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► **To cite this version:**

Yu Zhou, Andrey Polyakov, Gang Zheng. Generalized homogeneous control with integral action. International Journal of Robust and Nonlinear Control, 2023, 33 (8), pp.4345-4366. 10.1002/rnc.6612 . hal-03993724

HAL Id: hal-03993724

<https://inria.hal.science/hal-03993724v1>

Submitted on 17 Feb 2023

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ARTICLE TYPE

Generalized Homogeneous Control with Integral Action

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Summary

A generalized homogeneous control with integral action for a multiple-input plant operating under uncertainty conditions is designed. The stability analysis is essentially based on a special version of the non-smooth Lyapunov function theorem for differential equations with discontinuous right-hand sides. A Lyapunov function for analysis of the closed-loop system is presented. For negative homogeneity degree, this Lyapunov function becomes a strict Lyapunov function allowing an advanced analysis to be provided. In particular, the maximum control magnitude and the settling-time of the closed-loop system are estimated and a class of disturbances to be rejected by the control law is characterized. The control parameters are tuned by solving a system of Linear Matrix Inequalities (LMIs), whose feasibility is proved at least for small (close to zero) homogeneity degrees. The theoretical results are illustrated by numerical simulations.

KEYWORDS:

Homogeneous system, integral control, finite-time, strict Lyapunov

1 | INTRODUCTION

The presence of unknown non-vanishing perturbations and uncertainties can lead the system to an unknown equilibrium. A continuous static feedback controller cannot stabilize a system with a matched non-vanishing disturbance to the origin. Sliding mode control, disturbance observer, and integral control are three common methods for compensating the matched unknown non-vanishing (at the origin) perturbations. Theoretically sliding mode control can compensate matched bounded non-vanishing perturbations. However, in practice, discontinuous control action of the sliding mode control evokes a "chattering", which may even damage a plant in some cases. Integral control action is used in practice in order to eliminate an unknown constant perturbation. In some cases, it can be utilized to reduce the chattering of the sliding mode controller, as for example, in the case of the super-twisting algorithm¹.

Integral control has an outstanding history in the context of PID (Proportional-Integral-Derivative) control theory. Linear MIMO (Multiple-Input Multiple-Output) control systems with integral actions are studied, for example, in^{2,3,4}. Linear PI control for nonlinear system is considered in^{5,6}. Some nonlinear integral actions are proposed in^{7,8}.

Homogeneity is a dilation symmetry, which is widely used in control theory for system analysis, regulation and observer design^{9,10,11,12,13,14,15}. Such interest in the homogeneity concept is caused by various useful features of homogeneous systems. In particular, local stability of homogeneous system means the global one; the rate of convergence of homogeneous systems can be assessed by its homogeneity degree¹⁶; homogeneous systems are robust with respect to a large class of external perturbations^{17,18,19,20} and time delays²¹, etc. The most common type of homogeneity used in control design is the so-called weighted homogeneity^{13,19,22}.

In^{23,24}, a homogeneous proportional-integral-derivative control for a single-input-single-output(SISO) system is proposed and its performance is validated by experiments with a robot manipulator. In²⁵, a weighted homogeneous integral control with positive degree on second-order and third-order system is proposed. A weighted homogeneous integral control based on implicit Lyapunov function for a Multiple-Input Multiple-Output (MIMO) system is proposed in²⁶. The use of the weighted homogeneity-based technique for linear MIMO plants requires specific decomposition of the system²⁷ leading to some cumbersome constructions²⁶.

The so-called linear homogeneity studied in this paper covers the weighted homogeneity as a particular case. It is shown in²⁸, all important results known for standard and weighted homogeneous systems hold for linearly homogeneous systems as well. The linear homogeneity is shown to be useful for the homogeneous MIMO control systems design^{28, 29}. The generalized homogeneous control with integral action is proposed in¹⁵ and was utilized for an upgrade of existing linear control law to a generalized homogeneous one in³⁰. The stability of the system is studied based on LaSalle's invariance principle, which does not allow an advanced stability analysis such as settling time estimation and characterization of the perturbations to be rejected.

A strict non-smooth Lyapunov for the super-twisting system (being planar and weighted homogeneous) is proposed in³¹. It allows the algebraic stability/stabilization condition, the simple scheme for the settling-time estimation as well as for the robustness analysis to be provided. In this paper we extend the strict Lyapunov function from³¹ to MIMO generalized homogeneous system using the so-called canonical homogeneous norm (see Definition 3). Unlike the paper³¹, the singularity set of the non-smooth Lyapunov function in the MIMO case is an n -dimensional manifold of the space \mathbb{R}^{n+m} . This complicates the stability analysis and requires an application of a special non-smooth Lyapunov function theorem. However, similarly to³¹, this allows the procedure of the control parameters tuning to be formalized in terms of LMIs and an advanced convergence/robustness analysis to be provided.

The main contributions of the paper are as follows.

1. The stability analysis based on a non-smooth Lyapunov function, introduced in³¹ for a planar weighted homogeneous system, is extended to the multidimensional case using the canonical homogeneous norm.
2. A generalized homogeneous integral control is proposed for finite-time stabilization of a linear MIMO system under non-vanishing matched and vanishing (at the origin) mismatched perturbations. A strict non-smooth Lyapunov function is constructed. It is shown that control parameters can be tuned by means of solving a system of LMIs.
3. The settling time and the maximal magnitude of the control are estimated.
4. A class of perturbations, which can be rejected by means of the homogeneous integral control, is characterized.

The preliminary version of this paper was presented at the Conference on Decision and Control 2021³². The key differences of this paper are as follows:

- A new generalized homogeneous integral controller is introduced.
- A less conservative LMIs for control parameters tuning are obtained and their feasibility is proven.
- Detailed proofs of all claims are presented.
- New numerical examples, better illustrating theoretical results, are presented.

The paper is organized as follows: Section 2 presents the basic facts about the homogeneous systems and homogeneous control. The problem statement is given in Section 3. Section 4 presents the main results of this paper: a non-smooth Lyapunov function theorem for multi-dimensional differential equations with discontinuous right-hand sides; generalized homogeneous control with integral action and analysis of the closed-loop system via a new strict Lyapunov function. In Section 5, a third-order MIMO linear system with a generalized homogeneous integral controller is studied, and numerical simulations of the proposed control for an omnidirectional mobile robot are presented. Proofs of all claims are given in Appendix.

NOTATION

\mathbb{R} is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$; $\|x\| = \sqrt{x^\top P x}$ is the weighted Euclidean norm in \mathbb{R}^n , where a positive definite matrix $P = P^\top \in \mathbb{R}^{n \times n}$ defined dependently on the context; $\mathbf{0}$ denotes the zero element of a vector space (e.g.,

$\mathbf{0} \in \mathbb{R}^n$ is the zero vector but $\mathbf{0} \in \mathbb{R}^{n \times n}$ is the zero matrix; $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix; $\text{diag} \{ \sigma_i \}_{i=1}^n$ is the diagonal matrix with elements σ_i ; $P > 0$ ($< 0, \geq 0, \leq 0$) for $P \in \mathbb{R}^{n \times n}$ means that the matrix P is symmetric and positive (negative) definite (semidefinite); $C(X, Y)$ denotes the space of continuous functions $X \rightarrow Y$, where X, Y are subsets of normed vector spaces; $C^p(X, Y)$ is the space of functions continuously differentiable at least up to the order p ; $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ represent the minimal and maximal eigenvalue of a matrix $P = P^T$; $\max \{x, y\}$ means the maximal value among x and y ; for $P \geq 0$ the square root of P is a matrix $M = P^{\frac{1}{2}}$ such that $M^2 = P$; if $A \in \mathbb{R}^{n \times n}$ then $\|A\| = \sup_{y \neq 0} \frac{\|Ay\|}{\|y\|}$; a function $\sigma : [0, +\infty) \rightarrow [0, +\infty)$ belongs to the class \mathcal{K} if it is positive definite, continuous and strictly increasing; a class \mathcal{K} function σ belongs to the class \mathcal{K}_∞ if $\sigma(s) \rightarrow +\infty$ as $s \rightarrow +\infty$; a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{KL} if it is \mathcal{K} with respect to the first argument and strictly decreasing to zero with respect to the second argument; L^∞ denotes the set of essentially bounded measurable functions $\mathbb{R} \rightarrow \mathbb{R}^k$ and $\|\eta\|_{L^\infty(t_0, t_1)} = \text{ess sup}_{t \in (t_0, t_1)} \|\eta(t)\|$.

2 | PROBLEM STATEMENT

Let us consider the following system:

$$\dot{x} = Ax + B(u + \tilde{g}_2(t)) + g_1(t, x), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is a control to be designed, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are system matrices, the unknown function $g_1 : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be locally Lebesgue measurable and locally bounded, the unknown function $\tilde{g}_2 : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is assumed to be locally Lipschitz continuous. The function \tilde{g}_2 defines matched (possibly non-vanishing) perturbations. The function g_1 , which may contain mismatched uncertainties, is assumed to be vanishing as x goes to zero, e.g., drag or damping forces in the case of a mechanical system. Notice that the latter vanishing condition is necessary for the stabilizability of the system (otherwise $x = 0$ is not an equilibrium). More restrictions to admissible perturbations g_1 and \tilde{g}_2 are given in Section 4.

We study the considered system under the following main assumption.

Assumption 1. The pair of (A, B) is controllable and all states x are available/measurable.

Our aim is to design a control law with integral action

$$u = u_1(x) + \int_0^t u_2(x(\tau)) d\tau, \quad u_i : \mathbb{R}^n \rightarrow \mathbb{R}^m, i = 1, 2, \quad (2)$$

such that the closed-loop unperturbed system (1)-(2) is homogeneous of negative degree (in a generalized sense described below) and asymptotically stable. We also need to characterize perturbations g_1 and \tilde{g}_2 , which can be rejected by this controller.

3 | PRELIMINARIES

3.1 | Homogeneous Systems

By definition, the homogeneity is a dilation symmetry^{33,34,9,35,15}. A dilation^{9,36} is a one-parameter group $\mathbf{d}(s), s \in \mathbb{R}$ of transformations satisfying the limit property: $\lim_{s \rightarrow s^\infty} \|\mathbf{d}(s)x\| = e^{s^\infty}, s^\infty = \pm\infty, \forall x \neq \mathbf{0}$.

Examples of dilations in \mathbb{R}^n are

- Uniform dilation (L. Euler, 18th century): $\mathbf{d}(s) = e^s I_n$, where I is the identity matrix \mathbb{R}^n ;
- Weighted dilation³³: $\mathbf{d}(s) = \begin{pmatrix} e^{r_1 s} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{r_n s} \end{pmatrix}$, where $r_i > 0, i = 1, 2, \dots, n$.
- Geometric dilation is a flow generated by unstable C^1 vector field in \mathbb{R}^n (see^{34,9}).

In this paper we deal only with the so-called linear (geometric) dilation in \mathbb{R}^n , which is defined as follows

$$\mathbf{d}(s) = e^{sG_d} := \sum_{i=0}^{+\infty} \frac{s^i G_d^i}{i!}, \quad s \in \mathbb{R}^n, \quad (3)$$

where $G_d \in \mathbb{R}^n$ is said to be an anti-Hurwitz¹ matrix called the generator of the dilation \mathbf{d} .

Definition 1.²⁸ A dilation \mathbf{d} is monotone if $s \rightarrow \|\mathbf{d}(s)x\|$ is a monotone increasing function for any $x \neq \mathbf{0}$.

