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# The Gauss-Galerkin approximation method in nonlinear filtering

Fabien Campillo\*

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## Abstract

We study an approximation method for the one-dimensional nonlinear filtering problem, with discrete time and continuous time observation. We first present the method applied to the Fokker-Planck equation. The convergence of the approximation is established. We finally present a numerical example.

**Keywords:** nonlinear filtering, moment method, particle approximation.

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## 1 Introduction

Usual methods for numerical solutions of partial differential equations typically involve a large number of space discretization points. Moreover, in their classical form, these methods use time-fixed discretization grids.

The method proposed by Donald A. Dawson [4] for the numerical solution of the Fokker-Planck equation, called the Gauss-Galerkin method, combines the Gauss quadrature and Galerkin approximation methods. This method, which can be considered as a particle method [10], has the double advantage of giving acceptable results even with a small number of unknown variables to calculate and a discretization grid able to adapt to the evolution of the solution of the partial differential equation considered. However, in its current form, the method is limited to the case of a single dimension of space.

We will study the behavior of this method, applied to the nonlinear filtering problem. In Section 2, we present the Gauss-Galerkin approximation method applied to the Fokker-Planck equation, and we establish a convergence result. The results in this section are a reworking and development of the work of Donald A. Dawson [4].

In Section 3, we first consider the nonlinear filtering problem with discrete time observation: we present the Gauss-Galerkin approximation and prove its convergence. We then

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consider the nonlinear filtering problem with continuous time observation: In this case, before introducing the approximation, we go back to the previous case by discretizing the observation equation.

We define the following spaces:

$\mathcal{M}_1(\mathbb{R})$	probability measures on $\mathbb{R}$ ,
$\mathcal{M}_+(\mathbb{R})$	non-negative measures on $\mathbb{R}$ ,
$\mathcal{M}(\mathbb{R})$	signed measures on $\mathbb{R}$ ,
$\mathcal{C}_u(\mathbb{R})$	bounded and uniformly continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ ,
$\mathcal{C}_c^\infty(\mathbb{R})$	continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ of class $\mathcal{C}^\infty$ with compact support,
$\mathcal{C}_b^2(\mathbb{R})$	continuous and bounded functions $\mathbb{R} \rightarrow \mathbb{R}$ of class $\mathcal{C}^2$ ,
$\mathcal{C}[0, T]$	continuous functions $[0, T] \rightarrow \mathbb{R}$ ,
$\mathcal{P}_{2N-1}$	polynomial functions of degree at most $2N - 1$ .

## 2 Numerical solution of the Fokker-Planck equation

### 2.1 The Gauss-Galerkin approximation method

To introduce the Fokker-Planck equation, we consider the stochastic differential equation:

$$(1) \quad dX_t = b(X_t) dt + \sigma(X_t) dW_t, \quad 0 \leq t \leq T, X_0 \sim \mu_0,$$

where  $(X_t)_{t \leq T}$  takes values in  $\mathbb{R}$ ;  $(W_t)_{t \leq T}$  is a real standard Wiener process independent of  $X_0$ . Let  $a(x) = \sigma^2(x)$ ,  $a'(x) = da(x)/dx$ ,  $b'(x) = db(x)/dx$ , we make the following assumptions:

**(H1)**  $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ , are measurable and bounded applications;

**(H2)**  $a' \in L^\infty(\mathbb{R})$  and there exists  $\underline{a} > 0$  such that  $a(x) \geq \underline{a}$ , for all  $x \in \mathbb{R}$ ;

**(H3)**  $b'$  is measurable bounded, and  $a'$  is continuous.

Under Assumptions **(H1)**-**(H2)**, Equation (1) admits a unique solution in the weak sense [12]. Hypothesis **(H3)** will be used in the following to demonstrate the convergence of the approximation.

Let  $\mu_t \in \mathcal{M}_1(\mathbb{R})$  be the distribution law of  $X_t$  on  $\mathbb{R}$ , for all  $0 \leq t \leq T$ :

$$\langle \mu_t, \varphi \rangle = \mathbb{E}(\varphi(X_t)), \quad \forall \varphi \in \mathcal{C}_b^2(\mathbb{R}),$$

where:

$$\langle \mu_t, \varphi \rangle := \int_{\mathbb{R}} \varphi(x) \mu_t(dx).$$

It results from the Itô's formula that  $(\mu_t)_{t \leq T}$  is a solution of the Fokker-Planck equation (written in weak form) :

$$(2) \quad \langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_s, \mathcal{L}\varphi \rangle ds, \quad 0 \leq t \leq T, \quad \forall \varphi \in \mathcal{C}_b^2(\mathbb{R}),$$

where  $\mathcal{L}$  denotes the the infinitesimal generator of the Markov process  $X_t$ :

$$\mathcal{L}\varphi(x) := b(x) \varphi'(x) + \frac{1}{2} a(x) \varphi''(x).$$

For  $N \in \mathbb{N}$  given, the Gauss-Galerkin approximation method consists in approximating  $(\mu_t)_{t \leq T}$  by a family of probability measures  $(\mu_t^N)_{t \leq T}$  of the form :

$$\mu_t^N(\mathrm{d}x) = \sum_{i=1}^N w_t^{(i)} \delta_{x_t^{(i)}}(\mathrm{d}x) \in \mathcal{M}_1(\mathbb{R}).$$

The functions  $t \rightarrow w_t^{(i)}, x_t^{(i)}$  are determined by posing :

$$(3) \quad \langle \mu_t^N, \pi \rangle = \langle \mu_0, \pi \rangle + \int_0^t \langle \mu_s^N, \mathcal{L}\pi \rangle \mathrm{d}s, \quad 0 \leq t \leq T, \quad \forall \pi \in \mathcal{P}_{2N-1};$$

Note that:

$$\langle \mu_0^N, \pi \rangle = \langle \mu_0, \pi \rangle, \quad \text{for all } \pi \in \mathcal{P}_{2N-1},$$

i.e.  $\mu_0^N$  is the  $N$ -points Gauss-Christoffel approximation of the  $\mu_0$  (see Section 4.1.1).

## 2.2 Convergence of the approximation

Under an additional assumption, we will establish a convergence result.

**Lemma 2.1 (Billingsley [1])** *Let  $\mu \in \mathcal{M}_+(\mathbb{R})$  with finite moments of all orders  $m_n = \langle \mu, x^n \rangle$ . Suppose that the power series :*

$$\sum_{n \in \mathbb{N}} \frac{\theta^n}{n!} m_n$$

*admits a strictly positive radius of convergence, then if  $\nu \in \mathcal{M}_+(\mathbb{R})$  is s.t.  $\langle \nu, x^n \rangle = m_n$  for all  $n$  then  $\mu = \nu$ . In this case, we say that the moment problem for  $\mu$  is well posed. As of now, we make the abuse of notation  $x^n$  to designate the polynomial function  $x \rightarrow x^n$ .*

Let:

$$\begin{aligned} m_n(t) &= \langle \mu_t, x^n \rangle, & \dot{m}_n(t) &= \mathrm{d}m_n(t)/\mathrm{d}t, \\ m_n^N(t) &= \langle \mu_t^N, x^n \rangle, & \dot{m}_n^N(t) &= \mathrm{d}m_n^N(t)/\mathrm{d}t. \end{aligned}$$

We make the additional hypothesis:

$$(H4) \quad \limsup_{n \rightarrow \infty} \left( \frac{m_{2n}(0)}{(2n)!} \right)^{\frac{1}{2n}} < \infty.$$

This assumption ensures in particular the existence of moments of all orders for  $X_0$ , and thus for  $X_t$ , for all  $0 \leq t \leq T$ . Moreover, by using the Cauchy criterion on the convergence of series, (H4) implies that the power series  $\sum_{n \in \mathbb{N}} (\theta^n/n!) m_n(0)$  has a strictly positive radius of convergence, so that according to Lemma 2.1,  $\mu_0$  is the only nonnegative measure on  $\mathbb{R}$  admitting  $(m_n(0))_{n \in \mathbb{N}}$  as moments.

**Theorem 2.2** Under assumptions (H1)-(H4) the Gauss-Galerkin approximation is convergent:

$$\mu_t^N \xrightarrow[N \rightarrow \infty]{} \mu_t, \quad t \geq 0.$$

To prove this theorem, we use several lemmas. In the following we will reason for  $t \in [0, T]$ ; all results will be true for any  $T > 0$

**Lemma 2.3** There exist real numbers  $K_n, K'_n$  which do not depend on  $N$  such that :

- (i)  $|m_n^N(t)| \leq K_n$ , for all  $(n, N)$  s.t.  $n \leq 2N - 1, 0 \leq t \leq T$ ,
- (ii)  $|\dot{m}_n^N(t)| \leq K'_n$ , for all  $(n, N)$  s.t.  $n \leq 2N - 1, 0 \leq t \leq T$ ,
- (iii) the power series  $\sum_{n \in \mathbb{N}} (\theta^n / n!) K_n$  has a strictly positive radius of convergence.

**Proof** We show that there exist  $K_n$  and  $K'_n$  such that :

- (4)  $|m_n(t)| \leq K_n$ , for all  $n, 0 \leq t \leq T$ ,
- (5)  $|\dot{m}_n(t)| \leq K'_n$ , for all  $n, 0 \leq t \leq T$ ,
- (6) the power series  $\sum_{n \in \mathbb{N}} (\theta^n / n!) K_n$  has a strictly positive radius of convergence.

