of systems in (2.1). Suppose that there is a CLF V_i , satisfying Assumption 2.1, for every S_i , $i \in I$. Then the functions ψ_1 and ψ_2 are continuous on $\mathbb{R}^n \setminus \{0\}$. Moreover, if the functions V_i , $i \in I$, satisfy the SCP, then ψ_1 and ψ_2 are continuous on \mathbb{R}^n .

Proof. By Assumption 2.1 and the definition of k_i (2.12), $i \in I$, we have $k_i(x) = 0$ whenever $b_i(x) = 0$. From this and the continuity of k_i , it then follows that ψ_{1i} and ψ_{2i} , $i \in I$, are continuous on $\mathbb{R}^n \setminus \{0\}$, then so are ψ_1 and ψ_2 . If in addition, the functions V_i , $i \in I$, satisfy the SCP, then k_i , $i \in I$, are continuous on \mathbb{R}^n . So, as above, ψ_{1i} and ψ_{2i} , $i \in I$, are continuous on \mathbb{R}^n since $k_i(0) = b_i(0) = 0$, $i \in I$. Consequently, ψ_1 and ψ_2 are also continuous on \mathbb{R}^n . This completes the proof of Lemma 3.1. \square

To easily present the following results, define the sets

$$A_1(x) = \{i \in I \mid (a_i(x) < 0, b_i(x) > 0, \Delta_i(x) > 0) \\ \text{or } (a_i(x) > 0) \text{ or } (a_i(x) = 0, b_i(x) > 0)\},$$

$$A_2(x) = \{i \in I \mid (a_i(x) \leq 0, b_i(x) \leq 0) \\ \text{or } (a_i(x) < 0, b_i(x) > 0, \Delta_i(x) < 0)\},$$

$$A_3(x) = \{i \in I \mid a_i(x) < 0, b_i(x) > 0, \Delta_i(x) = 0\},$$

$$B_1(x) = \{i \in I \mid (a_i(x) < 0, b_i(x) < 0, \Delta_i(x) > 0) \\ \text{or } (a_i(x) > 0) \text{ or } (a_i(x) = 0, b_i(x) < 0)\},$$

$$B_2(x) = \{i \in I \mid (a_i(x) \leq 0, b_i(x) \geq 0) \\ \text{or } (a_i(x) < 0, b_i(x) < 0, \Delta_i(x) < 0)\},$$

$$B_3(x) = \{i \in I \mid a_i(x) < 0, b_i(x) < 0, \Delta_i(x) = 0\}.$$

Clearly, $I = A_1(x) \cup A_2(x) \cup A_3(x) = B_1(x) \cup B_2(x) \cup B_3(x)$.

For convenience, define

$$A_{31}(x) = \{i \in A_3(x) \mid \forall \eta > 0, \exists y \in B(x, \eta) \setminus \{x\} \text{ such that, } \Delta_i(y) \geq 0\},$$

and $A_{32}(x) = A_3(x) \setminus A_{31}(x)$. We can, similarly, define $B_{31}(x)$ and $B_{32}(x)$.

For guaranteeing the continuity of the simultaneous feedback laws (which will be derived later) for the collection of quadratic input nonlinear systems in (2.1), here we assume

Assumption 3.1. If $A_3(x) \neq \emptyset$,

$$\min(\psi_1(x), \min_{i \in A_1(x)} \varphi_{2i}(x)) \leq \min_{j \in A_3(x)} -\frac{b_j(x)}{2a_j(x)}.$$

Assumption 3.2. If $B_3(x) \neq \emptyset$,

$$\max_{j \in B_3(x)} -\frac{b_j(x)}{2a_j(x)} \leq \max(\psi_2(x), \max_{i \in B_1(x)} \varphi_{1i}(x)).$$

Lemma 3.2. Consider the collection of systems in (2.1). Suppose that there is a CLF V_i , satisfying Assumption 2.1 for every S_i , $i \in I$ and Assumptions 3.1, 3.2 hold, Then the functions w_1 and w_2 are continuous on $\mathbb{R}^n \setminus \{0\}$. If in addition the functions V_i , $i \in I$, satisfy the small control property, then w_1 and w_2 are continuous on \mathbb{R}^n .

Proof. To establish the continuity of w_1 at $x_0 \in \mathbb{R}^n \setminus \{0\}$, we will verify that the functions φ_{2i} , $i \in A_1(x_0)$, are continuous on a neighborhood of x_0 and:

- If $A_{31}(x_0) = \emptyset$, then $w_1(x) = \min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x))$ on a neighborhood of x_0 . The conclusion follows since ψ_1 and φ_{2i} , $i \in A_1(x_0)$, are continuous on a neighborhood of x_0 .
- If $A_{31}(x_0) \neq \emptyset$, then $w_1(x) = \min_{k \in A_{31}(x_0)} w_1^k(x)$, with w_1^k , $k \in A_{31}(x_0)$ are continuous functions on a neighborhood of x_0 .

To do this, we first study the behavior of the functions φ_{2i} , $i \in I$, in a neighborhood of x_0 according to the sign of $a_i(x_0)$, $i \in I$.

Case 1: If $a_i(x_0) > 0$, by continuity of the function a_i , $a_i(x) > 0$ in a neighborhood \mathcal{V}_{x_0} of x_0 . Then $\varphi_{2i}(x) = \lambda_{2i}(x)$ on \mathcal{V}_{x_0} , so φ_{2i} is continuous at x_0 since λ_{2i} is continuous on \mathcal{V}_{x_0} .

Case 2: If $a_i(x_0) = 0$ and $b_i(x_0) > 0$, then $\Delta_i(x_0) > 0$. By continuity, $b_i(x) > 0$ and $\Delta_i(x) > 0$ in a neighborhood \mathcal{V}_{x_0} of x_0 , together with (2.10), (2.11) and definition of φ_{2i} , we can easily verify that φ_{2i} is continuous at x_0 .

Case 3: If $a_i(x_0) = 0$ and $b_i(x_0) = 0$. Let x^j be a sequence of states that converges to x_0 and let us review the different possibilities according to the definition of φ_{2i} .

- If the sequence x^j satisfies $a_i(x^j) > 0$, it then follows from (2.7) and definition of φ_{2i}

$$\lim_{x^j \rightarrow x_0} \varphi_{2i}(x^j) = \lim_{x^j \rightarrow x_0} \lambda_{2i}(x^j) = +\infty.$$

- If the sequence x^j satisfies $a_i(x^j) < 0$, $b_i(x^j) > 0$ and $\Delta_i(x^j) \geq 0$, (2.9) yields

$$\lim_{x^j \rightarrow x_0} \varphi_{2i}(x^j) = +\infty.$$

- If the sequence x^j satisfies $a_i(x^j) = 0$ and $b_i(x^j) > 0$, recall that since V_i is a CLF of System S_i , we have $c_i(x_0) < 0$, hence by continuity, $c_i(x^j) < 0$ whenever x^j is close enough to x_0 . So

$$\lim_{x^j \rightarrow x_0} \varphi_{2i}(x^j) = \lim_{x^j \rightarrow x_0} -\frac{c_i(x^j)}{b_i(x^j)} = +\infty.$$

- If the sequence x^j does not satisfy any of the above situations, then we have $\varphi_{2i}(x) = +\infty$.

Therefore, in all cases, i.e., if $a_i(x_0) = 0$ and $b_i(x_0) = 0$, the following holds

$$\forall L > 0, \exists \lambda_L^i > 0 \text{ such that } \varphi_{2i}(x) > L, \forall x \in B(x_0, \lambda_L^i).$$

Case 4: If $a_i(x_0) = 0$ and $b_i(x_0) < 0$, by continuity, $b_i(x) < 0$ in a neighborhood \mathcal{V}_{x_0} of x_0 , so, $\varphi_{2i}(x) = +\infty$ on \mathcal{V}_{x_0} .

Case 5: If $a_i(x_0) < 0$, $b_i(x_0) > 0$ and $\Delta_i(x_0) > 0$, this case is similar to the Case 1.

Case 6: If $a_i(x_0) < 0$, $b_i(x_0) > 0$ and $\Delta_i(x_0) = 0$, that is $i \in A_3(x_0)$. If $i \in A_{32}(x_0)$, then there exists $\delta_i > 0$ such that $\Delta_i(x) < 0$ for all $x \in B(x_0, \delta_i) \setminus \{x_0\}$. So, $\varphi_{2i}(x) = +\infty$ on $B(x_0, \delta_i) \setminus \{x_0\}$.

If $i \in A_{31}(x_0)$, then for x close enough to x_0 we have, $\varphi_{2i}(x) = +\infty$ if $\Delta_i(x) < 0$ and $\varphi_{2i}(x) = \lambda_{2i}(x)$ if $\Delta_i(x) \geq 0$.