Monotonicity of the dilation may depend of the norm in \mathbb{R}^n . Any linear dilation in \mathbb{R}^n is monotone²⁸ with respect to the norm in \mathbb{R}^n is defined as follows

$$\|x\| = \sqrt{x^\top P x}, \quad P G_d + G_d^\top P > 0, \quad 0 < P = P^\top \in \mathbb{R}^{n \times n}. \quad (4)$$

The linear dilation introduces an alternative norm topology in \mathbb{R}^n by means of a homogeneous norm (see, e.g.,³⁷ for an example of a homogeneous norm induced by the weighted dilation).

Definition 2.²⁸ The function $\|\cdot\|_d : \mathbb{R}^n \rightarrow [0, +\infty)$ defined as $\|\mathbf{0}\|_d = 0$ and

$$\|x\|_d = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0}$$

is called the canonical homogeneous norm in \mathbb{R}^n , where \mathbf{d} is a monotone dilation in \mathbb{R}^n .

For any linear monotone dilation in \mathbb{R}^n , the canonical homogeneous norm is continuous on \mathbb{R}^n and locally Lipschitz continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Moreover²⁸, it is differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ provided that $\|\cdot\|$ is given by (4):

$$\frac{\partial \|x\|_d}{\partial x} = \|x\|_d \frac{x^\top \mathbf{d}^\top (-\ln \|x\|_d) P \mathbf{d} (-\ln \|x\|_d)}{x^\top \mathbf{d}^\top (-\ln \|x\|_d) P G_d \mathbf{d} (-\ln \|x\|_d) x}, \quad x \neq \mathbf{0}. \quad (5)$$

Definition 3.⁹ A vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (resp. a function $h : \mathbb{R}^n \rightarrow \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$ if

$$\begin{aligned} f(\mathbf{d}(s)x) &= e^{\mu s} \mathbf{d}(s) f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}, \\ (\text{resp. } h(\mathbf{d}(s)x)) &= e^{\mu s} h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}, \end{aligned}$$

where \mathbf{d} is a linear dilation in \mathbb{R}^n .

In²⁸ it is shown that any \mathbf{d} -homogeneous system of degree μ

$$\dot{x} = f(x), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n \quad (6)$$

is diffeomorphic on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ to a standard homogeneous system. This means that all important results known for standard and weighted homogeneous systems hold for linear homogeneous systems as well. For instance, the results obtained in^{16,13} imply that if the system (6) is asymptotically stable then this systems is

- globally uniformly finite-time stable² for $\mu < 0$;
- globally uniformly exponentially stable for $\mu = 0$;
- globally uniformly nearly fixed-time stable³ for $\mu > 0$.

3.2 | Homogeneous control algorithm

Let us consider the following linear system

$$\dot{x} = Ax + Bu(x) + g(t, x), \quad t > 0, \quad x(0) = x_0 \quad (7)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the feedback control to be designed, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are system matrices, a locally measurable function $g : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ models parametric uncertainties and exogenous perturbations.

The system (7) is said to be \mathbf{d} -homogeneously stabilizable of degree $\mu \in \mathbb{R}$ if there exists a feedback $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $f(x) = Ax + Bu(x)$ is \mathbf{d} -homogeneous of degree μ , and the closed-loop system is globally asymptotically stable. The disturbance-free (i.e., $g = \mathbf{0}$) system (7) is \mathbf{d} -homogeneously stabilizable with a degree $\mu \neq 0$ if and only if the pair $\{A, B\}$ is controllable²⁹.

Theorem 1.²⁹ If the linear equation

$$AG_0 + BY_0 = G_0A + A, \quad G_0B = \mathbf{0}. \quad (8)$$

¹A matrix $G \in \mathbb{R}^{n \times n}$ is anti-Hurwitz if the matrix $-G$ is Hurwitz.

²it is Lyapunov stable and $\exists T(x_0) : \|x(t)\| = 0, \forall t \geq T(x_0), \forall x_0 \in \mathbb{R}^n$.

³it is Lyapunov stable and $\forall r > 0, \exists T_r > 0 : \|x(t)\| < r, \forall t \geq T_r$, independently of $x_0 \in \mathbb{R}^n$.

has a solution $G_0 \in \mathbb{R}^{n \times n}$ and $Y_0 \in \mathbb{R}^{m \times n}$ such that $G_0 - I_n$ is invertible, then for any $\mu \geq -1$ such that $G_d = I_n + \mu G_0$ is anti-Hurwitz, the *disturbance-free* system (7) can always be homogeneously stabilized by the following control

$$u = u_{hom}(x) = K_0 x + \|x\|_d^{1+\mu} K d(-\ln \|x\|_d) x, \quad K = Y X^{-1} \quad (9)$$

with $K_0 = Y_0(G_0 - I_n)^{-1}$ and any $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{m \times n}$ satisfying the following linear system

$$\begin{cases} X A_0^\top + A_0 X + Y^\top B^\top + B Y + \rho (X G_d^\top + G_d X) = \mathbf{0} \\ X G_d^\top + G_d X > 0, \quad X > 0 \end{cases} \quad (10)$$

where $A_0 = A + B K_0$, $\rho > 0$. The canonical homogeneous norm $\|\cdot\|_d$ is induced by the norm $\|x\| = \sqrt{x^\top X^{-1} x}$. Moreover, if

$$2 \frac{g^\top(t, x) \mathbf{d}^\top (-\ln \|x\|_d) X^{-1} \mathbf{d} (-\ln \|x\|_d) x}{x^\top \mathbf{d}^\top (-\ln \|x\|_d) X^{-1} G_d \mathbf{d} (-\ln \|x\|_d) x} < \rho \|x\|_d^\mu, \quad \forall t \geq 0, \quad (11)$$

then the perturbed system (7) is

- globally uniformly finite-time stable for $\mu < 0$;
- globally uniformly exponentially stable for $\mu = 0$;
- globally nearly fixed-time stable for $\mu > 0$.

□

The proof of this theorem is based on the use of the canonical homogeneous norm as a Lyapunov function of the closed-loop system (7), (9) with $g = \mathbf{0}$:

$$\frac{d}{dt} \|x(t)\|_d = \|x\|_d \frac{x^\top \mathbf{d}^\top (-\ln \|x\|_d) X^{-1} \mathbf{d} (-\ln \|x\|_d) (A x + B u_{hom}(x))}{x^\top \mathbf{d}^\top (-\ln \|x\|_d) X^{-1} G_d \mathbf{d} (-\ln \|x\|_d) x} = \|x\|_d^{1+\mu} \frac{x^\top \mathbf{d}^\top (-\ln \|x\|_d) X^{-1} (A_0 + B K) \mathbf{d} (-\ln \|x\|_d) x}{x^\top \mathbf{d}^\top (-\ln \|x\|_d) X^{-1} G_d \mathbf{d} (-\ln \|x\|_d) x} = -\rho \|x(t)\|_d^{1+\mu}. \quad (12)$$

The disturbance g rejected by the control law is characterized by the inequality (11), which is obtained from (5) and (12) using

$$\frac{d \|x(t)\|_d}{dt} = -\rho \|x(t)\|_d^{1+\mu} + 2 \|x\|_d \frac{g^\top(t, x) \mathbf{d}^\top (-\ln \|x\|_d) X^{-1} \mathbf{d} (-\ln \|x\|_d) x}{x^\top \mathbf{d}^\top (-\ln \|x\|_d) X^{-1} G_d \mathbf{d} (-\ln \|x\|_d) x}.$$

The formula (12) implies that the canonical homogeneous norm is a Lyapunov function for the closed-loop system. The feasibility of the algebraic system is proven in²⁷. The existence of an appropriate solution for (8) is studied in²⁹. It can be shown that the matrix $G_0 - I$ is invertible for any solution of (8). Obviously, for a sufficiently small $|\mu|$ we have $G_d = I + \mu G_0$ is anti-Hurwitz.

4 | MAIN RESULTS

Our aim is to design a generalized homogeneous control with integral action for MIMO system (1). For this purpose we design a strict homogeneous non-smooth Lyapunov function.

4.1 | Non-smooth Lyapunov function theorem

In many cases (see, e.g.,^{12, 31, 38}), the use of a non-smooth Lyapunov function simplifies the stability analysis of a closed-loop control system. However, it may require special restrictions to the system. Inspired by^{39, 40} we prove (see Appendix) the following theorem.

Theorem 2. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that

- $\exists \alpha, \beta \in \mathcal{K}^\infty : \alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \forall x \in \mathbb{R}^n$;
- $V \in C(\mathbb{R}^n, \mathbb{R}) \cap C^1(\mathbb{R}^n \setminus S, \mathbb{R})$, where $S \subset \mathbb{R}^n$ is $n - m$ -dimensional manifold in \mathbb{R}^n , $1 \leq m < n$;
- $f \in C(\mathbb{R}^n \setminus S, \mathbb{R}^n)$ is locally bounded on \mathbb{R}^n ;
- $\frac{\partial V}{\partial x} f(x) \leq -W(x)$ for $x \in \mathbb{R}^n \setminus \{S \cup \mathbf{0}\}$, where $W \in C(\mathbb{R}^n, \mathbb{R})$ is a positive definite (resp., *semidefinite*) function.

Then the system

$$\dot{x} = f(x), \quad t > 0$$

is globally uniformly asymptotically (*resp.*, *Lyapunov*) stable⁴ provided that any trajectory of the system may cross the set $S \setminus \{\mathbf{0}\}$ only at isolated instances of time, namely, if $\exists \bar{t} > 0 : x(\bar{t}) \in S \setminus \{\mathbf{0}\}$ then $\exists r = r(\bar{t}) > 0$ such that $x(t) \notin S \setminus \{\mathbf{0}\}$ for all $t \in (\bar{t}, \bar{t} + r)$.

Notice that the simplest sufficient condition to guarantee that the manifold $S := \{x \in \mathbb{R}^n : s(x) = 0\}$, $s \in C^1(\mathbb{R}^n)$ does not contain non-zero trajectories of the system with $f \in C(\mathbb{R}^n)$ is

$$\frac{\partial s(x)}{\partial x} f(x) \neq 0 \quad \text{for } x = 0.$$

The latter condition simply means that trajectories of the systems cross the manifold S by avoiding to slide on it.

Since in Theorem 2 the vector field f may be discontinuous on the manifold S , then the presented theorem also holds for systems with discontinuous right-hand sides. In the latter case, the solutions of the system are understood in the sense of Filippov⁴¹.

The finite/fixed-time stability analysis by means of a non-Lipschitz Lyapunov function is based on the following corollary.

Corollary 1. If under conditions of Theorem 2 there exist $\alpha > 0$ and $\rho \geq 0$ such that

$$\frac{\partial V}{\partial x} f(x) \leq -qV^\rho(x) \quad \text{for } x \in \mathbb{R}^n \setminus \{S \cup \mathbf{0}\}$$

then the origin of the system is

- globally uniformly finite-time stable for $\rho < 1$ and $x(t) = 0$ for all $t \geq \frac{V^{1-\rho}(x(0))}{(1-\rho)q}$;
- globally uniformly asymptotically stable for $\rho = 1$ and $\alpha(\|x(t)\|) \leq e^{-qt} \beta(\|x(0)\|)$ for all $t \geq 0$;
- globally uniformly nearly fixed-time stable for $\rho > 1$ and $V(x(t)) \leq r$ for all $t \geq \frac{1}{r^{\rho-1}(\rho-1)\alpha}$ and for any $r > 0$.

□

Remark 1. Theorem 2 and Corollary 1 remain valid for a time-varying system $\dot{x} = f(t, x)$ provided that the term $\frac{\partial V}{\partial x} f(x)$ is replaced with $\sup_{t \in \mathbb{R}} \frac{\partial V}{\partial x} f(t, x)$.