Suppose that  $|m_{2n-2}(t)| \leq K_{2n-2}$ , for all  $0 \leq t \leq T$ , taking  $\varphi(x) = x^{2n}$  in (2) leads to:

$$m_{2n}(t) \leq m_{2n}(0) + c \int_0^t (m_{2n}(s) + n^2 K_{2n-2}) ds.$$

Then using Gronwall's lemma:

$$m_{2n}(t) \leq c (m_{2n}(0) + n^2 K_{2n-2}),$$

where  $c$  denotes a constant that depends on  $T$ , but not on  $n$ . Let  $K_0$  be such that  $|m_0(t)| \leq K_0$ , for all  $0 \leq t \leq T$ , we define by recurrence :

$$(7) \quad K_{2n} = c (m_{2n}(0) + n^2 K_{2n-2}),$$

then  $|m_{2n}(t)| \leq K_{2n}$ , for all  $n, 0 \leq t \leq T$ . Moreover:

$$|x|^{2n-1} \leq \frac{1}{2} \left( \frac{x^{2n}}{2n} + 2n x^{2n-2} \right),$$

then we can choose:

$$K_{2n-1} = \frac{1}{2} \left( \frac{K_{2n}}{2n} + 2n K_{2n-2} \right),$$

and (4) is thus proved. By explicitly writing  $K_{2n}$  from (7) we can show (6). (5) is verified without difficulty. To establish the lemma it suffices to note that the above argument remains valid for moments  $m_n^N(t)$  with the same constants  $K_n$  and  $K'_n$ .  $\square$

**Lemma 2.4** There exists a family of distribution laws  $(\nu_t; 0 \leq t \leq T)$ , and a subsequence  $(\nu_t^N; 0 \leq t \leq T)_{n \in \mathbb{N}}$  extracted from  $(\nu_t^N; 0 \leq t \leq T)_{n \in \mathbb{N}}$ , such that :

$$\nu_t^N \xrightarrow[N \rightarrow \infty]{} \nu_t, \quad 0 \leq t \leq T.$$

**Proof** According to Lemma 2.3 (i)-(ii), for all  $n$  fixed, the family  $(m_n^N(\cdot))$ ;  $N > (n + 1)/2$  is bounded and equicontinuous in  $\mathcal{C}[0, T]$ , and therefore relatively compact. By a Cantor diagonalization procedure we show that there exists an increasing sequence of integers  $(N_{n'})_{n' \in \mathbb{N}}$  and functions  $m_n^* \in \mathcal{C}[0, T]$ , such that:

$$(8) \quad m_n^{N_{n'}}(\cdot) \xrightarrow{n' \rightarrow \infty} m_n^*(\cdot) \quad \text{in } \mathcal{C}[0, T], \quad \forall n \in \mathbb{N}.$$

Moreover, we consider the following result [11]: Given a sequence of real numbers  $(m_p)_{p \in \mathbb{N}}$ , a necessary and sufficient condition for there to exist a non-negative measure which admits  $(m_p)_{p \in \mathbb{N}}$  for moments, is that

$$\forall P \in \mathbb{N}, C_0, C_1, \dots, C_P \in \mathbb{R} : \quad \left( \sum_{p=0}^P C_p x^p \geq 0, \quad \forall x \in \mathbb{R} \right) \Rightarrow \left( \sum_{p=0}^P C_p m_p \geq 0 \right).$$

This last property is satisfied by  $(m_n^{N_p}(t))_{n \in \mathbb{N}}$ , so is preserved at the limit  $p \rightarrow \infty$ . According to (8), there exists a nonnegative measure  $\nu_t$  which admits  $(m_n^*(t))_{n \in \mathbb{N}}$  for moments, for all  $n$  and  $0 \leq t \leq T$ . Hence for  $t \leq T$ :

$$(9) \quad \langle \mu_t^{N_p}, x^n \rangle \xrightarrow{p \rightarrow \infty} \langle \nu_t, x^n \rangle$$

According to Lemma 2.3, the power series  $\sum_{n \in \mathbb{N}} (\theta^n / n!) m_n^*(t)$  has a strictly positive radius of convergence  $\nu_t$  is the only law on  $\mathbb{R}$  which verifies (9) (see [1]), which makes it possible to assert that  $\nu_t^{N_p} \Rightarrow \nu_t$  as  $p \rightarrow \infty$  [2, p. 181].  $\square$

**Lemma 2.5** *Under the assumptions (H1)-(H3), the Fokker-Planck equation (2) has a unique solution  $t \rightarrow \mu_t$ , a function with values in  $\mathcal{M}_+(\mathbb{R})$ .*

**Proof** Using Itô's formula, we can easily verify that the law of  $X_t$  solves (2), hence the existence of a solution is proved. Let  $\psi(\cdot, \cdot) \in \mathcal{C}_b^{1,2}(\mathbb{R}_+ \times \mathbb{R})$  and  $\tilde{\mu}$  a solution of (2) with values in  $\mathcal{M}(\mathbb{R})$ . Then,

$$(10) \quad \langle \tilde{\mu}_t, \psi(t, \cdot) \rangle = \langle \tilde{\mu}_0, \psi(0, \cdot) \rangle + \int_0^t \langle \tilde{\mu}_s, \partial_s \psi(s, \cdot) + \mathcal{L} \psi(s, \cdot) \rangle ds.$$

Furthermore, we consider the backward partial differential equation:

$$(11) \quad \frac{\partial v(s, x)}{\partial s} + \mathcal{L} v(s, x) = 0, \quad s < t, \quad v(t, x) = \bar{v}(x), \quad \forall x \in \mathbb{R}$$

$(v'(s, x) := \partial v(s, x) / \partial s)$ . According to the assumptions made, and using regularity theorems for solutions of parabolic PDEs [8] we have : for all  $\bar{v} \in \mathcal{C}_c^\infty(\mathbb{R})$ , (11) admits a solution  $v \in \mathcal{C}_b^{1,2}([0, t] \times \mathbb{R})$ . After taking the difference between two solutions, to prove uniqueness it suffices to check that if  $\mu_0 = 0$  then  $\tilde{\mu}_t = 0$  for  $t \geq 0$ .

Let  $t \geq 0$  and  $\bar{v} \in \mathcal{C}_c^\infty(\mathbb{R})$ , by (11) we associate to  $\bar{v}$  an application  $v \in \mathcal{C}_b^{1,2}([0, t] \times \mathbb{R})$ . From (10), with  $\mu_0 = 0$ , and (11):

$$\langle \tilde{\mu}_t, v(t, \cdot) \rangle = \int_0^t \langle \tilde{\mu}_s, \partial_s v(s, \cdot) + \mathcal{L} v(s, \cdot) \rangle ds = 0,$$

so that  $\langle \tilde{\mu}_t, v(t) \rangle = \langle \tilde{\mu}_t, \bar{v} \rangle = 0$  for all  $\bar{v} \in \mathcal{C}_c^\infty(\mathbb{R})$ , hence  $\tilde{\mu}_t = 0$ .  $\square$

**Proof of Theorem 2.2** If we establish that

$$(12) \quad \langle \nu_t, \varphi \rangle = \langle \nu_0, \varphi \rangle + \int_0^t \langle \nu_s, \mathcal{L}\varphi \rangle ds, \quad t \leq T, \quad \forall \varphi \in \mathcal{C}_b^2(\mathbb{R}),$$

where  $(\nu_t)_{t \leq T}$  is the limit of a subsequence whose existence is guaranteed by Lemma 2.4, then by Lemma 2.5 we have  $\nu_t = \mu_t$ , for all  $0 \leq t \leq T$ . We deduce that a subsequence of  $\mu_t^N$  converges to  $\mu_t$ . But by redoing the demonstration, by uniqueness of the limit we show that the whole sequence converges. So we have to show that (12) is verified for all  $\varphi \in \mathcal{C}_b^2(\mathbb{R})$ . We will consider several steps.

*Step 1:* Suppose that  $\varphi$  is a polynomial function of degree  $d$ . For all  $N \geq (d+1)/2$ :

$$\langle \nu_t^N, \varphi \rangle = \langle \mu_0^N, \varphi \rangle + \int_0^t \langle \nu_s^N, \mathcal{L}\varphi \rangle ds,$$

so (12) is obtained by dominated convergence when  $N \rightarrow \infty$ .

*Step 2:* Suppose that  $\varphi(x) = e^{i\theta x} \pi(x)$ , where  $\theta \in \mathbb{R}$ ,  $\pi$  polynomial function, and  $i^2 = -1$ . Let us first take  $\varphi(x) = e^{i\theta x}$ ,  $|\theta| \leq \theta_1$ , where  $\theta_1$  is the radius of convergence given by Lemma 2.3-(iii). Let  $\varphi_n(x) = \sum_{k=0}^n (i\theta x)^k / k!$ ,  $\varphi_n$  verifies (12), so when  $n \rightarrow \infty$  we get  $\Phi(\theta) = 0$  for all  $|\theta| \leq \theta_1$  where:

$$\Phi(\theta) := \langle \nu_t, \varphi \rangle - \langle \mu_0, \varphi \rangle - \int_0^t \langle \nu_s, \mathcal{L}\varphi \rangle ds.$$

Thus for all  $j \geq 1$ ,  $\Phi^{(j)}(\theta) = 0$ ,  $|\theta| \leq \theta_1$ , where  $\Phi^{(j)}$  is the  $j$ th derivative of  $\Phi$  w.r.t.  $\theta$ , we deduce that (12) is true for any  $\varphi$  of the form  $e^{i\theta x} \pi(x)$ ,  $|\theta| \leq \theta_1$ ,  $\pi$  polynomial function. Using the inequality :

$$\left| \exp(i(\theta + \theta_1)x) + \exp(i\theta_1 x) \sum_{k=0}^n \frac{(i\theta x)^k}{k!} \right| \leq c \frac{|\theta|^{n+1}}{(n+1)!} |x|^{n+1},$$

and by the same argument, we show that (12) is verified for any  $\varphi$  of the form  $e^{i\theta x} \pi(x)$ ,  $|\theta| \leq 2\theta_1$ ,  $\pi$  polynomial function, and recursively for all  $\theta \in \mathbb{R}$ . Thus, Step 2 is proved.