Case 7: If $a_i(x_0) < 0$, $b_i(x_0) > 0$ and $\Delta_i(x_0) < 0$, by continuity, $\Delta_i(x) < 0$ in a neighborhood \mathcal{V}_{x_0} of x_0 , so, $\varphi_{2i}(x) = +\infty$ on \mathcal{V}_{x_0} .

Case 8: If $a_i(x_0) < 0$, $b_i(x_0) = 0$, this case is similar to Case 7 since from Assumption 2.1, we have $\Delta_i(x_0) < 0$.

Case 9: If $a_i(x_0) < 0$, $b_i(x_0) < 0$, this case is similar to Case 4.

Summarizing the above, we have:

- For $i \in A_1(x_0)$, the discussions of Cases 1, 2, 5 indicate that there exists $\varepsilon_1^i > 0$ such that, the function φ_{2i} is continuous on $B(x_0, \varepsilon_1^i)$. Take $\varepsilon_1 = \min_{i \in A_1(x_0)} \varepsilon_1^i$, it follows that the functions φ_{2i} , $i \in A_1(x_0)$, are continuous on $B(x_0, \varepsilon_1)$.
- For $i \in A_2(x_0)$, Cases 3, 4, 7, 8, 9 imply that the following holds

$$\forall L > 0, \exists \lambda_L^i > 0, \text{ s. t. } \varphi_{2i}(x) > L, \forall x \in B(x_0, \lambda_L^i).$$

Choosing $\lambda_L = \min_{i \in A_2(x_0)} \lambda_L^i$, it follows that

$$\forall L > 0, \exists \lambda_L > 0, \text{ such that } \varphi_{2i}(x) > L, \forall x \in B(x_0, \lambda_L), \forall i \in A_2(x_0). \quad (3.1)$$

- For $i \in A_{32}(x_0)$, Case 6 indicates that there exists $\delta_i > 0$ such that $\varphi_{2i}(x) = +\infty$ on $B(x_0, \delta_i) \setminus \{x_0\}$. Take $\delta = \min_{i \in A_{32}(x_0)} \delta_i$, it follows that

$$\varphi_{2i}(x) = +\infty, \forall x \in B(x_0, \delta) \setminus \{x_0\}, \forall i \in A_{32}(x_0).$$

From this and Assumption 3.1, we obtain, by taking $\mu = \min(\varepsilon_1, \delta)$, for all $x \in B(x_0, \mu)$

$$\min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x), \min_{i \in A_{32}(x_0)} \varphi_{2i}(x)) = \min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x)). \quad (3.2)$$

Let $\varepsilon > 0$ and take $L = |\psi_1(x_0)| + \varepsilon$. The continuity of the function ψ_1 and inequality (3.1) ensure the existence of $\lambda_L^\varepsilon > 0$ such that for all $x \in B(x_0, \lambda_L^\varepsilon)$

$$\psi_1(x) < |\psi_1(x_0)| + \varepsilon = L < \varphi_{2i}(x), \forall i \in A_2(x_0). \quad (3.3)$$

In terms of (3.2) and (3.3), choosing $\eta = \min(\mu, \lambda_L^\varepsilon)$, we have for all $x \in B(x_0, \eta)$

$$\min\left(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x), \min_{i \in A_2(x_0)} \varphi_{2i}(x), \min_{i \in A_{32}(x_0)} \varphi_{2i}(x)\right) = \min\left(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x)\right). \quad (3.4)$$

Now, if $A_{31}(x_0) = \emptyset$. From (3.4), we get

$$w_1(x) = \min\left(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x)\right), \forall x \in B(x_0, \eta).$$

Thus, w_1 is continuous at x_0 since ψ_1 and φ_{2i} , $i \in A_1(x_0)$, are continuous on $B(x_0, \eta)$.

If $A_{31}(x_0) \neq \emptyset$. For each $i \in A_{31}(x_0)$, we have $a_i(x_0) < 0$ and $b_i(x_0) > 0$. For reasons of continuity, $a_i(x) < 0$ and $b_i(x) > 0$ in a neighborhood $\mathcal{W}_{x_0}^i$ of x_0 . Recall that φ_{2i} , $i \in A_1(x_0)$, are continuous on $B(x_0, \varepsilon_1)$ and let

$$\mathcal{W}_{x_0} = \bigcap_{i \in A_3(x_0)} \mathcal{W}_{x_0}^i \bigcap B(x_0, \varepsilon_1).$$

Now, let $k \in A_{31}(x_0)$ and for convenience, define the function w_1^k on \mathcal{W}_{x_0} by:

$$w_1^k(x) = \min\left(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x), \varphi_{2k}(x)\right), \quad x \in \mathcal{W}_{x_0}.$$

Note that $\varphi_{2k}(x_0) = -\frac{b_k(x_0)}{2a_k(x_0)}$, so, Assumption 3.1 yields

$$w_1^k(x_0) = \min(\psi_1(x_0), \min_{i \in A_1(x_0)} \varphi_{2i}(x_0)).$$

Let $x^j \in \mathcal{W}_{x_0}$ and $y^j \in \mathcal{W}_{x_0}$ be two sequences of states that converges to x_0 satisfying $\Delta_k(x^j) \geq 0$ and $\Delta_k(y^j) < 0$. By definition of φ_{2k} , we have

$$\lim_{x^j \rightarrow x_0} \varphi_{2k}(x^j) = \lim_{x^j \rightarrow x_0} \lambda_{2k}(x^j) = -\frac{b_k(x_0)}{2a_k(x_0)} = \varphi_{2k}(x_0). \quad (3.5)$$

and

$$\varphi_{2k}(y^j) = +\infty. \quad (3.6)$$

On the one hand, Assumption 3.1, equality (3.5) and the continuity of the functions ψ_1 and φ_{2i} , $i \in A_1(x_0)$, on \mathcal{W}_{x_0} , along with the definition of w_1^k , we have

$$\begin{aligned} \lim_{x^j \rightarrow x_0} w_1^k(x^j) &= \min\left(\psi_1(x_0), \min_{i \in A_1(x_0)} \varphi_{2i}(x_0), -\frac{b_k(x_0)}{2a_k(x_0)}\right) \\ &= \min\left(\psi_1(x_0), \min_{i \in A_1(x_0)} \varphi_{2i}(x_0)\right) = w_1^k(x_0). \end{aligned} \quad (3.7)$$

On the other hand, equality (3.6) yields $w_1^k(y^j) = \min\left(\psi_1(y^j), \min_{i \in A_1(x_0)} \varphi_{2i}(y^j)\right)$, which implies

$$\lim_{y^j \rightarrow x_0} w_1^k(y^j) = \min\left(\psi_1(x_0), \min_{i \in A_1(x_0)} \varphi_{2i}(x_0)\right) = w_1^k(x_0). \quad (3.8)$$

In terms of (3.7) and (3.8), the functions w_1^k , $k \in A_{31}(x_0)$, are continuous at x_0 .

Finally, using (3.4) and choosing $\mathcal{W}'_{x_0} = \mathcal{W}_{x_0} \cap B(x_0, \eta)$, along with the definitions of w_1 and w_1^k , $k \in A_{31}(x_0)$, we obtain

$$\begin{aligned} w_1(x) &= \min\left(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x), \min_{k \in A_{31}(x_0)} \varphi_{2k}(x)\right) \\ &= \min_{k \in A_{31}(x_0)} w_1^k(x), \quad \forall x \in \mathcal{W}'_{x_0}. \end{aligned}$$

Thus, w_1 is continuous at x_0 since the functions w_1^k , $k \in A_{31}(x_0)$, are continuous at x_0 .

Now, we establish the continuity of w_1 at the origin in the case of the small control property of V_i , $i \in I$.

For each $i \in I$, take $\varepsilon > 0$, due to the small control property, there exists $\alpha_i > 0$ such that, for each $x \neq 0$ and $\|x\| < \alpha_i$, there exists $v \in]-\varepsilon, \varepsilon[$ that satisfies the inequality

$$\dot{V}_i(x) = a_i(x)v^2 + b_i(x)v + c_i(x) < 0. \quad (3.9)$$

With this inequality in mind, we review the different items in the definition of φ_{2i} .

- If $a_i(x) > 0$, then (3.9) leads to $v \in]-\lambda_{1i}(x), \lambda_{2i}(x)[$, that is, $-\varepsilon < v < \lambda_{2i}(x) = \varphi_{2i}(x)$.
- If $a_i(x) = 0$ and $b_i(x) > 0$, then (3.9) leads to $-\varepsilon < v < -\frac{c_i(x)}{b_i(x)} = \varphi_{2i}(x)$.

