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# Energetically optimal finite-time stabilization of generalized homogeneous linear systems

Andrey Polyakov, Denis Efimov, Xubin Ping

**Abstract**—The problem of finite-time stabilization of a linear plant with an optimization of both a settling time and an averaged/weighted control energy is studied using the concept of generalized homogeneity. It is shown that the optimal finite-time stabilizing control in this case can be designed by means of solving a simple linear algebraic equation. Robustness of the obtained control law is studied.

**Index Terms**—nonlinear control; stability of nonlinear systems; optimal control

## I. INTRODUCTION

The linear quadratic regulator (LQR) on an infinite horizon seems to be the most famous example of an optimal control design, when the infinite dimensional optimization problem can be reduced to solving a finite-dimensional algebraic (Riccati) equation. This LQR regulator is a static linear feedback which stabilizes exponentially the origin of a linear system and optimizes both the system trajectory and the control energy. This paper tries to answer the question: how a similar design can be developed for an *optimal finite-time stabilization*?

If the stabilization time  $T > 0$  is prescribed (fixed), the problem of an energetically optimal finite-time control is an LQR problem on the finite horizon  $(0, T)$ . An optimal solution in this case is a time-varying linear feedback known as minimum energy control (see, e.g., [1]). The minimum energy control is one of classical examples of prescribed-time stabilization algorithm [2]. To guarantee the prescribed-time convergence, the time-varying feedback gain tends to  $\infty$  as the time  $t$  tends to  $T$ . Such a time-varying controller cannot be utilized for a set-point tracking if  $t > T$ . The infinite gain also complicates a practical application of the minimum energy control and the robustness analysis of the closed-loop system. The model predictive control (MPC) [3] using a moving horizon minimization suggests another possible way to

implement the optimal control algorithms in practice, to improve the robustness of the feedforward optimal control and to fulfill additional state constraints.

Seemly, the first algorithm of the finite-time stabilization was introduced in the mid of 1930s as a solution to the so-called minimum time control problem (see, e.g., [4] for more details). In [5], a family of optimal finite-time stabilization algorithms was presented for the planar control system. The optimization of a settling time together with a quadratic cost functional was also studied in [6], [7]. The mentioned finite-time control algorithms are relay or more complicated discontinuous regulators, which are difficult to implement in practice due to the chattering phenomenon [8], [9]. Usually, the switching surfaces of the minimum time relay control algorithms do not have an explicit representation [6].

Continuous finite-time controllers are more appreciated for the set-point tracking than discontinuous ones. The design of such control algorithms is a rather popular research direction last years [10], [11], [12], [13]. The continuous finite-time controllers can also be solutions of optimal control problems [14]. In this paper, we deal with a particular optimal control problem on non-fixed horizon with a mixed cost functional, which optimizes both a stabilization time and a weighted/averaged control energy. Similarly to LQR case, we are interested in the optimal control design by solving an algebraic equation, so the practical implementation of the optimal control would not require a solving of a differential Riccati equation. We show that such a design is possible the so-called generalized homogeneous linear plants [15].

The homogeneity is a symmetry of an object (function, set, vector field, etc.) with respect to a group of dilations. A symmetry of differential equations with respect to a generalized (non-uniform) dilation of the initial state is studied in [16]. Some extensions of the homogeneity theory of finite-dimensional systems can be found in [17], [18]. Homogeneous differential equations/inclusions form an important class of control system models [19], [20], [12], [21]. They appear as local approximations [22] or set-valued extensions [23] of nonlinear systems and include models of process control [24], nonholonomic systems [25], mechanical

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models with frictions [26], etc. Stability and stabilizability problems were studied for both standard [27] and weighted homogeneous [28], [29], [12] systems which are the most popular today [26], [23], [20]. Asymptotically stable homogeneous system of negative degree is finite-time stable [11]. An introduction to homogeneous optimal control can be found in [15, Chapter 12]. The homogeneous MPC design is studied in [30]. This paper presents an application of the homogeneity theory to energetically optimal finite-time stabilization.

The preliminary version of the paper has been presented in [31]. The key differences are as follows: the multi-input optimal control problem is studied; the proofs of all claims are given; the homogeneity of the closed-loop optimal control system is proven; a new example is presented.

*Notation:*  $\mathbb{N}$  is the set of natural numbers with zero;  $\mathbb{R}$  is the field of real numbers;  $\|\cdot\|$  denotes a norm in  $\mathbb{R}^n$ ;  $\mathbf{0}$  is the zero element of a vector space;  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix;  $P \succ 0$  (resp.  $P \succeq 0$ ) denotes positive (semi-)definiteness of a matrix  $P = P^\top \in \mathbb{R}^{n \times n}$ ;  $\lambda_{\max}(P)$  is a maximum eigenvalue of the matrix  $P = P^\top$ ;  $\dim S$  denotes the dimension of a manifold  $S$ ;  $L_{loc}^2(X, Y)$  (resp.,  $C(X, Y)$  and  $C^1(X, Y)$ ) is the space of locally square-integrable (resp., continuous and continuously differentiable) functions  $X \subset \mathbb{R}^p \rightarrow Y \subset \mathbb{R}^q$ .

## II. PROBLEM STATEMENT

Let us consider the optimal control problem:

$$J(u, T) := T + \frac{1}{T^\eta} \int_0^T u^\top(\tau) \tilde{R}(T - \tau) u(\tau) d\tau \rightarrow \min \quad (1)$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \neq \mathbf{0}, \quad x(T) = \mathbf{0}, \quad (2)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u \in L_{loc}^2((0, T), \mathbb{R}^m)$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ , the time horizon is denoted by  $T \in (0, +\infty)$ , the function  $\tilde{R}: (0, +\infty) \rightarrow \mathbb{R}^{m \times m}$  is symmetric-valued  $\tilde{R}(\sigma) = \tilde{R}^\top(\sigma) \in \mathbb{R}^{m \times m}$  and positive definite  $\tilde{R}(\sigma) \succ 0$  for all  $\sigma > 0$ , and  $\eta \in \mathbb{R}$  is a constant parameter.

The cost functional  $J$  allows the minimization of both a control energy (possibly averaged and weighted) and an optimal time required for the stabilization of the system state to zero. The parameters  $\eta$  and  $\tilde{R}$  specify some restrictions to an admissible control. For  $\eta = 1$  and  $\tilde{R} = I_m$  the second term of the cost functional defines an average control energy, while a proper selection of  $\tilde{R}$  could guarantee  $\|u(\tau)\| \rightarrow 0$  as  $\tau \rightarrow T$ .

The time horizon  $T$  may be *fixed or non-fixed*. In the latter case, the functional  $J$  may have a finite value only if  $T < +\infty$ , i.e., the optimal trajectory (if it exists) always converges to  $\mathbf{0}$  in a finite time  $T^* = T^*(x_0)$ .

*Assumption 1:*  $\{A, B\}$  is controllable.

This assumption is necessary to guarantee the existence of an optimal solution for all  $x_0 \in \mathbb{R}^n$ . Otherwise there exists an uncontrollable linear subspace, where the system states cannot converge to zero in a finite time.

In this paper we consider a class of the so-called generalized homogeneous systems [16], [18], [19], [29], [12], [32], [30], [33] characterized by the assumption.

*Assumption 2:* There exist anti-Hurwitz matrices<sup>1</sup>  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  and  $G_{\tilde{\mathbf{d}}} \in \mathbb{R}^{m \times m}$  such that

$$AG_{\mathbf{d}} = (G_{\mathbf{d}} - I_n)A, \quad G_{\mathbf{d}}B = BG_{\tilde{\mathbf{d}}}. \quad (3)$$

Notice that for  $m = 1$  Assumptions 1, 2 imply that the system is equivalent to the integrator chain. So, optimal control design for nonlinear systems topologically equivalent to the integrator chain can be studied as well. To find a solution of the optimal control problem (1)–(2) in a rather simple explicit form, we impose a special restriction to the form of the matrix-valued function  $\tilde{R}$ .

