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Abderrahman Iggidr, Mohamed Oumoun

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Simultaneous stabilization of quadratic input nonlinear systems

A. Iggidr

Université de Lorraine, CNRS, Inria, IECL
F57000 Metz, France.

Email adress: Abderrahman.Iggidr@inria.fr

M. Oumoun

Cadi Ayyad University, National School of Applied Sciences,
Av. Abdelkrim Khattabi, B.P. 575, Marrakech, Morocco.

Email adress: m.oumoun@uca.ac.ma

Abstract

In this paper simultaneous stabilization of a collection of quadratic input nonlinear systems is considered. With the help of the technique of control Lyapunov functions (CLFs), necessary and sufficient conditions for the existence of simultaneously stabilizing state feedback controller are derived. Moreover, we present a constructive method for designing continuous simultaneously stabilizing controller when the provided sufficient conditions are satisfied. To verify the obtained results, an illustrative example is presented.

Keywords. Quadratic nonlinear systems, simultaneous stabilization, control Lyapunov function, state feedback.

2000 Mathematics Subject Classification: 93D15.

1 Introduction

The simultaneous stabilization problems arise due to several reasons: system's uncertainty, failure modes or systems with various modes of operation. The aim of this problem is to determine a single controller which simultaneously stabilizes a finite collection of systems. Current applications of simultaneous stabilization include many real systems, such as Hamiltonian systems [2, 18, 19], power converters [3], chemical reaction systems [25, 26], and descriptor systems [4] to name a few. Therefore, the investigation of simultaneous stabilization problems is quite relevant from a practical point of view, and many important results have been obtained for linear systems: see [8, 9, 13, 14, 16, 22] and references therein. For nonlinear deterministic and stochastic systems, the problem is more involved, and there are fewer results in the literature, see, for examples, [10, 20, 21, 24] for nonlinear deterministic systems case, and [11, 12] for nonlinear stochastic systems case. Recently, the problem of simultaneous stabilization has been addressed for nonlinear systems with delays in [5, 6, 7].

Till now, to our knowledge, no results have been reported about simultaneous stabilization of quadratic input nonlinear systems. Here, Motivated by the approach in [10, 12], in the context

of quadratic input nonlinear systems, we study the problem of designing constructively a simultaneously asymptotically stabilizing control for these systems. Based on the control Lyapunov functions approach (CLF), we first give a necessary condition for the existence of simultaneously asymptotically stabilizing feedback. After that, we present sufficient conditions for the existence of simultaneously stabilizing feedback. Furthermore, we give an explicit formula for constructing the simultaneously stabilizing controller. It is shown that, under the small control property (SCP) assumption, the obtained feedback is continuous everywhere in the state space.

2 Problem formulation and preliminaries

2.1 Problem formulation

Consider a collection of m quadratic input nonlinear systems described as follows:

$$S_i : \dot{x} = f_i(x) + ug_i(x) + u^2h_i(x), \quad i \in \{1, \dots, m\} = I, \quad (2.1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control, $f_i(\cdot)$, $g_i(\cdot)$ and $h_i(\cdot)$ are continuous functions with $f_i(0) = 0$.

The objective of this paper is to find a continuous function $k(\cdot)$ such that the state feedback controller $u = k(x)$ globally asymptotically stabilizes the collection of systems in (2.1) simultaneously, that is, the origin is a globally asymptotically stable equilibrium for the resulting closed-loop systems

$$S_i : \dot{x} = f_i(x) + k(x)g_i(x) + k(x)^2h_i(x), \quad i \in I. \quad (2.2)$$

2.2 Preliminaries

We first give some notations and basic definitions concerning control Lyapunov functions, then we recall some results on the stabilization of quadratic input nonlinear systems.

A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be positive definite if $V(x) \geq 0$ and $V(x) = 0 \iff x = 0$. V is said to be proper if for any $\rho \geq 0$, the set $\{x \in \mathbb{R}^n : V(x) \leq \rho\}$ is a compact set of \mathbb{R}^n .

For each $i = 1, 2, \dots, m$, let V_i be a C^1 positive definite and proper function defined on \mathbb{R}^n . Its derivative, denoted by \dot{V}_i , along the trajectories of system S_i can be written as

$$\dot{V}_i(x) = a_i(x)u^2 + b_i(x)u + c_i(x), \quad (2.3)$$

with, $a_i = \langle h_i, \nabla V_i \rangle$, $b_i = \langle g_i, \nabla V_i \rangle$, $c_i = \langle f_i, \nabla V_i \rangle$.

In the sequel, we will denote by $\Delta_i(x)$, $i \in I$, the discriminant of the right-hand side of (2.3), viewed as a polynomial in u , at every $x \in \mathbb{R}^n$, i.e., $\Delta_i(x) = b_i^2(x) - 4a_i(x)c_i(x)$. Moreover, when $\Delta_i(x) \geq 0$, we denote by $\lambda_{1i}(x)$ and $\lambda_{2i}(x)$ the two roots of the equation $a_i(x)\lambda^2 + b_i(x)\lambda + c_i(x) = 0$ given by

$$\lambda_{1i}(x) = \frac{-b_i(x) - \sqrt{\Delta_i(x)}}{2a_i(x)},$$

$$\lambda_{2i}(x) = \frac{-b_i(x) + \sqrt{\Delta_i(x)}}{2a_i(x)}.$$

We start by recalling the following definitions from [1, 17, 23].

Definition 2.1. ([1, 17, 23]) For $i \in I$. A C^1 positive definite and proper function V_i , defined on \mathbb{R}^n is a control Lyapunov function (CLF) for system S_i if for any $x \in \mathbb{R}^n \setminus \{0\}$

$$\inf_{u \in \mathbb{R}} \dot{V}_i(x) = \inf_{u \in \mathbb{R}} \left(a_i(x)u^2 + b_i(x)u + c_i(x) \right) < 0.$$

A CLF V_i for System S_i is said to satisfy the small control property (SCP), if for each $\varepsilon > 0$ there is $\delta > 0$ such that, if $x \neq 0$ satisfies $\|x\| < \delta$, then there is some u with $\|u\| < \varepsilon$ such that $\dot{V}_i(x) = a_i(x)u^2 + b_i(x)u + c_i(x) < 0$.

As far as possible, our objective is to explicitly design a common state feedback law u , which is continuous on $\mathbb{R}^n \setminus \{0\}$, and is such that the resulting closed-loop systems (2.2) are globally asymptotically stable. To be more precise, if V_i is a known collection of CLFs for the systems S_i , $i \in I$, we shall construct a feedback law u such that for each $i \in I$, V_i is a Lyapunov function for the closed-loop system $\dot{x} = f_i(x) + u(x)g_i(x) + u^2(x)h_i(x)$. This means to construct a continuous feedback such that, for $x \neq 0$, $i \in I$,

$$\dot{V}_i(x) = a_i(x)u^2(x) + b_i(x)u(x) + c_i(x) < 0. \quad (2.4)$$

Remark 2.1. I. Since V_i is a CLF for system S_i , $i \in I$, it is easy to see that for $x \neq 0$, if $a_i(x) > 0$, then $\Delta_i(x) > 0$.

II. Inequality (2.4) shows that in the setting of simultaneous quadratic input nonlinear systems, a continuous feedback u simultaneously stabilizes the collection of systems in (2.1) has to verify for each $i \in I$,

1. $u(x) \in]\lambda_{1i}(x), \lambda_{2i}(x)[$, if $a_i(x) > 0$,
2. $u(x) < -c_i(x)/b_i(x)$, if $a_i(x) = 0$ and $b_i(x) > 0$,
3. $u(x) > -c_i(x)/b_i(x)$, if $a_i(x) = 0$ and $b_i(x) < 0$,
4. $u(x) \notin [\lambda_{2i}(x), \lambda_{1i}(x)]$, if $a_i(x) < 0$ and $\Delta_i(x) \geq 0$.

In what follows, we recall some results (from [15]) that will be used in this paper.

Lemma 2.1. ([15]) Assume that V_i is a CLF for system S_i , $i \in I$. Then the following assertions hold:

- (1) the inequality $\Delta_i(x) < 0$ implies $a_i(x) < 0$;
- (2) $\Delta_i(x) = 0$, $a_i(x) \neq 0$ implies $a_i(x) < 0$;
- (3) $\Delta_i(x) = 0$, $a_i(x) = 0$ implies $b_i(x) = 0$ and $c_i(x) < 0$.

