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# Multi-Step Problem of Inventory Control with Returns

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**Abstract.** A new model of inventory control with returns is considered, when it is possible for consumers to return (under certain conditions) the products they have purchased. It proved to be that the optimal inventory control strategy in such a system turns out to be four-level.

**Keywords:** Inventory Control with Returns, Inventory Control Strategies

## 1 Introduction

A fundamentally new model of inventory control with returns is considered, which differs from the classical models of the theory of inventory control [1], [2], [3]. In this case, it is assumed that the consumer can not only purchase the goods stored in the warehouse or in the store, but also return it to the seller (possibly on terms different from the conditions of purchase). In the same way, it is assumed that the warehouse itself can not only submit an order for replenishment, but also return previously received consignments of goods to the same supplier (and also on different conditions). These models are not so common in the practice of managing warehouse systems. However, such situations are often typical for leasing schemes.

A few words about the main motivation for exploring this new inventory control model. It was due to the analogy identified between this formulation of the inventory control problem and management problems for one class of queuing systems firstly described in [4]<sup>1</sup>. For such queuing systems control strategy consisted of to enable or disable redundant service channels [5], [6]. Based on this analogy, for a completely applied class of problems in the theory of controlled queuing systems, real algorithms for optimal channel switching in queuing systems were constructed.

## 2 Inventory Control Model with Returns

A multistep model of inventory control is considered during the planning period  $T = (0, N\tau)$ , where  $N$  is a sufficiently large natural number, of one type of product with backlogging of the outstanding demand (so-called backorders). The demand at each of

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<sup>1</sup> In [4] the inventory control model with returns was called “fantasy”.

the steps of the process is described by a model of independent in the aggregate, identically distributed random variables  $\{z(n), n = 1, 2, \dots, N\}$  with the distribution function  $F(z)$ . We will also assume that the delivery lag time is equal to 0. In this case, it is customary to associate the state of the inventory management system not with the stock on hand, but with a so-called inventory position. An inventory position (with a delivery time equal to 0) is defined [7] as the stock on hand minus the backordered demand.

Such a model describes well the systems for managing the stocks of mass consumption products, the demand for which is not (or is not very susceptible) to seasonal changes. This type of products include staple foods (baked goods, meat, dairy products, etc.), medicines for chronic diseases, and many, many others.

Unlike the classical multi-step inventory control model, we will assume that the domain of the demand distribution function at one step  $F(z)$  is the entire real axis:  $-\infty < z < +\infty$ . Let also the mathematical expectation of this distribution be positive. The possibility of negative values of demand will be interpreted as the return by the consumer of the products purchased at the warehouse. We will also assume that when the goods are returned to the consumer, the entire amount that he paid when purchasing it is returned. The warehouse also gets the opportunity to return the goods to its supplier, paying a fixed amount  $A_2$  for this, but the price  $c_2$ , at which the money is returned by the supplier, is less than the price of its purchase from the supplier:  $c_2 < c_1$ .

Let the criterion for the optimal functioning of the warehouse be the minimum of the total average costs during the planning period  $[0, T]$ , where  $T = N\tau$ , and  $\tau$  is the duration of one-step between the moments of making order decisions. Let  $C_n^*(x)$  denote the minimum possible value of average costs for the inventory control system, which has  $n$  steps left until the end of the planning period  $[0, T]$  and which at the beginning of this interval has an inventory position  $x$ . Then we can write the following discrete dynamic programming equations:

$$C_0^*(x) \equiv 0; C_n^*(x) = \min_u \{(A_1 + c_1 u) \times \mathbf{1}(u) + (A_2 + c_2 u) \times \mathbf{1}(-u) + g(x + u) + \alpha \int_{-\infty}^{\infty} C_{n-1}^*(x + u - z) dF(z)\}, n = 1, 2, \dots, N \quad (1)$$

where  $g(y) = h \int_{-\infty}^{\max\{y, 0\}} (y - z) dF(z) + d \int_{\max\{y, 0\}}^{+\infty} (z - y) dF(z)$ ,  $\alpha$  is discount coefficient,  $0 \leq \alpha \leq 1$  and  $\mathbf{1}(u)$  is the unit jump function (Heaviside function).