4.2 | Generalized homogeneous integral control

Let us consider the following integral controller for the system (7):

$$\begin{aligned} u(t, x) &= K_0 x + u_{hom}(x) + \int_0^t u_{int}(x(s)) ds, \\ u_{hom}(x) &= \|x\|_{\mathbf{d}}^{1+\mu} \mathbf{K} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x, \\ u_{int}(x) &= \|x\|_{\mathbf{d}}^{1+2\mu} \tilde{\mathbf{K}} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x, \end{aligned} \quad (13)$$

where K_0 and K are given in Theorem 1, and $\tilde{\mathbf{K}} \in \mathbb{R}^{m \times n}$ is the gain to be defined.

Remark 2. The control law (13) differs from the algorithms considered in^{15,26,42}, since the gain $\tilde{\mathbf{K}}$, in our case, is independent of x . This simplifies the practical implementation of the algorithm and allows the classical linear PI algorithms to be easily obtained taking $\mu = 0$ (see below).

We consider a special class of perturbations as in the system (1). The disturbance function g in (7) is assumed to have the following representation

$$g(t, x(\cdot)) = g_1(t, x) + B \tilde{g}_2(t), \quad \tilde{g}_2(t) = p + \int_0^t g_2(s, x(s)) ds, \quad t \geq 0 \quad (14)$$

where $g_1 : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g_2 : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ are assumed to be locally measurable and locally bounded unknown functions (satisfying certain condition given below) and $p \in \mathbb{R}^m$ is an unknown constant vector. By Newton-Leibniz formula, the representation (14) holds for any locally Lipschitz continuous function \tilde{g}_2 taking $p = \tilde{g}_2(0)$ and $g_2 = \frac{d\tilde{g}_2}{dt}$. Notice that \tilde{g}_2

⁴In the case of discontinuous f solutions are understood in the sense of Filippov.

and g_2 may depend on the state x . For example, if $\tilde{g}_2(t) = \gamma(t, x(t))$, where the function $\gamma : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuously differentiable (or, at least, locally Lipschitz continuous), then the representation (14) holds for $p = \gamma(0, x_0)$ and $g_2(t, x) = \frac{\partial \gamma(t, x)}{\partial t} + \frac{\partial \gamma(t, x)}{\partial x} (Ax + B(u + \gamma(t, x)) + g_1(t, x))$.

With the notation

$$x_{n+1} = p + \int_0^t u_{int}(x(s)) + g_2(s, x(s)) ds \quad (15)$$

the closed-loop system becomes

$$\begin{aligned} \dot{x} &= Ax + B(K_0 x + u_{hom}(x) + x_{n+1}) + g_1(t, x), \\ \dot{x}_{n+1} &= u_{int}(x) + g_2(t, x). \end{aligned} \quad (16)$$

For $g_1 = \mathbf{0}$, $g_2 = \mathbf{0}$, the system (16) is $\tilde{\mathbf{d}}$ -homogeneous of degree $\mu \geq -0.5$ with

$$\tilde{\mathbf{d}}(s) = \begin{pmatrix} e^{sG_d} & \mathbf{0} \\ \mathbf{0} & e^{s(1+\mu)I_m} \end{pmatrix}. \quad (17)$$

Remark 3. For a single-input system $m = n = 1$, the homogeneous integral control (13) becomes a classical PI algorithm for $\mu = 0$, $G_d = I_n$. It becomes the super-twisting algorithm^{11, 1} for $n = 1$, $m = 1$, $\mu = -0.5$, $G_d = 1$, $G_d = \begin{pmatrix} e^s & 0 \\ 0 & e^{0.5s} \end{pmatrix}$:

$$\begin{aligned} \dot{x}_1 &= K \sqrt{|x_1|} \text{sign}(x_1) + x_2 + g_1(t, x), \\ \dot{x}_2 &= \tilde{K} \text{sign}(x_1) + g_2(t, x). \end{aligned}$$

The very simple (non-Lipschitz) Lyapunov function

$$V = \tilde{\xi}^\top \tilde{P} \tilde{\xi}, \quad \tilde{\xi} = \begin{pmatrix} |x_1|^{1+\mu} \text{sign}(x_1) \\ x_2 \end{pmatrix}, \quad 0 < \tilde{P} = \tilde{P}^\top \in \mathbb{R}^{2 \times 2} \quad (18)$$

was designed in³¹ for the stability and robustness analysis of the super-twisting system. We extend this result to the MIMO system (7) and (13) using the linear (geometric) homogeneity and the canonical homogeneous norm.

Theorem 3 (The case of a constant perturbation). Let $g_1 = \mathbf{0}$, $g_2 = \mathbf{0}$ and the matrices G_d , K_0 , A_0 , K , X , Y be defined as in Theorem 1 for some $\mu \geq -0.5$ and some $\rho > 0$. If the matrix $\tilde{Y} \in \mathbb{R}^{m \times n}$ and the symmetric matrix $Q \in \mathbb{R}^{m \times m}$ satisfy the following system of LMIs:

$$W_1 = \begin{bmatrix} \frac{P}{1+\mu} & \tilde{Y}^\top \\ \tilde{Y} & Q \end{bmatrix} > 0 \quad (19)$$

$$W(q_i) = \begin{bmatrix} -\rho P & \frac{PB}{q_i} + \frac{(\hat{F} - \rho \tilde{F} + I)^\top \tilde{Y}^\top}{2} & \tilde{Y}^\top \\ * & M_i & \mathbf{0} \\ * & * & -Q \end{bmatrix} < 0 \quad (20)$$

$$\begin{bmatrix} \gamma B^\top PB & \tilde{Y} \tilde{F} \\ \tilde{F}^\top \tilde{Y}^\top & \gamma P \end{bmatrix} \geq 0, \quad \gamma > 0 \quad (21)$$

where $P = X^{-1}$, $R = PG_d + G_d^\top P > 0$, $\hat{F} = A_0 + BK$, $\tilde{F} = ((1 + \mu)I_n - G_d)$, $q_1 = \lambda_{\min}(P^{-1/2}RP^{-1/2})$, $q_2 = \lambda_{\max}(P^{-1/2}RP^{-1/2})$ and $M_i = \frac{\gamma B^\top PB}{q_i} + \frac{B^\top \tilde{Y} \tilde{F} + \tilde{F}^\top \tilde{Y}^\top B}{2q_i} + \frac{\tilde{Y} B + B^\top \tilde{Y}^\top}{2}$, $i = 1, 2$, then the closed-loop system (7), (13) with $\tilde{K} = Q^{-1} \tilde{Y}$ is

- globally uniformly finite-time stable for $\mu < 0$;
- globally uniformly exponentially stable for $\mu = 0$;
- globally uniformly nearly fixed-time stable $\mu > 0$.

□

For the proof of this theorem (see Appendix) we show that the system (7) and (13) has a Lyapunov function of the form

$$V(\xi) = \frac{1}{2} \xi^\top \begin{bmatrix} \frac{P}{1+\mu} & \tilde{Y}^\top \\ \tilde{Y} & Q \end{bmatrix} \xi \quad (22)$$

where $\xi = \begin{pmatrix} \|x\|_{\mathbf{d}}^{1+\mu} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x \\ x_{n+1} \end{pmatrix}$ and

$$\dot{V}(\xi) \leq -\tilde{\rho} \|x\|_{\mathbf{d}}^{\mu} V(\xi), \quad \xi \neq \mathbf{0}, \quad (23)$$

$$\tilde{\rho} = -\max \left\{ \lambda_{\max} \left\{ \frac{1}{2} W_1^{-\frac{1}{2}} W(q_i) W_1^{-\frac{1}{2}} \right\}, i = 1, 2 \right\} > 0. \quad (24)$$

The considered Lyapunov function repeats the structure of the Lyapunov function (18), where the modulus $|\cdot|$ is replaced with the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ and the sign function is replaced with its vector-valued analog $\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$. Indeed, in the case of the standard homogeneity (i.e., for $\mathbf{d}(s) = e^s I_n$) we have $\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x = \frac{x}{\|x\|}$.

If $n = m$ and $A = \mathbf{0}$ then the homogeneous PI controller becomes a version of MIMO super-twisting algorithm.

For $\mu > 0$ the Lyapunov function V is a weak Lyapunov function, so the global asymptotic stability in this case is established by LaSalle invariance principle^{39, 43}. For $\mu < 0$ this Lyapunov function is a strict (but non-smooth) due to Theorem 2. For $n = 1, \mu = -0.5$ the Lyapunov function (22) coincides with (18) designed in³¹ for the super-twisting system.

Corollary 2. The system of LMIs (19)-(21) is feasible, at least, for $\rho < 1$ and a small enough $|\mu|$.

The settling time of the closed-loop system can be estimated using the strict Lyapunov function.

Corollary 3 (Finite-time Stability for $\mu < 0$). Under conditions of Theorem 3 with $\mu < 0$, the closed-loop system is finite-time stable and the settling-time $T : \mathbb{R}^n \rightarrow \mathbb{R}_+$ admits the estimate

$$T(x) \leq \frac{r^{-\frac{\mu}{2+2\mu}} V^{-\frac{\mu}{2+2\mu}}(\xi_0)}{\frac{-\mu}{2+2\mu} \tilde{\rho}}$$

where $r = \lambda_{\max}(P)/\tilde{\lambda}$,

$$\tilde{\lambda} = \frac{1}{2} \lambda_{\min} \begin{bmatrix} \frac{P}{1+\mu} & \tilde{Y}^T \\ \tilde{Y} & Q \end{bmatrix}$$

and

$$\xi_0 = \begin{pmatrix} \|x_0\|_{\mathbf{d}}^{1+\mu} \mathbf{d}(-\ln \|x_0\|_{\mathbf{d}})x_0 \\ p \end{pmatrix}.$$

Notice that the settling time estimate depends on the system's initial state $x_0 \in \mathbb{R}^n$ and the unknown constant perturbation $p \in \mathbb{R}^m$. Therefore, the settling time cannot be estimated without prior estimates of x_0 and p .

Remark 4. Notice that u_{int} is a continuous function of x for $-0.5 < \mu < 0$. In this case the control can reject a non-vanishing (at $x = \mathbf{0}$) matched perturbation only if this perturbation is a constant. Non-constant and non-vanishing (at $x = \mathbf{0}$) matched perturbation may be rejected by the control (13) only if $\mu = -0.5$. In the latter case u_{int} is discontinuous at $x = \mathbf{0}$.

In fact, the proposed integral control is capable to reject (at least, theoretically) a wide range of disturbances. Based on Theorem 3, we present a broader class of perturbations g that can be rejected by the proposed generalized homogeneous control with integral action.

Corollary 4 (The case of time and state dependent perturbations). Let under conditions of Theorem 3 the following inequalities hold:

$$\frac{2 |x^T \mathbf{d}^T(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) g_1(t, x)|}{x^T \mathbf{d}^T(-\ln \|x\|_{\mathbf{d}}) R \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x} \leq \gamma_1 \|x\|_{\mathbf{d}}^{\mu} \quad (25)$$

$$\frac{|g_1^T(t, x) \mathbf{d}^T(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) g_1(t, x)|}{4} \leq \gamma_3^2 \|x\|_{\mathbf{d}}^{2\mu} \quad (26)$$

$$|x^T \mathbf{d}^T(-\ln \|x\|_{\mathbf{d}}) P B g_2(t, x)| \leq \gamma_2 \|x\|_{\mathbf{d}}^{1+2\mu} \quad (27)$$

$$\frac{g_2^T(t, x) Q (B^T P B)^{-1} Q g_2(t, x)}{4} \leq \gamma_4^2 \|x\|_{\mathbf{d}}^{2+4\mu} \quad (28)$$

where the parameters $\gamma_1, \tilde{\gamma}_1, \gamma_2, \tilde{\gamma}_2 \geq 0$ satisfy the systems of LMIs (19-20) and

$$\begin{bmatrix} (\sum_{j=1}^4 \gamma_j - \rho)P & \frac{PB}{q_i} + \frac{(H \pm 2\gamma_1 \tilde{F})^T \tilde{Y}^T}{2} & \tilde{Y}^T & \mathbf{0} \\ * & \frac{\gamma_4 B^T P B}{q_i} + M_i & \mathbf{0} & \tilde{Y} \\ * & * & -Q & \mathbf{0} \\ * & * & * & -\frac{P}{\gamma_3} \end{bmatrix} \prec 0 \quad (29)$$

where $H = \hat{F} - \rho \tilde{F} + I$. Then the closed-loop system (7), (14), (13) is

- globally uniformly finite-time stable for $\mu < 0$;
- globally uniformly exponentially stable for $\mu = 0$;
- globally uniformly nearly fixed-time stable $\mu > 0$.