*Assumption 3:* Let  $0 \prec R = R^\top \in \mathbb{R}^{m \times m}$ ,  $\gamma \in \mathbb{R}$  define

$$\tilde{R}(\sigma) = \sigma^{2-\gamma} \tilde{\mathbf{d}}^\top(-\ln \sigma) R \tilde{\mathbf{d}}(-\ln \sigma), \quad \sigma > 0, \quad (4)$$

where  $\tilde{\mathbf{d}}(s) = e^{sG_{\tilde{\mathbf{d}}}}$ ,  $s \in \mathbb{R}$  and  $G_{\tilde{\mathbf{d}}}$  is by Assumption 2.

In practice, solutions of the optimization problem (1)–(2) can be utilized for energetically optimal finite-time stabilization and MPC design. Our goal is to derive the corresponding controllers as well as to analyze stability and robustness properties of the closed-loop system.

## III. HOMOGENEOUS OPTIMAL FINITE-TIME CONTROL

### A. Preliminaries

By definition, the homogeneity is a dilation symmetry [16], [22], [18]. In this paper, we only deal with the so-called linear dilations [15, Chapter 6] given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} \quad s \in \mathbb{R}, \quad (5)$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix (called *the generator of the dilation*  $\mathbf{d}$ ). The latter guarantees that  $\mathbf{d}$  satisfies the *limit property*  $\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)x\| = 0$ ,  $\lim_{s \rightarrow +\infty} \|\mathbf{d}(s)x\| = +\infty$ ,  $\forall x \neq \mathbf{0}$  required for a group  $\mathbf{d}$  to be a dilation in  $\mathbb{R}^n$  [18].

*Definition 1 ([18]):* A vector field  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  (resp. a function  $h: \mathbb{R}^n \rightarrow \mathbb{R}$ ) is said to be  $\mathbf{d}$ -homogeneous of degree  $\nu \in \mathbb{R}$  if  $f(\mathbf{d}(s)x) = e^{\nu s} \mathbf{d}(s) f(x)$ ,  $\forall x \in \mathbb{R}^n$ ,  $\forall s \in \mathbb{R}$  (resp.  $h(\mathbf{d}(s)x) = e^{\nu s} h(x)$ ,  $\forall x \in \mathbb{R}^n$ ,  $\forall s \in \mathbb{R}$ ).

The system  $\dot{x} = f(x)$  with  $\mathbf{d}$ -homogeneous vector field possess a certain symmetry as well [18]:  $x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\nu s}t, x_0)$ , where  $x(\cdot, z)$  denotes a solution of the considered system with the initial condition  $x(0) = z$ .

A dilation introduces an alternative norm topology in  $\mathbb{R}^n$  using the canonical  $\mathbf{d}$ -homogeneous norm [32]

$$\|x\|_{\mathbf{d}} = e^s \quad \text{with } s \in \mathbb{R} : \|\mathbf{d}(-s)x\| = 1 \quad (6)$$

<sup>1</sup>A matrix  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is anti-Hurwitz if  $-G_{\mathbf{d}}$  is Hurwitz.

and  $\|\mathbf{0}\|_{\mathbf{d}} = 0$ , where  $\mathbf{d}$  is a linear monotone<sup>2</sup> dilation. The function  $\|\cdot\|_{\mathbf{d}}$  is continuous on  $\mathbb{R}^n$ , locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  and  $\mathbf{d}$ -homogeneous of degree 1. Moreover, if  $\|\cdot\| \in C^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$  then  $\|\cdot\|_{\mathbf{d}} \in C^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$ .

If Assumption 1 holds then there exists a non-singular matrix  $\Phi \in \mathbb{R}^{n \times n}$ :

$$\Phi^{-1}A\Phi = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ A_{k1} & A_{k2} & A_{k3} & \dots & A_{kk} \end{pmatrix}, \Phi^{-1}B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_{kk+1} \end{pmatrix}, \quad (7)$$

where  $1 \leq k \leq n$  is a minimal number such that  $\text{rank}[A^{k-1}B, \dots, AB, B] = n$ ,  $A_{ii+1} \in \mathbb{R}^{n_i \times n_{i+1}}$ ,  $A_{ki} \in \mathbb{R}^{n_k \times n_i}$ ,  $\text{rank}A_{ii+1} = n_i$ ,  $1 \leq n_i \leq n_{i+1}$ ,  $i = 1, \dots, k$ ,  $n_1 + \dots + n_k = n$  and  $n_{k+1} = m$ . The transformation  $\Phi$  is presented in [21, Section 4.1].

*Lemma 1 ([15], page 207):* The claims are equivalent:

- the linear vector field  $x \rightarrow Ax$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $x \in \mathbb{R}^n$  is  $\mathbf{d}$ -homogeneous of degree  $\nu \neq 0$ ;
- there exist an anti-Hurwitz matrix  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  and a real number  $\nu \neq 0$  such that  $AG_{\mathbf{d}} = (\nu I_n + G_{\mathbf{d}})A$ ;
- $A$  is nilpotent.

From this lemma, we conclude that Assumption 2 may hold only if *the matrix  $A$  is nilpotent*. For  $m = 1$  and for a nilpotent matrix  $A$ , we have  $n_i = 1$ ,  $k = n$ ,  $A_{ni} = 0$ ,  $i = 1, \dots, n$ , so the identities (3) are fulfilled for  $G_{\mathbf{d}} = \Phi \text{diag}(n, n-1, \dots, 1) \Phi^{-1}$  and  $G_{\tilde{\mathbf{d}}} = 1$ .

*Corollary 1:* If Assumptions 1, 2 hold then

$$G_{\mathbf{d}} = \Phi \begin{pmatrix} G_{11} & 0 & \dots & 0 & 0 \\ G_{21} & G_{22} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ G_{k-11} & G_{k-12} & \dots & G_{k-1k-1} & 0 \\ G_{k1} & G_{k2} & \dots & G_{kk-1} & G_{kk} \end{pmatrix} \Phi^{-1}, \quad (8)$$

where  $G_{ij} \in \mathbb{R}^{n_i \times n_j}$ ,  $k \geq i \geq j \geq 1$ .

**Proof.** Denoting  $\tilde{G}_{\mathbf{d}} = \Phi^{-1}G_{\mathbf{d}}\Phi$ ,  $\tilde{A} = \Phi^{-1}A\Phi$  and  $\tilde{B} = \Phi^{-1}B$  we derive that the identities (3) are equivalent to  $\tilde{A}\tilde{G}_{\mathbf{d}} = (\tilde{G}_{\mathbf{d}} - I_n)\tilde{A}$  and  $\tilde{G}_{\mathbf{d}}\tilde{B} = \tilde{B}G_{\tilde{\mathbf{d}}}$ . Let  $\tilde{G}_{\mathbf{d}}$  be written in the block form:

$$\tilde{G}_{\mathbf{d}} = \begin{pmatrix} G_{11} & G_{12} & G_{13} & \dots & G_{1k-1} & G_{1k} \\ G_{21} & G_{22} & G_{23} & \dots & G_{2k-1} & G_{2k} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ G_{k-11} & G_{k-12} & G_{k-13} & \dots & G_{k-1k-1} & G_{k-1k} \\ G_{k1} & G_{k2} & G_{k3} & \dots & G_{kk-1} & G_{kk} \end{pmatrix},$$

where  $G_{ij} \in \mathbb{R}^{n_i \times n_j}$  are blocks of  $\tilde{G}_{\mathbf{d}}$ . From the the identity  $\tilde{G}_{\mathbf{d}}\tilde{B} = \tilde{B}G_{\tilde{\mathbf{d}}}$  we derive  $G_{ik}A_{kk+1} = \mathbf{0}$ ,  $i = 1, \dots, k-1$ . Hence, taking into account that the matrix  $A_{kk+1}$  is of full row rank and  $n_k \leq n_{k+1}$ , we conclude  $G_{ik} = \mathbf{0}$ ,  $\forall i = 1, 2, \dots, k-1$ . In this case, using the representation (7) we derive