Lemma 2.2. ([15]) Assume that V_i is a CLF for System S_i , $i \in I$. Then for any $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that $a_i(x_0) = 0$, one has:

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{1i}(x) = -\frac{c_i(x_0)}{b_i(x_0)}, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{2i}(x) = +\infty \text{ if } b_i(x_0) < 0. \quad (2.5)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0}} \lambda_{1i}(x) = -\frac{c_i(x_0)}{b_i(x_0)}, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0}} \lambda_{2i}(x) = -\infty \text{ if } b_i(x_0) < 0. \quad (2.6)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{1i}(x) = -\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{2i}(x) = +\infty \text{ if } b_i(x_0) = 0. \quad (2.7)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0 \\ b_i(x) < 0}} \lambda_{1i}(x) = -\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0 \\ b_i(x) < 0}} \lambda_{2i}(x) = -\infty \text{ if } b_i(x_0) = 0. \quad (2.8)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0 \\ b_i(x) > 0}} \lambda_{1i}(x) = +\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0 \\ b_i(x) > 0}} \lambda_{2i}(x) = +\infty \text{ if } b_i(x_0) = 0. \quad (2.9)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{1i}(x) = -\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) > 0}} \lambda_{2i}(x) = -\frac{c_i(x_0)}{b_i(x_0)} \text{ if } b_i(x_0) > 0. \quad (2.10)$$

$$\lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0}} \lambda_{1i}(x) = +\infty, \text{ and } \lim_{\substack{x \rightarrow x_0 \\ a_i(x) < 0}} \lambda_{2i}(x) = -\frac{c_i(x_0)}{b_i(x_0)} \text{ if } b_i(x_0) > 0. \quad (2.11)$$

Theorem 2.1. ([15]) For $i \in I$, assume V_i is a CLF for System S_i , and satisfies the following assumption:

Assumption 2.1. For every $x \in \mathbb{R}^n \setminus \{0\}$, $(a_i(x) < 0 \text{ and } \Delta_i(x) \geq 0) \implies b_i(x) \neq 0$.

Then, the feedback

$$k_i(x) = \begin{cases} v_i(x), & \text{if } \Delta_i(x) > 0, \\ 0, & \text{if } \Delta_i(x) \leq 0, \end{cases} \quad (2.12)$$

where

$$v_i(x) = \begin{cases} \min \left(\lambda_{2i}(x) - \frac{\sqrt{\Delta_i(x)}}{2(1+a_i^2(x))}, 0 \right) & \text{if } a_i(x) \neq 0 \text{ and } b_i(x) > 0, \\ \max \left(\lambda_{1i}(x) + \frac{\sqrt{\Delta_i(x)}}{2(1+a_i^2(x))}, 0 \right) & \text{if } a_i(x) \neq 0 \text{ and } b_i(x) < 0, \\ \min \left(-\frac{c_i(x)}{b_i(x)} - \frac{b_i(x)}{2}, 0 \right) & \text{if } a_i(x) = 0 \text{ and } b_i(x) > 0, \\ \max \left(-\frac{c_i(x)}{b_i(x)} - \frac{b_i(x)}{2}, 0 \right) & \text{if } a_i(x) = 0 \text{ and } b_i(x) < 0, \\ 0 & \text{if } a_i(x) > 0 \text{ and } b_i(x) = 0, \end{cases}$$

is continuous on $\mathbb{R}^n \setminus \{0\}$ and globally asymptotically stabilizes the system S_i . In addition, if V_i satisfies the small control property (SCP), then k_i is continuous all over \mathbb{R}^n .

Hereafter we give a result that shows how to verify the small control property (SCP).

Lemma 2.3. For $i \in I$, assume V_i is a CLF for System S_i . Then V_i satisfy the SCP if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in B(0, \delta) \setminus \{0\}$ and $\Delta_i(x) \geq 0$, we have

$$\begin{aligned} \lambda_{1i}(x) < \varepsilon \text{ and } -\varepsilon < \lambda_{2i}(x) & \text{ if } a_i(x) > 0, \\ \lambda_{2i}(x) < \lambda_{1i}(x) < \varepsilon & \text{ if } a_i(x) < 0 \text{ and } b_i(x) < 0, \\ -\varepsilon < \lambda_{2i}(x) < \lambda_{1i}(x) & \text{ if } a_i(x) < 0 \text{ and } b_i(x) > 0, \\ -\lambda_{2i}(x) = \lambda_{1i}(x) < \varepsilon & \text{ if } a_i(x) < 0 \text{ and } b_i(x) = 0, \\ -\frac{c_i(x)}{b_i(x)} < \varepsilon & \text{ if } a_i(x) = 0 \text{ and } b_i(x) < 0, \\ -\varepsilon < -\frac{c_i(x)}{b_i(x)} & \text{ if } a_i(x) = 0 \text{ and } b_i(x) > 0. \end{aligned}$$

Proof. The proof is an immediate consequence of the definition of the SCP and the study of the sign of the polynomial in u : $a_i(x)u^2 + b_i(x)u + c_i(x)$, according to the sign of $a_i(x)$, x in a neighborhood of the origin.

Note that for the second assertion, we take into account the fact that $\lambda_{2i}(x) < 0$ and $|\lambda_{1i}(x)| < |\lambda_{2i}(x)|$ if $a_i(x) < 0$ and $b_i(x) < 0$. Similarly, for the third one, we use the fact that $\lambda_{1i}(x) > 0$ and $|\lambda_{2i}(x)| < \lambda_{1i}(x)$ if $a_i(x) < 0$ and $b_i(x) > 0$. \square

Remark 2.2. Notice that if V_i , $i \in I$, is a CLF satisfying Assumption 2.1, then the set $\{x \in \mathbb{R}^n \mid a_i(x) < 0, \Delta_i(x) \geq 0, b_i(x) = 0\}$ is empty. So, in our case, to verify the SCP, it is sufficient to find functions $d_{ji} : \mathbb{R}^n \rightarrow \mathbb{R}$, $j = 1, \dots, 5$, continuous in a neighborhood of the origin and $d_{ji}(0) = 0$, such that for $x \neq 0$ close enough to 0 and $\Delta_i(x) \geq 0$, one has:

$$\begin{aligned} \lambda_{1i}(x) < d_{1i}(x) < \lambda_{2i}(x) & \quad \text{if } a_i(x) > 0, \\ \lambda_{1i}(x) < d_{2i}(x) & \quad \text{if } a_i(x) < 0 \text{ and } b_i(x) < 0, \\ d_{3i}(x) < \lambda_{2i}(x) & \quad \text{if } a_i(x) < 0 \text{ and } b_i(x) > 0, \\ -\frac{c_i(x)}{b_i(x)} < d_{4i}(x) & \quad \text{if } a_i(x) = 0 \text{ and } b_i(x) < 0, \\ d_{5i}(x) < -\frac{c_i(x)}{b_i(x)} & \quad \text{if } a_i(x) = 0 \text{ and } b_i(x) > 0. \end{aligned}$$

3 Main results: Design method of simultaneously stabilizing controller

As a first step towards the construction of a simultaneously stabilizing feedback law for the collection of quadratic systems (2.1), and for easily presenting the main results, we define the following sets and functions: For each $i \in I$, suppose that V_i is a CLF for System S_i and let for each $x \in \mathbb{R}^n$

$$I_1(x) = \{i \in I \mid a_i(x) > 0\},$$

$$I_2(x) = \{i \in I \mid a_i(x) < 0, b_i(x) < 0, \Delta_i(x) \geq 0\},$$

$$I_3(x) = \{i \in I \mid a_i(x) < 0, b_i(x) > 0, \Delta_i(x) \geq 0\},$$

$$I_4(x) = \{i \in I \mid a_i(x) = 0, b_i(x) < 0\}$$

$$I_5(x) = \{i \in I \mid a_i(x) = 0, b_i(x) > 0\}.$$

$$\varphi_{1i}(x) = \begin{cases} \lambda_{1i}(x) & \text{if } i \in I_1 \cup I_2, \\ -\frac{c_i(x)}{b_i(x)} & \text{if } i \in I_4, \\ -\infty & \text{otherwise,} \end{cases}$$

$$\varphi_{2i}(x) = \begin{cases} \lambda_{2i}(x) & \text{if } i \in I_1 \cup I_3, \\ -\frac{c_i(x)}{b_i(x)} & \text{if } i \in I_5, \\ +\infty & \text{otherwise,} \end{cases}$$

$$\psi_{1i}(x) = \begin{cases} k_i(x) & \text{if } b_i(x) < 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{2i}(x) = \begin{cases} k_i(x) & \text{if } b_i(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\varphi_1(x) = \max_{i \in I} \varphi_{1i}(x), \quad \varphi_2(x) = \min_{i \in I} \varphi_{2i}(x),$$

$$\psi_1(x) = \max_{i \in I} \psi_{1i}(x), \quad \psi_2(x) = \min_{i \in I} \psi_{2i}(x).$$

Finally, define

$$w_1(x) = \min(\psi_1(x), \varphi_2(x)) \quad \text{and} \quad w_2(x) = \max(\psi_2(x), \varphi_1(x)).$$

Remark 3.1. Thanks to definitions of ψ_{1i} and ψ_{2i} , it is easy to see that $\psi_{1i}(x) \geq 0$ and $\psi_{2i}(x) \leq 0$ for all $x \in \mathbb{R}^n$ and for all $i \in I$. Thus, $\psi_1(x) \geq 0$ and $\psi_2(x) \leq 0$, $\forall x \in \mathbb{R}^n$.

