Beating binary powering for polynomial matrices

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ABSTRACT

The Nth power of a polynomial matrix of fixed size and degree can be computed by binary powering as fast as multiplying two polynomials of linear degree in N. When Fast Fourier Transform (FFT) is available, the resulting arithmetic complexity is *softly linear* in N, i.e. linear in N with extra logarithmic factors. We show that it is possible to beat binary powering, by an algorithm whose complexity is *purely linear* in N, even in absence of FFT. The key result making this improvement possible is that the entries of the Nth power of a polynomial matrix satisfy linear differential equations with polynomial coefficients whose orders and degrees are independent of N. Similar algorithms are proposed for two related problems: computing the Nth term of a C-recursive sequence of polynomials, and modular exponentiation to the power N for bivariate polynomials.

CCS CONCEPTS

• **Computing methodologies** → **Algebraic algorithms**.

KEYWORDS

Algebraic Algorithms; Computational Complexity; FFT; Binary Powering; C-recursive Sequence; Rational Power Series; Linear Differential Equations; Creative Telescoping; Polynomial Matrices.

1 INTRODUCTION

A sequence $(u_n)_{n\geq 0}$ is called *C-recursive* if it satisfies a linear recurrence relation whose coefficients are constant with respect to n. The famous sequence $(f_n)_{n\geq 0}$ of Fibonacci numbers, defined by the recurrence $f_{n+2} = f_{n+1} + f_n$ and the initial values $f_0 = 0, f_1 = 1$, is perhaps the most basic example of a C-recursive sequence after the geometric ones $(q^n)_{n\geq 0}$. It is classical that the term f_N can be computed in $O(\log N)$ arithmetic operations, thus as fast as q^N. This can be achieved by *binary powering* for q^N, and in fact for f_N as well, since it is the top-right entry of C^N where $C = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is a 2×2 matrix. This idea generalizes to any C-recursive sequence $(u_n)_{n\geq 0}$: a recurrence of order $r \geq 1$ for $(u_n)_{n\geq 0}$ can be encoded, via its *companion matrix*, into an $r \times r$ matrix recurrence of order 1. Then the term u_N of the sequence appears as an entry of the Nth power of this $r \times r$ companion matrix multiplied by the vector of initial values $(u_0, \ldots, u_{r-1})$ [18, 31]. Then u_N can be computed in $O(\log N)$ arithmetic operations, and in $O(N \log(N))$ bit operations if $(u_n)_{n\geq 0}$ is an integer sequence, using fast integer multiplication [21]. Here r is considered as a constant parameter, i.e., $r \in O(1)$.

Fibonacci polynomials $F_n(x)$ are a natural generalization of Fibonacci numbers (see e.g. [10]). They are defined by the recurrence

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x) \quad \text{for } n \geq 0 \tag{1}$$

and the initial values $F_0(x) = 0, F_1(x) = 1$. The first few terms are

$$(F_n)_{n\geq 0} = (0, 1, x, x^2 + 1, x^3 + 2x, x^4 + 3x^2 + 1, \ldots).$$

Obviously, for all $n \geq 1$, the polynomial $F_n(x)$ is monic of degree $n - 1$ and the sum of its coefficients is $F_n(1) = f_n$.

Given $N \in \mathbb{N}$, the direct iterative algorithm for computing $F_N(x)$ has arithmetic complexity $O(N^2)$. It computes, for each $n \leq N$, all the n coefficients of the intermediate polynomial $F_n(x)$; in total this amounts to $\Theta(N^2)$ coefficients. Therefore, if one wants to compute all of $(F_0, \ldots, F_N)$ then this direct method is optimal with respect to the total arithmetic size of the output. However, it becomes quadratic if one is only interested in determining $F_N(x)$ alone.