$$\tilde{A}\tilde{G}_{\mathbf{d}} = \begin{pmatrix} A_{12}G_{21} & A_{12}G_{22} & \dots & A_{12}G_{2k-1} & 0 \\ A_{23}G_{31} & A_{23}G_{32} & \dots & A_{23}G_{3k-1} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_{k-1k}G_{k1} & A_{k-1k}G_{k2} & \dots & A_{k-1k}G_{kk-1} & A_{k-1k}G_{kk} \\ \sum_{i=1}^k A_{ki}G_{i1} & \sum_{i=1}^k A_{ki}G_{i2} & \dots & \sum_{i=1}^k A_{ki}G_{ik-1} & A_{kk}G_{kk} \end{pmatrix}$$

$$\tilde{G}_{\mathbf{d}}\tilde{A} = \begin{pmatrix} 0 & G_{11}A_{12} & \dots & G_{1k-2}A_{k-2k-1} & G_{1k-1}A_{k-1k} \\ 0 & G_{21}A_{12} & \dots & G_{2k-2}A_{k-2k-1} & G_{2k-1}A_{k-1k} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & G_{k-11}A_{12} & \dots & G_{k-1k-2}A_{k-2k-1} & G_{k-1k-1}A_{k-1k} \\ * & * & \dots & * & * \end{pmatrix},$$

<sup>2</sup>A dilation is monotone if  $s \rightarrow \|\mathbf{d}(s)x\|$  is a monotone function for any  $x$ . Any linear dilation in  $\mathbb{R}^n$  is monotone provided that the norm  $\|\cdot\|$  is properly selected [32].

where  $*$  denotes some (possibly) non-zero block. Since the matrices  $A_{ii+1}$  are of full row rank and the identity  $\tilde{A}\tilde{G}_{\mathbf{d}} = (\tilde{G}_{\mathbf{d}} - I_n)\tilde{A}$  holds, then the matrix  $\tilde{G}_{\mathbf{d}}$  has the block lower triangular form as shown in (8). Indeed, considering (starting from the last column) the block representation of  $\tilde{A}\tilde{G}_{\mathbf{d}} = (\tilde{G}_{\mathbf{d}} - I_n)\tilde{A}$  we derive  $G_{ik-1}A_{k-1k} = \mathbf{0} \Rightarrow G_{ik-1} = \mathbf{0}$ ,  $i = 1, \dots, k-2$ , and  $G_{ik-2}A_{k-1k} = \mathbf{0} \Rightarrow G_{ik-2} = \mathbf{0}$ ,  $i = 1, \dots, k-3$ , etc. ■

Under Assumptions 1, 2 the vector field  $x \rightarrow Ax$  is  $\mathbf{d}$ -homogeneous of degree  $\nu = -1$  with the generator of the dilation  $\mathbf{d}$  satisfying (3). Recall [11] that any asymptotically stable homogeneous system of negative degree is finite-time stable<sup>3</sup>. Below we show that an optimal solution of the problem (1) – (2) may have a form of a *homogeneous finite-time stabilizing feedback*.

### B. Optimal homogeneous finite-time stabilization

For fixed time horizon  $T$ , the problem (1)–(2) can be solved by means of the conventional LQR design (see, [6], [7]). The optimal control in this case is a linear feedback with a time-varying gain defined by a differential Riccati equation. The homogeneity of the system (Assumption 2) allows a solution of this Riccati equation to be found by solving a linear algebraic equation.

*Theorem 1:* Let Assumptions 1, 2, 3 be fulfilled and the matrices  $G_{\mathbf{d}} + \frac{\gamma-1}{2}I_n$  and  $A + G_{\mathbf{d}} + \frac{\gamma-1}{2}I_n$  be anti-Hurwitz. The optimal solution to the problem (1)–(2) with a *fixed horizon*  $T > 0$  is

$$u^*(t) = -\frac{\tilde{\mathbf{d}}(\ln(T-t))R^{-1}B^{\top}P^{-1}\mathbf{d}(-\ln(T-t))x(t)}{T-t}, \quad (9)$$

and the optimal cost is

$$J(u^*, T) = T + \frac{x_0^{\top}\tilde{\mathbf{d}}^{\top}(-\ln T)P^{-1}\mathbf{d}(-\ln T)x_0}{T^{\eta+\gamma-1}}, \quad (10)$$

where  $\tilde{\mathbf{d}}$  is given in Assumption 3,  $P \in \mathbb{R}^{n \times n}$  is the positive definite solution of the linear matrix equation:

$$(A + G_{\mathbf{d}})P + P(A + G_{\mathbf{d}})^{\top} + (\gamma - 1)P = BR^{-1}B^{\top}. \quad (11)$$

**Proof.** Since the pair  $\{A, B\}$  is controllable then, due to the representation (7), the pair  $\{A + \Phi\Delta\Phi^{-1}, B\}$  is controllable as well provided that  $\Delta \in \mathbb{R}^{n \times n}$  has a lower triangular block form with blocks having the sizes  $n_i \times n_j$ , where  $i, j = 1, 2, \dots, k$ . Hence, taking into account the representation (8) we conclude that the pair  $\{A + G_{\mathbf{d}} + \frac{\gamma-1}{2}I_n, B\}$  is controllable for any  $\gamma$ . Since  $A + G_{\mathbf{d}} + \frac{\gamma-1}{2}I_n$  is anti-Hurwitz then the equation (11) has a unique positive definite solution being the controllability Gramian defined as follows (see, e.g. [34, Lemma 9.1]):

$$P = \int_0^{\infty} e^{-\tau(A + G_{\mathbf{d}} + \frac{\gamma-1}{2}I_n)} BR^{-1}B^{\top} e^{-\tau(A + G_{\mathbf{d}} + \frac{\gamma-1}{2}I_n)^{\top}} d\tau.$$

<sup>3</sup>A system  $\dot{x} = f(x)$  is said to be globally uniformly finite-time stable if it is globally uniformly Lyapunov stable and there exists a locally bounded function  $T : \mathbb{R}^n \rightarrow [0, +\infty)$  such that  $x(t, x_0) = \mathbf{0}$  for any  $t \geq T(x_0)$ , for any  $x_0 \in \mathbb{R}^n$  and for any solution  $x(\cdot, x_0)$  of the considered system with the initial condition  $x(0) = x_0$ .

Let us denote  $N(t) := t^{\gamma-1} \mathbf{d}(\ln t) P \mathbf{d}^\top(\ln t)$  for  $t > 0$ . Using (11) we derive

$$\begin{aligned} \dot{N}(t) &= t^{\gamma-2} \mathbf{d}(\ln t) (G_{\mathbf{d}} P + P G_{\mathbf{d}}^\top + (\gamma - 1) P) \mathbf{d}^\top(\ln t) \\ &= t^{\gamma-2} \mathbf{d}(\ln t) (-AP - PA^\top + BR^{-1}B^\top) \mathbf{d}^\top(\ln t). \end{aligned}$$

The identities (3) imply that  $e^{-s} \mathbf{d}(s) A = A \mathbf{d}(s)$  and  $\mathbf{d}(s) B = B \tilde{\mathbf{d}}(s)$ ,  $\forall s \in \mathbb{R}$ . Since  $G_{\mathbf{d}} + \frac{\gamma-1}{2} I_n$  is an anti-Hurwitz matrix, then  $N(t) \rightarrow \mathbf{0}$  as  $t \rightarrow 0$ . Hence,

$$\dot{N}(t) = -AN(t) - N(t)A^\top + B\tilde{R}^{-1}(t)B^\top, \quad N(0) = \mathbf{0}. \quad (12)$$