The homogeneous systems are also robust in the sense of Input-to-State Stability⁵ with respect to a wide class of perturbations^{18, 20}. The ISS analysis of a nonlinear control system with the homogeneous PI control is given in the following corollary, which, in particular, proves that the presented controller is applicable for some non-linear systems as well.

Corollary 5. Let a function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ be a continuous and

$$f(x, x_{n+1}, \mathbf{0}) = \begin{pmatrix} Ax + B(K_0 x + u_{hom}(x) + x_{n+1}) \\ u_{int}(x) \end{pmatrix}, \quad x \in \mathbb{R}^n, \quad x_{n+1} \in \mathbb{R}^m,$$

where K_0, u_{hom}, u_{int} are components of the homogeneous PI controller (13) defined according to Theorem 3 for $\mu > -0.5$. If there exists a linear dilation \mathbf{d}_η in \mathbb{R}^k such that

$$f(\mathbf{d}(s)x, e^{s(1+\mu)}x_{n+1}, \mathbf{d}_\eta(s)\eta) = e^{\mu s} \tilde{\mathbf{d}}(s) f(x, x_{n+1}, \eta), \quad \forall s \in \mathbb{R}, \forall x \in \mathbb{R}^n, x_{n+1} \in \mathbb{R}^m, \forall \eta \in \mathbb{R}^k, \quad (30)$$

where $\tilde{\mathbf{d}}(s)$ is given by (17), then the system

$$\begin{pmatrix} \dot{x} \\ \dot{x}_{n+1} \end{pmatrix} = f(x, x_{n+1}, \eta), \quad t > 0, \quad x(0) = x_0, \quad \eta \in L^\infty$$

is input-to-state stable.

The latter corollary, in particular, guarantees the robustness (in the sense of the input-to-state stability) of the closed-loop system (1), (13) with respect to measurement noises. Indeed, all conditions of the corollary are fulfilled for $\mathbf{d}_\eta(s) = \begin{pmatrix} \mathbf{d}(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{d}(s) \end{pmatrix}$,

$$f(x, x_{n+1}, \eta) = \begin{pmatrix} Ax + B(K_0 x + u_{hom}(x + \eta_1) + x_{n+1}) + \eta_2 \\ u_{int}(x + \eta_1) \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad \eta_1, \eta_2 \in \mathbb{R}^n,$$

where $\eta \in \mathbb{R}^{2n}$ models perturbations. If $\eta_0 \in L^\infty$ is a measurement noise (i.e., we measure $\tilde{x} = x + \eta_0$) then selecting $\eta_1 = \eta_0$ and $\eta_2 = BK_0 \eta_0$ we conclude that the linear system (1) with the homogeneous PI controller (13) is ISS with respect to the measurement noise.

Finally, the estimate of the maximum magnitude of the controller (13) is given below.

Corollary 6 (The estimate of the control magnitude). Under conditions of Theorem 3 (or Corollary 4) one has

$$\|u(x)\|_{\mathbb{R}^m} \leq \|K_0\| \cdot \|x\| + \|K\| \cdot \|x\|^{1+\mu} + \sqrt{\frac{V(\xi_0)}{\tilde{\lambda}}} + \|p\|_{\mathbb{R}^m}, \quad (31)$$

where $\tilde{\lambda}$ and ξ_0 are defined in Corollary 3, the vector norm in \mathbb{R}^m is defined as $\|v\|_{\mathbb{R}^m} = \sqrt{v^T v}$, the vector norm in \mathbb{R}^n is given by $\|y\| = \sqrt{y^T P y}$ and the matrix norm for $Z \in \mathbb{R}^{m \times n}$ is defined as follows $\|Z\| = \sqrt{\lambda_{\max}(P^{-1/2} Z^T Z P^{-1/2})}$.

4.3 | Control tuning and "upgrading" from a existing linear PI controller

There are several parameters for homogeneous PI control tuning. The first one is the homogeneity degree $\mu \in [-0.5, 0)$. The case $\mu = 0$ corresponds to a linear PI algorithm, but for $\mu = -0.5$ we derive a MIMO version of the super-twisting algorithm.

⁵The system $\dot{x} = f(x, \eta), t > 0, x(0) = x_0, \eta \in L^\infty$ is said to be input-to-state stable⁴⁴ if there is a function $\beta \in \mathcal{KL}$ and a function $\gamma \in \mathcal{K}$ such that for all initial states $x(t_0) = x_0 \in \mathbb{R}^n$, all $t_0 \geq 0$ and for all admissible inputs $\eta \in L^\infty$ the following inequality holds: $\|x(t, x_0)\| \leq \beta(\|x_0\|, t - t_0) + \gamma(\|\eta\|_{L^\infty(t_0, t)})$.

The generator G_d and the "homogenization" gain K_0 are calculated via (9) and $G_d = I_n + \mu G_0$, where the matrices G_0 , K_0 are independent of the homogeneity degree μ (see Theorem 1) defined by means of the system matrices A and B only .

The second parameter $\rho > 0$ can be utilized for the tuning of the convergence time (the larger ρ , the faster convergence) and the robustness of the system (the larger ρ , the larger class of perturbations can be rejected). The parameters K and P can be determined by solving the LMIs (10) for the selected μ and ρ . The LMI (10) is always feasible.

On the final step, using the selected parameters, the gain \tilde{K} of the integral term can be found by solving the LMIs (19), (20) and (21). Corollary 2 guarantees that the homogeneous PI controller can always be designed via suggested procedure, at least, if μ is close to zero. To tune the finite-time controller, we may start from μ close to zero and, next, decrease μ while the LMIs (10) and (19)-(21) remain feasible.

The paper³⁰ proposes a method for "upgrading" a well-tuned linear PI controller

$$u_{lin}(t) = K_p x(t) + \int_0^t K_i x(s) ds, \quad t \geq 0$$

to a homogeneous one (13). The parameters K_p and K_i are suggested to be taken from the linear controller. So, to complete the upgrade, the parameters μ , ρ has to be selected such that the LMIs will be feasible with respect to X and Q ($Y = (K_p - K_0)X$ and $\tilde{Y} = K_i Q$). In the view of Corollary 2, such an "upgrade" is always possible, at least, for μ close to 0 and $\rho \in (0, 1)$.

5 | EXAMPLE

In this section, two examples of the proposed control design are provided. In the first example, we compare the homogeneous PI control with linear PI control. We also demonstrate the possibility of the "upgrading" of a linear PI algorithm to a homogeneous one. In the second example, we apply the homogeneous PI control to an omnidirectional mobile robot and compare it with a homogeneous control without integral action.

5.1 | MIMO linear system stabilization

Let us consider the system

$$\dot{x} = Ax + Bu(x + \eta) + g(t, x)$$

with the parameters

$$A = \begin{bmatrix} 0 & 1.0 & 0.5 \\ 0 & 0 & 1.0 \\ 1.0 & 2.0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1.0 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

and the perturbation g is defined as

$$g = g_1 + Bp + \int g_2, \quad g_1 = 0.01[1, 1, 1]^T \sqrt{\|x\|}, \quad g_2 = 0.01[1, 1]^T \sqrt{\|x\|}, \quad p = [0.2, 0.2]^T.$$

and η models the measurement noise. We have proven the robustness of the homogeneous PI controller with respect to some classes of matched and mismatched perturbations. The robustness (in the sense of Intut-to-State stability) of homogeneous control systems with respect to the measurement noises is well known^{18, 20}. For simulations, the noise η is generated by a normally (Gaussian) distributed random signal with the zero mean and the variance of 0.0005.

Then, selecting $\mu = -0.45$, $\rho = 0.5$ and by solving the LMIs (10) and (19-21) (we use the package YALMIP and solver SEDUMI in MATLAB), we obtain:

$$G_d = \begin{bmatrix} 1.45 & 0 & 0 \\ 0.0031 & 1 & 0 \\ -0.0062 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 3.1591 & 0.8489 & 0.4211 \\ 0.8489 & 1.4690 & -0.0154 \\ 0.4211 & -0.0154 & 1.4924 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 0 & 0.0069 & -0.9965 \\ -2 & -4.0346 & 0.9827 \end{bmatrix}, \quad K = \begin{bmatrix} -2.2737 & -1.0838 & -0.6828 \\ 0.5636 & 1.3007 & -0.5996 \end{bmatrix},$$

$$\tilde{K} = \begin{bmatrix} -0.1289 & -0.0674 & -0.0223 \\ 0.0108 & 0.0784 & -0.1608 \end{bmatrix}, \quad \tilde{Y} = \begin{bmatrix} -1.3539 & -0.5740 & -0.5323 \\ -0.1977 & 0.1583 & -0.6190 \end{bmatrix},$$

$$Q = \begin{bmatrix} 10.6540 & 1.8339 \\ 1.8339 & 3.5954 \end{bmatrix}, \quad \gamma = 0.5139$$

To illustrate the robustness of the proposed generalized homogeneous PI control, we design a linear PI control

$$u_{lin} = K_p x + \int K_i x d\tau,$$

$$K_p = \begin{bmatrix} -11.85 & -3.6471 & -3.2985 \\ 0.97 & 0.3504 & -1.0383 \end{bmatrix}, \quad K_i = \begin{bmatrix} -0.2385 & -0.08056 & -0.02665 \\ 0.02176 & 0.09374 & -0.1923 \end{bmatrix}.$$

in such a way, in the nominal (disturbance-free and noise-free) case, it provides approximately the same time response as the homogeneous one. For the initial state $x_0 = [0.1, 0.1, 0.1]^T$ both controllers guarantee $\|x\| \leq 0.001$ after approximately the same instant of time $t = 2.78$ (see Figures 1a, 1b). Both controllers have good quality for the system without perturbations. In the presence of additive disturbances g , the system driven by generalized homogeneous control with integral action converges much faster than the system with linear PI control (see Figures 1c, 1d). Homogeneous finite-time controllers (as any sliding mode or high gain algorithm) are more sensitive to the noise than linear algorithms. However, in the presence of both noises and perturbations, the homogeneous PI controller still provides the better control performance than the linear controller (see Figures 1e, 1f). We also demonstrate a possibility of an "upgrading" of a linear feedback algorithm to a homogeneous one with a better control quality. For this purpose we fix the gains K_p and K_i of the above linear controller and search for μ, ρ and P such that the system of LMI (10) and (19-21) is fulfilled. For the "upgrading", we select $\mu = -0.1, \rho = 0.5$, then the control parameters are determining by solving the LMIs 19-21

$$G_d^* = \begin{bmatrix} 1.45 & 0 & 0 \\ 0.0007 & 1 & 0 \\ -0.0014 & 0 & 1 \end{bmatrix}, \quad P^* = \begin{bmatrix} 0.0819 & 0.0164 & 0 \\ 0 & 0.01 & 0 \\ 0.0348 & 0.0069 & 0.01 \end{bmatrix},$$

The results of the simulation are shown in Figure 2. Comparing performance of the origin linear PI control(see Figures 1b, 1d,1f) and the "upgraded" (generalized homogeneous) control (see Figures 2a, 2b,2c), we can observe an essential improvement of the control quality (such as faster convergence and better robustness) in both perturbed and non-perturbed case.