- If $a_i(x) < 0$, $b_i(x) > 0$ and $\Delta_i(x) \geq 0$, then (3.9) leads to $v < \lambda_{2i}(x) \leq \lambda_{1i}(x)$ or $\lambda_{2i}(x) \leq \lambda_{1i}(x) < v$. But in this case, we have $\lambda_{1i}(x) > 0$ and $|\lambda_{2i}(x)| \leq \lambda_{1i}(x)$. So, we always have $-\varepsilon < -|v| < \lambda_{2i}(x) = \varphi_{2i}(x)$.
- Otherwise, $\varphi_{2i}(x) = +\infty$.

From the above discussion, we deduce that

$$\forall \varepsilon > 0, \exists \alpha_i > 0 \text{ such that, } -\varepsilon < \varphi_{2i}(x), \forall x \in B(0, \alpha_i).$$

Take $\alpha = \min_{i \in I} \alpha_i$, it follows

$$\forall \varepsilon > 0, \exists \alpha > 0 \text{ s. t., } -\varepsilon < \varphi_2(x), \forall x \in B(0, \alpha). \quad (3.10)$$

Continuity of ψ_1 at the origin leads to

$$\forall \varepsilon > 0, \exists \beta > 0 \text{ s. t., } -\varepsilon < \psi_1(x) < \varepsilon, \forall x \in B(0, \beta). \quad (3.11)$$

Take $\gamma = \min(\alpha, \beta)$, it follows from (3.10) and (3.11)

$$\forall \varepsilon > 0, \exists \gamma > 0 \text{ such that, } -\varepsilon < w_1(x) = \min(\psi_1(x), \varphi_2(x)) < \varepsilon, \forall x \in B(0, \gamma).$$

That is, w_1 is continuous at the origin and so on \mathbb{R}^n .

Using Assumptions 2.1, 3.2 and Lemma 2.2, the continuity of w_2 can be treated similarly and is omitted here. The proof of Lemma 3.2 is now completed. \square

Remark 3.2. Let V_i , $i \in I$ be a collection of CLFs for system S_i , respectively. It is easy to see that Assumptions 2.1, 3.1, 3.2 are trivial if $a_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $i \in I$. Indeed, in this case we have $A_3(x) = B_3(x) = \emptyset$ for all $x \in \mathbb{R}^n$ and, for each $i \in I$, the set $\{x \in \mathbb{R}^n, | a_i(x) < 0\}$ is empty. So, by Lemma 3.2, if $a_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $i \in I$, then the functions w_1 and w_2 are continuous on $\mathbb{R}^n \setminus \{0\}$ and continuous on \mathbb{R}^n if the functions V_i satisfy the SCP.

The following theorem presents a necessary condition for the existence of simultaneously stabilizing state feedback for the collection of systems in (2.1).

Theorem 3.3. Consider the collection of systems in (2.1). If there exists a continuous feedback $k : \mathbb{R}^n \rightarrow \mathbb{R}$, with $k(0) = 0$, so that for each $i \in I$, the closed-loop system $\dot{x} = f_i(x) + k(x)g_i(x) + k^2(x)h_i(x)$ is globally stabilizable, then there exists a collection of CLFs V_i for Systems S_i , respectively, satisfying for all $x \in \mathbb{R}^n$ such that $I_1(x) \cup I_4(x) \neq \emptyset$ and $I_1(x) \cup I_5(x) \neq \emptyset$

$$\max_{i \in I_1(x) \cup I_4(x)} \varphi_{1i}(x) < \min_{i \in I_1(x) \cup I_5(x)} \varphi_{2i}(x). \quad (3.12)$$

Proof. Suppose that there exists a continuous feedback k such that the closed-loop systems $\dot{x} = f_i(x) + k(x)g_i(x) + k^2(x)h_i(x)$, $i \in I$, (whose coefficients are continuous) are stabilizable, then by converse Lyapunov theorem (see (Massera, 1956, Theorem 14), (Kurzweil, 1956), one can also see a survey in (Kellett, 2015)), there exists a collection of smooth, positive definite and radially unbounded functions V_i , $i \in I$, such

that for each $i \in I$, $\dot{V}_i(x) = a_i(x)k^2(x) + b_i(x)k(x) + c_i(x) < 0$, for all $x \neq 0$. It then follows that for each $x \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} \lambda_{1i}(x) &< k(x) < \lambda_{2i}(x) \text{ if } a_i(x) > 0, \\ k(x) &> -\frac{c_i(x)}{b_i(x)} \text{ if } a_i(x) = 0 \text{ and } b_i(x) < 0, \\ k(x) &< -\frac{c_i(x)}{b_i(x)} \text{ if } a_i(x) = 0 \text{ and } b_i(x) > 0. \end{aligned}$$

That is, $\varphi_{1i}(x) < k(x)$ if $i \in I_1(x) \cup I_4(x)$ and, $k(x) < \varphi_{2i}(x)$ if $i \in I_1(x) \cup I_5(x)$, which is equivalent, for all $x \in \mathbb{R}^n$ such that $I_1(x) \cup I_4(x) \neq \emptyset$ and $I_1(x) \cup I_5(x) \neq \emptyset$, to

$$\max_{i \in I_1(x) \cup I_4(x)} \varphi_{1i}(x) < k(x) < \min_{i \in I_1(x) \cup I_5(x)} \varphi_{2i}(x), \quad (3.13)$$

and this completes the proof of Theorem 3.3. \square

The necessary condition (3.12) shows that a simultaneously stabilizing feedback u for the collection of systems in (2.1) verify items 1, 2 and 3 in Remark 2.1. The feedback u has also to verify item 4 in Remark 2.1, that is $u(x) < \lambda_{2i}(x) \leq \lambda_{1i}(x)$ or $\lambda_{2i}(x) \leq \lambda_{1i}(x) < u(x)$ for each $i \in I$ such that $a_i(x) < 0$ and $\Delta_i(x) \geq 0$.

But the following situation may occur: There exists an open set $\mathcal{V} \subset \mathbb{R}^n$ and $i \in I$ such that:

- (1) $a_i(x) < 0$, $b_i(x) > 0$ and $\Delta_i(x) \geq 0$, for every $x \in \mathcal{V}$;
- (2) there exists $x_0 \in \partial\mathcal{V}$ (the border of \mathcal{V}) such that $a_i(x_0) = 0$ and $b_i(x_0) > 0$.

Let $x_1 \in \mathcal{V}$, if we choose $\lambda_{2i}(x_1) \leq \lambda_{1i}(x_1) < u(x_1)$, for reasons of continuity we have $\lambda_{2i}(x) \leq \lambda_{1i}(x) < u(x)$ for all $x \in \mathcal{V}$. From this and using (2.11) we get,

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \mathcal{V}}} u(x) \geq \lim_{\substack{x \rightarrow x_0 \\ x \in \mathcal{V}}} \lambda_{1i}(x) = +\infty$$

which contradict the fact that u is continuous at x_0 . To eliminate this situation, we have to choose $u(x) < \lambda_{2i}(x) \leq \lambda_{1i}(x)$ if $a_i(x) < 0$, $b_i(x) > 0$ and $\Delta_i(x) \geq 0$. Similarly, by using (2.6), we have to choose $\lambda_{2i}(x) \leq \lambda_{1i}(x) < u(x)$ if $a_i(x) < 0$, $b_i(x) < 0$ and $\Delta_i(x) \geq 0$. That is, for each $x \in \mathbb{R}^n \setminus \{0\}$

$$\lambda_{1i}(x) < u(x), \quad \forall i \in I_2(x) \text{ and } u(x) < \lambda_{2i}(x), \quad \forall i \in I_3(x). \quad (3.14)$$

(3.14) and definitions of φ_{1i} and φ_{2i} lead to, for each $x \in \mathbb{R}^n$

$$\max_{i \in I_2(x)} \varphi_{1i}(x) < u(x) < \min_{i \in I_3(x)} \varphi_{2i}(x). \quad (3.15)$$

The discussion above and Theorem 3.12 show that in order to construct a continuous feedback u that simultaneously stabilizes the collection of systems in (2.1), the feedback u should verify (3.13) and (3.15), which means for all $x \in \mathbb{R}^n$

$$\max_{i \in I_N(x)} \varphi_{1i}(x) < u(x) < \min_{i \in I_P(x)} \varphi_{2i}(x), \quad (3.16)$$

with, $I_N(x) = I_1(x) \cup I_2(x) \cup I_4(x)$, $I_P(x) = I_1(x) \cup I_3(x) \cup I_5(x)$.

To guarantee that the feedback that will be constructed below verify (3.16), we assume

$$\max_{i \in I_N(x)} \varphi_{1i}(x) < \min_{i \in I_P(x)} \varphi_{2i}(x), \quad \forall x \in \mathbb{R}^n. \quad (3.17)$$

Remark 3.3. Note that for each $x \in \mathbb{R}^n$, since $\varphi_{1i}(x) = -\infty$ for all $i \in I \setminus I_N(x)$ and $\varphi_{2i}(x) = +\infty$ for all $i \in I \setminus I_P(x)$, then inequality (3.17) is equivalent to the following assumption.