Let us consider the LQR problem:

$$\tilde{J}(u) = \frac{x^\top(T)x(T)}{\epsilon} + \int_0^T u^\top(\tau) \tilde{R}(T-\tau) u(\tau) d\tau \rightarrow \min \quad (13)$$

$$\text{subject to } \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \quad (14)$$

where  $\epsilon > 0$  and  $t \in [0, T]$ . The optimal solution to this problem is given by (see, e.g. [7])

$$u_\epsilon^*(t) = -\tilde{R}^{-1}(T-t)B^\top P_\epsilon(t)x(t), \quad \tilde{J}(u_\epsilon^*) = x_0^\top P_\epsilon(0)x_0,$$

where  $P_\epsilon(t) = P_\epsilon^\top(t) \in \mathbb{R}^{n \times n}$  satisfies the following differential Riccati equation

$$\dot{P}_\epsilon(t) = A^\top P_\epsilon(t) + P_\epsilon(t)A - P_\epsilon(t)B^\top \tilde{R}^{-1}(T-t)BP_\epsilon(t)$$

with the initial condition  $P_\epsilon(T) = \frac{1}{\epsilon} I_n$ . Denoting  $N_\epsilon(t) = P_\epsilon^{-1}(T-t)$ , we derive

$$\dot{N}_\epsilon(t) = -AN_\epsilon(t) - N_\epsilon(t)A^\top + B\tilde{R}^{-1}(t)B^\top, \quad N_\epsilon(0) = \epsilon I_n,$$

$$u_\epsilon^*(t) = -\tilde{R}^{-1}(T-t)B^\top N_\epsilon^{-1}(T-t)x(t),$$

and the optimal cost is  $\tilde{J}(u_\epsilon^*) = x_0^\top N_\epsilon^{-1}(T)x_0$ . Taking  $\epsilon = 0$  we derive that  $N_\epsilon(t) = N(t)$  defines the optimal solution to the problem (13)–(14) with the terminal condition  $x(T) = 0$ . Hence, the optimal control for the original problem (1)–(2) with fixed horizon  $T$  is

$$u^*(t) = -\tilde{R}^{-1}(T-t)B^\top N^{-1}(T-t)x(t) \quad (15)$$

and the optimal cost is given by  $J(u^*, T) = T + \frac{1}{T^n} x_0^\top N^{-1}(T)x_0$ . Hence, we conclude that (10) is fulfilled. Finally, from (15) we derive (9) using  $\mathbf{d}(s)B = B\tilde{\mathbf{d}}(s)$ ,  $s \in \mathbb{R}$ . ■

The optimization of the time  $T$  in (10) results in the following straightforward corollary.

*Corollary 2:* If all conditions of Theorem 1 are fulfilled and  $G_{\mathbf{d}} + \frac{\mu}{2}$  is anti-Hurwitz, then the optimal time  $T = T^*(x_0)$  for the problem (1)–(2) with *non-fixed horizon* is given by

$$T^*(x) := \min \left\{ \operatorname{argmin}_{r>0} \frac{x^\top \mathbf{d}^\top(-\ln r) Q \mathbf{d}(-\ln r) x}{r^{\mu-1}} + r \right\} \quad (16)$$

and the unique optimal control is

$$u^*(t) = \frac{\tilde{\mathbf{d}}(\ln(T^*(x_0)-t)) K \mathbf{d}(-\ln(T^*(x_0)-t)) x(t)}{T^*(x_0)-t}, \quad (17)$$

where  $K = -R^{-1}B^\top P^{-1}$ ,  $Q = P^{-1}$  and  $\mu = \eta + \gamma$ .

The function (16) becomes the canonical homogeneous norm (6) under certain restrictions on  $Q$  and  $\mu$  (see, the formula (19) given below).

*Lemma 2:* Let  $\mathbf{d}$  be a linear dilation in  $\mathbb{R}^n$  with a generator  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  and  $\mathbf{d}_\mu$  be a linear dilation with the generator  $G_{\mathbf{d}_\mu} := G_{\mathbf{d}} + \frac{\mu}{2} I_n$  such that  $G_{\mathbf{d}_\mu} - \frac{1}{2} I_n$  is

an anti-Hurwitz matrix. If  $Q = Q^\top \in \mathbb{R}^{n \times n}$  is a positive definite matrix, then

1)  $T^*$  is well-defined on  $\mathbb{R}^n$ , positive definite and  $\mathbf{d}_\mu$ -homogeneous of degree 1;

2)  $T^*$  is continuously differentiable on  $\mathbb{R}^n \setminus \Sigma$ ,

$$\begin{aligned} \Sigma &\subset \{\mathbf{0}\} \cup_{s \in \mathbb{R}} \mathbf{d}_\mu(s) \Sigma_0, \\ \Sigma_0 &= \left\{ x : x^\top Z x = 1, x^\top (Z G_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Z) x = 0 \right\}, \quad (18) \end{aligned}$$

where  $Z := Q G_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Q - Q$  and  $\Sigma$  is a set of measure zero in  $\mathbb{R}^n$  provided that  $\dim \Sigma_0 \leq n - 2$ ;

3)  $T^* \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$  provided that

$$Z G_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Z \succ 0, \quad Z \succ 0; \quad (19)$$

4)  $T^*$  is locally bounded and continuous at  $\mathbf{0}$  if  $\mu \in \mathbb{N}$  and  $G_{\mathbf{d}} \in \mathbb{R}^n$  is diagonalizable:

$$\tilde{\Phi}^{-1} G_{\mathbf{d}} \tilde{\Phi} = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_n \end{pmatrix}, \quad p_i \in \mathbb{N}, \quad (20)$$

where  $\tilde{\Phi} \in \mathbb{R}^{n \times n}$  is a non-singular matrix.

**Proof.** 1) For  $x \in \mathbb{R}^n$ , let  $\Xi(x)$  be the set of minimum points of the function  $r \rightarrow \xi(r, x) := r + \frac{x^\top \mathbf{d}^\top(-\ln r) Q \mathbf{d}(-\ln r) x}{r^{\mu-1}}$ ,  $r \geq 0$  which is analytic on  $(0, +\infty)$ . Obviously,  $T^*(x) = \min_{\tilde{r} \in \Xi(x)} \tilde{r}$  and  $\Xi(\mathbf{0}) = \{\mathbf{0}\}$ . Since  $Q \succ 0$  then for any  $r > 0$  and any  $x \neq \mathbf{0}$  we have  $\xi(r, x) = r (1 + x^\top \mathbf{d}_\mu^\top(-\ln r) Q \mathbf{d}_\mu(-\ln r) x) > 0$ . Moreover, by the assumption, the matrix  $G_{\mathbf{d}_\mu} - \frac{1}{2} I_n$  is anti-Hurwitz and, consequently,  $\lim_{r \rightarrow 0} \xi(r, x) = \lim_{r \rightarrow +\infty} \xi(r, x) = +\infty$ . Hence, for any  $x \neq \mathbf{0}$ , the set  $\Xi(x)$  is non-empty, bounded and it belongs to the interior of  $(0, +\infty)$ . The latter means that the function  $T^*$  is well-defined and positive definite. On the one hand, due to the necessary condition for an extremum, for any  $\tilde{r} \in \Xi(x)$ , we have  $\frac{\partial \xi(r, x)}{\partial r} \Big|_{r=\tilde{r}} = 0$  or, equivalently,  $1 = x^\top \mathbf{d}_\mu^\top(-\ln \tilde{r}) Z \mathbf{d}_\mu(-\ln \tilde{r}) x$ . On the other hand, the same identity holds for  $x_s = \mathbf{d}(s)x$ , any  $s \in \mathbb{R}$  and any  $\tilde{r}_s \in \Xi(x_s)$ , namely,  $1 = x_s^\top \mathbf{d}_\mu^\top(-\ln \tilde{r}_s) Z \mathbf{d}_\mu(-\ln \tilde{r}_s) x_s$  or, equivalently,  $1 = x^\top \mathbf{d}_\mu^\top(-\ln(e^{-s} \tilde{r}_s)) Z \mathbf{d}_\mu(-\ln(e^{-s} \tilde{r}_s)) x$ , where the group property of the dilation  $\mathbf{d}(s_1 + s_2) = \mathbf{d}(s_1) \mathbf{d}(s_2)$ ,  $\forall s_1, s_2 \in \mathbb{R}$  is utilized on the last step. Therefore, we derive  $e^{-s} \Xi(\mathbf{d}(s)x) = \Xi(x)$ . Hence, the function  $T^*$  is  $\mathbf{d}_\mu$ -homogeneous of degree 1.