To compute the polynomial $F_N(x)$ faster, one can use, as in the scalar case, the reformulation of the second-order recurrence (1) as a first-order (polynomial) matrix recurrence:

$$\begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix} = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{2}$$

This shows that $F_n(x)$ is the top-right entry of the matrix $C(x)^n$, where $C(x)$ is the 2×2 polynomial matrix $C(x) = \left(\begin{smallmatrix} x & 1 \\ 1 & 0 \end{smallmatrix}\right)$. One can again compute $C(x)^N$ using binary powering, whose costliest step is the multiplication of two polynomial matrices each with degree about $N/2$. This leads to an algorithm that finds $F_N(x)$ in arithmetic complexity $O(\mathsf{M}(N))$, where $\mathsf{M}(N)$ denotes the arithmetic cost of polynomial multiplication in degree at most N.

Using FFT-based polynomial multiplication, this amounts to a number of arithmetic operations in the base field $\mathbb{K}$ which is quasi-linear in N [11]. Not only does this compare favorably to the complexity $O(N^2)$ of the direct iterative algorithm, but this is even quasi-optimal (i.e., optimal up to logarithmic factors) with respect to the arithmetic size $\Theta(N)$ of the output polynomial $F_N(x)$.

In this context, the idea also generalizes to any C-recursive sequence $(u_n(x))_{n\geq 0}$ of polynomials in $\mathbb{K}[x]$, which we will call *polynomial C-recursive sequences*. Indeed, one can encode any recurrence of arbitrary (but independent of n) order $r \geq 1$ and coefficients in $\mathbb{K}[x]$ into a polynomial $r \times r$ matrix recurrence of order 1, and the Nth term of the sequence, $u_N(x)$, can be computed as an element in the Nth power of an $r \times r$ polynomial matrix multiplied by the polynomial vector of initial values. Conversely, computing the Nth power of any polynomial matrix can be reduced to computing terms in polynomial C-recursive sequences (see the introduction of Section 4). Binary powering allows to solve both problems in $O(\mathsf{M}(N))$ arithmetic operations, and in $O(N^2 \log(N))$ bit operations if $\mathbb{K} = \mathbb{Q}$, considering both the recurrence order (or the matrix size) r and the recurrence degree (or the matrix degree) d as constant parameters, i.e., $r, d \in O(1)$. The main question addressed in this article is:

Can one achieve a better complexity for these tasks?

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As far as scalar C-recursive sequences are concerned, the arithmetic complexity $O(\log N)$ seems very difficult (if not impossible) to beat, but it is perhaps not impossible to improve the bit complexity $O(N \log(N))$ towards $O(N)$. While we do not achieve this, our results provide polynomial analogues for this type of improvement. As frequently noticed in computer algebra, polynomials are "computationally easier" to deal with than integers. In our case, philosophically, this comes from the fact that we can benefit from an additional operation on polynomials: differentiation.

This possibly cryptic remark will hopefully become clear throughout the next section. There, using the specific case of Fibonacci polynomials as a test bench, we argue why it is indeed legitimate to hope for algorithms of arithmetic complexity $O(N)$ for computing the Nth term of a polynomial C-recursive sequence.

Main result. Recall that a *C-finite sequence* is a sequence $(u_n)_{n\geq 0}$ of elements u_n in some ring R which satisfies a recurrence equation

$$u_{n+r} = c_{r-1}u_{n+r-1} + \cdots + c_0 u_n \quad \text{for all } n \geq 0, \tag{3}$$

for $c_0, \ldots, c_{r-1} \in R$. In this work we consider *polynomial* C-finite sequences, i.e., the case $R = \mathbb{K}[x]$ for some (effective) field $\mathbb{K}$ of characteristic zero; thus $u_n = u_n(x) \in \mathbb{K}[x]$. The customary data structure for representing such a sequence consists of the polynomials $c_0(x), \ldots, c_{r-1}(x)$ defining the recurrence and the r initial conditions $u_0(x), \ldots, u_{r-1}(x) \in \mathbb{K}[x]$. The *order* of the recurrence is r while its *degree* is the maximum of the degrees of the c_i's.