5.2 | Mobile robot stabilization

Let us consider an omnidirectional mobile robot⁴⁵ modeled by the following ordinary differential equation :

$$M\ddot{\zeta} + C(\dot{\zeta})\dot{\zeta} + D\dot{\zeta} = \tau + \delta \quad (32)$$

where $M \in \mathbb{R}^{3 \times 3}$, $C \in \mathbb{R}^{3 \times 3}$ and $D \in \mathbb{R}^{3 \times 3}$ are system matrices, $\zeta = [x, y, \theta]^T$, (x, y) is the coordinates of robot on the plane \mathbb{R}^2 , θ is the orientation angle with respect to x -axis of the inertial frame, $\tau \in \mathbb{R}^3$ is the control input, and $\delta \in \mathbb{R}^3$ denotes the matched additive disturbance. Denoting $A = \begin{bmatrix} \mathbf{0} & I_3 \\ \mathbf{0} & -M^{-1}D \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{0} \\ M^{-1} \end{bmatrix}$, the model can be rewritten as

$$\begin{bmatrix} \dot{\zeta} \\ \ddot{\zeta} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_3 \\ \mathbf{0} & -M^{-1}D \end{bmatrix} \begin{bmatrix} \zeta \\ \dot{\zeta} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ M^{-1} \end{bmatrix} (u + \delta) \iff \dot{z}_\zeta = Az_\zeta + B(u + \delta) \quad (33)$$

where $z_\zeta = [\zeta^T, \dot{\zeta}^T]^T$, $u = \tau - C(\dot{\zeta})\dot{\zeta}$.

$$M = \begin{bmatrix} 3.9364 & -0.5189 & -0.5189 \\ -0.5189 & 3.9364 & -0.5189 \\ -0.5189 & -0.5189 & 1.1491 \end{bmatrix}, \quad C(\dot{\zeta}) = [0, 0, 4.254]\dot{\zeta} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 165.9787 & -80.0129 & -80.0129 \\ -80.0129 & 165.9787 & -80.0129 \\ -80.0129 & -80.0129 & 165.9787 \end{bmatrix}.$$

The parameters of homogeneous controller are obtained by solving the LMIs via MATLAB package YALMIP and the solver SEDUMI. The simulation was done in Simulink with the solver ODE45 (Dormand-Prince), and the relative tolerance is 10^{-5} . For $\mu = -0.4, \rho = 0.5$, the controller parameters are given as follows:

$$K_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 165.9787 & -80.0129 & -80.0129 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -80.0129 & 165.9787 & -80.0129 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -80.0129 & -80.0129 & 165.9787 \end{bmatrix},$$

$$K = \begin{bmatrix} -9.1082 & 2.8498 & -1.9469 & -4.9963 & 3.0147 & -3.8379 \\ -1.0112 & -9.2387 & 0.9243 & -2.5011 & -5.1827 & 0.2639 \\ 2.3453 & 1.1399 & -2.0910 & 2.2939 & 0.5719 & -0.6473 \end{bmatrix},$$

$$\tilde{K} = \begin{bmatrix} -0.1428 & 0.0272 & -0.1100 & -0.0855 & 0.0056 & -0.0418 \\ -0.0451 & -0.1623 & -0.0668 & -0.0167 & -0.0989 & -0.0288 \\ 0.1226 & -0.0160 & -0.3299 & 0.0378 & -0.0154 & -0.1707 \end{bmatrix},$$

$$P = \begin{bmatrix} 3.3129I_3 & 1.0237I_3 \\ 1.8965I_3 & 1.4624I_3 \end{bmatrix}, \quad G_d = \begin{bmatrix} 1.4I_3 & \mathbf{0} \\ \mathbf{0} & I_3 \end{bmatrix}$$

For the simulation, we consider a constant perturbation $\delta = [2, 2, 0.2]^\top$ and the initial state $\zeta(0) = [1.5, 1.5, 1, 0, 0, 0]^\top$. The simulation results are presented in Figure 3, where the homogeneous controllers with and without integral action are compared. Note that in Figure 3a, the robot governed by the homogeneous PI control (with integral action) can reach the origin, but the controller without integral action has a static error. According to Figures 3b,3c and 3d the integral action almost has no effect in the transition phase, but it helps to improve precision in the steady state.

6 | CONCLUSION

In this paper, based on a non-smooth Lyapunov function method, we present a strict Lyapunov function for analysis and design of the generalized homogeneous integral controller with negative homogeneity degree.

- The controller is designed in a simpler form that makes it easier to implement in practice and to cover both linear PI and super-twisting SM algorithms as particular cases.
- The controller gain can be easily tuned by LMIs, which is beneficial when dealing with high-dimension system. In comparison to the preceding integral homogeneous controller^{46, 15}, the generalized integral homogeneous control does not require the system's specific structure, thus it can avoid cumbersome constructions.
- Furthermore, the generalized integral control can reject matched constant and Lipschitz perturbations as some vanishing at zero mismatched perturbations.

A class of perturbations to be rejected by the mentioned controller is characterized. Besides, the settling time and the magnitude of the controller are estimated. All estimates are obtained in terms of the so-called canonical homogeneous norm, which can be computed numerically or explicitly estimated analytically (see,^{15, Chapter 9}) The simulations demonstrate that the generalized homogeneous integral controller has a faster response than a linear PI controller with the same gains and better precision than a generalized homogeneous controller without integral action. A further study about the case of a positive homogeneity degree, which implies fixed-time convergence properties of the system, will be an interesting problem for our future research.

7 | APPENDIX

7.1 | Proof of Theorem 2

First of all notice that continuity of W on \mathbb{R}^n implies that $\limsup_{x \rightarrow S} \frac{\partial V}{\partial x} f(x) \leq -W(x)$, where the limit $x \rightarrow S$ is understood in the Hausdorff topology (see, e.g.⁴¹).

Let $x(t)$ be an arbitrary trajectory of the system and let $\bar{t} > 0$ be such that $x(\bar{t}) \in S \setminus \{\mathbf{0}\}$. On the one hand, by assumption of the theorem, \bar{t} is isolated (from the right), i.e. there exists $r > 0$ such that $x(t) \notin S \setminus \{\mathbf{0}\}$ for all $t \in (\bar{t}, \bar{t} + r)$. In this case, the function $t \rightarrow V(x(t))$ is continuously differentiable on $(\bar{t}, \bar{t} + r]$.

On the other hand, since any continuous function is uniformly continuous on any compact, then the function $t \rightarrow V(x(t))$ is uniformly continuous on $[\bar{t}, \bar{t} + r]$, i.e. there exists $\omega \in \mathcal{K}$ such that $|V(x(t')) - V(x(t''))| \leq \omega(|t' - t''|)$, $\forall t', t'' \in [\bar{t}, \bar{t} + r]$. For

any $t \in (\bar{t}, \bar{t} + r)$ and any $h \in (0, r)$ one has $[t, t + h] \subset (\bar{t}, \bar{t} + r)$ and

$$\begin{aligned} \frac{V(x(\bar{t} + h)) - V(x(\bar{t}))}{h} &= \frac{V(x(\bar{t} + h)) - V(x(t + h)) + V(x(t + h))}{h} + \frac{-V(x(t)) + V(x(t)) - V(x(\bar{t}))}{h} \\ &\leq \frac{2\omega(|\bar{t} - t|) + V(x(t + h)) - V(x(t))}{h} \\ &= \frac{2\omega(|\bar{t} - t|) + \int_t^{t+h} \dot{V}(x(s)) ds}{h} \\ &= \frac{2\omega(|\bar{t} - t|)}{h} + \dot{V}(x(s^*)) \Big|_{s^* \in [t, t+h]} \end{aligned}$$

where the mean value theorem is utilized on the last step. Since $\omega \in \mathcal{K}$ then for any $h \in (0, r)$ we can always select $t = t(h) \in (\bar{t}, \bar{t} + r)$ such that $\omega(|\bar{t} - t|) \leq h^2$. The latter means that $t(h) \rightarrow \bar{t}$ as $h \rightarrow 0^+$ and

$$\begin{aligned} D^+V(x(\bar{t})) &:= \limsup_{h \rightarrow 0^+} \frac{V(x(\bar{t} + h)) - V(x(\bar{t}))}{h} \\ &\leq \limsup_{h \rightarrow 0^+} 2h + \frac{\partial V}{\partial x} f(x) \Big|_{x=x(s^*): s^* \in [t(h), t(h)+h]} \leq -W(x(\bar{t})). \end{aligned}$$

Hence, we derive $D^+V(x(t)) \leq -W(x(t))$ as long as $x(t) \neq 0$ and by^{47, Corollary 2, page 11}, the function $t \rightarrow V(x(t))$ is monotone decreasing. The latter means $V(x(t)) \leq V(x(0))$ and

$$\|x(t)\| \leq \alpha^{-1}(\beta(\|x(0)\|)), \quad \forall t > 0,$$

i.e. the system is Lyapunov stable⁴⁸, where α^{-1} denotes the inverse function of $\alpha \in \mathcal{K}^\infty$.

Since V is monotone then $\exists \lim_{t \rightarrow +\infty} V(x(t)) = V^*$. To complete the proof it is sufficient to show that $V^* = 0$. Suppose the contrary, i.e. $V^* > 0$. In this case, $V(x(t)) \geq V^*$ for all $t \geq 0$ and $\|x(t)\| \geq \beta^{-1}(V^*)$, where β^{-1} denotes the inverse function of $\beta \in \mathcal{K}^\infty$. Hence,

$$W(x(t)) \geq W^* := \inf_{\|x\| \geq \beta^{-1}(V^*)} W(x) > 0, \quad \forall t \geq 0$$

and

$$D^+V(x(t)) \leq -W^*, \quad \forall t > 0.$$

Using^{47, Theorem 9.3, page 25} we derive the contradiction: $V(x(t)) \leq -W^*t \leq 0, \forall t > 1/W^*$.

7.2 | Proof of Corollary 1

Since

$$D^+V(x(t)) \leq -\alpha V^\rho(x(t)), \forall t \geq 0$$

then using^{47, Theorem 9.3, page 25} we derive

$$(1 - \rho) (V^{1-\rho}(x(t)) - V^{1-\rho}(x(0))) \leq -qt \text{ for } \rho \neq 0,$$

$$\ln V(x(t)) - \ln V(x(0)) \leq -qt \text{ for } \rho = 0$$

and complete the proof.

7.3 | Supporting result

The proof of our main theorem uses the following result.

Lemma 1. For any $v_1 \in \mathbb{R}^n, v_2 \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with $\|y\| = 1$, the inequality

$$(y^\top v_1) (y^\top v_2) \leq \frac{1}{2} \|v_1\| \|v_2\| + \frac{1}{2} v_1^\top v_2$$

holds.

Proof. For $v_1 = v_2$ the result is a trivial consequence of the Cauchy-Swartz inequality.