2) Notice that the function  $T^*$  is implicitly defined by the equation  $\frac{\partial \xi(r, x)}{\partial r} \Big|_{r=T^*} = 0$ . Since the function  $(r, x) \rightarrow \frac{\partial \xi(r, x)}{\partial r}$  is analytic on  $(0, +\infty) \times \mathbb{R}^n$  then by Inverse Function Theorem (see, e.g., [35, Theorem 19.28], the function  $T^*$  is continuously differentiable at any point  $x \in \mathbb{R}^n \setminus \Sigma$  with  $\Sigma = \left\{ x : \frac{\partial \xi(\tilde{r}, x)}{\partial \tilde{r}} = 0, \frac{\partial^2 \xi(\tilde{r}, x)}{\partial \tilde{r}^2} = 0, \tilde{r} = T^*(x) \right\}$  or, equivalently,

$$\Sigma = \left\{ x \in \mathbb{R}^n : 0 = x^\top \mathbf{d}_\mu^\top(-\ln \tilde{r}) (Z G_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Z) \mathbf{d}_\mu(-\ln \tilde{r}) x, \text{ where } \tilde{r} = T^*(x). \right\}$$

Using the  $\mathbf{d}_\mu$ -homogeneity of  $T^*$ , we derive (18).

Since  $\mathbf{d}$  is a continuous one-parameter group of linear transformations in  $\mathbb{R}^n$  then  $\dim \Sigma = 1 + \dim \Sigma_0$  provided that  $\Sigma_0 \neq \emptyset$ . If  $\dim \Sigma_0 \leq n - 2$  then  $\dim \Sigma \leq n - 1$  and the set  $\Sigma$  is of measure zero in  $\mathbb{R}^n$ .

3) The condition (19) implies  $\Sigma_0 = \emptyset$  and  $\Sigma = \{\mathbf{0}\}$ , i.e.  $T^* \in C^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$ . Since  $T^*$  is  $\mathbf{d}_\mu$ -homogeneous of degree 1 then  $T^*(x) = \|x\|_{\mathbf{d}_\mu} T^*(\mathbf{d}_\mu(-\ln \|x\|_{\mathbf{d}_\mu})x)$ . Using  $\|\mathbf{d}_\mu(-\ln \|x\|_{\mathbf{d}_\mu})x\| = 1$  we derive  $T^*(x) \leq \|x\|_{\mathbf{d}_\mu} \sup_{\|y\|=1} T^*(y)$ , i.e.,  $T^*$  is continuous at  $\mathbf{0}$ .

4) Finally, if  $G_{\mathbf{d}}$  satisfies (20) and  $\mu \geq 0$  is an integer, then for a fixed  $x \in \mathbb{R}^n$  the expression  $\frac{\partial \xi(r, x)}{\partial r} r^{\mu+2} \max p_i$  is a polynomial of degree  $\mu + 2 \max p_i$  with respect to  $r$ . The roots (in the complex plane) of any polynomial depend continuously on coefficients and they are bounded [36] by  $L^\infty$ -norm of coefficients divided by the highest degree coefficient (which equals 1 in our case). The latter means that the function  $T^*$  is locally bounded on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  and  $T^*(x) \rightarrow \mathbf{0}$  as  $\|x\| \rightarrow \mathbf{0}$ . ■

Assumptions 1–3 specify a class of generalized homogeneous optimal control problems (1)–(2).

*Corollary 3 (Homogeneity of optimal control system):* If all conditions of Corollary 2 are fulfilled then solutions of the optimal control problem (1)–(2) are symmetric with respect to a dilation of the initial condition:  $x^*(t, \mathbf{d}_\mu(s)x_0) = \mathbf{d}_\mu(s)x^*(e^{-s}t, x_0)$ ,  $u^*(t, \mathbf{d}_\mu(s)x_0) = e^{(\frac{\mu}{2}-1)s} \tilde{\mathbf{d}}(s)u^*(e^{-s}t, x_0)$ ,  $\forall t \in (0, e^s T^*(x_0))$ ,  $\forall s \in \mathbb{R}$ ,  $\forall x \in \mathbb{R}^n$ , where the function  $t \rightarrow (x^*(t, z), u^*(t, z))$  with the domain  $[0, T^*(z))$  defines the optimal pair (i.e., the optimal trajectory and the optimal control) for the problem (1)–(2) with the initial condition  $x(0) = z \in \mathbb{R}^n$ .

**Proof.** By Lemma 2 the function  $T^*$  is  $\mathbf{d}_\mu$ -homogeneous of degree 1, i.e.,  $T^*(\mathbf{d}_\mu(s)x_0) = e^s T^*(x_0)$ . This means that the domains of the functions  $t \rightarrow u^*(t, \mathbf{d}_\mu(s)x_0)$  and  $t \rightarrow \tilde{u}^*(e^{-s}t, x_0)$  (resp.  $t \rightarrow x^*(t, \mathbf{d}_\mu(s)x_0)$  and  $t \rightarrow x^*(e^{-s}t, x_0)$ ) coincide and are given by the interval  $[0, e^{-s}T^*(x_0))$ .

Let  $\{x^*(t, x_0), u^*(t, x_0)\}$  be an optimal pair for the problem (1)–(2) with the initial condition  $x(0) = x_0 \in \mathbb{R}^n$ . Let us denote  $x_s(t) = \mathbf{d}_\mu(s)x^*(e^{-s}t, x_0)$ , where  $t \in [0, e^s T^*(x_0))$ . Since the pair  $\{x^*(t, x_0), u^*(t, x_0)\}$  satisfies (2), then  $\dot{x}_s(t) = \mathbf{d}(s)e^{-s} \frac{\partial x^*(\tau, x_0)}{\partial \tau} \Big|_{\tau=e^{-s}t} = \mathbf{d}_\mu(s)e^{-s} (Ax^*(e^{-s}t, x_0) + Bu^*(e^{-s}t, x_0))$ . On the one hand, the identities  $\mathbf{d}_\mu(s) = e^{\frac{\mu}{2}s} \mathbf{d}(s)$  and  $A\mathbf{d}_\mu(s) = e^{-s} \mathbf{d}_\mu(s)A$ ,  $\mathbf{d}_\mu(s)B = e^{\frac{\mu}{2}s} B\tilde{\mathbf{d}}(s)$ ,  $\forall s \in \mathbb{R}$  imply that  $\dot{x}_s(t) = Ax_s(t) + Bu_s(t)$ ,  $x_s(0) = \mathbf{d}_\mu(s)x_0$ , where  $u_s(t) = e^{(\frac{\mu}{2}-1)s} \tilde{\mathbf{d}}(s)u^*(e^{-s}t, x_0)$ .