In preparation for the construction of a continuous simultaneously stabilizing control u , we need the following lemma.

Lemma 3.1. Consider the collection of systems in (2.1). Suppose that there is a CLF V_i , satisfying Assumption 2.1, for every S_i , $i \in I$. Then the functions ψ_1 and ψ_2 are continuous on $\mathbb{R}^n \setminus \{0\}$. Moreover, if the functions V_i , $i \in I$, satisfy the SCP, then ψ_1 and ψ_2 are continuous on \mathbb{R}^n .

Proof. By Assumption 2.1 and the definition of k_i (2.12), $i \in I$, we have $k_i(x) = 0$ whenever $b_i(x) = 0$. From this and the continuity of k_i , it then follows that ψ_{1i} and ψ_{2i} , $i \in I$, are continuous on $\mathbb{R}^n \setminus \{0\}$, then so are ψ_1 and ψ_2 . If in addition, the functions V_i , $i \in I$, satisfy the SCP, then k_i , $i \in I$, are continuous on \mathbb{R}^n . So, as above, ψ_{1i} and ψ_{2i} , $i \in I$, are continuous on \mathbb{R}^n since $k_i(0) = b_i(0) = 0$, $i \in I$. Consequently, ψ_1 and ψ_2 are also continuous on \mathbb{R}^n . This completes the proof of Lemma 3.1. \square

To easily present the following results, define the sets

$$A_1(x) = \{i \in I \mid (a_i(x) < 0, b_i(x) > 0, \Delta_i(x) > 0) \\ \text{or } (a_i(x) > 0) \text{ or } (a_i(x) = 0, b_i(x) > 0)\},$$

$$A_2(x) = \{i \in I \mid (a_i(x) \leq 0, b_i(x) \leq 0) \\ \text{or } (a_i(x) < 0, b_i(x) > 0, \Delta_i(x) < 0)\},$$

$$A_3(x) = \{i \in I \mid a_i(x) < 0, b_i(x) > 0, \Delta_i(x) = 0\},$$

$$B_1(x) = \{i \in I \mid (a_i(x) < 0, b_i(x) < 0, \Delta_i(x) > 0) \\ \text{or } (a_i(x) > 0) \text{ or } (a_i(x) = 0, b_i(x) < 0)\},$$

$$B_2(x) = \{i \in I \mid (a_i(x) \leq 0, b_i(x) \geq 0) \\ \text{or } (a_i(x) < 0, b_i(x) < 0, \Delta_i(x) < 0)\},$$

$$B_3(x) = \{i \in I \mid a_i(x) < 0, b_i(x) < 0, \Delta_i(x) = 0\}.$$

Clearly, $I = A_1(x) \cup A_2(x) \cup A_3(x) = B_1(x) \cup B_2(x) \cup B_3(x)$.

For convenience, define

$$A_{31}(x) = \{i \in A_3(x) \mid \forall \eta > 0, \exists y \in B(x, \eta) \setminus \{x\} \text{ such that, } \Delta_i(y) \geq 0\},$$

and $A_{32}(x) = A_3(x) \setminus A_{31}(x)$.

We can, similarly, define $B_{31}(x)$ and $B_{32}(x)$.

For guaranteeing the continuity of the simultaneous feedback laws (which will be derived later) for the collection of quadratic input nonlinear systems in (2.1), here we assume

Assumption 3.1. *If $A_3(x) \neq \emptyset$,*

$$\min\left(\psi_1(x), \min_{i \in A_1(x)} \varphi_{2i}(x)\right) \leq \min_{j \in A_3(x)} -\frac{b_j(x)}{2a_j(x)}.$$

Assumption 3.2. *If $B_3(x) \neq \emptyset$,*

$$\max_{j \in B_3(x)} -\frac{b_j(x)}{2a_j(x)} \leq \max\left(\psi_2(x), \max_{i \in B_1(x)} \varphi_{1i}(x)\right).$$

Lemma 3.2. *Consider the collection of systems in (2.1). Suppose that there is a CLF V_i , satisfying Assumption 2.1 for every S_i , $i \in I$ and Assumptions 3.1, 3.2 hold, Then the functions w_1 and w_2 are continuous on $\mathbb{R}^n \setminus \{0\}$. If in addition the functions V_i , $i \in I$, satisfy the small control property, then w_1 and w_2 are continuous on \mathbb{R}^n .*

Proof. To establish the continuity of w_1 at $x_0 \in \mathbb{R}^n \setminus \{0\}$, we will verify that the functions φ_{2i} , $i \in A_1(x_0)$, are continuous on a neighborhood of x_0 and:

- If $A_{31}(x_0) = \emptyset$, then $w_1(x) = \min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x))$ on a neighborhood of x_0 . The conclusion follows since ψ_1 and φ_{2i} , $i \in A_1(x_0)$, are continuous on a neighborhood of x_0 .
- If $A_{31}(x_0) \neq \emptyset$, then $w_1(x) = \min_{k \in A_{31}(x_0)} w_1^k(x)$, with w_1^k , $k \in A_{31}(x_0)$ are continuous functions on a neighborhood of x_0 .

To do this, we first study the behavior of the functions φ_{2i} , $i \in I$, in a neighborhood of x_0 according to the sign of $a_i(x_0)$, $i \in I$.

Case 1: If $a_i(x_0) > 0$, by continuity of the function a_i , $a_i(x) > 0$ in a neighborhood \mathcal{V}_{x_0} of x_0 . Then $\varphi_{2i}(x) = \lambda_{2i}(x)$ on \mathcal{V}_{x_0} , so φ_{2i} is continuous at x_0 since λ_{2i} is continuous on \mathcal{V}_{x_0} .

Case 2: If $a_i(x_0) = 0$ and $b_i(x_0) > 0$, then $\Delta_i(x_0) > 0$. By continuity, $b_i(x) > 0$ and $\Delta_i(x) > 0$ in a neighborhood \mathcal{V}_{x_0} of x_0 , together with (2.10), (2.11) and definition of φ_{2i} , we can easily verify that φ_{2i} is continuous at x_0 .

Case 3: If $a_i(x_0) = 0$ and $b_i(x_0) = 0$. Let x^j be a sequence of states that converges to x_0 and let us review the different possibilities according to the definition of φ_{2i} .

- If the sequence x^j satisfies $a_i(x^j) > 0$, it then follows from (2.7) and definition of φ_{2i}

$$\lim_{x^j \rightarrow x_0} \varphi_{2i}(x^j) = \lim_{x^j \rightarrow x_0} \lambda_{2i}(x^j) = +\infty.$$

- If the sequence x^j satisfies $a_i(x^j) < 0$, $b_i(x^j) > 0$ and $\Delta_i(x^j) \geq 0$, (2.9) yields

$$\lim_{x^j \rightarrow x_0} \varphi_{2i}(x^j) = +\infty.$$

- If the sequence x^j satisfies $a_i(x^j) = 0$ and $b_i(x^j) > 0$, recall that since V_i is a CLF of System S_i , we have $c_i(x_0) < 0$, hence by continuity, $c_i(x^j) < 0$ whenever x^j is close enough to x_0 . So

$$\lim_{x^j \rightarrow x_0} \varphi_{2i}(x^j) = \lim_{x^j \rightarrow x_0} -\frac{c_i(x^j)}{b_i(x^j)} = +\infty.$$

- If the sequence x^j does not satisfy any of the above situations, then we have $\varphi_{2i}(x) = +\infty$.

Therefore, in all cases, i.e., if $a_i(x_0) = 0$ and $b_i(x_0) = 0$, the following holds

$$\forall L > 0, \exists \lambda_L^i > 0 \text{ such that } \varphi_{2i}(x) > L, \forall x \in B(x_0, \lambda_L^i).$$

Case 4: If $a_i(x_0) = 0$ and $b_i(x_0) < 0$, by continuity, $b_i(x) < 0$ in a neighborhood \mathcal{V}_{x_0} of x_0 , so, $\varphi_{2i}(x) = +\infty$ on \mathcal{V}_{x_0} .

Case 5: If $a_i(x_0) < 0$, $b_i(x_0) > 0$ and $\Delta_i(x_0) > 0$, this case is similar to the Case 1.