Assumption 3.4. For every $x \in \mathbb{R}^n$, $\varphi_1(x) < \varphi_2(x)$.

Remark 3.4. If the collection of CLFs in Theorem 3.3 verify $a_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $i \in I$, then $I_2(x) = I_3(x) = \emptyset$ for all $x \in \mathbb{R}^n$. So, inequalities (3.12) and (3.17) are equivalent. That is, by Theorem 3.3 and Remark 3.3, Assumption 3.4 becomes necessary in this case.

Now, we state the main result of this paper.

Theorem 3.5. Consider the collection of systems in (2.1). If there exists a collection of CLFs V_i , $i = 1, \dots, m$, satisfying Assumption 2.1, such that Assumptions 3.1, 3.2 and 3.4 hold, then the collection of systems in (2.1) is simultaneously almost stabilizable with the feedback

$$k(x) = \frac{w_1(x) + w_2(x)}{2} \quad (3.18)$$

Moreover, the collection of systems in (2.1) is simultaneously stabilizable by the feedback k if the functions V_i , $i = 1, 2, \dots, m$, satisfy the small control property.

Proof. According to Lemmas 3.1 and 3.2, the continuity of k is immediate.

For the asymptotic stability, we only have to verify that for each $i \in I$

$$\dot{V}_i(x) = a_i(x)k^2(x) + b_i(x)k(x) + c_i(x) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

according to the sign of $\Delta_i(x)$.

Case 1. If $\Delta_i(x) < 0$, then $a_i(x) < 0$ according to Lemma 2.1. Thus, the sign of $\dot{V}_i(x)$ regarded as a quadratic polynomial in k is that of $a_i(x)$. So, we have $\dot{V}_i(x) < 0$ in this case.

Case 2. If $\Delta_i(x) = 0$, from Lemma 2.1, we know that $a_i(x) = b_i(x) = 0$ and $c_i(x) < 0$ or $a_i(x) < 0$.

- i. If $a_i(x) = b_i(x) = 0$ and $c_i(x) < 0$, then $\dot{V}_i(x) = c_i(x) < 0$.
- ii. If $a_i(x) < 0$, from Assumption 2.1 we have $b_i(x) \neq 0$.
 - o If $b_i(x) > 0$. On the one hand, From the definitions of w_1 , φ_2 and φ_{2i} , we obtain

$$w_1(x) = \min(\psi_1(x), \varphi_2(x)) \leq \varphi_2(x) \leq \varphi_{2i}(x) = \lambda_{2i}(x) = -\frac{b_i(x)}{2a_i(x)}. \quad (3.19)$$

On the other hand, definitions of ψ_2 , ψ_{2i} and k_i give, $\psi_2(x) \leq \psi_{2i}(x) = k_i(x) < \varphi_{2i}(x) = \lambda_{2i}(x) = -\frac{b_i(x)}{2a_i(x)}$. From Assumption 3.4, we get $\varphi_1(x) < \varphi_2(x) \leq -\frac{b_i(x)}{2a_i(x)}$. Hence,

$$w_2(x) = \max(\psi_2(x), \varphi_1(x)) < \lambda_{2i}(x) = -\frac{b_i(x)}{2a_i(x)}. \quad (3.20)$$

(3.19) and (3.20) lead to

$$k(x) = \frac{w_1(x) + w_2(x)}{2} < \lambda_{2i}(x) = \lambda_{1i}(x) = -\frac{b_i(x)}{2a_i(x)}.$$

Thus, $\dot{V}_i(x) < 0$ since $a_i(x) < 0$.

- If $b_i(x) < 0$, then $\varphi_{1i}(x) = \lambda_{1i}(x) < k_i(x) = \psi_{1i}(x) \leq \psi_1(x)$. Assumption 3.4 yields $\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) < \varphi_2(x)$. So

$$\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) \leq \max(\psi_2(x), \varphi_1(x)) = w_2(x),$$

and

$$\lambda_{1i}(x) < \min(\psi_1(x), \varphi_2(x)) = w_1(x).$$

That is,

$$\lambda_{2i}(x) = \lambda_{1i}(x) < \frac{w_1(x) + w_2(x)}{2} = k(x).$$

Thus, $\dot{V}_i(x) < 0$ since $a_i(x) < 0$.

Case 3. If $\Delta_i(x) > 0$. In this case, we consider the three following cases according to the sign of $b_i(x)$.

- i. If $b_i(x) < 0$, we consider again three cases according to the sign of $a_i(x)$.
 - If $a_i(x) > 0$. On the one hand, since k_i stabilizes System S_i , we have $\lambda_{1i}(x) < k_i(x) = \psi_{1i}(x) \leq \psi_1(x)$. By Assumption 3.4, we get $\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) < \varphi_2(x)$. So $\lambda_{1i}(x) < \min(\psi_1(x), \varphi_2(x)) = w_1(x)$. Together with the fact that $w_1(x) \leq \varphi_2(x) \leq \varphi_{2i}(x) = \lambda_{2i}(x)$, it then follows that

$$\lambda_{1i}(x) < w_1(x) \leq \lambda_{2i}(x). \quad (3.21)$$

On the other hand, note that in this case $0 < \lambda_{2i}(x)$, hence $\psi_2(x) \leq \psi_{2i}(x) = 0 < \lambda_{2i}(x)$. As above, by Assumption 3.4, we get $\varphi_1(x) < \varphi_2(x) \leq \varphi_{2i}(x) = \lambda_{2i}(x)$. So $w_2(x) = \max(\psi_2(x), \varphi_1(x)) < \lambda_{2i}(x)$, and since $\lambda_{1i}(x) = \varphi_{1i}(x) \leq$

$\varphi_1(x) \leq w_2(x)$, it then follows that

$$\lambda_{1i}(x) \leq w_2(x) < \lambda_{2i}(x). \quad (3.22)$$

(3.21) and (3.22) give

$$\lambda_{1i}(x) < \frac{w_1(x) + w_2(x)}{2} = k(x) < \lambda_{2i}(x).$$

Thus, $\dot{V}_i(x) < 0$ since $a_i(x) > 0$.

- If $a_i(x) = 0$. Then $-\frac{c_i(x)}{b_i(x)} < k_i(x) = \psi_{1i}(x) \leq \psi_1(x)$. Assumption 3.4, yields $-\frac{c_i(x)}{b_i(x)} = \varphi_{1i}(x) \leq \varphi_1(x) < \varphi_2(x)$. So $-\frac{c_i(x)}{b_i(x)} < w_1(x)$ and $-\frac{c_i(x)}{b_i(x)} \leq w_2(x)$ since $\varphi_1(x) \leq w_2(x)$. It then follows that

$$-\frac{c_i(x)}{b_i(x)} < \frac{w_1(x) + w_2(x)}{2} = k(x).$$

Thus, $\dot{V}_i(x) = b_i(x)k(x) + c_i(x) < 0$ since $b_i(x) < 0$.

- If $a_i(x) < 0$. As above, $\lambda_{1i}(x) < k_i(x) = \psi_{1i}(x) \leq \psi_1(x)$. Assumption 3.4 yields $\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) < \varphi_2(x)$. Hence $\lambda_{1i}(x) < w_1(x)$ and $\lambda_{1i}(x) \leq \varphi_1(x) \leq w_2(x)$. So

$$\lambda_{2i}(x) < \lambda_{1i}(x) < \frac{w_1(x) + w_2(x)}{2} = k(x).$$

That is, $\dot{V}_i(x) < 0$ since $a_i(x) < 0$.

- ii. If $b_i(x) > 0$, this case can be treated similarly to the previous one.
- iii. If $b_i(x) = 0$, by Assumption 2.1 and since $\Delta_i(x) > 0$, we have $a_i(x) > 0$, so, this case is similar to the case $a_i(x) > 0$ in Case 3, item i.

Thus, the proof of Theorem 3.5 is completed. \square

Remark 3.5. 1. Notice that if the collection of CLFs in Theorem 3.5 verify $a_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $i \in I$, then, by Remark 3.2, the feedback (3.18) simultaneously stabilizes all the systems in (2.1) if only Assumption 3.4 holds.

2. Notice also that if V_i is a CLF for the system S_i such that $a_i(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, then, according to Theorem 3.4 in Maniar et al. (2017), the system S_i is almost stabilizable with the feedback, ($\varepsilon > 0$).

$$u_i(x) = \begin{cases} \frac{-b_i(x) - \sqrt{|\Delta_i(x)|}}{2a_i(x)} + \varepsilon & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

So, if V_i is a collection of CLFs V_i for systems S_i satisfying $a_i(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $i \in I$, then the collection of systems in (2.1) is simultaneously almost stabilizable with the feedback

$$u(x) = \max_{i \in I} u_i(x).$$

4. Illustrative example

In this section, we present an example to illustrate the effectiveness of the results obtained in this paper to design a simultaneously stabilizing feedback for some nonlinear quadratic systems.