On the other hand, taking into account the  $\mathbf{d}_\mu$ -homogeneity of the function  $T^*$ , we derive

$$u^*\left(\frac{t}{e^s}, x_0\right) = \frac{\tilde{\mathbf{d}}(\ln(T^*(x_0) - \frac{t}{e^s}))K\mathbf{d}(-\ln(T^*(x_0) - \frac{t}{e^s}))x^*\left(\frac{t}{e^s}, x_0\right)}{T^*(x_0) - e^{-s}t} =$$

$$\frac{\tilde{\mathbf{d}}(-s)\tilde{\mathbf{d}}(\ln(T^*(\mathbf{d}_\mu(s)x_0) - t))K\mathbf{d}(-\ln(T^*(\mathbf{d}(s)x_0) - t))\mathbf{d}(s)x^*\left(\frac{t}{e^s}, x_0\right)}{e^{-s}(T^*(\mathbf{d}(s)x_0) - t)},$$

where the group properties  $\tilde{\mathbf{d}}(s_1 + s_2) = \tilde{\mathbf{d}}(s_1)\tilde{\mathbf{d}}(s_2)$  and  $\mathbf{d}(s_1 + s_2) = \mathbf{d}(s_1)\mathbf{d}(s_2)$ ,  $\forall s_1, s_2 \in \mathbb{R}$  are utilized on the last step. Hence, we conclude

$$u_s(t) = \frac{\tilde{\mathbf{d}}(\ln(T^*(\mathbf{d}_\mu(s)x_0) - t))K\mathbf{d}(-\ln(T^*(\mathbf{d}(s)x_0) - t))x_s(t)}{T^*(\mathbf{d}(s)x_0) - t}.$$

By the formula (17), the obtained control  $t \rightarrow u_s(t)$  is the optimal for the problem (1)–(2) with the initial condition  $x(0) = \mathbf{d}_\mu(s)x_0$ . ■

### C. Analysis of the nonlinear homogeneous feedback

Inspired by MPC ideas [3], we consider also a feedback law of the form  $u(x_0) := u^*(0)$ , where  $t \rightarrow u^*(t)$  is a continuous-time optimal feedforward control. In the view of Corollary 2, the corresponding feedback has the form

$$u(x) = \frac{\tilde{\mathbf{d}}(\ln T^*(x))K\mathbf{d}(-\ln T^*(x))x}{T^*(x)}. \quad (21)$$

If the optimal control problem (1)–(2) satisfies the Bellman principle<sup>4</sup> (e.g., if  $\eta = 0$  and  $\tilde{R} = \text{const}$ ) then (21) corresponds to the *optimal feedback control* for (1)–(2). If, for the closed-loop system (2), (21), we have  $\dot{T}^*(x(t, x_0)) \approx -1$  or, equivalently,  $T^*(x(t, x_0)) \approx T^*(x_0) - t$  then the non-linear feedback (21) can also be interpreted as an approximation of the optimal control (17). In the general case, the formula (21) just defines a nonlinear control inspired by MPC ideas, so its stability, robustness and applicability needs to be studied.

*Corollary 4:* If all conditions of Corollary 2 are fulfilled, then the control  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  given by (21) is

- $\mathbf{d}_\mu$ -homogeneous, namely,  $\forall x \neq \mathbf{0}, \forall s \in \mathbb{R}$ , one has
$$Bu(\mathbf{d}_\mu(s)x) = e^{-s} \mathbf{d}_\mu(s)Bu(x), \quad (22)$$
- where  $\mathbf{d}_\mu(s) = e^{\frac{\mu}{2}s} \mathbf{d}(s)$  and  $\mu = \eta + \gamma$ ;
- continuously differentiable on  $\mathbb{R}^n \setminus \Sigma$  and
$$u^\top(x)u(x) \leq \rho(T^*(x))\lambda_{\max}(Z^{-\frac{1}{2}}K^\top KZ^{-\frac{1}{2}}) \quad (23)$$
- provided that  $Z := P^{-1}G_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top P^{-1} - P^{-1} \succ 0$ , where  $\Sigma$  is defined in Lemma 2 and
$$\rho(\sigma) = \begin{cases} \sigma^{\mu-2+\lambda_{\max}(G_{\mathbf{d}}^\top + G_{\mathbf{d}})} & \text{if } \sigma \geq 1, \\ \sigma^{\mu-2+\lambda_{\min}(G_{\mathbf{d}}^\top + G_{\mathbf{d}})} & \text{if } \sigma < 1; \end{cases}$$
- locally bounded on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  if  $\mu \in \mathbb{N}$  and (20) holds;
- continuous at  $\mathbf{0}$  if  $Z \succ 0$  and  $\mu > 2 - \lambda_{\min}(G_{\mathbf{d}}^\top + G_{\mathbf{d}})$ ;
- globally bounded if  $Z \succ 0$  and
$$\lambda_{\min}(G_{\mathbf{d}}^\top + G_{\mathbf{d}}) = \lambda_{\max}(G_{\mathbf{d}}^\top + G_{\mathbf{d}}) = 2 - \mu;$$
- belongs to  $C^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$  provided that (19) holds.

**Proof.** By Lemma 2 we have  $T^*$  is  $\mathbf{d}_\mu$ -homogeneous of degree 1, i.e.,  $T^*(\mathbf{d}_\mu(s)x) = e^s T^*(x)$ . Since  $u(x) = (T^*(x))^{\frac{\mu}{2}-1} \tilde{\mathbf{d}}(\ln T^*(x))K\mathbf{d}_\mu(-\ln T^*(x))x$  then using  $\mathbf{d}(s)B = B\tilde{\mathbf{d}}(s)$  we conclude that the identity (22) holds.

Notice that for any  $z \in \mathbb{R}^m$  we have that the function  $s \rightarrow \psi(s) := \tilde{z}^\top \tilde{\mathbf{d}}^\top(s)\tilde{\mathbf{d}}(s)z$  satisfies

<sup>4</sup>A tail of any optimal trajectory is optimal too.

$\lambda_{\min}(G_{\tilde{\mathbf{d}}}^\top + G_{\tilde{\mathbf{d}}})\psi(s) \leq \frac{d\psi(s)}{ds} \leq \lambda_{\max}(G_{\tilde{\mathbf{d}}}^\top + G_{\tilde{\mathbf{d}}})\psi(s)$ , and  $\psi(0) = z^\top z$ . Hence, we derive that  $\tilde{\mathbf{d}}^\top(s)\tilde{\mathbf{d}}(s) \leq e^{s\lambda_{\max}(G_{\tilde{\mathbf{d}}}^\top + G_{\tilde{\mathbf{d}}})}I_m$  for  $s \geq 0$  and  $\tilde{\mathbf{d}}^\top(s)\tilde{\mathbf{d}}(s) \leq e^{s\lambda_{\min}(G_{\tilde{\mathbf{d}}}^\top + G_{\tilde{\mathbf{d}}})}I_m$  for  $s < 0$ . Since  $x^\top \mathbf{d}_\mu^\top(-\ln T^*(x))Z\mathbf{d}_\mu(-\ln T^*(x))x = 1$  then  $x^\top \mathbf{d}_\mu^\top(-\ln T^*(x))K^\top K\mathbf{d}_\mu(-\ln T^*(x))x \leq \lambda_{\max}(Z^{-\frac{1}{2}}K^\top KZ^{-\frac{1}{2}})$  and  $u^\top(x)u(x) \leq \rho(T^*(x))\lambda_{\max}(Z^{-\frac{1}{2}}K^\top KZ^{-\frac{1}{2}})$ . Finally, using (23) we derive  $\|u(x)\| \rightarrow 0$  as  $\|x\| \rightarrow \mathbf{0}$  for  $\mu - 2 + \lambda_{\min}(G_{\tilde{\mathbf{d}}}^\top + G_{\tilde{\mathbf{d}}}) > 0$  and  $\sup_{x \in \mathbb{R}^n} \|u\| < +\infty$  for  $\lambda_{\min}(G_{\tilde{\mathbf{d}}}^\top + G_{\tilde{\mathbf{d}}}) = \lambda_{\max}(G_{\tilde{\mathbf{d}}}^\top + G_{\tilde{\mathbf{d}}}) = 2 - \mu$ .

From Lemma 2, case 4), we conclude that  $u$  is locally bounded on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  provided that  $\mu$  is an integer and  $G_{\tilde{\mathbf{d}}}$  admits the representation (20). Finally, using Lemma 2, case 3), we complete the proof. ■

According to the latter corollary, the feedback control (21) may be a discontinuous function of the state. In this paper, we study only the regular feedback laws  $u \in C(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{\mathbf{0}\})$ .