*Step 3:* Suppose that  $\varphi \in \mathcal{C}^2(\mathbb{R})$  with compact support. There exists  $\varphi_n$  of the form:

$$\varphi_n(x) = \sum_{k=-n}^n a_k^n \exp(ib_k^n x) \quad \text{such that} \quad \|\varphi^{(j)} - \varphi_n^{(j)}\|_\infty \xrightarrow{n \rightarrow \infty} 0 \quad (j = 0, 1, 2),$$

where  $\varphi^{(j)}$  is the  $j$ th derivative of  $\varphi$ , (12) is verified for  $\varphi_n$  for all  $n$ , and therefore, by taking the limit  $n \rightarrow \infty$ , also for  $\varphi$ .

Step 4: Suppose  $\varphi \in \mathcal{C}_b^2(\mathbb{R})$ . Let  $\tau_n \in \mathcal{C}_b^2(\mathbb{R})$  s.t.

$$\begin{aligned} 0 &\leq \tau_n^{(j)} \leq 1, \quad j = 0, 1, 2, \\ \tau_n &= 1 \text{ on } [-n, n], \\ \tau_n &= 0 \text{ on } (-\infty, -n-1] \cup (n+1, \infty). \end{aligned}$$

Then we can apply the previous step to  $\varphi_n := \tau_n \varphi$  and by dominated convergence ( $n \rightarrow \infty$ ) we prove that  $\varphi$  satisfies (12), which ends the proof.  $\square$

**Remark 2.6** We proved the conservation of the Cauchy criterion :

$$\text{if } \lim_{n \rightarrow \infty} \left( \frac{m_{2n}(0)}{(2n)!} \right)^{\frac{1}{2n}} < \infty, \text{ then } \overline{\lim}_{n \rightarrow \infty} \left( \frac{m_{2n}(t)}{(2n)!} \right)^{\frac{1}{2n}} < \infty, \quad \forall t \in [0, T].$$

This result will be used to prove convergence in the case of nonlinear filtering.

### 3 Numerical solution of the Zakai equation

#### 3.1 Filtering with discrete time observation

We consider the system :

$$\begin{aligned} dX_t &= b(X_t) dt + \sigma(X_t) dW_t, \quad X_0 \sim \mu_0, \\ y_k &= h(X_{t_k}) + v_k, \end{aligned}$$

where  $0 \leq t \leq T$ ,  $0 < t_1 < \dots < t_K = T$  is a sequence of given instants, to simplify we take:

$$t_k = k \Delta, \text{ with } \Delta = \frac{T}{K} \text{ for some } K \in \mathbb{N}.$$

$(X_t)_{t \leq T}$ ,  $(W_t)_{t \leq T}$ ,  $(y_k)_{k \leq K}$  et  $(v_k)_{k \leq K}$  are processes with values in  $\mathbb{R}$ ;  $(v_k)_{k \leq K}$  is a sequence of independent Gaussian variables,  $v_k \sim N(0, R)$ ;  $(W_t)_{t \leq T}$  is a standard standard Wiener process independent of  $(v_k)_{k \leq K}$ ;  $X_0$  is independent of  $(W_t)_{t \leq T}$  and  $(v_k)_{k \leq K}$ . Note that the case where the observation  $y_k$  takes values in  $\mathbb{R}^d$  is treated in exactly the same way.

Let us assume Hypotheses (H1)-(H4) satisfied, as well as the hypothesis :

**(H5)**  $h : \mathbb{R} \rightarrow \mathbb{R}$  is measurable and bounded.

$X_t$  describes the evolution of a physical system, and  $y_k$  its discrete time observation. The filtering problem consists in determining  $\eta_t$ , the conditional law of  $X_t$  given  $(y_k)_{k; t_k \leq t} = (y_1, \dots, y_{\lfloor t/\Delta \rfloor})$ , that is:

$$\langle \nu_t, \varphi \rangle = \mathbb{E}(\varphi(X_t) | y_1, \dots, y_{\lfloor t/\Delta \rfloor}), \quad \forall \varphi \in \mathcal{C}_b^2(\mathbb{R}),$$

where  $\lfloor t/\Delta \rfloor$  is the integer part of  $t/\Delta$ . Between two moments of observation, i.e.  $t_{k-1} < t < t_k$ , the evolution of  $\eta_t$  is described by the (weak form of the) Fokker-Planck equation :

$$\frac{d}{dt} \langle \eta_t, \varphi \rangle = \langle \eta_t, \mathcal{L}\varphi \rangle, \quad \forall \varphi \in \mathcal{C}_b^2(\mathbb{R}).$$



At the time of observation  $t = t_k$ , by using the Bayes formula:

$$\langle \eta_{t_k}, \varphi \rangle = \frac{\langle \eta_{t_k^-}, f(\cdot, y_k) \varphi \rangle}{\langle \eta_{t_k^-}, f(\cdot, y_k) \rangle},$$

where

$$\langle \eta_{t_k^-}, \varphi \rangle := \lim_{\substack{t \rightarrow t_k \\ t < t_k}} \langle \eta_t, \varphi \rangle,$$

and  $f(x, y)$  is the local likelihood function:

$$f(x, y) := \exp\left(\frac{1}{R} h(x) y - \frac{1}{2R} h(x)^2\right).$$

Thus  $(\eta_t)_{t \leq T}$  is a solution of the equation :

$$(13) \quad \langle \eta_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \eta_s, \mathcal{L}\varphi \rangle ds + \sum_{k=1}^{\lfloor t/\Delta \rfloor} \left\{ \frac{\langle \eta_{t_k^-}, f(\cdot, y_k) \varphi \rangle}{\langle \eta_{t_k^-}, f(\cdot, y_k) \rangle} - \langle \eta_{t_k^-}, \varphi \rangle \right\}, \quad \forall \varphi \in \mathcal{C}_b^2(\mathbb{R}).$$

We propose to approximate  $\eta_t$  by a probability measure of the form:

$$\eta_t^N(dx) = \sum_{i=1}^N w_t^{(i)} \delta_{x_t^{(i)}}(dx),$$

where the stochastic processes  $w_t^{(i)}$  and  $x_t^{(i)}$  are determined by posing:

$$(14) \quad \langle \eta_t^N, \pi \rangle = \langle \mu_0, \pi \rangle + \int_0^t \langle \eta_s^N, \mathcal{L}\pi \rangle ds + \sum_{k=1}^{\lfloor t/\Delta \rfloor} \left\{ \frac{\langle \eta_{t_k^-}^N, f(\cdot, y_k) \pi \rangle}{\langle \eta_{t_k^-}^N, f(\cdot, y_k) \rangle} - \langle \eta_{t_k^-}^N, \pi \rangle \right\}, \quad \forall \pi \in \mathcal{P}_{2N-1}.$$

**Theorem 3.1** *Under assumptions (H1)-(H5), for any given trajectory  $(y_1, \dots, y_K)$ , the Gauss-Galerkin approximation is convergent :  $\eta_t^N \Rightarrow \eta_t$  as  $N \rightarrow \infty$ , for all  $0 \leq t \leq T$ .*

**Proof** Let  $m_n(t) = \langle \eta_t, x^n \rangle$ , let's assume that the hypotheses:

$$(15) \quad \eta_t^N \Rightarrow \eta_t, \text{ as } N \rightarrow \infty,$$

$$(16) \quad \overline{\lim}_{n \rightarrow \infty} \left( \frac{m_{2n}(t)}{(2n)!} \right)^{\frac{1}{2n}} < \infty$$

are verified for  $t = t_{k-1}$ ; we will show that (15)-(16) are verified for  $t \in [t_{k-1}, t_k]$ . To prove the theorem it will be enough for us to establish (15)-(16) for  $t = 0$ .

For  $t \in (t_{k-1}, t_k)$ , the evolution of  $\eta_t$  is described by the Fokker-Planck equation, we deduce from Theorem 2.2, and from (16) in  $t = t_{k-1}$ , that (15) is satisfied for all  $t \in (t_{k-1}, t_k)$ . Since:

$$\langle \eta_{t_k}^N, \varphi \rangle = \frac{\langle \eta_{t_k}^N, f(\cdot, y_k) \varphi \rangle}{\langle \eta_{t_k}^N, f(\cdot, y_k) \rangle},$$

we deduce that (15) is also true for  $t = t_k$ . Moreover,

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \left( \frac{m_{2n}(t_k)}{(2n)!} \right)^{\frac{1}{2n}} &= \overline{\lim}_{n \rightarrow \infty} \left( \frac{1}{(2n)!} \frac{\langle \eta_{t_k}^-, f(\cdot, y_k) x^{2n} \rangle}{\langle \eta_{t_k}^-, f(\cdot, y_k) \rangle} \right)^{\frac{1}{2n}} \\ &\leq \overline{\lim}_{n \rightarrow \infty} \left( \frac{m_{2n}(t_k^-)}{(2n)!} \right)^{\frac{1}{2n}}. \end{aligned}$$

Using (15), for  $t = t_{k-1}$ , and Remark 2.6, we show that the latter expression is finite. We deduce that (16) is true for  $t = t_k$ . To end the demonstration, we just need to check (15)-(16) for  $t = 0$ . From Equation (14),  $\eta_0^N$  is the Gauss-Christoffel approximation of  $\eta_0 = \mu_0$ , and the convergence  $\eta_0^N \Rightarrow \eta_0$  can be deduced from Theorem 2.2. By Moreover (16) in  $t = 0$  is exactly Hypothesis (H4).  $\square$

### 3.2 Filtering with continuous time observation

We consider the nonlinear system :

$$(17) \quad \begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dW_t, & X_0 \sim \mu_0, \\ dY_t = h(X_t) dt + dV_t, & Y_0 = 0, \end{cases}$$

for  $0 \leq t \leq T$ . The assumptions of the previous sections are assumed to be satisfied. The observation  $(Y_t)_{t \leq T}$  with values in  $\mathbb{R}$ , is here in continuous time,  $(V_t)_{t \leq T}$  is a standard Wiener process independent of  $X_0$  and  $(W_t)_{t \leq T}$ .