Let $v_1 \neq v_2$. Select the basis in \mathbb{R}^n as follows $e_1 = \frac{v_1}{\|v_1\|}, e_2 = \frac{v_2}{\|v_2\|}, \{i, j \in \mathbb{Z}|, e_i^\top e_j = 0, 3 \leq i \leq n, 1 \leq j \leq n\}$ then for any element $y \in \mathbb{R}^n$, we have

$$y = a_1 e_1 + a_2 e_2 + \dots + a_n e_n.$$

Due to the fact that $\|y\| = 1$, then we obtain

$$a_1^2 + a_2^2 + 2a_1a_2e_1^\top e_2 + \sum_{i=3}^n a_i^2 = 1$$

and

$$\begin{aligned} y^\top v_1 y^\top v_2 &= \|v_1\| \|v_2\| \|y^\top e_1 y^\top e_2\| = \|v_1\| \|v_2\| (a_1 + a_2 e_2^\top e_1)(a_1 e_1^\top e_2 + a_2) \\ &= \|v_1\| \|v_2\| \left[a_1 a_2 + (a_1^2 + a_2^2) e_1^\top e_2 + a_1 a_2 (e_1^\top e_2)^2 \right] = \|v_1\| \|v_2\| \left[\left(1 - \sum_{i=3}^n a_i^2 \right) e_1^\top e_2 + a_1 a_2 (1 - (e_1^\top e_2)^2) \right] \end{aligned}$$

On the other hand, since

$$2a_1a_2(1 + e_1^\top e_2) \leq a_1^2 + a_2^2 + 2a_1a_2e_1^\top e_2 = 1 - \sum_{i=3}^n a_i^2$$

then

$$a_1a_2(1 - (e_1^\top e_2)^2) = (1 - e_1^\top e_2) a_1a_2(1 + e_1^\top e_2) \leq (1 - e_1^\top e_2) \frac{1 - \sum_{i=3}^n a_i^2}{2}$$

and

$$y^\top v_1 y^\top v_2 \leq \|v_1\| \|v_2\| \left[\frac{1}{2} (1 + e_1 \cdot e_2) \left(1 - \sum_{i=3}^n a_i^2 \right) \right] \leq \left(\frac{1}{2} \|v_1\| \|v_2\| + \frac{1}{2} v_1 v_2 \right) \left(1 - \sum_{i=3}^n a_i^2 \right) \leq \left(\frac{1}{2} \|v_1\| \|v_2\| + \frac{1}{2} v_1^\top v_2 \right).$$

7.4 | The proof of Theorem 3

Let us consider the Lyapunov function candidate:

$$V = \frac{\|x\|_{\mathbf{d}}^{2+2\mu}}{2+2\mu} + \frac{2\|x\|_{\mathbf{d}}^{1+\mu} x_{n+1}^\top \tilde{Y} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x}{2} + \frac{x_{n+1}^\top Q x_{n+1}}{2}.$$

The canonical homogeneous is defined in the implicit manner: $x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x = 1$, and its derivative respect to x is given by (5).

For $g_1 = \mathbf{0}$ and $g_2 = \mathbf{0}$, let calculate the derivative of V . Using (12) and the formula (5) we derive

$$\dot{V}(\xi) = -\rho \|x\|_{\mathbf{d}}^{2+3\mu} + \|x\|_{\mathbf{d}}^{1+2\mu} \frac{2x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) P B x_{n+1}}{q(x)} + \|x\|_{\mathbf{d}}^{1+2\mu} x_{n+1}^\top Q \tilde{K} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x + \frac{d\tilde{V}(\xi)}{dt}$$

where

$$q(x) := x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) R \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x$$

$$\tilde{V}(\xi) := \|x\|_{\mathbf{d}}^{1+\mu} x_{n+1}^\top \tilde{Y} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x.$$

Denoting $R = P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P$, $\tilde{F} = (1 + \mu) I_n - G_{\mathbf{d}}$, $\hat{F} = A_0 + B K$ and $\tilde{K} = Q^{-1} \tilde{Y}$, we obtain

$$\frac{d\tilde{V}(\xi)}{dt} = \|x\|_{\mathbf{d}}^\mu \frac{\partial \|x\|_{\mathbf{d}}}{\partial x} \dot{x} x_{n+1}^\top \tilde{Y} \tilde{F} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x + \|x\|_{\mathbf{d}}^{1+\mu} \dot{x}_{n+1}^\top \tilde{Y} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x + \|x\|_{\mathbf{d}}^{1+\mu} x_{n+1}^\top \tilde{Y} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) \dot{x}$$

For $A_0 = A + B K_0$ we have the identities $A_0 \mathbf{d}(s) = e^{\mu s} \mathbf{d}(s) A_0$, $\mathbf{d}(s) B = e^s B$, $s \in \mathbb{R}$. It is equivalent to:

$$\mathbf{d}(-\ln \|x\|_{\mathbf{d}}) A_0 = \|x\|_{\mathbf{d}}^\mu A_0 \mathbf{d}(-\ln \|x\|_{\mathbf{d}}), \quad \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) B = \|x\|_{\mathbf{d}}^{-1} B.$$

Utilizing the latter identities, it yields that

$$\begin{aligned} \frac{d\tilde{V}(\xi)}{dt} &= 2\|x\|_{\mathbf{d}}^\mu \frac{x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) P B x_{n+1} x_{n+1}^\top \tilde{Y} \tilde{F} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x}{q(x)} - \rho \|x\|_{\mathbf{d}}^{1+2\mu} x_{n+1}^\top \tilde{Y} \tilde{F} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x \\ &\quad + \|x\|_{\mathbf{d}}^{2+3\mu} x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) \tilde{Y}^\top Q^{-1} \tilde{Y} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x + \|x\|_{\mathbf{d}}^{1+2\mu} x_{n+1}^\top \tilde{Y} \hat{F} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x + \|x\|_{\mathbf{d}}^\mu x_{n+1}^\top \tilde{Y} B x_{n+1} \end{aligned}$$

Denoting $y = P^{1/2} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x$, $v = P^{1/2} B x_{n+1}$ and $w = P^{-1/2} \tilde{F}^\top \tilde{Y}^\top x_{n+1}$. Taking into account $y^\top y = 1$ and using Lemma 1 we derive

$$\beta := (y^\top v)(y^\top w) \leq \left(\frac{1}{2} \|v\| \|w\| + \frac{1}{2} v^\top w \right)$$

or, equivalently,

$$\beta \leq \frac{1}{2} \sqrt{x_{n+1}^\top B^\top P B x_{n+1}} \sqrt{x_{n+1}^\top \tilde{Y} \tilde{F} P^{-1} \tilde{F}^\top \tilde{Y}^\top x_{n+1}} + \frac{1}{2} x_{n+1}^\top \tilde{Y} \tilde{F} B x_{n+1}$$

If $\gamma^2 B^T P B \geq \tilde{Y} \tilde{F} P^{-1} \tilde{F}^T \tilde{Y}^T$, then we have

$$\beta \leq \frac{1}{2} x_{n+1}^T (\gamma B^T P B + \tilde{Y} \tilde{F} B) x_{n+1}$$

Notice that the inequality $\gamma^2 B^T P B \geq \tilde{Y} \tilde{F} P^{-1} \tilde{F}^T \tilde{Y}^T$ is equivalent (by Schur complement) to

$$\begin{bmatrix} \gamma B^T P B & \tilde{Y} \tilde{F} \\ \tilde{F}^T \tilde{Y}^T & \gamma P \end{bmatrix} \geq 0, \quad \gamma > 0$$

Therefore, we have

$$\dot{V}(\xi) \leq \|x\|_{\mathbf{d}}^\mu \xi^T W(q) \xi \quad (34)$$

where

$$W(q_i) = \begin{bmatrix} -\rho P + \tilde{Y}^T Q^{-1} \tilde{Y} & \frac{PB}{q_i} + \frac{(\hat{F} - \rho \tilde{F} + I)^T \tilde{Y}^T}{2} \\ * & M_i \end{bmatrix}$$

with $M_i = \frac{\gamma B^T P B}{q_i} + \frac{B \tilde{Y} \tilde{F} + \tilde{F}^T \tilde{Y}^T B}{2q_i} + \frac{\tilde{Y} B + B^T \tilde{Y}^T}{2}$. Taking into account $q_1 \leq q(x) \leq q_2$ we complete the proof using the convex embedding $W(q) = \tau W(q_1) + (1 - \tau) W(q_2)$ and using the Schur complement for the transformation of $W(q_i) < 0$ to (20).

Notice that $\|\cdot\|_{\mathbf{d}}$ is not differentiable at $\mathbf{0}$. However, for $\mu > 0$ using the formula (5) it can be shown that the function $x \rightarrow \|x\|_{\mathbf{d}}^{1+\mu} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$ is differentiable on \mathbb{R}^n and its partial derivatives vanish at zero. In this case, V satisfies LaSalle invariance principle⁴³ implying that the closed-loop system is globally uniformly asymptotically stable. If $\mu = 0$ then $\|x\|_{\mathbf{d}}^{1+\mu} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x = x$ and V is a quadratic Lyapunov function such that $\dot{V}(\xi) \leq \xi W(q) \xi$. For $-0.5 \leq \mu < 0$ the Lyapunov function candidate V is continuous but not differentiable on the surface $x = \mathbf{0}$ of $n + m$ dimensional space. By Theorem 2 the system is globally uniformly asymptotically stable. Finally, using the $\tilde{\mathbf{d}}$ -homogeneity of the system we complete the proof.

7.5 | Proof of Corollary 2

The feasibility of (10) for any $\mu \in [-1, 1/k]$ is proven in^{49, 15}. Taking into account the representation $G_{\mathbf{d}} = I_n + \mu G_0$ we conclude that $G_{\mathbf{d}} \rightarrow I$ as $\mu \rightarrow 0$. In this case $q(x) \rightarrow 2$ uniformly on x and $\tilde{F} \rightarrow \mathbf{0}$ as $\mu \rightarrow 0$. In the limit case the system of LMIs (19)-(21) becomes:

$$\begin{aligned} W_1 &= \begin{bmatrix} P & \tilde{Y}^T \\ \tilde{Y} & Q \end{bmatrix} > 0 \\ W &= \begin{bmatrix} -\rho P & \frac{PB}{2} + \frac{(\hat{F}+I)^T \tilde{Y}^T}{2} & \tilde{Y}^T \\ * & \frac{\gamma B^T P B}{2} + \frac{\tilde{Y} B + B^T \tilde{Y}^T}{2} & \mathbf{0} \\ * & * & -Q \end{bmatrix} < 0 \\ &\begin{bmatrix} \gamma B^T P B & \mathbf{0} \\ \mathbf{0} & \gamma P \end{bmatrix} \geq 0, \quad \gamma > 0 \end{aligned}$$

On the one hand, from (10), we have

$$\hat{F}^T P + P \hat{F} + \rho (G_{\mathbf{d}}^T P + P G_{\mathbf{d}}) = 0 \quad (35)$$

considering the generator $G_{\mathbf{d}} = I_n + \mu G_0$, then

$$\begin{aligned} \hat{F}^T P + P \hat{F} + \rho (2P + \mu (G_0^T P + P G_0)) &= 0 \\ \Leftrightarrow (\hat{F} + I)^T P + P (\hat{F} + I) + 2(\rho - 1)P + 2\rho\mu (G_0^T P + P G_0) &= 0 \end{aligned}$$

For $\rho < 1$ and μ sufficiently close to zero we have

$$(\hat{F} + I)^T P + P (\hat{F} + I) > 0 \quad (36)$$

The latter means that $\hat{F} + I$ is anti-Hurwitz, and consequently invertible. Let $\tilde{Y}^T = -(\hat{F} + I)^{-T} P B$. For a sufficiently small γ we have

$$W < 0 \Leftrightarrow \hat{W} < 0 \Leftrightarrow \frac{\tilde{Y} B + B^T \tilde{Y}^T}{2} < 0 \Leftrightarrow -B^T \left((\hat{F} + I)^{-T} P + P (\hat{F} + I)^{-1} \right) B < 0$$

where

$$\hat{W} = \begin{bmatrix} \rho P & \frac{PB}{2} + \frac{(\hat{F}+I)^T \tilde{Y}^T}{2} \\ * & \frac{\tilde{Y} B + B^T \tilde{Y}^T}{2} \end{bmatrix}, \quad P B + (\hat{F} + I)^T \tilde{Y}^T = \mathbf{0}.$$

The inequality (36) yields $-B^T \left((\hat{F} + I)^{-T} P + P (\hat{F} + I)^{-1} \right) B < 0$, so the system of LMIs (19)-(21) is feasible at least for a sufficiently small $|\mu|$ and $\rho < 1$.