Equations (1) can be rewritten as follows

$$C_0^*(x) \equiv 0; C_n^*(x) = -c_1 x + \min \begin{cases} A_1 + \min_{y > x} G_n(y), \\ \min\{G_n(x), \tilde{G}_n(x)\}, \\ A_2 + (c_1 + c_2)x + \min_{y < x} \tilde{G}_n(y), \end{cases} n = 1, 2, \dots, N. \quad (2)$$

where the function  $G_n(y)$  is given by the formula

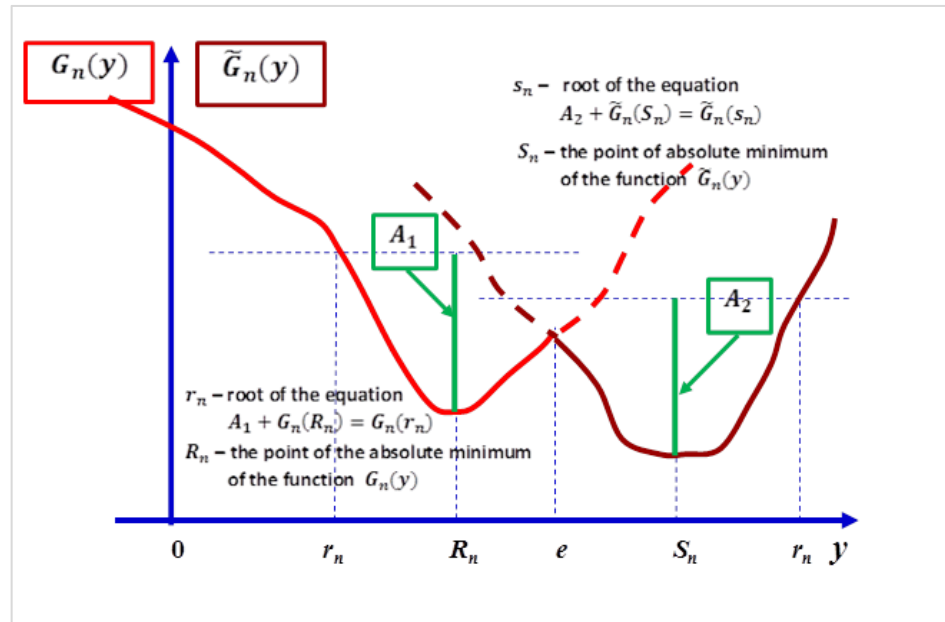
$$G_n(y) = c_1 y + h \int_{-\infty}^{\max\{y, 0\}} (y - z) dF(z) + d \int_{\max\{y, 0\}}^{+\infty} (z - y) dF(z) + \alpha \int_{-\infty}^{+\infty} C_{n-1}^*(y - z) dF(z), \quad (3)$$

and the function  $\tilde{G}_n(y)$  by the formula

$$\tilde{G}_n(y) = -c_2 y + h \int_{-\infty}^{\max\{y, 0\}} (y - z) dF(z) + d \int_{\max\{y, 0\}}^{+\infty} (z - y) dF(z) + \alpha \int_{-\infty}^{\infty} C_{n-1}^*(y - z) dF(z). \quad (4)$$

In the framework of the economics of calculating costs, the function  $\min_{y>x} G_n(y)$  describes the minimum variable part of the costs when making a decision to place an order (excluding the fixed part  $A_1$  and the current inventory position  $x$  and including the case of failure to submit an order when  $u = 0$ ), and the function  $\min_{y<x} \tilde{G}_n(y)$  is equal to variable part of the minimum cost when deciding whether to return the goods to the supplier (excluding the fixed part  $A_2$  and the current inventory position  $x$ ). Note also that, due to the form of the functions  $G_n(y)$  and  $\tilde{G}_n(y)$ , the point  $S_n$  of the absolute minimum of the function  $\tilde{G}_n(y)$  is located to the right of the point  $R_n$  of the absolute minimum of the function  $G_n(y)$ <sup>2</sup>.

In the next Fig. 1 shows the hypothetical form and relative position of the functions  $G_n(y)$  and  $\tilde{G}_n(y)$ .



**Fig. 1.** Type and arrangement of functions  $G_n(y)$  и  $\tilde{G}_n(y)$ .

For the functions  $G_n(y)$  and  $\tilde{G}_n(y)$  shown in Fig. 1 the optimal rule of inventory control is given by the formula:

$$u_n^*(x) = \begin{cases} R_n - x, & \text{if } x \leq r_n, \\ 0, & \text{if } r_n < x < s_n, \\ x - S_n, & \text{if } \geq s_n. \end{cases} \quad (5)$$

<sup>2</sup> This statement will be proved in the next section.

So, if the functions  $G_n(y)$  and  $\tilde{G}_n(y)$  for all values of  $n$  have the properties that are qualitatively characterized in Fig. 1, then the optimal inventory control strategy turns out to be a four-level  $(R_n, r_n, S_n, s_n)$ -strategy ( $r_n < R_n < S_n < s_n$ ). This strategy is arranged so that:

(1) If the inventory position  $x$  at the time of making decisions  $n$  steps before the end of the planning period is less than or equal to  $r_n$ , then an order is submitted that replenishes the inventory position in the warehouse to the value  $R_n$ .