Consider a system with the following two possible modes

$$S_1 : \begin{cases} \dot{x}_1 = -x_1 + \frac{x_1 x_2^4}{4} + \frac{x_1}{1+x_1^2} u^2 \\ \dot{x}_2 = -x_2 + x_1^2 x_2 u, \end{cases} \quad (4.1)$$

$$S_2 : \begin{cases} \dot{x}_1 = -x_1 + x_1^3 x_2^4 + (4x_1 - 4x_2)u - 4x_1 u^2 \\ \dot{x}_2 = -x_2 + x_1 u. \end{cases} \quad (4.2)$$

Let

$$V_1(x) = \frac{x_1^2 + x_2^2}{2} \quad \text{and} \quad V_2(x) = \frac{x_1^2 + 4x_2^2}{8}.$$

The Lie derivative of V_1 and V_2 along the trajectories of Systems S_1 and S_2 , respectively, write:

$$\dot{V}_1(x) = a_1(x)u^2 + b_1(x)u + c_1(x),$$

$$\dot{V}_2(x) = a_2(x)u^2 + b_2(x)u + c_2(x),$$

with $a_1(x) = x_1^2/(1+x_1^2)$, $b_1(x) = x_1^2 x_2^2$, $c_1(x) = -x_1^2 - x_2^2 + (1/4)x_1^2 x_2^4$, $a_2(x) = -x_1^2$, $b_2(x) = x_1^2$, and $c_2(x) = (1/4)(-x_1^2 + x_1^4 x_2^4) - x_2^2$.

By the definitions in Section 2, we have

$$\Delta_1(x) = \frac{x_1^6 x_2^4 + 4x_1^2 x_2^2 + 4x_1^4}{1+x_1^2}, \quad \Delta_2(x) = x_1^2 x_2^2 (x_1^4 x_2^2 - 4),$$

$$\lambda_{11}(x) = \frac{-x_1^2 x_2^2 - x_2^2}{2} - \frac{\sqrt{(1+x_1^2)(x_1^6 x_2^4 + 4x_1^2 x_2^2 + 4x_1^4)}}{2x_1^2},$$

$$\lambda_{21}(x) = \frac{-x_1^2 x_2^2 - x_2^2}{2} + \frac{\sqrt{(1+x_1^2)(x_1^6 x_2^4 + 4x_1^2 x_2^2 + 4x_1^4)}}{2x_1^2},$$

$$\lambda_{12}(x) = \frac{1}{2} + \frac{\sqrt{x_1^2 x_2^2 (x_1^4 x_2^2 - 4)}}{2x_1^2},$$

$$\lambda_{22}(x) = \frac{1}{2} - \frac{\sqrt{x_1^2 x_2^2 (x_1^4 x_2^2 - 4)}}{2x_1^2}.$$

We can prove that V_1 and V_2 are CLFs for Systems S_1 and S_2 , respectively, and satisfy the small control property. In fact, V_2 satisfies the small control property since $\Delta_2(x) < 0$ whenever $x \neq 0$ is close enough to 0 and then $a_2(x) < 0$ if $x_1 \neq 0$ or $c_2(x) < 0$ if $a_2(x) = b_2(x) = 0$.

Using notations of the Section 3, we have

$$\varphi_1(x) = \begin{cases} \lambda_{11}(x), & \text{if } x_1 \neq 0, \\ -\infty, & \text{if } x_1 = 0, \end{cases}$$

$$\varphi_2(x) = \begin{cases} \min(\lambda_{21}(x), \lambda_{22}(x)), & \text{if } x_1^4 x_2^2 - 4 \geq 0 \text{ or } x_1 \neq 0 \text{ and } x_2 = 0, \\ \lambda_{21}(x), & \text{if } x_1^4 x_2^2 - 4 < 0 \text{ and } x_1 \neq 0 \text{ and } x_2 \neq 0, \\ +\infty, & \text{if } x_1 = 0, \end{cases}$$

$$\psi_1(x) = 0, \quad \forall x \in \mathbb{R}^2,$$

$$\psi_2(x) = \begin{cases} \min(\lambda_{21}(x) - \frac{\sqrt{\Delta_1(x)}}{2(1+a_1^2(x))}, \lambda_{22}(x) - \frac{\sqrt{\Delta_2(x)}}{2(1+a_2^2(x))}, 0), & \text{if } x_1 \neq 0, x_2 \neq 0, \\ \min(\lambda_{22}(x) - \frac{\sqrt{\Delta_2(x)}}{2(1+a_2^2(x))}, 0), & \text{if } x_1 \neq 0, x_2 = 0, \\ 0, & \text{if } x_1 = 0. \end{cases}$$

It is not difficult to verify that Systems S_1 and S_2 satisfy Assumptions 2.1, 3.2.

To verify Assumption 3.1, define $S = \{x \in \mathbb{R}^2 \mid x_1 = 0 \text{ and } x_2 = 0, \text{ or } x_1^4 x_2^2 = 4\}$.

- For $x \notin S$, $A_3(x) = \emptyset$ and then Assumption 3.1 holds.
- For $x \in S$, we have $A_1(x) = \{1\}$, $A_3(x) = \{2\}$, $\psi_1(x) = 0$ and $-\frac{b_2(x)}{2a_2(x)} = \frac{1}{2}$.

Hence, $\min(\psi_1(x), \varphi_{21}(x)) \leq \psi_1(x) < -\frac{b_2(x)}{2a_2(x)}$, that is Assumption 3.1 holds.

It remains to verify Assumption 3.4. To this end define $D_1 = \{x \in \mathbb{R}^2 \mid x_1 = 0 \text{ or, } x_1^4 x_2^2 < 4 \text{ and } x_1 x_2 \neq 0\}$ and $D_2 = \{x \in \mathbb{R}^2 \mid x_1 \neq 0 \text{ and } x_2 = 0 \text{ or, } x_1^4 x_2^2 \geq 4\}$. Clearly, $\mathbb{R}^2 = D_1 \cup D_2$.

- For $x \in D_1$, from definitions of φ_1 and φ_2 , we have $\varphi_1(x) < \varphi_2(x)$.
- For $x \in D_2$, we have $\varphi_1(x) = \lambda_{11}(x)$ and $\varphi_2(x) = \min(\lambda_{21}(x), \lambda_{22}(x))$. Notice that $\lambda_{11}(x) < \lambda_{21}(x)$ for all $x \in D_2$. Notice also that since $-\sqrt{(1+x_1^2)(x_1^6 x_2^4 + 4x_1^2 x_2^2 + 4x_1^4)} \leq -\sqrt{x_1^2 x_2^2 (x_1^4 x_2^2 - 4)}$ for all $x \in D_2$, it follows that $\lambda_{11}(x) < \lambda_{22}(x)$ for all $x \in D_2$. Thus $\varphi_1(x) < \varphi_2(x)$ for all $x \in D_2$.

In both cases, $\varphi_1(x) < \varphi_2(x)$ and thus Assumption 3.4 holds.

Therefore, by Theorem 3.5, the feedback

$$u(x) = \frac{w_1(x) + w_2(x)}{2} \tag{4.3}$$

is continuous on \mathbb{R}^2 and globally asymptotically stabilizes the Systems S_1 and S_2 simultaneously, with $w_1(x) = \min(\psi_1(x), \varphi_2(x))$ and $w_2(x) = \max(\psi_2(x), \varphi_1(x))$.

The behaviors of the closed-loop systems (4.1) and (4.2) under the action of the simultaneously stabilizing feedback are drawn in Figures 1-2. The evolution of the stabilizing feedback along the closed-loop systems (4.1) and (4.2) is shown in Figures 3-4.

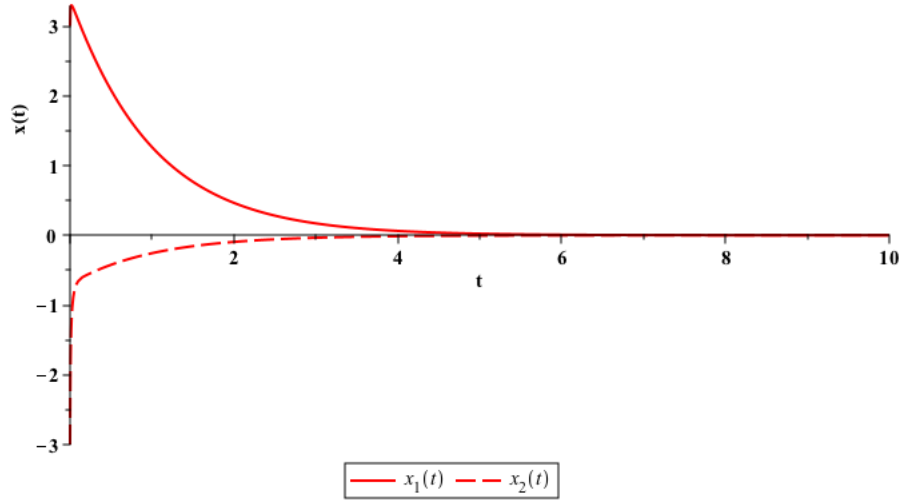


Figure 1. The responses of the closed-loop system S_1 and (4.3) (initial state $x(0) = (3, -3)$).