Theorem 1.1. *Let $\mathbb{K}$ be an effective field of characteristic 0. Let d and r be fixed positive integers. For each of the following problems, there exists an algorithm solving it in $O(N)$ operations $(\pm, \times, \div)$ in $\mathbb{K}$:*

- *SeqTerm: Given a polynomial C-finite sequence $(u_n(x))_{n\geq 0}$ of order and degree at most r and d, compute the Nth term $u_N(x)$.*
- *BivModPow: Given polynomials $Q(x, y)$ and $P(x, y)$ in $\mathbb{K}[x, y]$ of degrees in y and x at most r and d, with $P(x, y)$ monic in y, compute $Q(x, y)^N \bmod P(x, y)$.*
- *PolMatPow: Given a square polynomial matrix $M(x)$ over $\mathbb{K}[x]$ of size and degree at most r and d, compute $M(x)^N$.*

Our algorithms for SeqTerm, BivModPow, and PolMatPow make essential use of divisions in $\mathbb{K}$. We do not know if the complexity $O(N)$ can be achieved using only operations $(+, -, \times)$ in $\mathbb{K}$.

Previous work. As already mentioned, the classical way of computing the Nth term of a given C-finite sequence goes via binary powering of the companion matrix, see e.g. [31]. Another algorithm, due to Fiduccia [18], works via binary powering in a polynomial quotient ring, and has a better complexity with respect the order r, but not with respect to N. The fastest existing algorithm [9] for computing the Nth term of a C-finite sequence (counting arithmetic operations in $\mathbb{K}$) beats Fiduccia's algorithm by a constant factor. In our polynomial C-finite case, and under the assumption $r, d \in O(1)$, all these algorithms achieve an arithmetic complexity in $O(\mathsf{M}(N))$.

Beyond this classical approach, the previous work on the aforementioned problems consists of two distinct directions. The special case of Chebyshev polynomials of the second kind $U_n(x) = (-i)^n F_{n+1}(2ix)$ (with $F_n(x)$ the nth Fibonacci polynomial and i the imaginary unit) was considered in [27] (and later in [16]). These references present various methods for the computation of the Chebyshev polynomials (of the first and second kind) with arithmetic complexity ranging from $O(N)$ to $O(N^3)$. The results in [16, 27] exploit the particular structure of these polynomials; except for possibly other families of classical orthogonal polynomials, for which explicit (hypergeometric) formulas exist, the methods in [16, 27] do not admit obvious generalizations.

An idea closely connected to a fundamental building block of our algorithms is explained in [19, Pbm. 4]. There, Flajolet and Salvy exploit the fact that, given a polynomial $P(x)$ in $\mathbb{K}[x]$, the coefficient sequence of the nth power $P(x)^n$ satisfies a linear recurrence of order independent of n, and with coefficients in $\mathbb{K}[x, n]$ of degree independent of n; this recurrence allows them to compute (a selected coefficient of) $P(x)^N$ more efficiently than by binary powering. This idea has been applied in [7, §8] to count points on hyperelliptic curves over finite fields, with applications to cryptography. The technique also yields a general solution to SeqTerm when $r = 1$.

Outline. The following observation generalizes that in [19]: the coefficient sequence of the nth power of any *algebraic function* satisfies a recurrence of order and degree independent of n. From this, in Section 3, we give algorithms for SeqTerm with cost $O(N)$.

To complete the proof of Theorem 1.1, we design reductions between the three problems. Obviously PolMatPow $\Rightarrow$ SeqTerm, i.e., any algorithm for PolMatPow with cost $O(N)$ induces one for SeqTerm with cost $O(N)$ as well. Indeed, the Nth term of a polynomial C-recursive sequence is equal to an entry of the product of the Nth power of a companion matrix and the vector of initial values, and this polynomial matrix-vector multiplication costs $O(N)$. Conversely, it also holds that SeqTerm $\Rightarrow$ PolMatPow. One natural way to see this is to consider r^2 sequences corresponding to each entry of $M(x)^n$, with recurrence given by the characteristic polynomial of $M(x)$; see the introduction of Section 4. In Section 4.2, we give a more efficient algorithm for this reduction, based on an algorithm for SeqTerm $\Rightarrow$ BivModPow described in Section 4.1.