(2) If the inventory position  $x$  at the time of making decisions  $n$  steps before the end of the planning period is greater than or equal to  $s_n$ , then part of the stock from the warehouse is returned to the supplier so as to bring the inventory position in the warehouse to the value  $S_n$ .

### 3 Proof of Optimality of $(R_n, r_n, S_n, s_n)$ -strategies

A multistep model of inventory control is considered during the planning period  $T = (0, N\tau)$ , where  $N$  is a sufficiently large natural number, of one type of product with backlogging of the outstanding demand (so-called backorders). The demand at each of the steps of the process is described by a model of independent in the aggregate, identically distributed random variables  $\{z(n), n = 1, 2, \dots, N\}$  with the distribution function  $F(z)$ . We will also assume that the delivery lag time is equal to 0. In this case, it is customary to associate the state of the inventory management system not with the stock on hand, but with a so-called inventory position. An inventory position (with a delivery time equal to 0) is defined [1] as the stock on hand minus the backordered demand.

We transform equation (2) by introducing the auxiliary functions  $G_{\text{order}}^{(n)}(y)$  and  $G_{\text{return}}^{(n)}(y)$ . The function  $G_{\text{order}}^{(n)}(y)$  describes the average cost for  $n$  steps before the end of the planning period at the current inventory position  $x$  and when deciding on a positive order size  $u$  (after which the inventory position becomes equal to  $y = x + u$ ) and choosing the optimal inventory control strategy at subsequent  $(n - 1)$ <sup>th</sup> steps without taking into account the constant component of costs when placing an order  $A_1$  (this is the subscript ‘‘order’’). The function  $G_{\text{return}}^{(n)}(y)$  describes the average cost for  $n$  steps before the end of the planning period when deciding on a negative order size  $u$  (after which the inventory level also becomes equal to  $y = x + u$ ) and the choice of the optimal inventory management strategy at the subsequent  $(n - 1)$ <sup>th</sup> steps without taking into account the constant component of costs when placing an order  $A_2$  (this is the subscript ‘‘return’’). Let

$$G_{\text{order}}^{(n)}(y) = c_1 y + g(y) + \alpha \int_{-\infty}^{+\infty} C_{n-1}^*(y - z) dF(z), \quad n \in \overline{1, N}, \quad (6)$$

$$G_{\text{return}}^{(n)}(y) = -c_2 y + g(y) + \alpha \int_{-\infty}^{+\infty} C_{n-1}^*(y - z) dF(z), \quad n \in \overline{1, N}. \quad (7)$$

Using functions (6)–(7), one can rewrite equations (2) in the following form:

$$C_0^*(x) \equiv 0; \quad C_n^*(x) = \min \begin{cases} -c_1 x + \min \begin{cases} A_1 + \min_{u>0} G_{\text{order}}^{(n)}(x+u), \\ G_{\text{order}}^{(n)}(x), \end{cases} \\ c_2 x + \min \begin{cases} A_2 + \min_{u<0} G_{\text{return}}^{(n)}(x+u), \\ G_{\text{return}}^{(n)}(x). \end{cases} \end{cases} \quad n \in \overline{1, N}. \quad (8)$$

*Proposition 1.*  $R_n < S_n$ .

*Proof.* Due to the fact that  $c_1 > c_2$  and using formulas (7)–(8), it is easy to show that the points of absolute minima in  $y$  of the functions  $G_{\text{order}}^{(n)}(y)$  and  $G_{\text{return}}^{(n)}(y)$  lead to such values of  $y$  denoted by  $R_n$  and  $S_n$  respectively (see Section 2) are ordered in accordance with the inequality  $R_n < S_n$ . *Q.E.D.*

In order to use the technique of proving the optimality of two-level  $(S, s)$ -strategies proposed in [7], [1] recall the basic notion of the  $A$ -convexity, on which this proof was based.

*Definition 1.* An everywhere differentiable function  $f(y)$  is called  $A$ -convex ( $A \geq 0$ ) if, for any  $a > 0$ , the next inequality holds

$$A + f(x+a) - f(x) - f'(x)a \geq 0. \quad (9)$$

It turns out that in order to prove the optimality of  $(R_n, r_n, S_n, s_n)$ -strategies, it is necessary to introduce a generalization of this concept.

*Definition 2.* Let there exist a real number  $a$ , which has the property that it is to the right of the point of absolute minimum of the everywhere differentiable function  $f(x)$  and for all real  $y$  and  $z$  such that the points  $y$  and  $y+z$  are to the left of the point  $a$ , that is belong to the interval  $(-\infty, a)$ , the next inequality holds

$$A + f(y+z) - f(y) - f'(y)z \geq 0, \quad A \geq 0. \quad (10)$$

Then the function  $f(x)$  is called  $A$ -convex on the left in the interval  $(-\infty, a]$ .