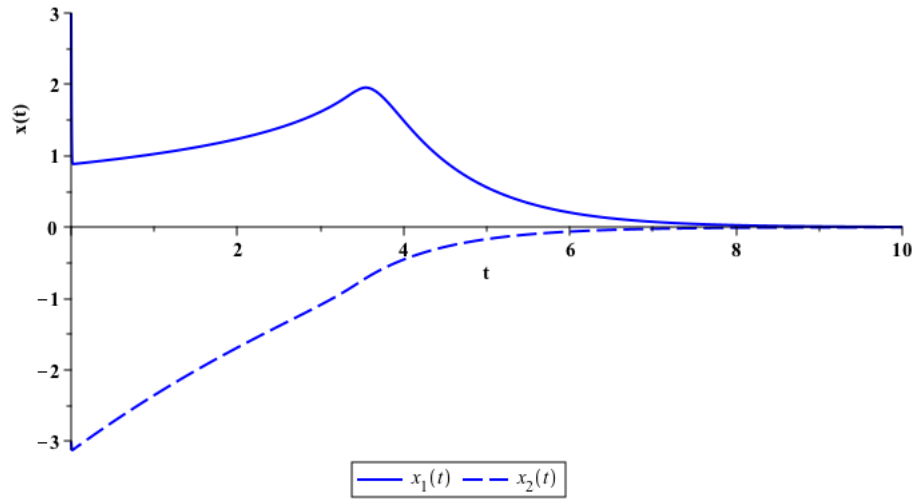


Figure 2. The responses of the closed-loop system S_2 and (4.3) (initial state $x(0) = (3, -3)$).

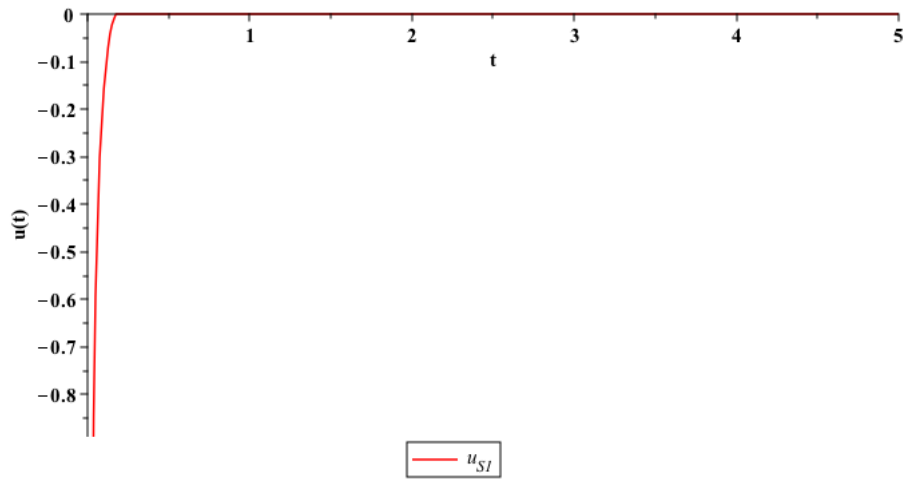


Figure 3. The evolution of the stabilizing control along the solutions of the closed-loop system S_1 and (4.3) (initial state $x(0) = (3, -3)$).

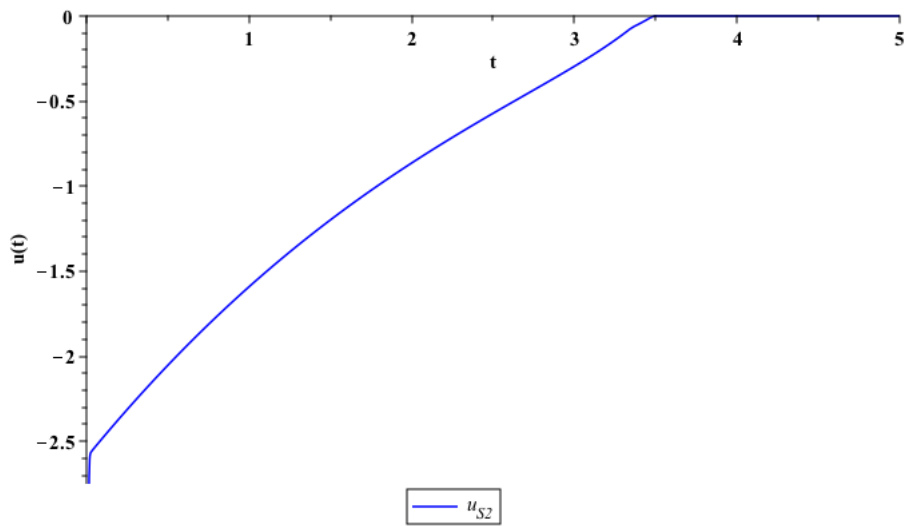


Figure 4. The evolution of the stabilizing control along the solutions of the closed-loop system S_2 and (4.3) (initial state $x(0) = (3, -3)$).

5. Simultaneous stabilization by discontinuous controller: The magnetic ball levitation systems

Remark 5.1. *In the following, using the sliding mode (SM) technique, we will show that to achieve the simultaneous stabilization, by a discontinuous feedback, of two levitation systems is reduced to solving a quadratic inequalities of the form (2.4). So, the method and notations are similar to those in Section 3.*

Also, note that since the feedback is discontinuous, we will focus on the negativity of quadratic inequalities, that is we only use Assumptions 2.1 and 3.4.

Consider two ferromagnetic balls suspended in a voltage-controlled magnetic field. Only the vertical motion is considered. The goal is to keep each ball in levitation at a desired position with respect to the ground.

The two levitation systems, Σ_1 and Σ_2 modeling, respectively, the motion of the ball 1 and ball 2, are described by the following equations (see (Bethoux et al., 2003; Moulay & Perruquetti, 2005) for more details):

$$\Sigma_i : \begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = g - \frac{k_i}{(l_i + e_1 + x_{1refi})^2} u^2. \end{cases} \quad (5.1)$$

where, for each system Σ_i , $i = 1, 2$, $e = (e_1, e_2) \in \mathbb{R}^2$. e_1 denotes the distance between x_1 the vertical position of the ball i with respect to the ground and x_{1refi} (a constant desired position), e_2 its speed, $g = 9.8$ the acceleration of gravity, u the coil current is the control and, k_i and l_i are some positive parameters depending on the electromagnet, the mass and the permeability of the free space (Bethoux et al., 2003).

The objective is to find a common control law u which stabilizes each ball i at a desired position x_{refi} . Toward this end, and in view of the sliding mode (SM) technique, we first design two appropriate sliding surfaces σ_1 and σ_2 on which, respectively, systems Σ_1 and Σ_2 are asymptotically stable. Second, we deal with the design of a single feedback law by explicit formulas in such a way that the so-called η -reachability condition $\dot{\sigma}_i \sigma_i < -\eta |\sigma_i|$ is satisfied, with $\eta > 0$.

Notice that, for each system Σ_i , $i = 1, 2$, the η -reachability condition guarantees that all trajectories are directed toward the switching surface $\sigma_i = 0$ in a finite time, and are maintained on this surface once it has been reached. For more details on this subject, we refer the reader to (Khalil, 2002; Perruquetti & Barbot, 2002).

Now, let us define the sliding surfaces as: $\sigma_1 = \sigma_2 = \beta e_1 + e_2$, where β is a positive scalar.

Note that the choice of the switching surfaces guarantees that $e = (e_1, e_2)$ converges to 0 as $t \rightarrow \infty$ when we have sliding (i.e., $\sigma_i = 0$), since on this manifold, the motion is governed by $\dot{e}_1 = -\beta e_1$. Note also that σ_1 and σ_2 have the same sign.