Case 6: If $a_i(x_0) < 0$, $b_i(x_0) > 0$ and $\Delta_i(x_0) = 0$, that is $i \in A_3(x_0)$. If $i \in A_{32}(x_0)$, then there exists $\delta_i > 0$ such that $\Delta_i(x) < 0$ for all $x \in B(x_0, \delta_i) \setminus \{x_0\}$. So, $\varphi_{2i}(x) = +\infty$ on $B(x_0, \delta_i) \setminus \{x_0\}$. If $i \in A_{31}(x_0)$, then for x close enough to x_0 we have, $\varphi_{2i}(x) = +\infty$ if $\Delta_i(x) < 0$ and $\varphi_{2i}(x) = \lambda_{2i}(x)$ if $\Delta_i(x) \geq 0$.

Case 7: If $a_i(x_0) < 0$, $b_i(x_0) > 0$ and $\Delta_i(x_0) < 0$, by continuity, $\Delta_i(x) < 0$ in a neighborhood \mathcal{V}_{x_0} of x_0 , so, $\varphi_{2i}(x) = +\infty$ on \mathcal{V}_{x_0} .

Case 8: If $a_i(x_0) < 0$, $b_i(x_0) = 0$, this case is similar to Case 7 since from Assumption 2.1, we have $\Delta_i(x_0) < 0$.

Case 9: If $a_i(x_0) < 0$, $b_i(x_0) < 0$, this case is similar to Case 4. Summarizing the above, we have:

- For $i \in A_1(x_0)$, the discussions of Cases 1, 2, 5 indicate that there exists $\varepsilon_1^i > 0$ such that, the function φ_{2i} is continuous on $B(x_0, \varepsilon_1^i)$. Take $\varepsilon_1 = \min_{i \in A_1(x_0)} \varepsilon_1^i$, it follows that the functions φ_{2i} , $i \in A_1(x_0)$, are continuous on $B(x_0, \varepsilon_1)$.

- For $i \in A_2(x_0)$, Cases 3, 4, 7, 8, 9 imply that the following holds

$$\forall L > 0, \exists \lambda_L^i > 0, \text{ s. t. } \varphi_{2i}(x) > L, \forall x \in B(x_0, \lambda_L^i).$$

Choosing $\lambda_L = \min_{i \in A_2(x_0)} \lambda_L^i$, it follows that

$$\forall L > 0, \exists \lambda_L > 0, \text{ such that } \varphi_{2i}(x) > L, \forall x \in B(x_0, \lambda_L), \forall i \in A_2(x_0). \quad (3.1)$$

- For $i \in A_{32}(x_0)$, Case 6 indicates that there exists $\delta_i > 0$ such that $\varphi_{2i}(x) = +\infty$ on $B(x_0, \delta_i) \setminus \{x_0\}$. Take $\delta = \min_{i \in A_{32}(x_0)} \delta_i$, it follows that

$$\varphi_{2i}(x) = +\infty, \forall x \in B(x_0, \delta) \setminus \{x_0\}, \forall i \in A_{32}(x_0).$$

From this and Assumption 3.1, we obtain, by taking $\mu = \min(\varepsilon_1, \delta)$, for all $x \in B(x_0, \mu)$

$$\min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x), \min_{i \in A_{32}(x_0)} \varphi_{2i}(x)) = \min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x)). \quad (3.2)$$

Let $\varepsilon > 0$ and take $L = |\psi_1(x_0)| + \varepsilon$. The continuity of the function ψ_1 and inequality (3.1) ensure the existence of $\lambda_L^\varepsilon > 0$ such that for all $x \in B(x_0, \lambda_L^\varepsilon)$

$$\psi_1(x) < |\psi_1(x_0)| + \varepsilon = L < \varphi_{2i}(x), \forall i \in A_2(x_0). \quad (3.3)$$

In terms of (3.2) and (3.3), choosing $\eta = \min(\mu, \lambda_L^\varepsilon)$, we have for all $x \in B(x_0, \eta)$

$$\min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x), \min_{i \in A_2(x_0)} \varphi_{2i}(x), \min_{i \in A_{32}(x_0)} \varphi_{2i}(x)) = \min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x)). \quad (3.4)$$

Now, if $A_{31}(x_0) = \emptyset$. From (3.4), we get

$$w_1(x) = \min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x)), \forall x \in B(x_0, \eta).$$

Thus, w_1 is continuous at x_0 since ψ_1 and φ_{2i} , $i \in A_1(x_0)$, are continuous on $B(x_0, \eta)$.

If $A_{31}(x_0) \neq \emptyset$. For each $i \in A_{31}(x_0)$, we have $a_i(x_0) < 0$ and $b_i(x_0) > 0$. For reasons of continuity, $a_i(x) < 0$ and $b_i(x) > 0$ in a neighborhood $\mathcal{W}_{x_0}^i$ of x_0 . Recall that φ_{2i} , $i \in A_1(x_0)$, are continuous on $B(x_0, \varepsilon_1)$ and let

$$\mathcal{W}_{x_0} = \bigcap_{i \in A_3(x_0)} \mathcal{W}_{x_0}^i \cap B(x_0, \varepsilon_1).$$

Now, let $k \in A_{31}(x_0)$ and for convenience, define the function w_1^k on \mathcal{W}_{x_0} by:

$$w_1^k(x) = \min(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x), \varphi_{2k}(x)), \quad x \in \mathcal{W}_{x_0}.$$

Note that $\varphi_{2k}(x_0) = -\frac{b_k(x_0)}{2a_k(x_0)}$, so, Assumption 3.1 yields

$$w_1^k(x_0) = \min(\psi_1(x_0), \min_{i \in A_1(x_0)} \varphi_{2i}(x_0)).$$

Let $x^j \in \mathcal{W}_{x_0}$ and $y^j \in \mathcal{W}_{x_0}$ be two sequences of states that converges to x_0 satisfying $\Delta_k(x^j) \geq 0$ and $\Delta_k(y^j) < 0$. By definition of φ_{2k} , we have

$$\lim_{x^j \rightarrow x_0} \varphi_{2k}(x^j) = \lim_{x^j \rightarrow x_0} \lambda_{2k}(x^j) = -\frac{b_k(x_0)}{2a_k(x_0)} = \varphi_{2k}(x_0). \quad (3.5)$$

and

$$\varphi_{2k}(y^j) = +\infty. \quad (3.6)$$

On the one hand, Assumption 3.1, equality (3.5) and the continuity of the functions ψ_1 and φ_{2i} , $i \in A_1(x_0)$, on \mathcal{W}_{x_0} , along with the definition of w_1^k , we have

$$\begin{aligned} \lim_{x^j \rightarrow x_0} w_1^k(x^j) &= \min\left(\psi_1(x_0), \min_{i \in A_1(x_0)} \varphi_{2i}(x_0), -\frac{b_k(x_0)}{2a_k(x_0)}\right) \\ &= \min\left(\psi_1(x_0), \min_{i \in A_1(x_0)} \varphi_{2i}(x_0)\right) = w_1^k(x_0). \end{aligned} \quad (3.7)$$

On the other hand, equality (3.6) yields $w_1^k(y^j) = \min\left(\psi_1(y^j), \min_{i \in A_1(x_0)} \varphi_{2i}(y^j)\right)$, which implies

$$\lim_{y^j \rightarrow x_0} w_1^k(y^j) = \min\left(\psi_1(x_0), \min_{i \in A_1(x_0)} \varphi_{2i}(x_0)\right) = w_1^k(x_0). \quad (3.8)$$

In terms of (3.7) and (3.8), the functions w_1^k , $k \in A_{31}(x_0)$, are continuous at x_0 .

Finally, using (3.4) and choosing $\mathcal{W}'_{x_0} = \mathcal{W}_{x_0} \cap B(x_0, \eta)$, along with the definitions of w_1 and w_1^k , $k \in A_{31}(x_0)$, we obtain

$$\begin{aligned} w_1(x) &= \min\left(\psi_1(x), \min_{i \in A_1(x_0)} \varphi_{2i}(x), \min_{k \in A_{31}(x_0)} \varphi_{2k}(x)\right) \\ &= \min_{k \in A_{31}(x_0)} w_1^k(x), \quad \forall x \in \mathcal{W}'_{x_0}. \end{aligned}$$

Thus, w_1 is continuous at x_0 since the functions w_1^k , $k \in A_{31}(x_0)$, are continuous at x_0 .

Now, we establish the continuity of w_1 at the origin in the case of the small control property of V_i , $i \in I$.

Far each $i \in I$, take $\varepsilon > 0$, due to the small control property, there exists $\alpha_i > 0$ such that, for each $x \neq 0$ and $\|x\| < \alpha_i$, there exists $v \in]-\varepsilon, \varepsilon[$ that satisfies the inequality

$$\dot{V}_i(x) = a_i(x)v^2 + b_i(x)v + c_i(x) < 0. \quad (3.9)$$

With this inequality in mind, we review the different items in the definition of φ_{2i} .