Basics of complexity and holonomic functions. Throughout the text, $\mathbb{K}$ denotes an effective field of characteristic zero. For analyzing the performance of algorithms we will use the arithmetic complexity model, meaning that arithmetic operations $(+, -, \times, \div)$ in the base field $\mathbb{K}$ are counted at unit cost. As before, $\mathsf{M}(N)$ stands for the complexity of multiplication of two polynomials in $\mathbb{K}[x]$ of degree at most N. With FFT-based multiplication $\mathsf{M}(N) \in O(N \log(N) \log\log(N))$ [11], improved to $O(N \log(N))$ if $\mathbb{K}$ contains suitable roots of unity [14] or if $\mathbb{K}$ is any finite field [22].

A power series $f(x) \in \mathbb{K}[\![x]\!]$ is called *D-finite* (or *holonomic*) if it satisfies a linear differential equation (ODE) of the form

$$q_\ell(x) f^{(\ell)}(x) + \cdots + q_0(x) f(x) = 0, \tag{4}$$

for some $q_0(x), \ldots, q_\ell(x) \in \mathbb{K}[x]$ with $q_\ell(x) \neq 0$. Equivalently, writing $f(x) = \sum_{k\geq 0} f_k x^k$, the sequence $(f_k)_{k\geq 0}$ is *P-recursive*, i.e., it satisfies a linear recurrence equation (LRE)

$$p_s(k) f_{k+s} + \cdots + p_0(k) f_k = 0 \quad \text{for all } k \geq 0,$$

with polynomial coefficients $p_0(x), \ldots, p_s(x) \in \mathbb{K}[x]$, and $p_s \neq 0$. Note that s and ℓ may differ in general, but $s \leq \ell + \max_i(\deg q_i(x))$. It also holds that $\max_i(\deg p_i(x)) \leq \ell$.

It is often useful to write (4) as $Lf(x) = 0$, where

$$L = q_\ell(x)\partial_x^\ell + \cdots + q_0(x)$$

is an element in the non-commutative Weyl algebra $\mathbb{K}[x]\langle\partial_x\rangle$ of linear differential operators with multiplication governed by the Leibniz rule $\partial_x x = x\partial_x + 1$. The *order* ℓ of a differential operator L is the highest power of ∂_x occurring in L, and the *degree* of L is the highest power of x. We recall that the least common left multiple (LCLM) of two operators $L_1, L_2 \in \mathbb{K}[x]\langle\partial_x\rangle$ is the unique monic operator $L = \mathrm{LCLM}(L_1, L_2)$ of minimal order such that there exist two nonzero operators $A, B \in \mathbb{K}(x)\langle\partial_x\rangle$ with $L = AL_1 = BL_2$. The LCLM can be computed efficiently [6].

2 THE CASE OF FIBONACCI POLYNOMIALS

Before solving the first part (SeqTerm) of Theorem 1.1 in general, we propose in this section three different approaches that can be used to compute the Nth Fibonacci polynomial $F_N(x)$ in arithmetic complexity $O(N)$. Two of these methods have the advantage that they generalize to the case of arbitrary C-finite sequences.

The starting point of all that follows is the observation that the generating function $F(x, y) := \sum_{n\geq 0} F_n(x)y^n \in \mathbb{K}[x][\![y]\!]$ of the sequence $(F_n(x))_{n\geq 0}$ is *rational*, and equal to $y/(1 - xy - y^2)$.