The filtering problem consists in determining  $\nu_t$  the conditional distribution of  $X_t$  given  $\mathcal{F}_t := \sigma(Y_s; s \leq t)$ , that is:

$$\langle \nu_t, \varphi \rangle = \mathbb{E}(\varphi(X_t) | Y_s, 0 \leq s \leq t), \quad \forall \varphi \in \mathcal{C}_b^2(\mathbb{R}).$$

To characterize  $\nu_t$ , we can use the method of the reference probability. Let  $\overset{\circ}{\mathbb{P}}$  be the law determined by :

$$\frac{d\overset{\circ}{\mathbb{P}}}{d\mathbb{P}} = Z_T^{-1}, \quad \text{with} \quad Z_t = \exp \int_0^t \left( h(X_s) dY_s - \frac{1}{2} h(X_s)^2 ds \right).$$

The computation of the conditional distribution of  $X_t$  given  $\mathcal{F}_t$  under  $\mathbb{P}$ , is related to an expression computed under  $\overset{\circ}{\mathbb{P}}$  by the Kallianpur-Striebel formula :

$$\mathbb{E}(\varphi(X_t) | \mathcal{F}_t) = \frac{\overset{\circ}{\mathbb{E}}(\varphi(X_t) Z_t | \mathcal{F}_t)}{\overset{\circ}{\mathbb{E}}(Z_t | \mathcal{F}_t)}.$$

We define  $\tilde{\nu}_t$  the unnormalized conditional distribution of  $X_t$  given  $\mathcal{F}_t$ , by posing:

$$\langle \tilde{\nu}_t, \varphi \rangle := \mathbb{E}(\varphi(X_t) Z_t | \mathcal{F}_t).$$

$\tilde{\nu}_t$  is a solution of the (weak form) Zakai equation:

$$(18) \quad \langle \tilde{\nu}_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \tilde{\nu}_s, \mathcal{L}\varphi \rangle ds + \int_0^t \langle \tilde{\nu}_s, h\varphi \rangle dY_s, \quad \forall \varphi \in C_b^2(\mathbb{R}).$$

We can determine  $\tilde{\nu}_t^N$  the Gauss-Galerkin approximation of  $\tilde{\nu}_t$ , but after discretization in time, the equation of  $\tilde{\nu}_t^N$  involves only discrete time observations. It is therefore preferable to discretize the observation equation in (17) directly:

$$y_k = h(X_{t_k}) + v_k,$$

with  $t_k = k\Delta$ , and where  $v_k := (Y_{t_{k+1}} - Y_{t_k})/\Delta$  and  $y_k$  is the approximation of  $(Y_{t_{k+1}} - Y_{t_k})/\Delta$ .

We define  $\mathcal{F}_t^\Delta := \sigma(y_1, \dots, y_{\lfloor t/\Delta \rfloor})$  et  ${}^\Delta\nu_t$  the conditional distribution of  $X_t$  given  $\mathcal{F}_t^\Delta$ . As we saw in Section 3.1, the evolution of  $({}^\Delta\nu_t)$  is described by the equation:

$$(19) \quad \langle {}^\Delta\nu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle {}^\Delta\nu_s, \mathcal{L}\varphi \rangle ds \\ + \sum_{k=1}^{\lfloor t/\Delta \rfloor} \left\{ \frac{\langle {}^\Delta\nu_{t_k^-}, f_\Delta(\cdot, y_k) \varphi \rangle}{\langle {}^\Delta\nu_{t_k^-}, f_\Delta(\cdot, y_k) \rangle} - \langle {}^\Delta\nu_{t_k^-}, \varphi \rangle \right\}, \quad \forall \varphi \in C_b^2(\mathbb{R}),$$

with:

$$f_\Delta(x, y) := \exp\left(h(x)y\Delta - \frac{1}{2}h(x)^2\Delta\right).$$

We have the following result:

**Theorem 3.2** *In addition to assumptions (H1)-(H5), suppose that  $h \in C_b^2(\mathbb{R})$ , then for any observed trajectory  $(Y_s)_{s \leq t}$ :*

$${}^\Delta\nu_t \xrightarrow[\Delta \rightarrow 0]{} \nu_t, \quad 0 \leq t \leq T$$

(provided that  $\nu_t$  is defined in “robust form” cf. for example [9]).

**Proof** Consider a probability space  $(\Omega, \mathcal{F}, \mathring{\mathbb{P}})$  and the following SDE on this space:

$$\begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dW_t, \\ dY_t = d\mathring{V}_t, \end{cases}$$

where  $(W_t, \mathring{V}_t)$  is a standard Wiener process with values in  $\mathbb{R} \times \mathbb{R}$  independent from  $X_0$  and  $Y_0 = 0$ .

For  $Y$  we adopt the canonical representation  $(\mathcal{C}[0, T], \mathcal{B}, \mathcal{W}, Y)$ , i.e.  $(\mathcal{C}[0, T], \mathcal{B})$  is the space of continuous functions  $[0, T] \rightarrow \mathbb{R}$  equipped with the Borel  $\sigma$ -algebra  $\mathcal{B}$ ,  $\mathcal{W}$  is

the Wiener measure on this space and  $Y$  is the canonical process: for all  $\omega \in \mathcal{C}[0, T]$ ,  $Y_t(\omega) := \omega(t)$ . Moreover, let  $\bar{\mathbb{P}}$  be the marginal distribution of  $X$  on a space  $(\bar{\Omega}, \bar{\mathbb{P}}, \bar{\mathcal{F}})$ .

Under  $\overset{\circ}{\mathbb{P}}$ ,  $X$  and  $Y$  are independent:

$$(20) \quad \overset{\circ}{\mathbb{P}}(dX, dY) = \bar{\mathbb{P}}(dX) \times \mathcal{W}(dY).$$

Let

$$\Delta_K := \frac{T}{K},$$

and  $t_k^K := k \Delta_K$  which we will denote  $t_k$ . Define also:

$$h^K(t, x) := h(t_k), \quad \text{for } t \in [t_k, t_{k+1}).$$

Consider the following  $\overset{\circ}{\mathbb{P}}$ -exponential martingales:

$$\begin{cases} Z_t := \exp \int_0^t \left( h(X_s) dY_s - \frac{1}{2} h(X_s)^2 ds \right), \\ Z_t^K := \exp \int_0^t \left( h^K(X_s) dY_s - \frac{1}{2} h^K(X_s)^2 ds \right). \end{cases}$$

Let:

$$dM_s := \mathcal{L}h(X_s) ds + (h' \sigma)(X_s) dW_s,$$

integration by part in the Itô integral leads to:

$$(21) \quad Z_t = \exp \left( h(X_t) Y_t - \int_0^t \left( h(X_s) dM_s - \frac{1}{2} h(X_s)^2 ds \right) \right).$$

In addition, as  $t \rightarrow h^K(t, x)$  is piecewise constant:

$$(22) \quad Z_t^K = \exp \left( h(X_{t_k}) (Y_t - Y_{t_k}) - \frac{1}{2} h(X_{t_k})^2 (t - t_k) \right. \\ \left. + \sum_{j=0}^{k-1} \left\{ h(X_{t_j}) (Y_{t_{j+1}} - Y_{t_j}) - \frac{1}{2} h(X_{t_j})^2 \Delta_k \right\} \right), \quad \text{for all } t \in [t_k, t_{k+1}).$$

Representations (21) and (22) allow to consider  $Z_t$  and  $Z_t^K$  for any fixed trajectory of  $Y$ .

We define the distribution:

$$(23) \quad \frac{d\mathbb{P}}{d\overset{\circ}{\mathbb{P}}} := Z_T, \quad \frac{d\mathbb{P}^K}{d\overset{\circ}{\mathbb{P}}} := Z_T^K,$$

and  $\mathbb{E}, \mathbb{E}^K$  the associated expectations. Under  $\mathbb{P}$  (resp.  $\mathbb{P}^K$ ),  $(X, Y)$  admits the representation:

$$\begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dW_t, \\ dY_t = h(X_t) dt + dV_t, \end{cases} \quad (\text{resp. } dY_t = h^K(X_t) dt + dV_t^K)$$

where  $V$  (resp.  $V^K$ ) is a  $\mathbb{P}$  standard Wiener process (resp.  $\mathbb{P}^K$  standard Wiener process) defined by:

$$V_t := \overset{\circ}{V}_t + \int_0^t h(X_s) ds \quad (\text{resp. } V_t^K := \overset{\circ}{V}_t + \int_0^t h^K(s, X_s) ds).$$

Consider now the system with discrete time observation:

$$\begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dW_t, \\ y_k^K = h(X_{t_k}) + v_k^K, \end{cases}$$

with

$$v_k^K := \frac{1}{\Delta_k} (V_{t_{k+1}} - V_{t_k}).$$

Clearly, under  $\mathbb{P}^K$ , the conditional distribution of  $X_t$  given  $\sigma(y_k^K; k \text{ s.t. } t_k \leq t)$  is equal to the conditional distribution of  $X_t$  given  $\mathcal{F}_t := \sigma(Y_s; s \leq t)$ . Our goal is therefore to demonstrate the convergence of expressions  $\mathbb{E}^K(\varphi(X_t)|\mathcal{F}_t)$  for any continuous and bounded function  $\varphi$ .