- If $a_i(x) > 0$, then (3.9) leads to $v \in]-\lambda_{1i}(x), \lambda_{2i}(x)[$, that is, $-\varepsilon < v < \lambda_{2i}(x) = \varphi_{2i}(x)$.
- If $a_i(x) = 0$ and $b_i(x) > 0$, then (3.9) leads to $-\varepsilon < v < -\frac{c_i(x)}{b_i(x)} = \varphi_{2i}(x)$.
- If $a_i(x) < 0$, $b_i(x) > 0$ and $\Delta_i(x) \geq 0$, then (3.9) leads to $v < \lambda_{2i}(x) \leq \lambda_{1i}(x)$ or $\lambda_{2i}(x) \leq \lambda_{1i}(x) < v$. But in this case, we have $\lambda_{1i}(x) > 0$ and $|\lambda_{2i}(x)| \leq \lambda_{1i}(x)$. So, we always have $-\varepsilon < -|v| < \lambda_{2i}(x) = \varphi_{2i}(x)$.
- Otherwise, $\varphi_{2i}(x) = +\infty$.

From the above discussion, we deduce that

$$\forall \varepsilon > 0, \exists \alpha_i > 0 \text{ such that, } -\varepsilon < \varphi_{2i}(x), \forall x \in B(0, \alpha_i).$$

Take $\alpha = \min_{i \in I} \alpha_i$, it follows

$$\forall \varepsilon > 0, \exists \alpha > 0 \text{ s. t., } -\varepsilon < \varphi_2(x), \forall x \in B(0, \alpha). \quad (3.10)$$

Continuity of ψ_1 at the origin leads to

$$\forall \varepsilon > 0, \exists \beta > 0 \text{ s. t., } -\varepsilon < \psi_1(x) < \varepsilon, \forall x \in B(0, \beta). \quad (3.11)$$

Take $\gamma = \min(\alpha, \beta)$, it follows from (3.10) and (3.11)

$$\forall \varepsilon > 0, \exists \gamma > 0 \text{ such that, } -\varepsilon < w_1(x) = \min(\psi_1(x), \varphi_2(x)) < \varepsilon, \forall x \in B(0, \gamma).$$

That is, w_1 is continuous at the origin and so on \mathbb{R}^n .

Using Assumptions 2.1, 3.2 and Lemma 2.2, the continuity of w_2 can be treated similarly and is omitted here. The proof of Lemma 3.2 is now completed. \square

Remark 3.2. Let $V_i, i \in I$ be a collection of CLFs for system S_i , respectively. It is easy to see that Assumptions 2.1, 3.1, 3.2 are trivial if $a_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $i \in I$. Indeed, in this case we have $A_3(x) = B_3(x) = \emptyset$ for all $x \in \mathbb{R}^n$ and, for each $i \in I$, the set $\{x \in \mathbb{R}^n, | a_i(x) < 0\}$ is empty. So, by Lemma 3.2, if $a_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $i \in I$, then the functions w_1 and w_2 are continuous on $\mathbb{R}^n \setminus \{0\}$ and continuous on \mathbb{R}^n if the functions V_i satisfy the SCP.

The following theorem presents a necessary condition for the existence of simultaneously asymptotically stabilizing state feedback for the collection of systems in (2.1).

Theorem 3.1. Consider the collection of systems in (2.1). If there exists a feedback $k : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for each $i \in I$, the closed-loop system $\dot{x} = f_i(x) + k(x)g_i(x) + k^2(x)h_i(x)$ is globally asymptotically stable, then there exists a collection of CLFs V_i for Systems S_i , respectively, satisfying for all $x \in \mathbb{R}^n$ such that $I_1(x) \cup I_4(x) \neq \emptyset$ and $I_1(x) \cup I_5(x) \neq \emptyset$

$$\max_{i \in I_1(x) \cup I_4(x)} \varphi_{1i}(x) < \min_{i \in I_1(x) \cup I_5(x)} \varphi_{2i}(x). \quad (3.12)$$

Proof. Suppose that there exists a feedback k such that the closed-loop systems $\dot{x} = f_i(x) + k(x)g_i(x) + k^2(x)h_i(x)$, $i \in I$, are globally asymptotically stable, then by converse Lyapunov theorem, there exist a collection of smooth, positive definite and radially unbounded functions V_i , $i \in I$, such that for each $i \in I$, $\dot{V}_i(x) = a_i(x)k^2(x) + b_i(x)k(x) + c_i(x) < 0$, for all $x \neq 0$. It then follows that for each $x \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} \lambda_{1i}(x) < k(x) < \lambda_{2i}(x) \text{ if } a_i(x) > 0, \\ k(x) > -\frac{c_i(x)}{b_i(x)} \text{ if } a_i(x) = 0 \text{ and } b_i(x) < 0, \\ k(x) < -\frac{c_i(x)}{b_i(x)} \text{ if } a_i(x) = 0 \text{ and } b_i(x) > 0. \end{aligned}$$

That is, $\varphi_{1i}(x) < k(x)$ if $i \in I_1(x) \cup I_4(x)$ and, $k(x) < \varphi_{2i}(x)$ if $i \in I_1(x) \cup I_5(x)$, which is equivalent, for all $x \in \mathbb{R}^n$ such that $I_1(x) \cup I_4(x) \neq \emptyset$ and $I_1(x) \cup I_5(x) \neq \emptyset$, to

$$\max_{i \in I_1(x) \cup I_4(x)} \varphi_{1i}(x) < k(x) < \min_{i \in I_1(x) \cup I_5(x)} \varphi_{2i}(x), \quad (3.13)$$

and this completes the proof of Theorem 3.1. \square

The necessary condition 3.12 shows that a simultaneously stabilizing feedback u for the collection of systems in (2.1) verify items 1, 2 and 3 in Remark 2.1. The feedback u has also to verify item 4 in Remark 2.1, that is $u(x) < \lambda_{2i}(x) \leq \lambda_{1i}(x)$ or $\lambda_{2i}(x) \leq \lambda_{1i}(x) < u(x)$ for each $i \in I$ such that $a_i(x) < 0$ and $\Delta_i(x) \geq 0$.

But the following situation may occur: There exists an open set $\mathcal{V} \subset \mathbb{R}^n$ and $i \in I$ such that:

- (1) $a_i(x) < 0$, $b_i(x) > 0$ and $\Delta_i(x) \geq 0$, for every $x \in \mathcal{V}$;
- (2) there exists $x_0 \in \partial\mathcal{V}$ (the border of \mathcal{V}) such that $a_i(x_0) = 0$ and $b_i(x_0) > 0$.

Let $x_1 \in \mathcal{V}$, if we choose $\lambda_{2i}(x_1) \leq \lambda_{1i}(x_1) < u(x_1)$, for reasons of continuity we have $\lambda_{2i}(x) \leq \lambda_{1i}(x) < u(x)$ for all $x \in \mathcal{V}$. From this and using (2.11) we get,

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \mathcal{V}}} u(x) \geq \lim_{\substack{x \rightarrow x_0 \\ x \in \mathcal{V}}} \lambda_{1i}(x) = +\infty$$

which contradict the fact that u is continuous at x_0 . To eliminate this situation, we have to choose $u(x) < \lambda_{2i}(x) \leq \lambda_{1i}(x)$ if $a_i(x) < 0$, $b_i(x) > 0$ and $\Delta_i(x) \geq 0$. Similarly, by using (2.6), we have to choose $\lambda_{2i}(x) \leq \lambda_{1i}(x) < u(x)$ if $a_i(x) < 0$, $b_i(x) < 0$ and $\Delta_i(x) \geq 0$. That is, for each $x \in \mathbb{R}^n \setminus \{0\}$

$$\lambda_{1i}(x) < u(x), \forall i \in I_2(x) \text{ and } u(x) < \lambda_{2i}(x), \forall i \in I_3(x). \quad (3.14)$$

(3.14) and definitions of φ_{1i} and φ_{2i} lead to, for each $x \in \mathbb{R}^n$

$$\max_{i \in I_2(x)} \varphi_{1i}(x) < u(x) < \min_{i \in I_3(x)} \varphi_{2i}(x). \quad (3.15)$$

The discussion above and Theorem 3.1 (condition (3.12)) show that in order to construct a continuous feedback u that simultaneously stabilizes the collection of systems in (2.1), the feedback u should verify (3.13) and (3.15), which means for all $x \in \mathbb{R}^n$

$$\max_{i \in I_N(x)} \varphi_{1i}(x) < u(x) < \min_{i \in I_P(x)} \varphi_{2i}(x), \quad (3.16)$$

with, $I_N(x) = I_1(x) \cup I_2(x) \cup I_4(x)$, $I_P(x) = I_1(x) \cup I_3(x) \cup I_5(x)$.