*Definition 3.* Let there exist a real number  $b$ , which has the property that it is to the left of the point of absolute minimum of the everywhere differentiable function  $f(x)$  and for all real  $y$  and  $z$  such that the points  $y$  and  $y+z$  are to the right of the point  $b$ , that is belong to the interval  $(b, \infty)$ , the inequality (11) holds then the function  $f(x)$  is called  $A$ -convex on the right in the interval  $[b, \infty)$ .

*Proposition 2.* All functions  $C_n^*(x)$  are  $A_1$ -convex on the left and  $A_2$ -convex on the right.

*Proof.* Using formulas (6)–(7) for the case  $n = 1$ , it is easy to show that the functions  $G_{\text{order}}^{(n)}(y)$  and  $G_{\text{return}}^{(n)}(y)$  are convex, that is, 0-convex, and by the characteristic properties of the  $A$ -convexity construction [1], are  $A_1$ - и  $A_2$ -convex, respectively. Hence, as in ([1], by virtue of formula (8) for the case  $n = 1$ , it is established that the function  $C_1^*(x)$  is simultaneously  $A_1$ -convex on the left and  $A_2$ -convex on the right.

Let us state the hypothesis of mathematical induction that for some number  $n > 1$  the functions  $C_n^*(x)$  are  $A_1$ -convex on the left and  $A_2$ -convex on the right. Further progress consists in establishing the fact that the  $G_{\text{order}}^{(n+1)}(y)$  function is  $A_1$ -convex and the  $G_{\text{return}}^{(n+1)}(y)$  function is  $A_2$ -convex. This follows from formulas (6) and (7) and the

scheme of the proof, as in the case  $n = 1$ , repeats the scheme of the proof given in [1]. Then it will follow from Eq. (9) and *Proposition 1* that the function  $C_{n+1}^*(x)$  will also be  $A_1$ -convex on the left and  $A_1$ -convex on the right. In this case, the role of points  $a$  and  $b$  from *Definitions 2* and *3* is played by the common point  $e$  of intersection of the graphs of the curves (see Fig. 1). *Q.E.D.*

As a result, the optimal inventory management strategy will be determined by the rule established by formula (6). In this case, the parameters  $r_n$  and  $s_n$  in formula (6) are solutions of the following equations (see Fig. 1):

$$A_1 + G_{\text{order}}^{(n)}(R_n) = G_{\text{order}}^{(n)}(r_n), \text{ for } r_n < R_n, \quad (11)$$

$$A_2 + G_{\text{return}}^{(n)}(S_n) = G_{\text{return}}^{(n)}(s_n) \text{ for } s_n > S_n. \quad (12)$$

It is easy to see that the proof of the above mathematical induction hypothesis methodologically repeats the proof of a similar proposition (for two-level strategies) in the classical theory of inventory management [1], adjusted for the alternativeness (orders and returns) of the objective functional.

## 4 Summary

A fundamentally new model of inventory control with returns is considered. It is proved that the optimal inventory control strategy according to this model turns out to be a four-level  $(R, r, S, s)$ -strategy. This strategy is an essential generalization of the well-known two-level  $(S, s)$ -strategies. The analogies between the optimization problems of the inventory control theory and the theory of controlled queuing systems are noted.

## References

1. Hadley, G; Whitin, T.M. Analysis of Inventory Systems. Prentice-Hall, Inc. Englewood Clifs, New Jersey (1963).
2. Pervozvansky, A. (). Mathematical Models of Inventory and Production Control. Moscow, Science Publ. House (In Russian, 1975).
3. Lototsky, V., Mandel, A. Inventory Control Models and Methods. Moscow, Science Publ. House (In Russian, 1992).
4. Mandel, A., Granin, S. Investigation of Analogies between the Problems of Inventory Control and the Problems of the Controlled Queuing Systems. In: Proceedings of the 11<sup>th</sup> International Conference "Management of Large-Scale System Development" (MLSD). M.: IEEE, 2018, pp. 1–4. Moscow. IEEE. DOI: 10.1109/MLSD.2018.855185 (2018).
5. Mandel, A., Laptin, V. Myopic Channel Switching Strategies for Stationary Mode: Threshold Calculation Algorithms In: Communications in Computer and Information Science. Vol. 919. pp. 410–420. Geneva, Springer Link (2018).
6. Mandel, A., Laptin, V. Channel Switching Threshold Strategies for Multichannel Controllable Queuing Systems. In: Communications in Computer and Information Science. Vol. 1337, pp. 259–270. link.springer.com/chapter – DOI: 10.1007/978-3-030-66242-4\_21 (2020).
7. Karlin, S. Mathematical Methods and Theory in Games, Programming, and Economics. New York, Addison-Wesley Publishing Company (1959).