7.6 | The proof of Corollary 3

Since

$$\|x\|_{\mathbf{d}}^{2+2\mu} = \|x\|_{\mathbf{d}}^{2+2\mu} x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x \leq rV$$

then for $-0.5 \leq \mu < 0$ one has $\|x\|_{\mathbf{d}}^{\mu} \geq r^{\frac{\mu}{2+2\mu}} V^{\frac{\mu}{2+2\mu}}$ and

$$\dot{V}(\xi) \leq -\tilde{\rho} r^{\frac{\mu}{2+2\mu}} V^{1+\frac{\mu}{2+2\mu}}(\xi), \quad \forall t \geq 0.$$

Taking into account $x_{n+1}(0) = \rho$ we complete the proof.

7.7 | The proof of Corollary 4

If $g_1 \neq \mathbf{0}$ and $g_2 \neq \mathbf{0}$ then (25) implies that

$$\frac{2 |x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) g_1(t, x)|}{q_i} \leq \gamma_1 \|x\|_{\mathbf{d}}^{\mu}$$

and repeating the considerations of the proof of Theorem 3 we derive

$$\dot{V} \leq \|x\|_{\mathbf{d}}^{\mu} \bar{\xi}^{\top} \bar{W} \bar{\xi}$$

where

$$\bar{\xi} = \begin{pmatrix} \|x\|_{\mathbf{d}}^{1+\mu} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x \\ x_{n+1} \\ \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) g_1(t, x) \\ g_2(t, x) \end{pmatrix}, \quad \bar{W} = \begin{bmatrix} -\rho P + H_Q & \frac{PB}{q_i} + \frac{(H \pm 2\gamma_1 \bar{F})^{\top} \bar{Y}^{\top}}{2} & \frac{P}{q_j} & \frac{\bar{Y}^{\top}}{2} \\ * & M & \frac{\bar{Y}}{2} & \frac{Q}{2} \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}$$

where $H_Q = \bar{Y} Q^{-1} \bar{Y}^{\top}$. Notice that

$$\frac{2x_{n+1}^{\top} \bar{Y} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) g_1}{2\gamma_3 x_{n+1}^{\top} \bar{Y} P^{-1} \bar{Y}^{\top} x_{n+1}} + \frac{\|x\|_{\mathbf{d}} g_1^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) g_1}{2\gamma_3 \|x\|_{\mathbf{d}}}$$

and

$$2x_{n+1}^{\top} Q g_2 \leq \frac{2\gamma_4 \|x\|_{\mathbf{d}}^{\mu} x_{n+1}^{\top} B^{\top} P B x_{n+1}}{q(x)} + q(x) \frac{g_2^{\top} Q (B^{\top} P B)^{-1} Q g_2}{2\gamma_4 \|x\|_{\mathbf{d}}^{\mu}}.$$

Using the inequalities (26), (27), (28) we derive $\dot{V} \leq \|x\|_{\mathbf{d}}^{\mu} \bar{\xi}^{\top} \tilde{W}(q) \bar{\xi}$, where

$$\tilde{W} = \begin{bmatrix} (\sum_{j=1}^4 \gamma_j - \rho) P + H_Q & \frac{PB}{q_i} + \frac{(H \pm 2\gamma_1 \bar{F})^{\top} \bar{Y}^{\top}}{2} \\ * & M + \frac{\gamma_4 B^{\top} P B}{q_i} + H_P \end{bmatrix}$$

with $H_P = \bar{Y} P^{-1} \bar{Y}^{\top}$.

Applying the convex embedding and Schur complement we complete the proof.

7.8 | Proof of Corollary 5

The proof is inspired by¹⁸. Since, by Theorem 3, the unperturbed system $[\dot{x}^{\top}, \dot{x}_{n+1}^{\top}]^{\top} = f(x, x_{n+1}, \mathbf{0})$ is asymptotically stable, then according to the converse Lyapunov function theory for homogeneous systems⁵⁰, there exist a $\tilde{\mathbf{d}}$ -homogeneous Lyapunov function $V \in C(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ of degree 1 such that

$$\alpha(\zeta) = \frac{\partial V}{\partial \zeta} \cdot f(\zeta, \mathbf{0}) < 0, \quad \zeta = [\dot{x}^{\top}, \dot{x}_{n+1}^{\top}]^{\top}.$$

For the perturbed system, the time derivative of V is given by

$$\dot{V} = \nabla V \cdot f(\zeta, \eta) - \nabla V \cdot f(\zeta, \mathbf{0}) + \nabla V \cdot f(\zeta, \mathbf{0}) = \nabla V \cdot (f(\zeta, \eta) - f(\zeta, \mathbf{0})) + \alpha(\zeta).$$

Using the identity for the partial derivative of homogeneous functions (see, e.g.²⁸),

$$\left. \frac{\partial V(z)}{\partial z} \right|_{z=\tilde{\mathbf{d}}(s)\zeta} \tilde{\mathbf{d}}(s)y = e^s \frac{\partial V}{\partial \zeta} y, \quad \forall y \in \mathbb{R}^n,$$

and a generalized homogeneity of f (see (30)), we derive

$$\dot{V} = \|\zeta\|_{\bar{\mathbf{d}}}^{(1+\mu)} \left[\frac{\partial V(z)}{\partial z} (f(z, \mathbf{d}_\eta(-\ln \|\zeta\|_{\bar{\mathbf{d}}})\eta) - f(z, \mathbf{0})) + \frac{\partial V(z)}{\partial z} f(z, \mathbf{0}) \right].$$

where $z := \bar{\mathbf{d}}(-\ln \|\zeta\|_{\bar{\mathbf{d}}})\zeta$. Since $\|z\| = 1$ then $a = \sup_{\|z\|=1} \frac{\partial V(z)}{\partial z} f(z, \mathbf{0}) < 0$, there exists $b > 0$ such that

$$\dot{V} \leq \|\zeta\|_{\bar{\mathbf{d}}}^{(1+\mu)} [b\|f(z, \mathbf{d}_\eta(-\ln \|\zeta\|_{\bar{\mathbf{d}}})\eta) - f(z, \mathbf{0})\| - a].$$

Since the function f is continuous, there exist $\sigma \in \mathcal{K}_\infty$ such that

$$\|f(z, \mathbf{d}_\eta(-\ln \|\zeta\|_{\bar{\mathbf{d}}})\eta) - f(z, \mathbf{0})\| \leq \sigma(\|\mathbf{d}_\eta(-\ln \|\zeta\|_{\bar{\mathbf{d}}})\eta\|)$$

By assumption, \mathbf{d}_η is a linear dilation. Due to the group property of the dilation, we have

$$\|\mathbf{d}_\eta(-\ln \|\zeta\|_{\bar{\mathbf{d}}})\eta\| = \left\| \mathbf{d}_\eta \left(-\ln \frac{\|\zeta\|_{\bar{\mathbf{d}}}}{\|\eta\|_{\mathbf{d}_\eta}} \right) \mathbf{d}_\eta(-\ln \|\eta\|_{\mathbf{d}_\eta})\eta \right\| \leq \left\| \mathbf{d}_\eta \left(-\ln \frac{\|\zeta\|_{\bar{\mathbf{d}}}}{\|\eta\|_{\mathbf{d}_\eta}} \right) \right\|.$$

Since the dilation \mathbf{d}_η is monotone, there exists a positive constant β_η such that^{15, Corollary 6.5 page 179}

$$\left\| \mathbf{d}_\eta \left(-\ln \frac{\|\zeta\|_{\bar{\mathbf{d}}}}{\|\eta\|_{\mathbf{d}_\eta}} \right) \right\| \leq e^{-\beta_\eta \ln \frac{\|\zeta\|_{\bar{\mathbf{d}}}}{\|\eta\|_{\mathbf{d}_\eta}}} = \frac{\|\eta\|_{\mathbf{d}_\eta}^{\beta_\eta}}{\|\zeta\|_{\bar{\mathbf{d}}}^{\beta_\eta}},$$

for $\|\zeta\|_{\bar{\mathbf{d}}} \geq \|\eta\|_{\mathbf{d}_\eta}$. Then,

$$\frac{d}{dt} V \leq \|\zeta\|_{\bar{\mathbf{d}}}^{\mu+1} \left(b\sigma \left(\|\eta\|_{\mathbf{d}_\eta}^{\beta_\eta} / \|\zeta\|_{\bar{\mathbf{d}}}^{\beta_\eta} \right) - a \right).$$

It is clear that there exist positive constants $\gamma > 0$ and $c = \max \left\{ 1, \frac{1}{\sigma^{-1}((a/2b)^{1/\beta_\eta})} \right\}$, where $\sigma^{-1}(\cdot)$ is the inverse function to σ , such that

$$\frac{d}{dt} V(\zeta) < -\gamma \|\zeta\|_{\bar{\mathbf{d}}}^{1+\mu},$$

provided that $\|\zeta\|_{\bar{\mathbf{d}}} \geq c\|\eta\|_{\mathbf{d}_\eta}$. Taking into account the topological equivalence of the Euclidean and \mathbf{d} -homogeneous norms^{15, Lemma 7.2, page 186} we conclude that V is the ISS Lyapunov function⁵¹. The proof is complete.

7.9 | Proof of Corollary 6

Notice that $\|u\|_m \leq \|K_0 x\|_m + \|u_{\text{hom}}(x)\|_m + \|x_{n+1} - p\|_m$. Then, by taking into account

$$\begin{aligned} y^\top Z^\top Z y &= x^\top P^{1/2} P^{-1/2} Z^\top Z P^{-1/2} P^{1/2} x \\ &\leq \lambda_{\max}(P^{-1/2} Z^\top Z P^{-1/2}) x^\top P x \end{aligned}$$

we derive $\|K_0 x\|_m \leq \|K_0\| \cdot \|x\|$ and $\|u_{\text{hom}}(x)\|_m \leq \|K\| \cdot \|x\|_{\bar{\mathbf{d}}}^{1+\mu}$. Since $V(\xi(t)) \leq V(\xi_0)$ then $x_{n+1}^\top(t) x_{n+1}(t) \leq \frac{V(\xi_0)}{\lambda}$. The proof is complete.