Therefore, to guarantee that the feedback that will be constructed below verify (3.16), we assume

$$\max_{i \in I_N(x)} \varphi_{1i}(x) < \min_{i \in I_P(x)} \varphi_{2i}(x), \forall x \in \mathbb{R}^n. \quad (3.17)$$

Remark 3.3. Note that for each $x \in \mathbb{R}^n$, since $\varphi_{1i}(x) = -\infty$ for all $i \in I \setminus I_N(x)$ and $\varphi_{2i}(x) = +\infty$ for all $i \in I \setminus I_P(x)$, then inequality (3.17) is equivalent to the following assumption.

Assumption 3.3. For every $x \in \mathbb{R}^n$, $\varphi_1(x) < \varphi_2(x)$.

Remark 3.4. If the collection of CLFs in Theorem 3.1 verify $a_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $i \in I$, then $I_2(x) = I_3(x) = \emptyset$ for all $x \in \mathbb{R}^n$. So, inequalities (3.12) and (3.17) are equivalent. That is, by Theorem 3.1 and Remark 3.3, Assumption 3.3 becomes necessary in this case.

Now, we can state our main result that gives the simultaneous stabilizing feedback law.

Theorem 3.2. Consider the collection of systems in (2.1). If there exists a collection of CLFs V_i , $i = 1, \dots, m$, satisfying Assumption 2.1, such that Assumptions 3.1, 3.2 and 3.3 hold, then the feedback

$$k(x) = \frac{w_1(x) + w_2(x)}{2} \quad (3.18)$$

is continuous on $\mathbb{R}^n \setminus \{0\}$ and globally asymptotically stabilizes the collection of systems in (2.1) simultaneously. Moreover, the feedback k is continuous on \mathbb{R}^n if the functions V_i , $i = 1, 2, \dots, m$, satisfy the small control property.

Proof. According to Lemmas 3.1 and 3.2, the continuity of k is immediate. For the asymptotic stability, we only have to verify that for each $i \in I$

$$\dot{V}_i(x) = a_i(x)k^2(x) + b_i(x)k(x) + c_i(x) < 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

according to the sign of $\Delta_i(x)$.

Case 1. If $\Delta_i(x) < 0$, then $a_i(x) < 0$ according to Lemma 2.1. Thus, the sign of $\dot{V}_i(x)$ regarded as a quadratic polynomial in k is that of $a_i(x)$. So, we have $\dot{V}_i(x) < 0$ in this case.

Case 2. If $\Delta_i(x) = 0$, from Lemma 2.1, we know that $a_i(x) = b_i(x) = 0$ and $c_i(x) < 0$ or $a_i(x) < 0$.

i. If $a_i(x) = b_i(x) = 0$ and $c_i(x) < 0$, then $\dot{V}_i(x) = c_i(x) < 0$.

ii. If $a_i(x) < 0$, from Assumption 2.1 we have $b_i(x) \neq 0$.

– If $b_i(x) > 0$. On the one hand, From the definitions of w_1 , φ_2 and φ_{2i} , we obtain

$$w_1(x) = \min(\psi_1(x), \varphi_2(x)) \leq \varphi_2(x) \leq \varphi_{2i}(x) = \lambda_{2i}(x) = -\frac{b_i(x)}{2a_i(x)}. \quad (3.19)$$

On the other hand, definitions of ψ_2 , ψ_{2i} and k_i give, $\psi_2(x) \leq \psi_{2i}(x) = k_i(x) < \varphi_{2i}(x) = \lambda_{2i}(x) = -\frac{b_i(x)}{2a_i(x)}$. From Assumption 3.3, we get $\varphi_1(x) < \varphi_2(x) \leq -\frac{b_i(x)}{2a_i(x)}$. Hence,

$$w_2(x) = \max(\psi_2(x), \varphi_1(x)) < \lambda_{2i}(x) = -\frac{b_i(x)}{2a_i(x)}. \quad (3.20)$$

(3.19) and (3.20) lead to

$$k(x) = \frac{w_1(x) + w_2(x)}{2} < \lambda_{2i}(x) = \lambda_{1i}(x) = -\frac{b_i(x)}{2a_i(x)}.$$

Thus, $\dot{V}_i(x) < 0$ since $a_i(x) < 0$.

– If $b_i(x) < 0$, then $\varphi_{1i}(x) = \lambda_{1i}(x) < k_i(x) = \psi_{1i}(x) \leq \psi_1(x)$. Assumption 3.3 yields $\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) < \varphi_2(x)$. So

$$\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) \leq \max(\psi_2(x), \varphi_1(x)) = w_2(x),$$

and

$$\lambda_{1i}(x) < \min(\psi_1(x), \varphi_2(x)) = w_1(x).$$

That is,

$$\lambda_{2i}(x) = \lambda_{1i}(x) < \frac{w_1(x) + w_2(x)}{2} = k(x).$$

Thus, $\dot{V}_i(x) < 0$ since $a_i(x) < 0$.

Case 3. If $\Delta_i(x) > 0$. In this case, we consider the three following cases according to the sign of $b_i(x)$.

i. If $b_i(x) < 0$, we consider again three cases according to the sign of $a_i(x)$.

- If $a_i(x) > 0$. On the one hand, since k_i stabilizes System S_i , we have $\lambda_{1i}(x) < k_i(x) = \psi_{1i}(x) \leq \psi_1(x)$. By Assumption 3.3, we get $\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) < \varphi_2(x)$. So $\lambda_{1i}(x) < \min(\psi_1(x), \varphi_2(x)) = w_1(x)$. Together with the fact that $w_1(x) \leq \varphi_2(x) \leq \varphi_{2i}(x) = \lambda_{2i}(x)$, it then follows that

$$\lambda_{1i}(x) < w_1(x) \leq \lambda_{2i}(x). \quad (3.21)$$

On the other hand, note that in this case $0 < \lambda_{2i}(x)$, hence $\psi_2(x) \leq \psi_{2i}(x) = 0 < \lambda_{2i}(x)$. As above, by Assumption 3.3, we get $\varphi_1(x) < \varphi_2(x) \leq \varphi_{2i}(x) = \lambda_{2i}(x)$. So $w_2(x) = \max(\psi_2(x), \varphi_1(x)) < \lambda_{2i}(x)$, and since $\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) \leq w_2(x)$, it then follows that

$$\lambda_{1i}(x) \leq w_2(x) < \lambda_{2i}(x). \quad (3.22)$$

(3.21) and (3.22) give

$$\lambda_{1i}(x) < \frac{w_1(x) + w_2(x)}{2} = k(x) < \lambda_{2i}(x).$$

Thus, $\dot{V}_i(x) < 0$ since $a_i(x) > 0$.

- If $a_i(x) = 0$. Then $-\frac{c_i(x)}{b_i(x)} < k_i(x) = \psi_{1i}(x) \leq \psi_1(x)$. Assumption 3.3, yields $-\frac{c_i(x)}{b_i(x)} = \varphi_{1i}(x) \leq \varphi_1(x) < \varphi_2(x)$. So $-\frac{c_i(x)}{b_i(x)} < w_1(x)$ and $-\frac{c_i(x)}{b_i(x)} \leq w_2(x)$ since $\varphi_1(x) \leq w_2(x)$. It then follows that

$$-\frac{c_i(x)}{b_i(x)} < \frac{w_1(x) + w_2(x)}{2} = k(x).$$

Thus, $\dot{V}_i(x) = b_i(x)k(x) + c_i(x) < 0$ since $b_i(x) < 0$.

- If $a_i(x) < 0$. As above, $\lambda_{1i}(x) < k_i(x) = \psi_{1i}(x) \leq \psi_1(x)$. Assumption 3.3 yields $\lambda_{1i}(x) = \varphi_{1i}(x) \leq \varphi_1(x) < \varphi_2(x)$. Hence $\lambda_{1i}(x) < w_1(x)$ and $\lambda_{1i}(x) \leq \varphi_1(x) \leq w_2(x)$. So

$$\lambda_{2i}(x) < \lambda_{1i}(x) < \frac{w_1(x) + w_2(x)}{2} = k(x).$$

That is, $\dot{V}_i(x) < 0$ since $a_i(x) < 0$.

ii. If $b_i(x) > 0$, this case can be treated similarly to the previous one.

iii. If $b_i(x) = 0$, by Assumption 2.1 and since $\Delta_i(x) > 0$, we have $a_i(x) > 0$, so, this case is similar to the case $a_i(x) > 0$ in Case 3, item i.