Differentiating σ_1 and σ_2 with respect to time along the trajectories, respectively,

of Σ_1 and Σ_2 leads to, for each $i = 1, 2$

$$\dot{\sigma}_i = \beta \dot{e}_1 + \dot{e}_2 = -\frac{k_i}{(l_i + e_1 + x_{1refi})^2} u^2 + \beta e_2 + g. \quad (5.2)$$

Take $\eta > 0$, from (5.2) we have

$$\dot{\sigma}_i < -\frac{k_i}{(l_i + e_1 + x_{1refi})^2} u^2 + \beta e_2 + g + \eta. \quad (5.3)$$

and

$$-\dot{\sigma}_i < \frac{k_i}{(l_i + e_1 + x_{1refi})^2} u^2 - \beta e_2 - g + \eta. \quad (5.4)$$

Now, to construct a control u that constrain both systems Σ_1 and Σ_2 to verify the η -reachability $\dot{\sigma}_i \sigma_i < -\eta |\sigma_i|$, one sees that it is sufficient that the right-hand side of (5.3) and (5.4), viewed as a polynomials in u , have negative values. For this, consider two cases for each system Σ_i , $i = 1, 2$, according to the sign of σ_i .

- If $\sigma_i > 0$, $i = 1, 2$. Consider the right-hand side of (5.3). These polynomials are of the form $a_i(e)u^2 + c_i(e)$ with

$$a_i(e) = -\frac{k_i}{(l_i + e_1 + x_{1refi})^2}, \quad c_i(e) = \beta e_2 + g + \eta,$$

and $\Delta_{1i}(e) = -4a_i(e)c_i(e)$, $i = 1, 2$, their discriminants which are always positive in a neighborhood of the origin.

Since $a_i(e) < 0$, according to second item in Remark 3.5, the control

$$u_p(e) = \max_{i=1,2} -\frac{\sqrt{\Delta_{1i}(e)}}{2a_i(e)} + \varepsilon, \quad (\varepsilon > 0),$$

renders both polynomials, $i = 1, 2$, in (5.3) negative, that is $\dot{\sigma}_i < -\eta$ and so, $\dot{\sigma}_i \sigma_i < -\eta |\sigma_i|$ as desired.

- If $\sigma_i < 0$, $i = 1, 2$. As above, consider polynomials in the right-hand side of (5.4) which are of the form $a'_i(e)u^2 + c'_i(e)$ with

$$a'_i(e) = \frac{k_i}{(l_i + e_1 + x_{1refi})^2} \quad \text{and} \quad c'_i(e) = -\beta e_2 - g + \eta.$$

These polynomials have negative values if their discriminants $\Delta_{2i}(e) = -4a'_i(e)c'_i(e)$ are positive, that is, $-\beta e_2 < g - \eta$, which occurs in a neighborhood of the origin if $g - \eta > 0$. It then follows that, in this situation, one can process as in Section 3 to construct u_n such that $a'_i(e)u_n^2(e) + c'_i(e) < 0$.

In this case, the two roots of each polynomial $a'_i(e)u^2 + c'_i(e)$, $i = 1, 2$ are

$$\lambda_{1i}(e) = -\frac{\sqrt{\Delta_{2i}(e)}}{2a'_i(e)} \quad \text{and} \quad \lambda_{2i}(e) = \frac{\sqrt{\Delta_{2i}(e)}}{2a'_i(e)}.$$

It is easy to verify that (notations below are those in Section 3)

$$\varphi_{1i}(e) = \lambda_{1i}(e), \quad \varphi_{2i}(e) = \lambda_{2i}(e) \quad \text{and} \quad \psi_{1i}(e) = \psi_{2i}(e) = 0.$$

Thus

$$\varphi_1(e) = \max(\lambda_{11}(e), \lambda_{12}(e)), \quad \varphi_2(e) = \min(\lambda_{21}(e), \lambda_{22}(e)) \quad \text{and} \quad \psi_1(e) = \psi_2(e) = 0.$$

One sees that $\varphi_1(e) < 0 < \varphi_2(e)$ that is, Assumption 3.4 holds. It is obvious that Assumption 2.1 holds too since $a'_i(e) > 0$. One sees also that

$$w_1(x) = \min(\psi_1(x), \varphi_2(x)) = 0 \quad \text{and} \quad w_2(x) = \max(\psi_2(x), \varphi_1(x)) = 0.$$

Consequently, the control

$$u_n(e) = \frac{w_1(e) + w_2(e)}{2} = 0$$

renders both polynomials, $i = 1, 2$, in (5.4) negative, that is $-\dot{\sigma}_i < -\eta$ which leads to $\dot{\sigma}_i \sigma_i < -\eta|\sigma_i|$.

Finally, the discontinuous control

$$u(e) = \begin{cases} 0 & \text{if } \sigma_i < 0 \\ \max_{i=1,2} -\frac{\sqrt{\Delta_{1i}(e)}}{2a_i(e)} + \varepsilon & \text{if } \sigma_i \geq 0, \end{cases} \quad (5.5)$$

guarantees the η -reachability condition and, from the discussion above, simultaneously stabilizes Σ_1 and Σ_2 .

Simulations:

Σ_1 : $l_1 = 0.01$, $k_1 = 1$, $x_{ref1} = 5$, $\beta = 1$, $\eta = 0.01$, $\varepsilon = 0.0001$.

Σ_2 : $l_2 = 0.03$, $k_2 = 1.5$, $x_{ref2} = 4$, $\beta = 1$, $\eta = 0.01$, $\varepsilon = 0.0001$.

References

- Bethoux, O., Floquet, T., & Barbot, J. P.(2003). Advanced sliding mode stabilization of a Levitation system. *European Control Conference (ECC)*, Cambridge, UK, pp. 501-506. doi: 10.23919/ECC.2003.7085005.
- Cai, L. C. (2015). Simultaneous stabilization of Port-Hamiltonian systems subject to actuation saturation and input delay, *Int. J. Autom. Comput.*, . <https://doi.org/10.1007/s11633-015-0928-4>.
- Chaudhuri, B., Korba, P., & Pal, B. C.(2006). Simultaneous stabilization technique for damping controller design in power systems, *Intelligent Automation & Soft Computing*, 12, 41–49.
- Chen, Y., Xu, T., Zeng, C., Zhou, Z., & Zhang, Q. (2012). Simultaneous stabilization for uncertain descriptor systems with input saturation, *International Journal of Robust and Nonlinear Control*, 22, 1938–1951.
- Dastaviz, A. & Binazadeh, T. (2019). Simultaneous stabilization of a collection of uncertain discrete-time systems with time-varying state-delay via discrete-time sliding mode control. *Journal of Vibration and Control*, 25.

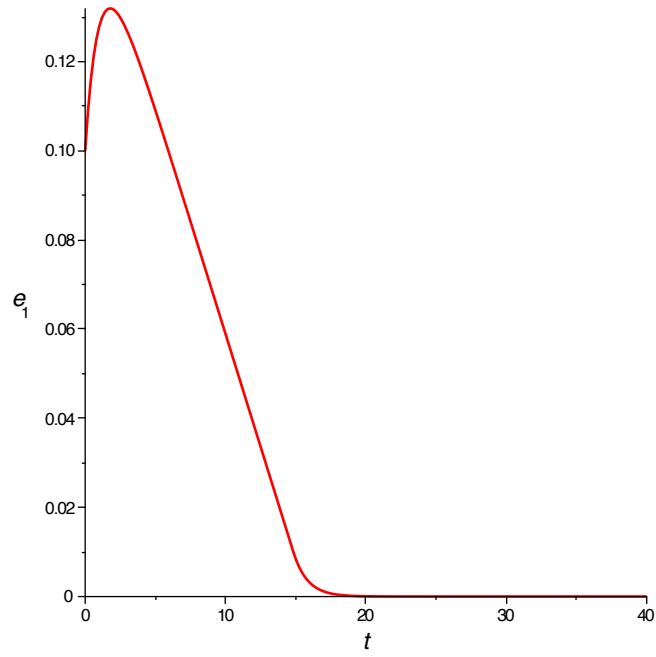


Figure 5. Evolution of the state $e_1(t)$ of the system Σ_1 and (5.5) (initial state $e(0) = (0.1, 0.05)$).

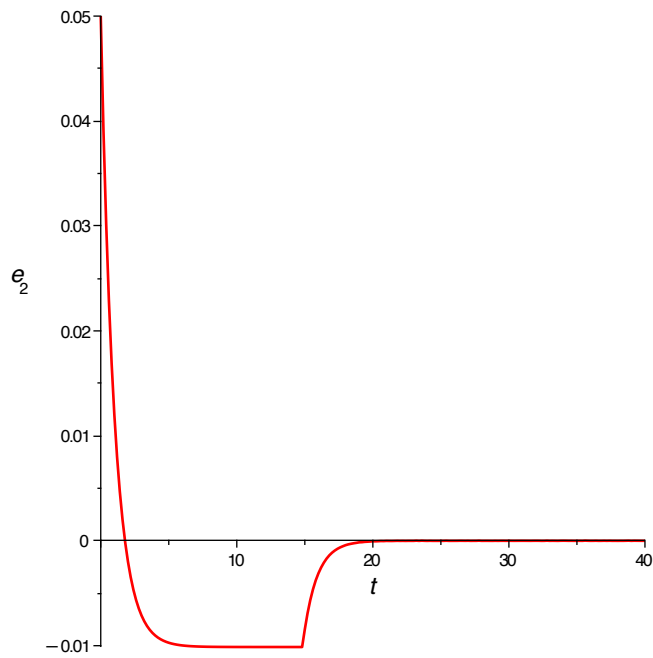


Figure 6. Evolution of the state $e_2(t)$ of the system Σ_1 and (5.5) (initial state $e(0) = (0.1, 0.05)$).