Thanks to the Kallianpur-Striebel formula, (23) gives:

$$\mathbb{E}(\varphi(X_t)|\mathcal{F}_t) = \frac{\mathbb{E}(\varphi(X_t) Z_t | \mathcal{F}_t)}{\mathbb{E}(Z_t | \mathcal{F}_t)}, \quad \mathbb{E}^K(\varphi(X_t)|\mathcal{F}_t) = \frac{\mathbb{E}(\varphi(X_t) Z_t^K | \mathcal{F}_t)}{\mathbb{E}(Z_t^K | \mathcal{F}_t)}, \quad \overset{\circ}{\mathbb{P}}\text{-a.s..}$$

But, according to (20):

$$\mathbb{E}(\varphi(X_t) Z_t | \mathcal{F}_t) = \bar{\mathbb{E}}(\varphi(X_t) Z_t), \quad \mathbb{E}(\varphi(X_t) Z_t^K | \mathcal{F}_t) = \bar{\mathbb{E}}(\varphi(X_t) Z_t^K), \quad \mathcal{W}\text{-a.s..}$$

For a given trajectory  $(Y_s; s \leq t)$  of the observation process, it is thus necessary to prove:

$$\mathbb{E}(\varphi(X_t) Z_t^K | \mathcal{F}_t) \xrightarrow{K \rightarrow \infty} \mathbb{E}(\varphi(X_t) Z_t | \mathcal{F}_t) \quad \text{a.s..}$$

Since  $\varphi$  is bounded, it is sufficient to show that:

$$(24) \quad Z_t^K \xrightarrow{K \rightarrow \infty} Z_t \quad \text{in } L^1(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$$

For any given  $t$ ,  $Z_t^K$  and  $Z_t$  are positive random variables with mean 1, for all  $K$ , so a sufficient condition for (24) is:

$$Z_t^K \xrightarrow{K \rightarrow \infty} Z_t \quad \text{in } \bar{\mathbb{P}}\text{-probability,}$$

this result can be deduced from definitions (21) and (22) of  $Z_t$  and  $Z_t^K$ , which completes the proof of Theorem 3.2.  $\square$

We use the Gauss-Galerkin method to approximate  $\Delta \nu_t$  by a probability measure  $\Delta \nu_t^N$  of the form:

$$\Delta \nu_t^N(dx) = \sum_{i=1}^N w_t^{(i)} \delta_{x_t^{(i)}}(dx),$$

where the stochastic processes  $(w_t^{(i)})_{t \leq T}$  and  $(x_t^{(i)})_{t \leq T}$ , which depend on  $\Delta$  and  $N$ , are determined by posing:

$$\langle \Delta \nu_t^N, \pi \rangle = \langle \mu_0, \pi \rangle + \int_0^t \langle \Delta \nu_s^N, \mathcal{L}\pi \rangle ds + \sum_{k=1}^{\lfloor t/\Delta \rfloor} \frac{\langle \Delta \nu_{t_k^-}^N, (f_\Delta(\cdot, y_k) - 1) \pi \rangle}{\langle \Delta \nu_{t_k^-}^N, f_\Delta(\cdot, y_k) \rangle}, \quad \forall \pi \in \mathcal{P}_{2N-1}.$$

According to Theorem 3.1, for any  $\Delta$ , we have the following convergence:

$$(25) \quad \Delta \nu_t^N(\omega) \xrightarrow[N \rightarrow \infty]{} \Delta \nu_t(\omega), \quad \text{for almost all } \omega, \text{ and } 0 \leq t \leq T.$$

Let  $(f_p)_{p \in \mathbb{N}}$  be a dense sequence in  $\mathcal{C}_u(\mathbb{R})$ , the set of bounded and uniformly continuous functions. We define :

$$d(\mu, \nu) := \sum_{p \in \mathbb{N}} \frac{1}{2^p} \frac{|\langle \mu, f_p \rangle - \langle \nu, f_p \rangle|}{\|f_p\|_\infty},$$

with  $\|f\|_\infty = \sup\{|f(x)|; x \in \mathbb{R}\}$ ;  $d(\cdot, \cdot)$  is a metric on  $\mathcal{M}_+(\mathbb{R})$ , which induces a topology equivalent to the one induced by the weak convergence of measures [12]. Thus (25) and Theorem 3.2 implies that for all  $\Delta > 0$  we can associate  $N(\Delta) \in \mathbb{N}$  such that:

$$\Delta \nu_t^{N(\Delta)}(\omega) \xrightarrow[\Delta \rightarrow 0]{} \nu_t(\omega), \quad \text{for all } \omega \text{ a.s. and } 0 \leq t \leq T.$$

This last convergence result is not entirely satisfactory, we do not know how to explicitly choose  $N(\Delta)$ , but as in practice the observation equation is always in discrete time, for a given discretization step  $\Delta$  the convergence (25) is satisfactory.

We could have obtained a root mean square convergence in the case of continuous time observations, however this is not of great interest.

To obtain a convergence for each observed trajectory, one could think of using the ‘‘robust form’’ of the Zakai equation, this was not feasible, because the multiplication by  $\exp(-h(x)Y_t)$  brings out of the space of polynomials of degree at most equal to  $2N - 1$ .

## 4 Numerical study

### 4.1 Presentation of the algorithm

We will use the following notations:

$$\begin{array}{lll} (w_k^{(1:N)}, x_k^{(1:N)}) & \text{will denote} & (w_k^{(i)}, x_k^{(i)})_{i=1, \dots, N}, \\ w_{0:K}^{(i)} & \dots & (w_k^{(i)})_{k=0, \dots, K}, \\ w_k^{(1:N)} = \tilde{w}_k^{(1:N)} & \dots & w_k^{(i)} = \tilde{w}_k^{(i)} \text{ for } i = 1, \dots, N, \\ i = 1 : N & \dots & i = 1, \dots, N, \quad \text{etc.} \end{array}$$

#### 4.1.1 A reminder on Gauss-Christoffel quadrature methods

All the results of this section come from Wheeler [13] and Gautschi [6]. Given  $N \in \mathbb{N}$  and a nonnegative measure  $\nu \in \mathcal{M}_+(\mathbb{R})$ , we want to find  $(w_{1:N}, x_{1:N})$  such that:

$$(26) \quad \sum_{i=1}^N w_i \pi(x_i) = \langle \nu, \pi \rangle, \quad \forall \pi \in \mathcal{P}_{2N-1}.$$

It is well known that, if  $x \rightarrow \nu((-\infty, x])$  admits at least  $N$  increasing points, then (26) admits a unique solution, where the particle  $x_i$  are two by two distinct and the weights are strictly positive,  $w_i > 0$ . The empirical measure:

$$\nu^N \stackrel{\text{def}}{=} \sum_{i=1}^N w_i \delta_{x_i}$$

is the Gauss-Christoffel approximation of  $\nu$ ;  $\nu^N$  and  $\nu$  have the same  $2N$  first moments:

$$\langle \nu^N, x^p \rangle = m_p := \langle \nu, x^p \rangle, \quad p = 0 : 2N - 1.$$

To compute  $(w_{1:N}, x_{1:N})$  from the moments  $m_{0:2N-1}$  we use a classical method. We introduce  $\pi_{0:2N-1}$ , the family of orthogonal polynomial functions relative to the measure  $\nu$ , i.e.  $\pi_p$  is of degree  $p$  and  $\langle \nu, \pi_p \pi_q \rangle = 0$  if  $p \neq q$ . These polynomial functions are defined up to a multiplicative constant, we can decide for example that the coefficient of the highest degree monomial in  $\pi_p$  is 1. In this case the family  $\pi_{0:2N-1}$  satisfies a recurrence relation of the form:

$$\begin{aligned} \pi_{-1}(x) &= 0, & & \text{(by convention),} \\ \pi_0(x) &= 1, \\ \pi_{p+1}(x) &= (x - \alpha_p) \pi_p(x) - \beta_p \pi_{p-1}(x), & & p = 0 : 2N - 2, \end{aligned}$$

for all  $x \in \mathbb{R}$ , for some  $(\alpha_{0:2N-2}, \beta_{0:2N-2}) \in \mathbb{R}^{2(2N-1)}$  with  $\beta_p > 0$  for  $p \geq 1$  and  $\beta_0 = 0$ .

The calculation of  $(w_{1:N}, x_{1:N})$  is reduced to the calculation of the coefficients  $(\alpha_{0:N-1}, \beta_{0:N-1})$ , with by convention  $\beta_0 = 0$ , in the following way, let:

$$(27) \quad J_N = \begin{pmatrix} \alpha_0 & \sqrt{\beta_1} & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & (0) \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{\beta_{N-2}} & \alpha_{N-2} & \sqrt{\beta_{N-1}} \\ (0) & & & \sqrt{\beta_{N-1}} & \alpha_{N-1} \end{pmatrix}.$$

$J_N$  has  $N$  real eigenvalues  $\lambda_{1:N}$ , two by two distinct; let  $\mathbf{v}_{1:N}$  the respectively associated orthonormal eigenvectors. We have the following result :

$$(28) \quad (w_{1:N}, x_{1:N}) = (\mathbf{v}_{1,1:N}^2, \lambda_{1:N})$$

where  $\mathbf{v}_{1,i}$  denotes the first component of the vector  $\mathbf{v}_i$ .