Thus, the proof of Theorem 3.2 is completed. □

Remark 3.5. *If the collection of CLFs in Theorem 3.2 verify $a_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $i \in I$, then, by Remark 3.2, the feedback (3.18) simultaneously stabilizes all the systems in (2.1) if only Assumption 3.3 holds.*

4 Illustrative example

In this section, we present an example to illustrate the effectiveness of the results obtained in this paper to design a simultaneously stabilizing feedback for some nonlinear quadratic systems.

Consider a system with the following two possible modes

$$S_1 : \begin{cases} \dot{x}_1 = -x_1 + \frac{x_1 x_2^4}{4} + \frac{x_1}{1+x_1^2} u^2 \\ \dot{x}_2 = -x_2 + x_1^2 x_2 u, \end{cases} \quad (4.1)$$

$$S_2 : \begin{cases} \dot{x}_1 = -x_1 + x_1^3 x_2^4 + (4x_1 - 4x_2)u - 4x_1 u^2 \\ \dot{x}_2 = -x_2 + x_1 u. \end{cases} \quad (4.2)$$

Let

$$V_1(x) = \frac{x_1^2 + x_2^2}{2} \quad \text{and} \quad V_2(x) = \frac{x_1^2 + 4x_2^2}{8}.$$

The Lie derivatives of V_1 and V_2 along the trajectories of Systems S_1 and S_2 , respectively, write:

$$\dot{V}_1(x) = a_1(x)u^2 + b_1(x)u + c_1(x),$$

$$\dot{V}_2(x) = a_2(x)u^2 + b_2(x)u + c_2(x),$$

with $a_1(x) = x_1^2/(1+x_1^2)$, $b_1(x) = x_1^2 x_2^2$, $c_1(x) = -x_1^2 - x_2^2 + (1/4)x_1^2 x_2^4$, $a_2(x) = -x_1^2$, $b_2(x) = x_1^2$, and $c_2(x) = (1/4)(-x_1^2 + x_1^4 x_2^4) - x_2^2$.

By the definitions in Section 2, we have

$$\Delta_1(x) = \frac{x_1^6 x_2^4 + 4x_1^2 x_2^2 + 4x_1^4}{1+x_1^2}, \quad \Delta_2(x) = x_1^2 x_2^2 (x_1^4 x_2^2 - 4),$$

$$\lambda_{11}(x) = \frac{-x_1^2 x_2^2 - x_2^2}{2} - \frac{\sqrt{(1+x_1^2)(x_1^6 x_2^4 + 4x_1^2 x_2^2 + 4x_1^4)}}{2x_1^2},$$

$$\lambda_{21}(x) = \frac{-x_1^2 x_2^2 - x_2^2}{2} + \frac{\sqrt{(1+x_1^2)(x_1^6 x_2^4 + 4x_1^2 x_2^2 + 4x_1^4)}}{2x_1^2},$$

$$\lambda_{12}(x) = \frac{1}{2} + \frac{\sqrt{x_1^2 x_2^2 (x_1^4 x_2^2 - 4)}}{2x_1^2},$$

$$\lambda_{22}(x) = \frac{1}{2} - \frac{\sqrt{x_1^2 x_2^2 (x_1^4 x_2^2 - 4)}}{2x_1^2}.$$

We can prove that V_1 and V_2 are CLFs for Systems S_1 and S_2 , respectively, and satisfy the small control property. In fact, V_2 satisfies the small control property since $\Delta_2(x) < 0$ whenever $x \neq 0$ is close enough to 0 and then $a_2(x) < 0$ if $x_1 \neq 0$ or $c_2(x) < 0$ if $a_2(x) = b_2(x) = 0$.

Using notations of the Section 3, we have

$$\begin{aligned} \varphi_1(x) &= \begin{cases} \lambda_{11}(x), & \text{if } x_1 \neq 0, \\ -\infty, & \text{if } x_1 = 0, \end{cases} \\ \varphi_2(x) &= \begin{cases} \min(\lambda_{21}(x), \lambda_{22}(x)), & \text{if } x_1^4 x_2^2 - 4 \geq 0 \text{ or } x_1 \neq 0 \text{ and } x_2 = 0, \\ \lambda_{21}(x), & \text{if } x_1^4 x_2^2 - 4 < 0 \text{ and } x_1 \neq 0 \text{ and } x_2 \neq 0, \\ +\infty, & \text{if } x_1 = 0, \end{cases} \\ \psi_1(x) &= 0, \quad \forall x \in \mathbb{R}^2, \\ \psi_2(x) &= \begin{cases} \min(\lambda_{21}(x) - \frac{\sqrt{\Delta_1(x)}}{2(1+a_1^2(x))}, \lambda_{22}(x) - \frac{\sqrt{\Delta_2(x)}}{2(1+a_2^2(x))}, 0), & \text{if } x_1 \neq 0, x_2 \neq 0, \\ \min(\lambda_{22}(x) - \frac{\sqrt{\Delta_2(x)}}{2(1+a_2^2(x))}, 0), & \text{if } x_1 \neq 0, x_2 = 0, \\ 0, & \text{if } x_1 = 0. \end{cases} \end{aligned}$$

It is not difficult to verify that Systems S_1 and S_2 satisfy Assumptions 2.1, 3.2.

To verify Assumption 3.1, define $S = \{x \in \mathbb{R}^2 \mid x_1 = 0 \text{ and } x_2 = 0, \text{ or } x_1^4 x_2^2 = 4\}$.

- For $x \notin S$, $A_3(x) = \emptyset$ and then Assumption 3.1 holds.
- For $x \in S$, we have $A_1(x) = \{1\}$, $A_3(x) = \{2\}$, $\psi_1(x) = 0$ and $-\frac{b_2(x)}{2a_2(x)} = \frac{1}{2}$. Hence, $\min(\psi_1(x), \varphi_{21}(x)) \leq \psi_1(x) < -\frac{b_2(x)}{2a_2(x)}$, that is Assumption 3.1 holds.

It remains to verify Assumption 3.3. To this end define $D_1 = \{x \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_1^4 x_2^2 < 4 \text{ and } x_1 x_2 \neq 0\}$ and $D_2 = \{x \in \mathbb{R}^2 \mid x_1 \neq 0 \text{ and } x_2 = 0 \text{ or } x_1^4 x_2^2 \geq 4\}$. Clearly, $\mathbb{R}^2 = D_1 \cup D_2$.

- For $x \in D_1$, from definitions of φ_1 and φ_2 , we have $\varphi_1(x) < \varphi_2(x)$.
- For $x \in D_2$, we have $\varphi_1(x) = \lambda_{11}(x)$ and $\varphi_2(x) = \min(\lambda_{21}(x), \lambda_{22}(x))$. Notice that $\lambda_{11}(x) < \lambda_{21}(x)$ for all $x \in D_2$. Notice also that since $-\sqrt{(1+x_1^2)(x_1^6 x_2^4 + 4x_1^2 x_2^2 + 4x_1^4)} \leq -\sqrt{x_1^2 x_2^2 (x_1^4 x_2^2 - 4)}$ for all $x \in D_2$, it follows that $\lambda_{11}(x) < \lambda_{22}(x)$ for all $x \in D_2$. Thus $\varphi_1(x) < \varphi_2(x)$ for all $x \in D_2$.

In both cases, $\varphi_1(x) < \varphi_2(x)$ and thus Assumption 3.3 holds.

Therefore, by Theorem 3.2, the feedback

$$u(x) = \frac{w_1(x) + w_2(x)}{2} \quad (4.3)$$

is continuous on \mathbb{R}^2 and globally asymptotically stabilizes the Systems S_1 and S_2 simultaneously, with $w_1(x) = \min(\psi_1(x), \varphi_2(x))$ and $w_2(x) = \max(\psi_2(x), \varphi_1(x))$.

The behaviors of the closed-loop systems (4.1) and (4.2) under the action of the simultaneously stabilizing feedback are drawn in Figures 1-2. The evolution of the stabilizing feedback along the closed-loop systems (4.1) and (4.2) is shown in Figures 3-4.