*Corollary 5 (Finite-time stability):* Let all conditions of Theorem 1 are fulfilled,  $\mu > 2 - \lambda_{\min}(G_{\tilde{\mathbf{d}}}^\top + G_{\tilde{\mathbf{d}}})$  and the condition (19) hold. If there exists  $q > 0$  such that

$$Z(A + BK + qG_{\mathbf{d}_\mu}) + (A + BK + qG_{\mathbf{d}_\mu})^\top Z \preceq 0 \quad (24)$$

then the closed-loop system (2), (21) is globally uniformly finite-time stable and  $x \rightarrow T^*(x)$  is a Lyapunov function:  $\dot{T}^*(x) \leq -q, \forall x \neq \mathbf{0}$ . Moreover, (24) holds and  $q = 1 - \lambda^*$  provided that  $G_{\tilde{\mathbf{d}}} = I_m$  and

$$\lambda^* := \lambda_{\max}\left(U^{-\frac{1}{2}}WU^{-\frac{1}{2}}\right) < 1, \quad (25)$$

$W = \eta P^{-1} + \eta Z - \mu P^{-1}BR^{-1}B^\top P^{-1}$ ,  $U = G_{\mathbf{d}_\mu}^\top Z + ZG_{\mathbf{d}_\mu}$ .

**Proof.** Since, under conditions of this corollary,  $T^*(x) = \|x\|_{\mathbf{d}_\mu, Z}$  is the canonical homogeneous norm induced by  $\|x\|_Z = \sqrt{x^\top Zx}$ , then by [32, Proposition 1] we have  $\frac{\partial T^*}{\partial x} = \frac{2x^\top \mathbf{d}_\mu^\top(-\ln T^*)Z\mathbf{d}_\mu(-\ln T^*)}{x^\top \mathbf{d}_\mu^\top(-\ln T^*)(ZG_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Z)\mathbf{d}_\mu(-\ln T^*)x} T^*$ . In this case, the time derivative of  $T^*$  along trajectories of the system (2), (21) is given by

$$\dot{T}^*(x) = \frac{2x^\top \mathbf{d}_\mu^\top(-\ln T^*)Z\mathbf{d}_\mu(-\ln T^*)(Ax + Bu(x))}{x^\top \mathbf{d}_\mu^\top(-\ln T^*)(ZG_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Z)\mathbf{d}_\mu(-\ln T^*)x} T^*.$$

Using  $\mathbf{d}_\mu(s)A = e^s A\mathbf{d}_\mu(s)$  and  $\mathbf{d}_\mu(s)B = e^{\frac{\mu}{2}s} B\tilde{\mathbf{d}}(s)$ ,  $\forall s \in \mathbb{R}$ , we derive  $\dot{T}^*(x) = \frac{2x^\top \mathbf{d}_\mu^\top(-\ln T^*)Z(A+BK)\mathbf{d}_\mu(-\ln T^*)x}{x^\top \mathbf{d}_\mu^\top(-\ln T^*)(ZG_{\mathbf{d}_\mu} + G_{\mathbf{d}_\mu}^\top Z)\mathbf{d}_\mu(-\ln T^*)x}$  and  $\dot{T}^*(x) \leq -q$  provided that (19) and (24) are fulfilled. Using the identity (11) one can be shown that  $Z(A + BK + G_{\mathbf{d}_\mu}) + (A + BK + G_{\mathbf{d}_\mu})^\top Z = \eta Z + \eta P^{-1} - \mu P^{-1}BR^{-1}B^\top P^{-1}$ . Hence, the inequality (24) holds for  $q = 1 - \lambda^*$  provided that (25) is fulfilled. ■

This corollary guarantees that the system (2), (21) is finite-time stable provided that  $\eta \leq 0$  and  $G_{\tilde{\mathbf{d}}} = I_m$ .

The robustness of the closed-loop system can be investigated using the concept of Input-to-State Stability (ISS) [37]. Recall that the system

$$\dot{x} = f(x, d), \quad (26)$$

is ISS if there exists  $\gamma \in \mathcal{K}$  and  $\beta \in \mathcal{KL}$  such that

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_{L^\infty((0,t), \mathbb{R}^m)}),$$

for all  $t \geq 0$ , for all  $x \in \mathbb{R}^n$  and for all  $d \in L^\infty$  where  $L^\infty$  denotes the space of Lebesgue measurable uniformly essentially bounded functions and the classes  $\mathcal{K}$  and  $\mathcal{KL}$  are defined in [37].

*Corollary 6 (Robustness):* Let all conditions of Corollary 5 be fulfilled and  $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be given by (21). Let  $f : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^n$  be a continuous function such that  $f(x, \mathbf{0}) = Ax + Bu(x), \forall x \in \mathbb{R}^n$ . If there exists a dilation  $\hat{\mathbf{d}}$  in  $\mathbb{R}^k$  such that  $f(\mathbf{d}_\mu(s)x, \hat{\mathbf{d}}(s)d) = e^{-s}\mathbf{d}_\mu(s)f(x, d), \forall x \in \mathbb{R}^n, \forall d \in \mathbb{R}^k, \forall s \in \mathbb{R}$ , then the system (26) is ISS.

This result immediately follows from [12, Corollary 2.21] and [32, Theorem 4]. Corollary 6 allows robustness of a rather large class of systems to be analyzed. For example, selecting  $f(x, d) = Ax + Bu(x + d_1) + d_2, d = (d_1^\top, d_2^\top)^\top$  we conclude that the nonlinear control system (2), (21) is always ISS of with respect to measurement noise and additive perturbations if the matrix  $G_{\mathbf{d}} + (\frac{\mu}{2} - 1)I_n$  is anti-Hurwitz.

*Remark 1:* Some issues of the digital implementation of the controller (21) with  $T^*(x) = \|x\|_{\mathbf{d}_\mu, Z}$  are already studied in [13], [21], where a specific numerical algorithm for computation of the canonical homogeneous norm is developed based on the bi-section method. Recently [33], this control scheme was successfully validated in practice for quad-rotor control.

#### IV. NUMERICAL EXAMPLE

Let us consider the optimal control problem (1)–(2) with  $n = 5, m = 2, \eta = 0, \tilde{R}(T - t) = \frac{1}{T-t}I_2$ ,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 1 \\ -2 & -5 & 0 & -2 & 0 \\ -2 & 0 & -1 & 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, G_{\mathbf{d}} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ -2 & -2 & 0 & 1 & 0 \\ -6 & 1 & -1 & 0 & 1 \end{pmatrix}.$$

The pair  $\{A, B\}$  is controllable and Assumption 2 is fulfilled for  $G_{\tilde{\mathbf{d}}} = I_2$ . The solution of (11) is

$$P = \frac{1}{120} \begin{pmatrix} 6 & -15 & 0 & 44 & -33 \\ -15 & 40 & 0 & -125 & 85 \\ 0 & 0 & 40 & 0 & -100 \\ 44 & -125 & 0 & 426 & -257 \\ -33 & 85 & -100 & -257 & 464 \end{pmatrix}.$$

and by Corollary 2, the optimal control is given by

$$u^*(t) = K\mathbf{d}(-\ln(\|x(0)\|_{\mathbf{d}_\mu, Z} - t))x, \quad (27)$$

where  $x \in \mathbb{R}^5, K = -B^\top P^{-1}$  and  $\|\cdot\|_{\mathbf{d}_\mu, Z}$  is the canonical  $\mathbf{d}_\mu$ -homogeneous norm induced by the norm  $\|x\|_Z = \sqrt{x^\top Zx}$  with  $G_{\mathbf{d}_\mu} = G_{\mathbf{d}} + \frac{1}{2}I_5$  and

$$Z = P^{-1}G_d + G_d P^{-1} = 12 \cdot \begin{pmatrix} 1291 & 496 & 12 & 42 & 3 \\ 496 & 209 & -4 & 19 & -1 \\ 12 & -4 & 19 & 0 & 4 \\ 42 & 19 & 0 & 2 & 0 \\ 3 & -1 & 4 & 0 & 1 \end{pmatrix} \succ 0.$$

The feedback (21) becomes

$$\tilde{u}(x) = Kd(-\ln(\|x\|_{d_\mu, Z}))x. \quad (28)$$

Since  $Z \succ 0$  and  $ZG_{d_\mu} + G_{d_\mu}^\top Z \succ 0$  then, Corollary 4 yields  $\tilde{u} \in C(\mathbb{R}^5) \cap C^1(\mathbb{R}^5 \setminus \{0\})$ . Corollary 5 holds with  $\lambda^* = 0$ , i.e. (28) is a finite-time stabilizing feedback.