Next, we have to notice that using the standard moments  $m_p = \langle \nu, x^p \rangle$  is numerically not a good idea, it leads to ill-conditioned algorithms. A classical method is instead to use *modified moments*, that is:

$$\tilde{m}_p \stackrel{\text{def}}{=} \langle \nu, \tilde{\pi}_p \rangle, \quad p = 0 : 2N - 1.$$

where  $\tilde{\pi}_{0:2N-1}$  is a *given* basis of  $\mathcal{P}_{2N-1}$  formed by orthogonal vectors, these kind of polynomial functions are defined by a recurrence:

$$\begin{aligned} \tilde{\pi}_{-1}(x) &= 0, & (\text{by convention}), \\ \tilde{\pi}_0(x) &= 1, \\ \tilde{\pi}_{p+1}(x) &= (x - \tilde{\alpha}_p) \tilde{\pi}_p(x) - \tilde{\beta}_p \tilde{\pi}_{p-1}(x), & p = 1 : 2N - 2, \end{aligned}$$

(for all  $x \in \mathbb{R}$ ) where the recurrence coefficients  $(\tilde{\alpha}_{0:2N-2}, \tilde{\beta}_{0:2N-2})$  are given with  $\tilde{\beta}_p > 0$  for  $p \geq 1$  and  $\tilde{\beta}_0 \equiv 0$ . In practice, we can use the Hermite polynomials.

#### Initialization

$$\begin{aligned} \sigma_{-1,0} &\leftarrow 0 \\ \sigma_{0,p} &\leftarrow \tilde{m}_p, \quad p = 0 : 2N - 1 \\ \alpha_0 &\leftarrow \tilde{\alpha}_0 + \tilde{m}_1 / \tilde{m}_0 \\ \beta_0 &\leftarrow 0 \end{aligned}$$

#### Iterations

$$\begin{aligned} &\text{for } p = 1 : N - 1 \text{ do} \\ &\quad \text{for } q = p : 2N - p + 1 \text{ do} \\ &\quad\quad \sigma_{p,q} \leftarrow \sigma_{p-1,q+1} - (\alpha_{p-1} - \tilde{\alpha}_q) \sigma_{p-1,q} - \beta_{p-1} \sigma_{p-2,q} + \tilde{\beta}_q \sigma_{p-1,q-1} \\ &\quad \text{end for} \\ &\quad \alpha_p \leftarrow \tilde{\alpha}_p - \frac{\sigma_{p-1,p}}{\sigma_{p-1,p-1}} + \frac{\sigma_{p,p+1}}{\sigma_{p,p}} \\ &\quad \beta_p \leftarrow \frac{\sigma_{p,p}}{\sigma_{p-1,p-1}} \\ &\quad \text{end for} \end{aligned}$$

Algorithm 1: *This modified Chebyshev algorithm allows us to compute  $(\alpha_{0:N-1}, \beta_{0:N-1})$  from  $(\tilde{m}_{0:2N-1}, \tilde{\alpha}_{0:2N-2}, \tilde{\beta}_{0:2N-2})$ , see [13].*

Finally, the computation of  $(\alpha_{0:N-1}, \beta_{0:N-1})$  from  $(\tilde{m}_{0:2N-1}, \tilde{\alpha}_{0:2N-2}, \tilde{\beta}_{0:2N-2})$  is performed using the modified Chebyshev Algorithm 1.

#### 4.1.2 Fokker-Planck equation

We first consider the approximation algorithm of the Fokker-Planck equation. The practical implementation of this algorithm requires a time discretization of the equation:

$$(29) \quad \langle \mu_t^N, \tilde{\pi}_p \rangle = \langle \mu_0, \tilde{\pi}_p \rangle + \int_0^t \langle \mu_s^N, \mathcal{L} \tilde{\pi}_p \rangle ds, \quad 0 \leq t \leq T, \quad p = 0 : 2N - 1,$$

where  $\tilde{\pi}_{0:2N-1}$  denotes a basis of  $\mathcal{P}_{2N-1}$ . In  $t = 0$ , (29) leads to the fact that  $\mu_0^N$  is the Gauss-Christoffel approximation of  $\mu_0$  so that  $\mu_0$  can be replaced by  $\mu_0^N$ .



All time discretization schemes could be considered, but in order to simplify the presentation we will use the Euler scheme with a time step  $\delta = T/L$ , with  $L \in \mathbb{N}$ . In order to simplify the notations, in the case of nonlinear filtering discussed later, we will assume that  $L$  is a multiple of  $K$ , so that the observation instants  $t_k = k \Delta$  are included in  $(\ell\delta)_{\ell=0:L}$ . Also to simplify the notation,  $\mu_{\ell\delta}^N, w_{\ell\delta}^{(i)}$  (etc.) will be noted  $\mu_\ell^N, w_\ell^{(i)}$  (etc.).

The time-discretized equation (29) is thus written:

$$(30) \quad \begin{aligned} & \langle \mu_0^N, \tilde{\pi}_p \rangle \leftarrow \langle \mu_0, \tilde{\pi}_p \rangle, \quad p = 0 : 2N - 1 \\ & \mathbf{for} \ell = 1 : L \mathbf{ do} \\ & \quad \langle \mu_\ell^N, \tilde{\pi}_p \rangle \leftarrow \langle \mu_{\ell-1}^N, \tilde{\pi}_p + \mathcal{L}\tilde{\pi}_p \delta \rangle, \quad p = 0 : 2N - 1 \\ & \mathbf{end for} \end{aligned}$$

where:

$$\mu_\ell^N(dx) = \sum_{i=1}^N w_\ell^{(i)} \delta_{x_\ell^{(i)}}(dx)$$

is the approximation of  $\mu_t^N$  at time  $t = \ell \delta$ .

From  $(w_{\ell-1}^{(1:N)}, x_{\ell-1}^{(1:N)})$ , the recurrence (30) allows us to approximate the modified moments of  $\mu_\ell^N$  with respect to the basis  $\tilde{\pi}_{0:2N-1}$ , that is :

$$\tilde{m}_p(\ell) \simeq \tilde{m}_p(\ell-1) + \delta \sum_{i=1}^N w_{\ell-1}^{(i)} \mathcal{L}\tilde{\pi}_p(x_{\ell-1}^{(i)}) \quad p = 0 : 2N - 1.$$

Given  $\tilde{m}_{0:2N-1}(\ell)$ , we now want to calculate  $(w_\ell^{(1:N)}, x_\ell^{(1:N)})$  such that :

$$(31) \quad \sum_{i=1}^N w_\ell^{(i)} \tilde{\pi}_p(x_\ell^{(i)}) = \tilde{m}_p(\ell), \quad p = 0 : 2N - 1.$$

To solve this problem, we use the Gauss-Christoffel quadrature method presented in Section 4.1.1. The Gauss-Galerkin approximation algorithm, for the Fokker-Planck equation, is given by see Algorithm 2.

### 4.1.3 Nonlinear filtering equation

For the nonlinear filtering problem, we have to solve numerically an equation of the form (cf. (14) et (18)) :

$$(32) \quad \begin{aligned} \langle \nu_t^N, \tilde{\pi}_p \rangle &= \langle \mu_0, \tilde{\pi}_p \rangle + \int_0^t \langle \nu_s^N, \mathcal{L}\tilde{\pi}_p \rangle ds \\ &+ \sum_{k=1}^{\lfloor t/\Delta \rfloor} \left\{ \frac{\langle \nu_{t_k^-}^N, f(\cdot, y_k) \tilde{\pi}_p \rangle}{\langle \nu_{t_k^-}^N, f(\cdot, y_k) \rangle} - \langle \nu_{t_k^-}^N, \tilde{\pi}_p \rangle \right\}, \quad p = 0 : 2N - 1. \end{aligned}$$

**Inputs**

$$\begin{aligned} \tilde{\alpha}_p, \tilde{\beta}_p, \quad p = 0 : 2N - 2 \\ \tilde{m}_p(0) := \langle \mu_0, \tilde{\pi}_p \rangle, \quad p = 0 : 2N - 1 \end{aligned}$$

**Iterations****for**  $\ell = 0 : L$  **do**Computation of  $(\alpha_{0:N-1}, \beta_{0:N-1})$  from  $(\tilde{m}_{0:2N-1}(\ell), \tilde{\alpha}_{0:2N-2}, \tilde{\beta}_{0:2N-2})$  (cf. Algo. 1)Computation of eigenvalues and orthonormal eigenvectors of  $J_N$  defined in (27)Computation of  $(w_\ell^{(1:N)}, x_\ell^{(1:N)})$  from (28)

$$\tilde{m}_p(\ell + 1) \leftarrow \tilde{m}_p(\ell) + \delta \sum_{i=1}^N w_\ell^{(i)} \mathcal{L} \tilde{\pi}_p(x_\ell^{(i)}), \quad p = 0 : 2N - 1$$

**end for**

Algorithm 2: *The Gauss-Galerkin approximation algorithm for the Fokker-Planck equation (2) presented with an Euler scheme (any other scheme can be used, see Section 4.1.4).*

Equation (32) after discretization using the Euler scheme, is written :

$$\langle v_0^N, \tilde{\pi}_p \rangle \leftarrow \langle \mu_0, \tilde{\pi}_p \rangle \text{ for } p = 0 : 2N - 1$$

**for**  $\ell = 1 : L$  **do**

$$\langle \nu_\ell^N, \tilde{\pi}_p \rangle \leftarrow \langle \nu_{\ell-1}^N, \tilde{\pi}_p + \Delta \mathcal{L} \tilde{\pi}_p \rangle \text{ for } p = 0 : 2N - 1 \quad \{\text{prediction}\}$$

**if**  $(\ell \text{ modulo } \frac{L}{K}) = 0$  **then**

$$k \leftarrow \ell K / L \quad \{\text{observation index}\}$$

$$\langle \nu_\ell^N, \tilde{\pi}_p \rangle \leftarrow \frac{\langle \nu_\ell^N, f(\cdot, y_k) \tilde{\pi}_p \rangle}{\langle \nu_\ell^N, f(\cdot, y_k) \rangle} \text{ for } p = 0 : 2N - 1 \quad \{\text{correction}\}$$

**end if****end for**

The complete algorithm is then equivalent to the one presented for the of Fokker-Planck.