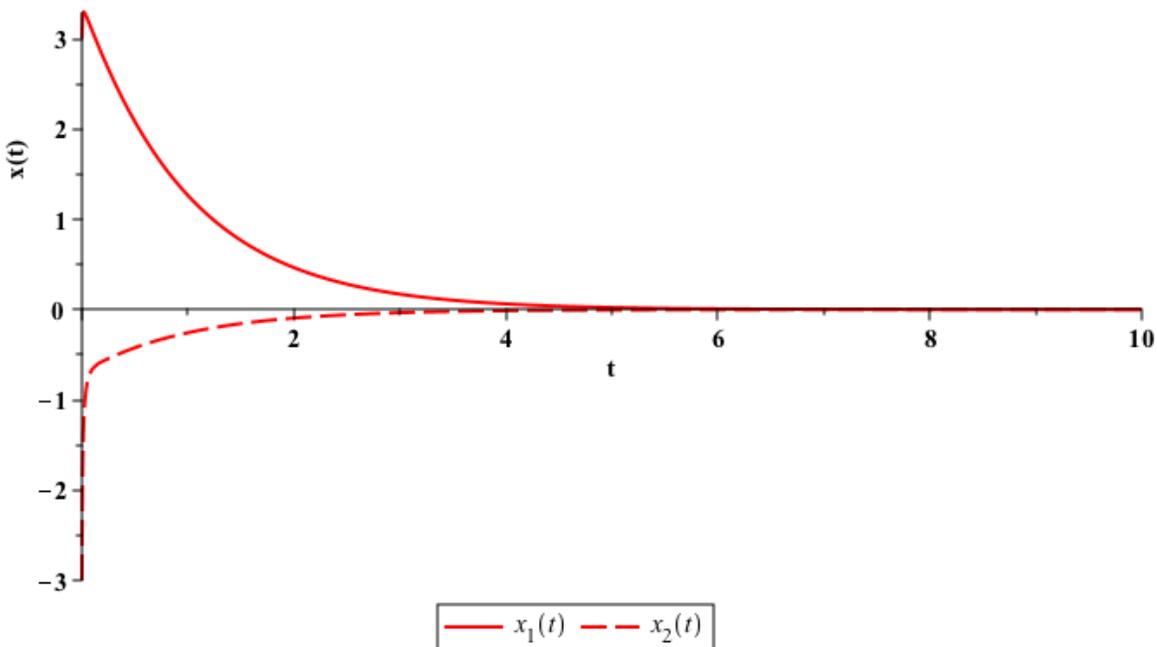


Figure 1: The responses of the closed-loop system S_1 and (4.3) (initial state $x(0) = (3, -3)$).

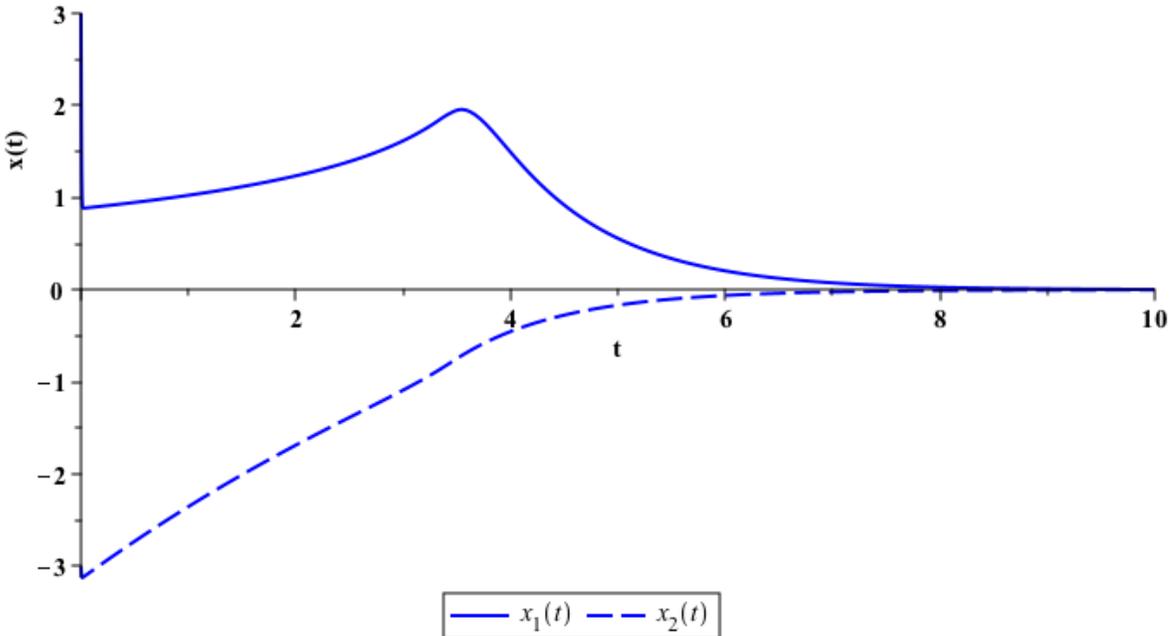


Figure 2: The responses of the closed-loop system S_2 and (4.3) (initial state $x(0) = (3, -3)$).

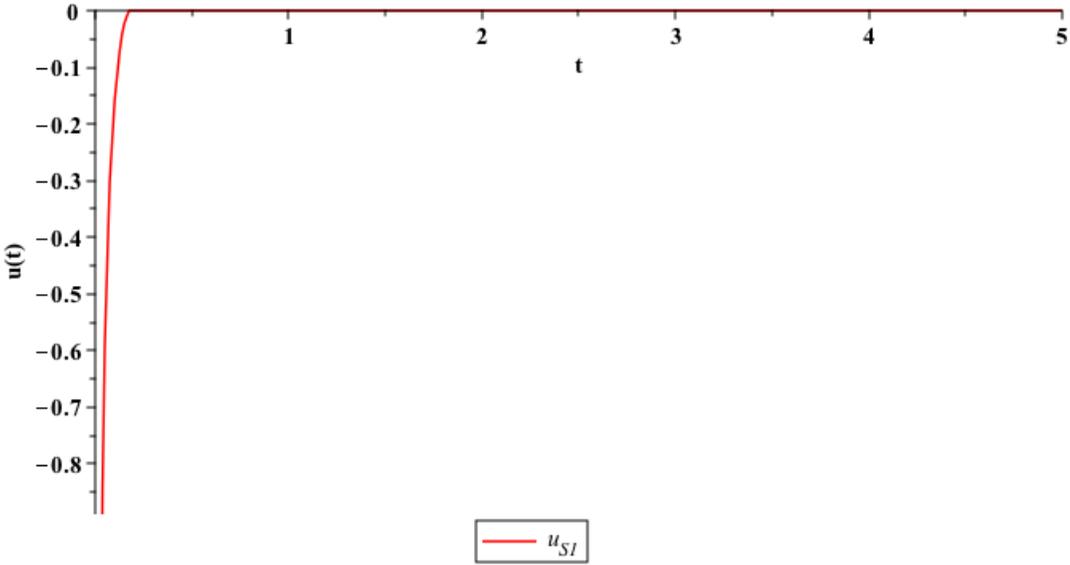


Figure 3: The evolution of the stabilizing control along the solutions of the closed-loop system S_1 and (4.3) (initial state $x(0) = (3, -3)$).

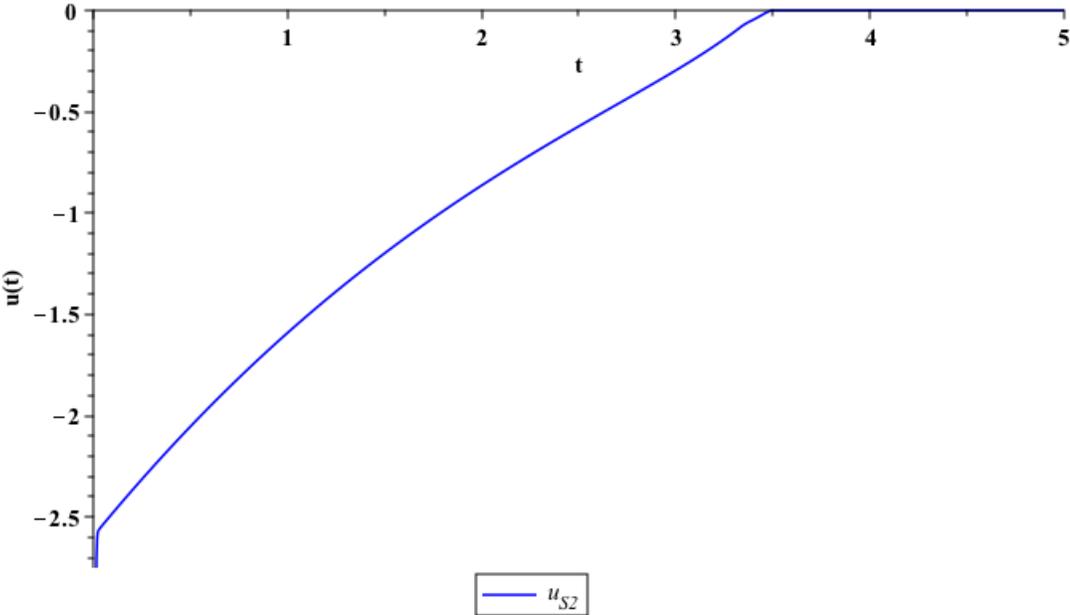


Figure 4: The evolution of the stabilizing control along the solutions of the closed-loop system S_2 and (4.3) (initial state $x(0) = (3, -3)$).

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