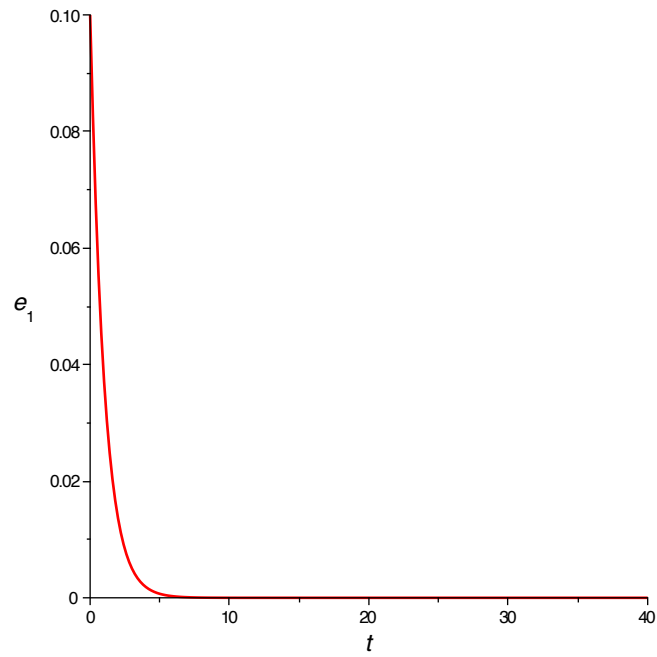


Figure 7. Evolution of the state $e_1(t)$ of the system Σ_2 and (5.5) (initial state $e(0) = (0.1, 0.05)$).

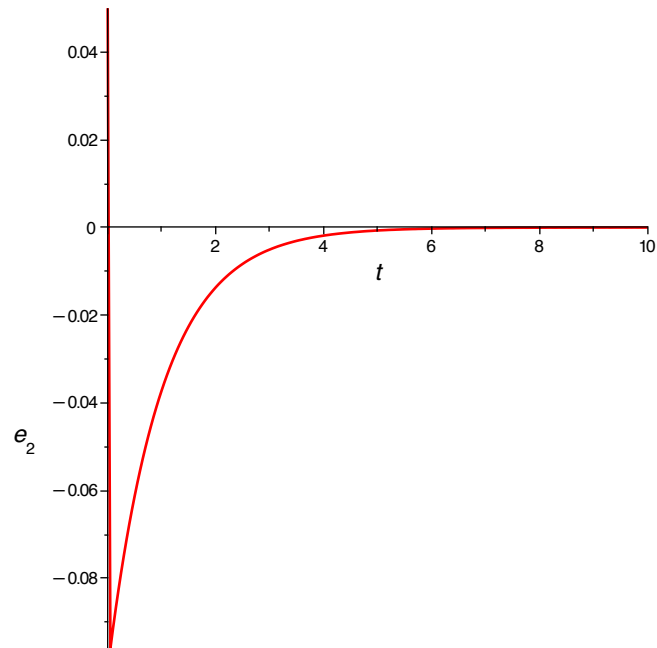


Figure 8. Evolution of the state $e_2(t)$ of the system Σ_2 and (5.5) (initial state $e(0) = (0.1, 0.05)$).

- Dastaviz, A. & Binazadeh, T. (2020). Simultaneous stabilization for a collection of uncertain time-delay systems using sliding-mode output feedback control. *International Journal of Control*, *93*, 2135–2144.
- Dastaviz, A., Binazadeh, T., & Cui, Y. (2021). A novel control lyapunov-krasovskii functional methodology for simultaneous stabilization of a set of multiple time-delays nonlinear systems. *Journal of the Franklin Institute*, *358*, 6101–6120.
- Galindo, R. (2012). Simultaneous stabilization of full state information Linear Time-Invariant systems. *Asian Journal of Control*, *14*, 1382–1386.
- Guan, Q., He, G., Wang, L., & Yu, W. (2011). Simultaneous stabilization of linear systems. *Control Theory and Applications*, *28*, 58–73.
- Iggidr, A., & Oumoun, M. (2020). Simultaneous Stabilization of Single-Input Nonlinear Systems With Bounded Controls. *International Journal of Dynamics and Control*. <https://doi.org/10.1007/s40435-020-00658-3>
- Iggidr, A., & Oumoun, M. (2019). Simultaneous stabilization for a collection of stochastic nonlinear systems based on CLFs. *International Conference on Control, Automation and Diagnosis (ICCAD)*, Grenoble, France, (2019). <https://doi.org/10.1109/ICCAD46983.2019.9037892>
- Iggidr, A., Oumoun, M., & Souza, M. O. (2020). Continuous simultaneous stabilization of single-input nonlinear stochastic systems. *Asian Journal of Control*. <https://doi.org/10.1002/asjc.2401>
- Khalil, H. K. (2002). *Nonlinear systems*. Upper Saddle River, NJ: Prentice-Hall.
- Kohan-Sedgh, P., Khayatian, A., & Asemami, M. H. (2016). Conservatism reduction in simultaneous output feedback stabilisation of linear systems. *IET Control Theory & Applications*. 2243–2250.
- Kellett, C. M. (2015). Classical converse theorems in Lyapunov’s second method. *Discrete and Continuous Dynamical Systems - Series B*, *20*(8):2333–2360.
- Kurzweil, J., (1956). On the inversion of Ljapunov’s second theorem on stability of motion. (Russian) *Czechoslovak Mathematical Journal*, *81*. 217–259, 455–484; English translation in *American Mathematical Society Translations (2)*, *24*, 19–77.
- Li, W., Wang, L., & Yu, W. (2016). Some Open Problems on Simultaneous Stabilization of Linear Systems. *Journal of Systems Science and Complexity*, *29*, 289–299.
- Maniar, L., Oumoun, M., & Vivalda, J. C. (2017). On the stabilization of quadratic nonlinear systems. *European Journal of Control*, *35*. 28–33.
- Massera, J. L. (1956). Contributions to stability theory. *Ann. Math. (2)*, *64*:182–206.
- Miller, D. E., & Chen, T. (2002). Simultaneous stabilization with near-optimal H^∞ performance. *IEEE Trans. Autom. Control*, *47*, 1986–1998.
- Moulay, E., & Perruquetti, W. (2005). Stabilization of nonaffine systems, A constructive method for polynomial systems. *IEEE Transactions on Automatic Control*, *50*, 520–526.
- Perruquetti, W., & Barbot, J. P. (2002). *Sliding Mode Control In Engineering*. Marcel Dekker, New York.
- Sane, H. S., & Bernstein, D. S. (2002). Robust nonlinear control of the electromagnetically controlled oscillator. Proceedings of the ACC, 809–814.
- Wang, Y. Z., Feng, G., & Cheng, D. Z. (2007). Simultaneous stabilization of a set of nonlinear port-controlled Hamiltonian systems. *Automatica*, 403–415.
- Wei, A., & Wang, Y. (2014). Adaptive parallel simultaneous stabilization of a set of uncertain port-controlled hamiltonian systems subject to actuator saturation. *International Journal of Adaptive Control and Signal Processing*, *28*, 1128–1144.
- Wu, J. L. (2010). Simultaneous quadratic stabilization for discrete-time nonlinear systems. *IEEE Trans. Autom. Control*, *55*, 1443–1448.
- Wu, J. L. (2005). Simultaneous stabilization for a collection of single-input nonlinear systems. *IEEE Trans. Autom. Control*, *50*, 328–337.
- Yu, T., & Chi, W. (2014). Sufficient conditions for simultaneous stabilization of three linear systems within the framework of nest algebras. *Journal of the Franklin Institute*, *351*, 5310–5325.

- Zhang, W., Han, Z. Z., & Cai, X. S. (2011). Simultaneous stabilization of a collection of single-input nonlinear systems based on CLFs. *Asian Journal of Control*, *13*, 582–589.
- Zhu, Y., & Yang, F. (2016). Simultaneous H_2/H_∞ stabilization for chemical reaction systems based on orthogonal complement space. *International Journal of Automation and Computing*, *13*, 19–30.
- Zhu, Y., & Yang, F. (2017). Network-based simultaneous H_∞ stabilization for chemical reaction systems with multiple packet dropouts. *International Journal of Control, Automation and Systems*, *15*, 104–112.