The control laws are assumed to be implemented in a digital device as follows:

$$u(t) = u^*(t_i) \quad (\text{resp. } u(t) = \tilde{u}(x(t_i))) \quad t \in [t_i, t_{i+1}),$$

where  $t_i = ih, i = 0, 1, \dots$  and  $h > 0$  is a sampling period.

For  $x(0) = x_0 := Z^{-\frac{1}{2}}(10000)^\top$  the optimal time is  $T^*(x_0) = 1$  and the optimal cost is  $J^* = 1 + x_0^\top P^{-1}x_0 \approx 1.1353$ . The system (2) with the control laws given by (27) and (28) is simulated for the sampling period  $h = 0.001$ . The controller (27) is defined only on  $[0, 1]$ . The feedback law (28) is defined for all  $t > 0$  and simulated on the time interval  $[0, 1.5]$ . In the latter case, the cost value and the settling time are estimated numerically  $J \approx 1.14$  and  $T \approx 0.983$ . The canonical homogeneous norm in (28) is computed using the bisection method (see, [13] for more details). First, the non-perturbed case is studied (Fig. 1 and 2). The non-linear feedback control (28) is sub-optimal, so the closed-loop system has a trajectory close to the optimal one generated by the control (27). Next, to study the robustness of the controllers, the system with measurement noise  $\hat{x}_i = x_i + 10^{-5}\cos(100t), i = 1, \dots, 5$  is simulated (Fig. 3, 4). The optimal linear time-varying feedback (27) is very sensitive to noises, while the non-linear one (28) shows reasonable behavior. The advantages of the non-linear feedback law are: 1) it can be implemented in practice for set-point tracking; 2) the closed-loop system with the control (28) is robust (ISS) with respect to a rather large class of perturbations (Corollary 6). The main disadvantage is the necessity to compute the homogeneous norm  $\|x\|_{d_\mu, Z}$  numerically if  $n \geq 3$ .

## V. CONCLUSIONS

The paper presents a rather simple solution of an energetically optimal finite-time stabilization problem for a class of the generalized homogeneous linear systems. Theorem 1 can be utilized for an optimal prescribed-time stabilization in the sense of [2]. A sub-optimal feedback inspired by MPC ideas is studied as well. In particular, the finite-time stability and the robustness (ISS property) of the closed-loop system are analyzed. The theoretical results are supported by numerical simulations presenting a comparison of optimal and sub-optimal controllers.

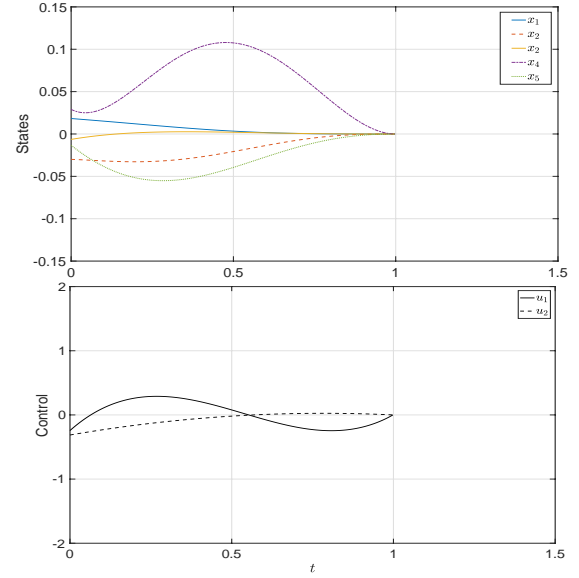


Fig. 1. The optimal control system (2), (27) (unperturbed case)

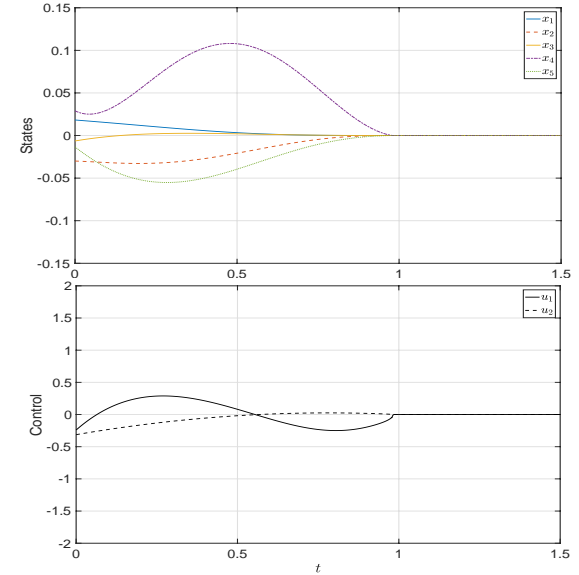


Fig. 2. The sub-optimal control system (2), (28) (unperturbed case)

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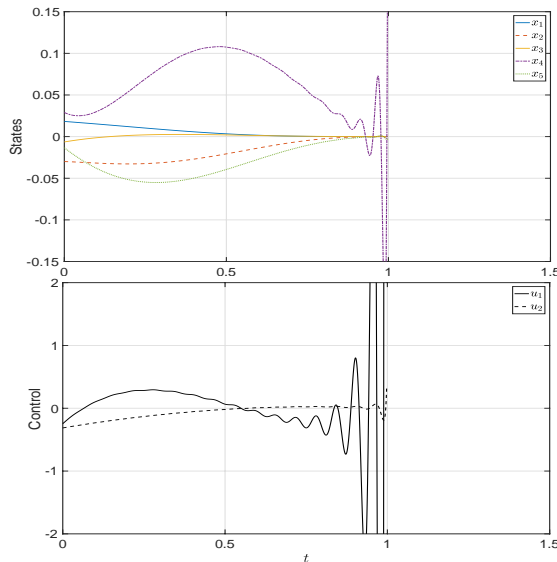


Fig. 3. The optimal control system (2), (27) (perturbed case)

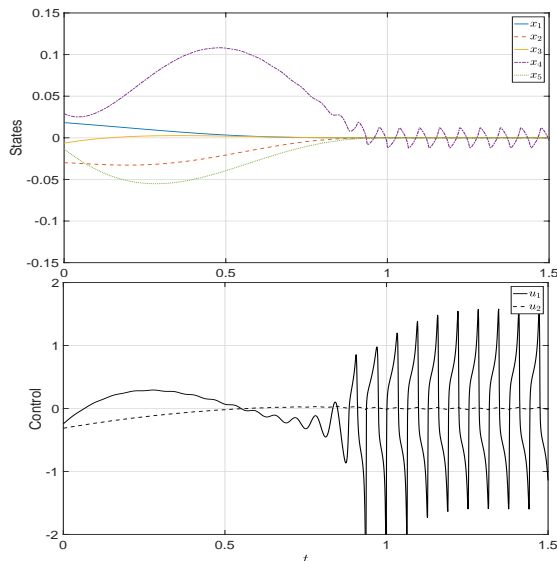


Fig. 4. The sub-optimal control system (2), (28) (perturbed case)

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