ACKNOWLEDGEMENTS

This work is partially supported by Chinese Science Council (CSC) 202006030019, the National Natural Science Foundation of China under Grant 62073081 and Grant 62050410352.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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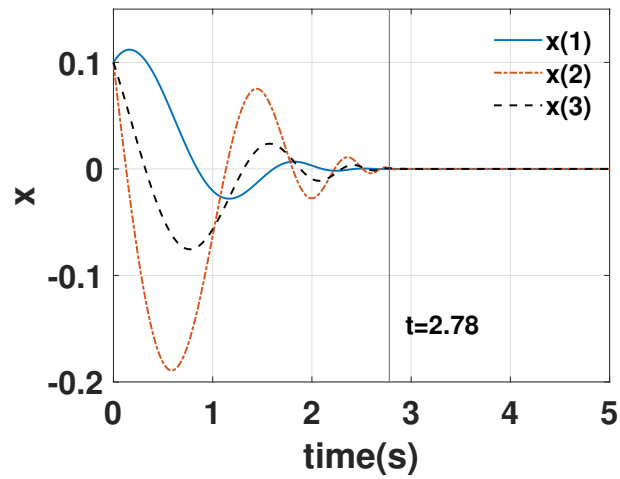
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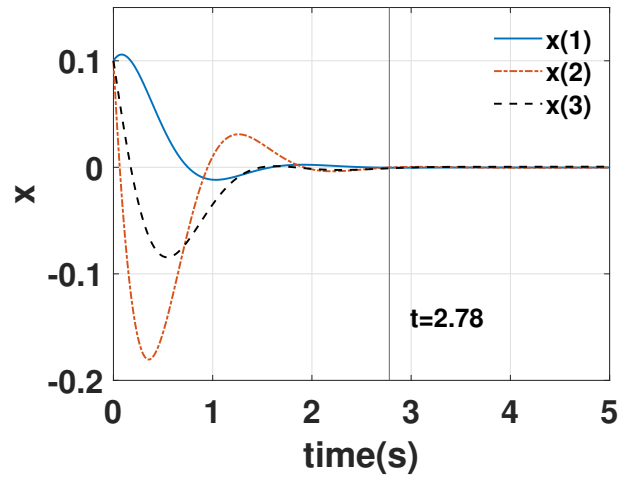
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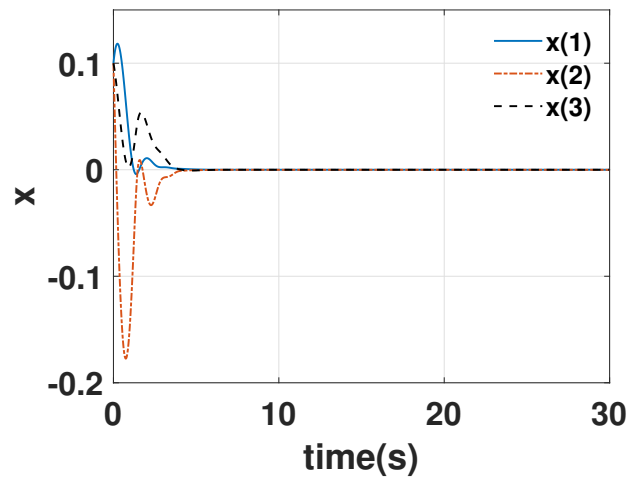
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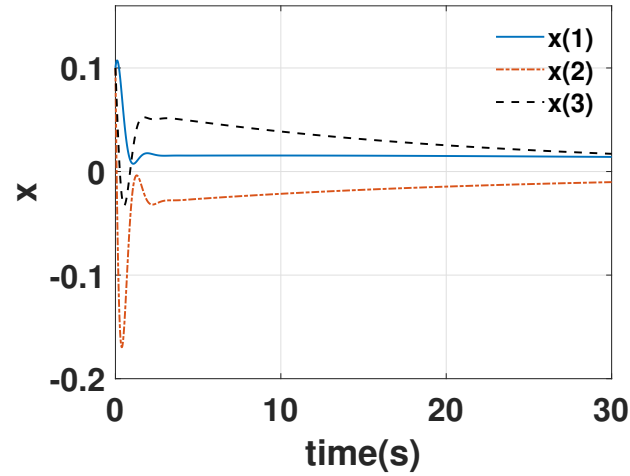
(a) Homogeneous PI control without perturbations



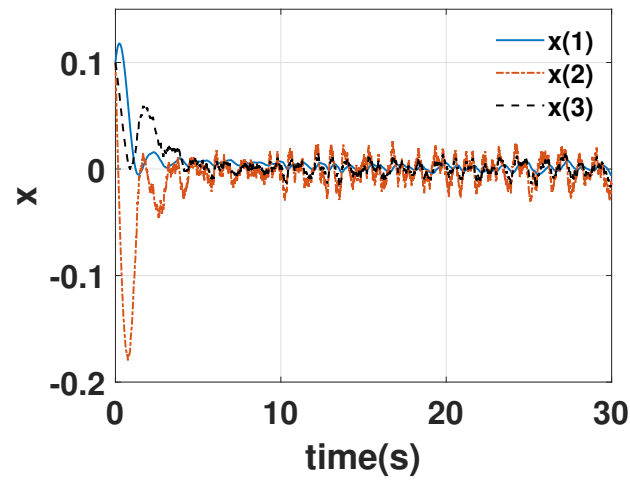
(b) Linear PI control without perturbations.



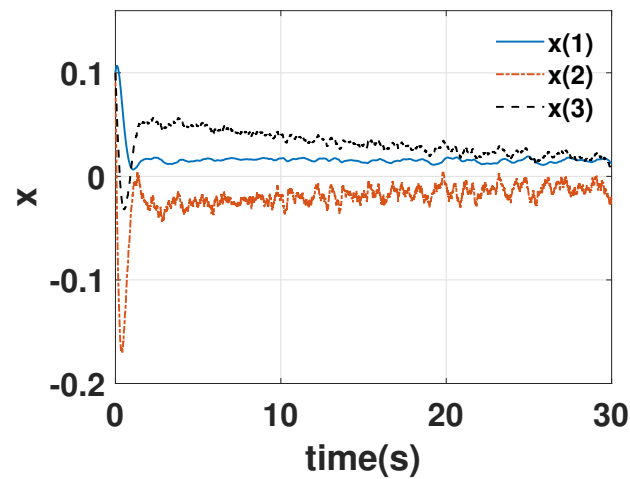
(c) Homogeneous PI control with additive disturbance



(d) Linear PI control with additive disturbance

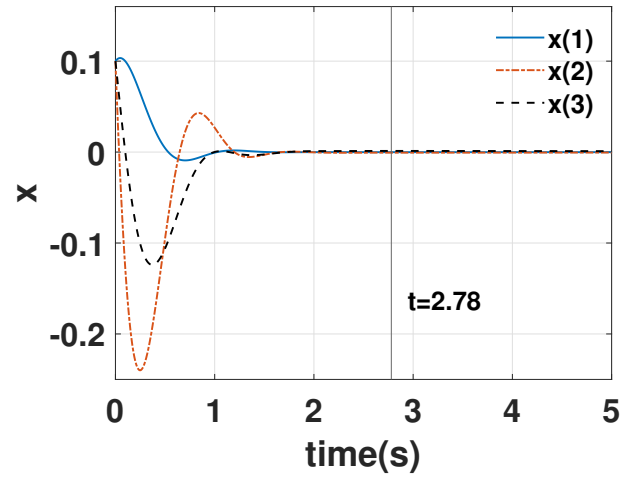


(e) Homogeneous PI control with disturbance and noise

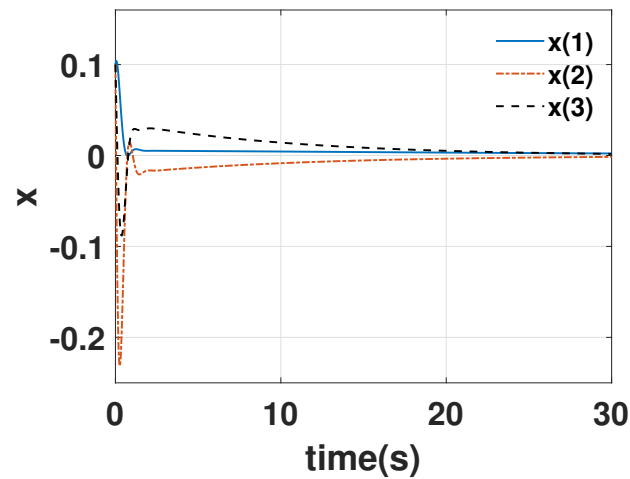


(f) Linear PI control with disturbance and noise

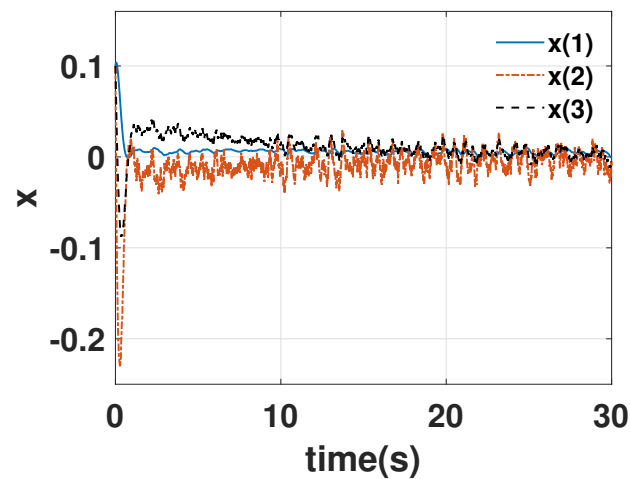
FIGURE 1 Comparison of homogeneous PI control and linear PI control without and with the perturbations and noises.



(a) System without perturbations.

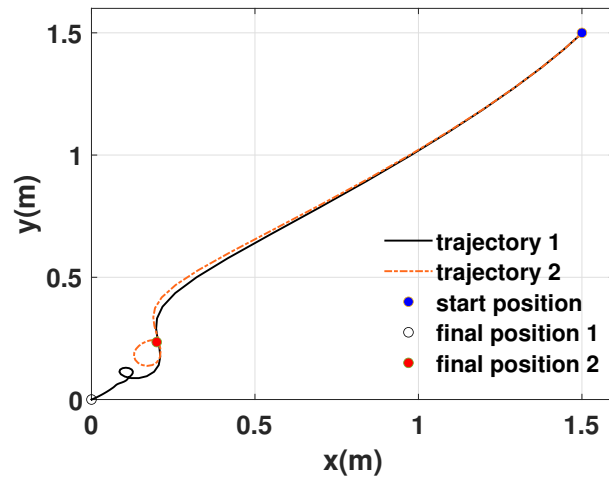


(b) System with additive disturbance.



(c) System with additive disturbance and measurement noise

FIGURE 2 Performance of the homogeneous PI control obtained by an "upgrading" of linear controller



(a) The trajectory 1 is generated by homogeneous control *with* integral action, while the trajectory 2 is generated by homogeneous control *without* integral action.

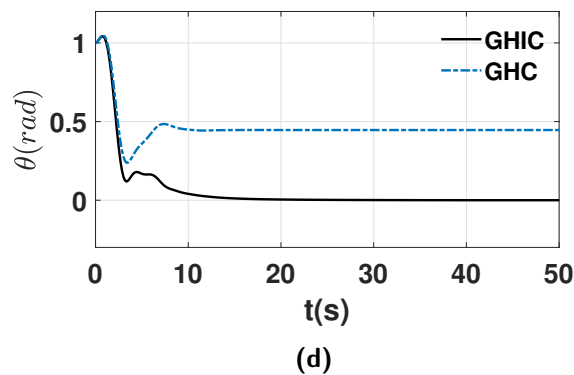
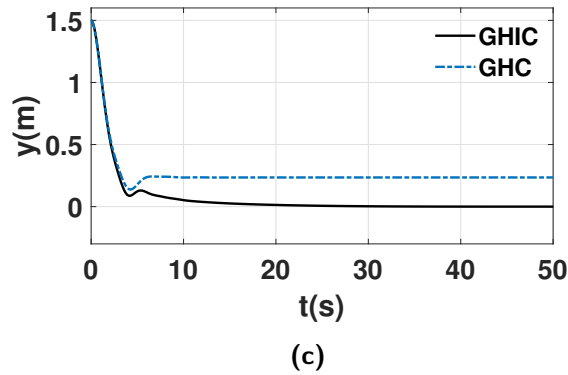
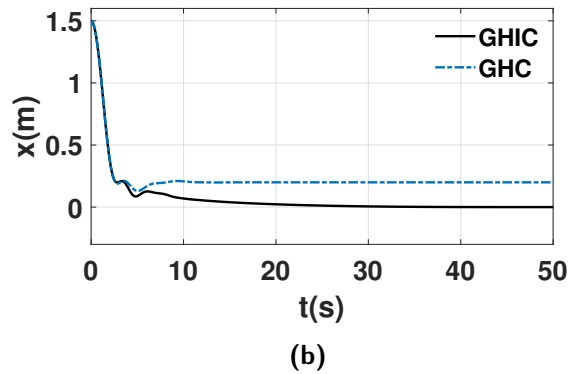


FIGURE 3 Simulation results for the omnidirectional mobile robot regulated by generalized homogeneous control *with* integral action (GHIC) and generalized homogeneous control (GHC) *without* integral action.