**4.1.4 Numerical tools**

For the approximation of the Fokker-Planck equation and of the prediction part of the nonlinear filter, we use a Runge-Kutta algorithm of order 2; one could of course use more efficient schemes if the nature of the considered problem requires it.

In practice, the basis  $\tilde{\pi}_{0:2N-1}$  of  $\mathcal{P}_{2N-1}$  used is that of the Hermite polynomial functions. For the computation of the eigenvalues of  $J_N$ , we used a variant of the *QL* algorithm for symmetric and tridiagonal matrices from the EISPACK software library [5].

**4.2 Example**

We present an example of application of the Gauss-Galerkin method in nonlinear filtering. The computations have been done on a VAX 730 computer in double precision FORTRAN 77. The approximation method applied to the Fokker-Planck equation on nonlinear examples gave good results up to  $N = 10$  ( $N$ : number of Gauss points). Beyond that, we run into problems of ill-conditioning. For the filtering problem, we first tested the method on linear examples. We compared the results obtained with those given by a Kalman-Bucy

filter. Here again, we obtained good results, even with very few Gauss points ( $N = 3$  or  $4$ ). We now present a numerical example; let us consider the nonlinear filtering problem:

$$(33) \quad \begin{cases} dX_t = -X_t dt + \sqrt{2} dW_t, & X_0 \sim N(0, 1), \\ dY_t = \exp(iX_t) dt + \rho dV_t, & Y_0 = 0, \end{cases}$$

$0 \leq t \leq T$ . The standard Wiener process  $(V_t)_{t \leq T}$  and the observation process  $(Y_t)_{t \leq T}$  take values in the complex plan ( $i^2 = -1$ );  $(W_t)_{t \leq T}$  is a real standard Wiener process independent of  $(V_t)_{t \leq T}$ ;  $X_0$  is independent of  $(W_t)_{t \leq T}$  and  $(V_t)_{t \leq T}$ . Let  $\nu_t$  be the conditional distribution of  $X_t$  given  $\mathcal{F}_t = \sigma(Y_s; s \leq t)$ . We implemented three methods of approximation of  $\nu_t$ :

**GGA Gauss-Galerkin approximation :**  $\nu_t$  is approximated by a distribution law of the form  $\nu_t^{\text{GGA}} = \sum_{i=1}^N w_t^{(i)} \delta_{x_t^{(i)}}$ , where  $N$  is the number of Gauss points. The calculation of  $\nu_t^{\text{GGA}}$  was presented in Section 4.1.3.

**FD Finite Differences :** we use a finite difference scheme in space, in order to solve numerically the Zakai equation for the unnormalized conditional density of  $\nu_t$ .  $\nu_t$  is thus approximated by a law  $\nu_t^{\text{FD}}$  of the form  $\nu_t^{\text{FD}}(dx) = p(t, x) dx$ ; for details of this method cf. Le Gland [7].

**EKF Extended Kalman filter :**  $\nu_t$  is approximated by the Gaussian distribution  $\nu_t^{\text{EKF}} = N(\hat{X}_t^{\text{EKF}}, Q_t^{\text{EKF}})$  where  $\hat{X}_t^{\text{EKF}}$  and  $Q_t^{\text{EKF}}$  are the outputs of the extended Kalman filter associated to (33).

**Remarks 4.1** (i) *These three methods are in fact implemented after discretization in time of the system (33).*

(ii) *The initial condition  $X_0$  as well as the Wiener processes (in discrete time: the Gaussian white noise)  $W_t$  and  $V_t$  have been simulated on a computer.*

(iii) *The FD method is used as a reference method: we will compare the conditional moments computed by GGA with those computed by FD. However, FD has the disadvantage that it cannot be applied in a simple way in the case where the support of  $\nu_t$  does not remain, when  $t$  varies, in a bounded and fixed domain of  $\mathbb{R}$ . Indeed in FD the conditional density  $p(t, x)$  is computed on a domain  $[-M, M]$  fixed in advance.*

For the simulation we take  $T = 10$  and  $\Delta = 0.01$ . In a first set of simulations we take  $N = 10$  and  $\rho = 0.5$ , see Figures 1-4. In a second set of simulations we take  $N = 2$  and  $\rho = 1$ , see Figure 5. In view of the numerical examples (two of which are presented at the end of the section) we can make several observations:

(i) The estimators  $\hat{X}_t^{\text{method}} := \langle \nu_t^{\text{method}}, x \rangle$  (method = GGA, FD, EKF) of  $X_t$ , given by the three methods, are equivalent (cf. Fig. 1). On the other hand, contrary to GGA, EKF gives a poor estimate of the conditional variance  $Q(t) := \langle \nu_t, x^2 \rangle - \langle \nu_t, x \rangle^2$  (cf. Fig. 2).

(ii) GGA correctly follows the evolution of the conditional moments for the first set of parameters ( $N = 10, \rho = 0.5$ ), the first 14 moments are estimated in a satisfactory way).

(iii) Even for a small number of Gauss points ( $N = 2$  in the second set of parameters), GGA gives significant results (cf. Fig. 5).

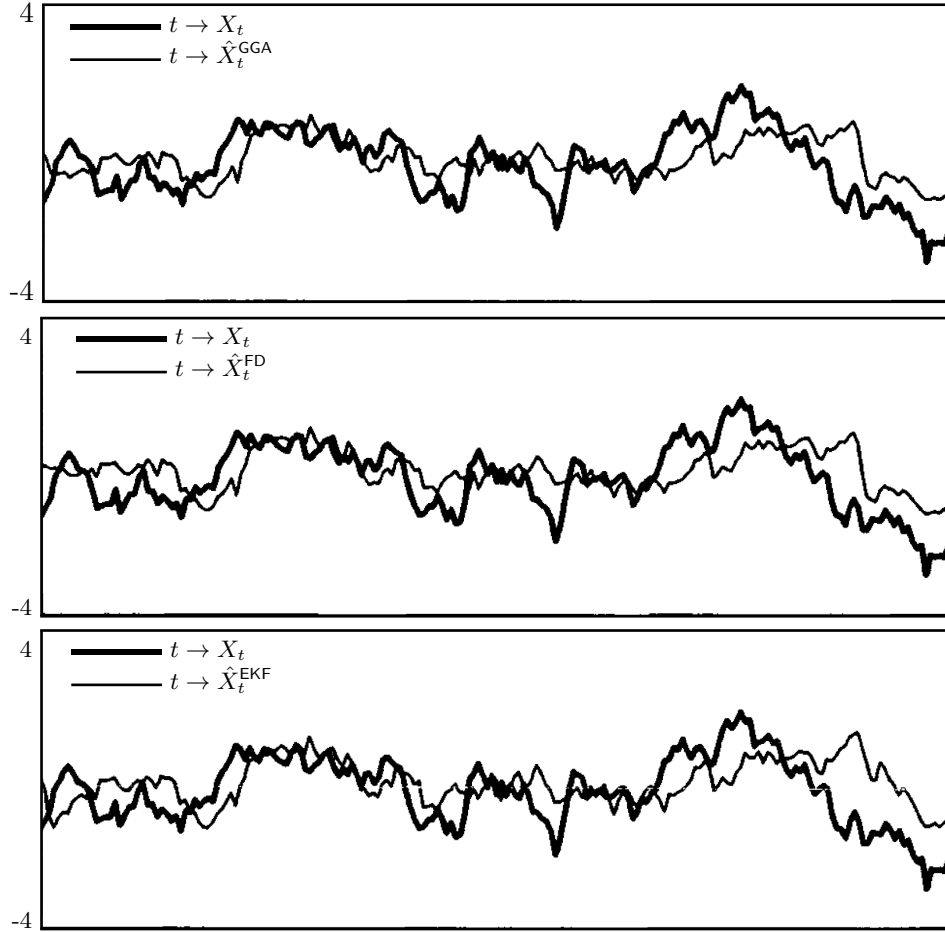


Figure 1: First set of parameters ( $N = 10$ ,  $\rho = 2$ ); comparison of the real state trajectory  $t \rightarrow X_t$  and of the estimators  $t \rightarrow \hat{X}_t^{\text{method}} := \langle \nu_t^{\text{method}}, x \rangle$  with method = GGA, FD, EKF.

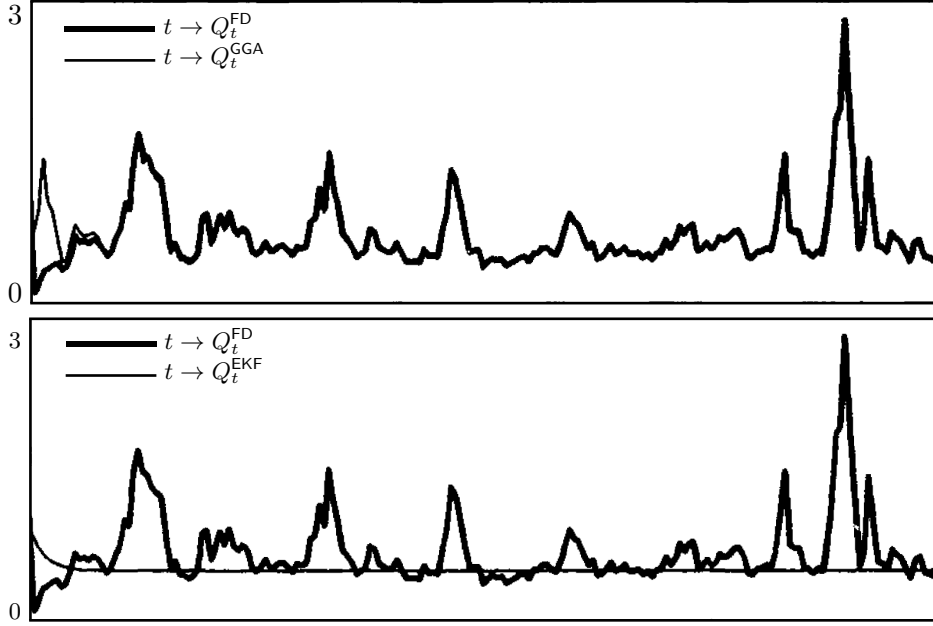


Figure 2: First set of parameters ( $N = 10$ ,  $\rho = 2$ ); comparison of conditional variances,  $t \rightarrow Q^{\text{method}}(t) := \langle \nu_t^{\text{method}}, x^2 \rangle - \langle \nu_t^{\text{method}}, x \rangle^2$  with method = GGA, FD, EKF.

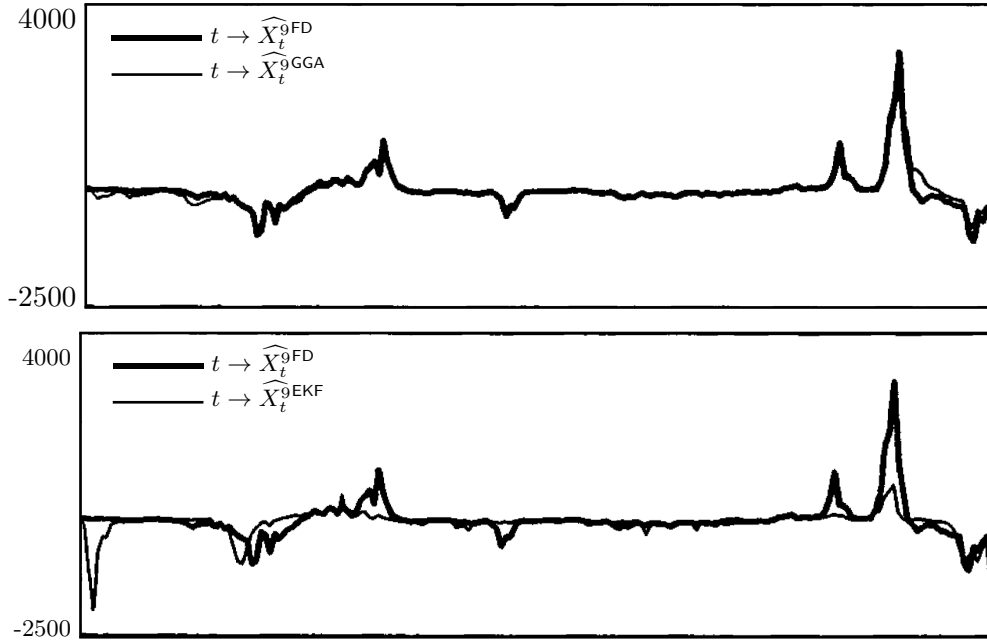


Figure 3: First set of parameters ( $N = 10$ ,  $\rho = 2$ ); comparison of conditional moments of order 9,  $t \rightarrow \widehat{X}_t^{9\text{method}} = \langle \nu_t^{\text{method}}, x^9 \rangle$  with method = GGA, FD, EKF.

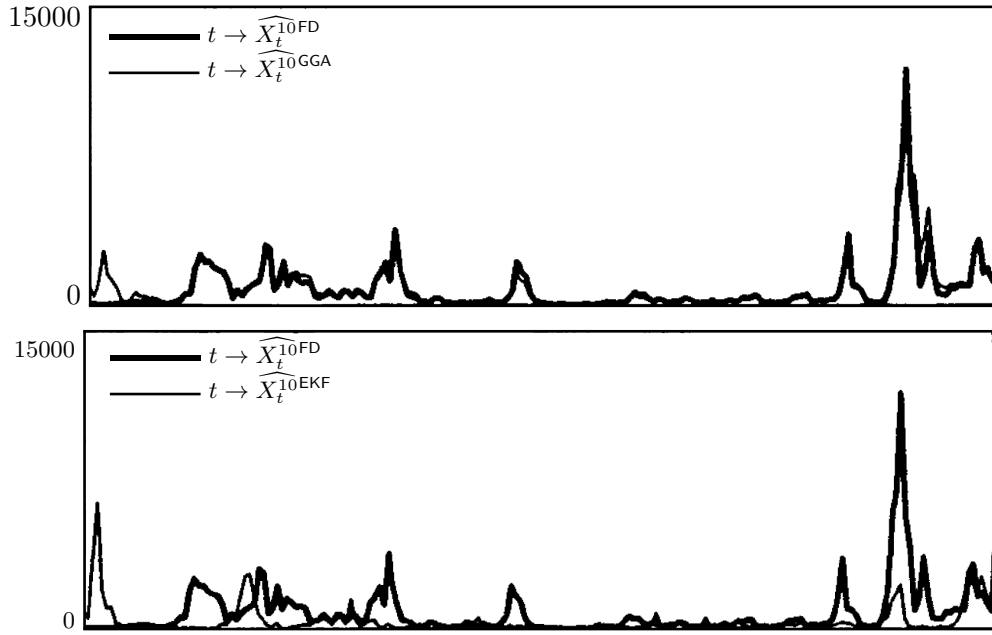


Figure 4: First set of parameters ( $N = 10, \rho = 2$ ); comparison of conditional moments of order 10,  $t \rightarrow \widehat{X}_t^{10\text{method}} = \langle \nu_t^{\text{method}}, x^{10} \rangle$  with method = GGA, FD, EKF.

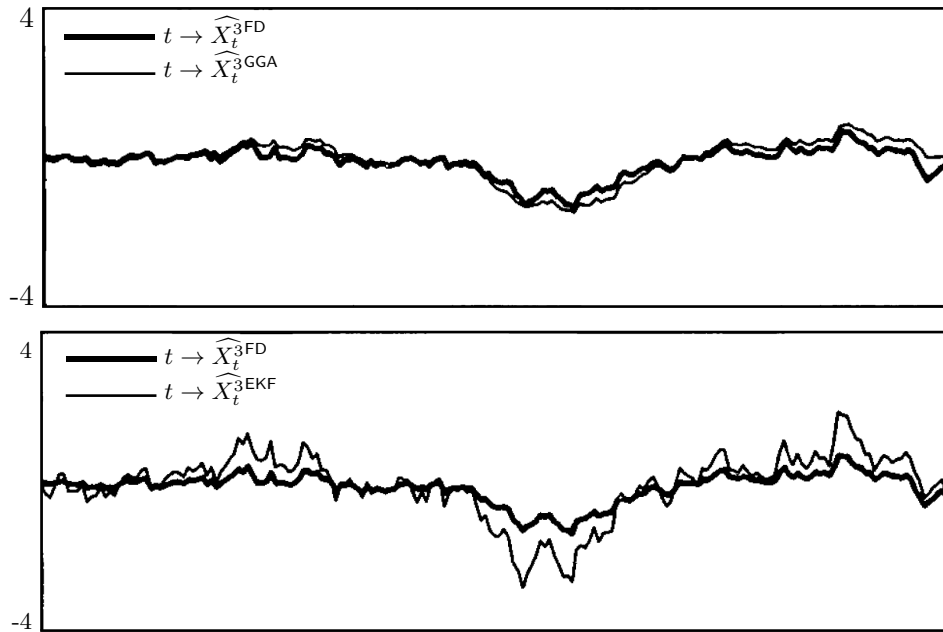


Figure 5: Second set of parameters ( $N = 2, \rho = 1$ ); comparison of conditional moments of order 3,  $t \rightarrow \widehat{X}_t^{3\text{method}} = \langle \nu_t^{\text{method}}, x^3 \rangle$  with method = GGA, FD, EKF.

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## Addendum

This is the English translation of the paper [B]. A number of typos (many...) have been corrected, some notations and demonstrations have been clarified. Also some elements from the original works [3], that was not detailed or present in the 1986 version, such as the proof of Theorem 3.2, are developed here in order to obtain a self-contained version.

This article contains, to my knowledge, the first occurrence of the term “**particle approximation**” in the context of nonlinear filtering. Indeed, the conditional law  $\eta_t$  of the state given the observations is approximated by an empirical law of the form:

$$\eta_t(dx) \simeq \eta_t^N(dx) = \sum_{i=1}^N w_t^{(i)} \delta_{x_t^{(i)}}(dx),$$

where  $w_t^{(i)} \geq 0$ ,  $\sum_{i=1}^N w_t^{(i)} = 1$ , and  $\delta_{x_t^{(i)}}(dx)$  is the Dirac measure on the particle  $x_t^{(i)}$ .

I coined this term in reference to the recent work at the time of Pierre-Arnaud Raviart on the approximation of solutions of first-order PDEs: “the exact solution is approximated by a linear combination of Dirac measures in the space variables” [C] and [10].

In the following years we proposed another particle approximation method in nonlinear filtering limited to the noise-free state equation case. In this case, the infinitesimal generator  $\mathcal{L}$  is of first order making it possible to use the particle approximation methods proposed by P.A. Raviart [C]. Although proposed in a rather limited case, the proposed approximation method constitutes one of the premises of what will be called later “particle filtering” or “sequential Monte Carlo”. In our approach, a crucial step was however missing, the famous bootstrap step! This idea, in the context of nonlinear filtering, came to the table later, in the beginning of the 90s [E-F].

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*In [B], I regrettably forgot to thank Walter Gautschi. Summer of 1984, a few months before the defense of my thesis, I indeed needed some additional elements concerning the Gauss quadrature methods using orthogonal polynomial functions. As Walter Gautschi was visiting Europe, I had invited him to Marseille. He completely clarified the situation for me. To thank him I proposed him to visit Aix-en-Provence... but my car broke down on the highway, Walter Gautschi finally got to visit Aix-en-Provence in record time ! I warmly thank him.*