2.1 First method via a closed-form expression

By using the partial fraction decomposition

$$\frac{y}{1-xy-y^2} = \frac{1}{\varphi_+(x)-\varphi_-(x)} \cdot \left(\frac{1}{1-\varphi_+(x)y} - \frac{1}{1-\varphi_-(x)y}\right)$$

where $\varphi_\pm(x) = (x \pm \sqrt{x^2+4})/2$ are the roots of $\varphi^2 - x\varphi - 1 = 0$, and by applying the geometric series, we get the closed form expression

$$F_n(x) = \frac{\varphi_+(x)^n - \varphi_-(x)^n}{\varphi_+(x) - \varphi_-(x)} \quad \text{for all } n \geq 0. \tag{5}$$

Now, using the binomial formula twice, we obtain the formula

$$F_n(x) = \frac{1}{2^{n-1}} \cdot \sum_{\ell\geq 0} 4^\ell \left(\sum_{k\geq 0} \binom{n}{2k+1}\binom{k}{\ell}\right) x^{n-2\ell-1}. \tag{6}$$

The identity [20, 3.121] implies a "magic" simplification:

$$\sum_{k\geq 0} \binom{n}{2k+1}\binom{k}{\ell} = 2^{n-1-2\ell}\binom{n-\ell-1}{\ell}. \tag{7}$$

In conclusion, from (6) and (7) it follows that

$$F_n(x) = \sum_{\ell\geq 0} \binom{n-\ell-1}{\ell} x^{n-2\ell-1}. \tag{8}$$

With this expression at hand, it becomes transparent that one can compute $F_N(x)$ efficiently. Indeed, by writing $F_N(x) = \sum_{k=0}^{N-1} f_k x^k$, it follows from (8) that $(f_n)_{n\geq 0}$ satisfies the recurrence relation

$$f_{k+2} = \frac{(N+k+1)(N-k-1)}{4(k+1)(k+2)} f_k \quad \text{for all } k \geq 0. \tag{9}$$

Moreover, (8) also gives $(f_0, f_1) = (1, 0)$ for odd N and otherwise $(f_0, f_1) = (0, N/2)$. With these initial conditions, it is now clear that $F_N(x)$ can be computed in $O(N)$ by unrolling the recurrence (9).

As mentioned in the introduction, the analogue of formula (8) for the case of Chebyshev polynomials of the first kind $T_n(x)$ was already exploited in [27, §1.9]. The disadvantage of this approach is that for general polynomial C-finite sequences there is no hope for a closed form expression like (8).

2.2 Second method via algebraic substitution

There is another method for computing $F_N(x)$ in $O(N)$, which has the advantage that it generalizes to any C-recursive sequence, as we will show in Section 3.1.

The crucial remark (Lemma 3.2) is that since $\varphi_\pm(x)$ is algebraic, $\varphi_\pm(x)^n$ satisfies a "small" linear differential equation: an ODE of order and degree independent of n. The same holds true for $1/(\varphi_+(x) - \varphi_-(x))$, therefore for $F_n(x)$ as well. More precisely, $\varphi_\pm(x)^n$ satisfies the linear differential equation

$$(x^2+4)y''(x) + xy'(x) - n^2 y(x) = 0,$$

and $1/(\varphi_+(x) - \varphi_-(x)) = (x^2+4)^{-1/2}$ satisfies the ODE

$$(x^2+4)y'(x) + xy(x) = 0.$$

Using (5), it then follows that the polynomial $F_n(x)$ satisfies

$$(x^2+4)y''(x) + 3xy'(x) + (1-n^2)y(x) = 0. \tag{10}$$

Writing $F_N(x) = \sum_{k=0}^{N-1} f_k x^k$, plugging into (10) for $n = N$ and extracting the $(k+2)$nd coefficient, it now follows that the sequence $(f_k)_{k\geq 0}$ satisfies recurrence (9). The initial conditions f_0, f_1 are given by $F_N(x) \bmod x^2$ which can be found in complexity $O(\log N)$ by computing the Nth power of the companion matrix (2) in $\mathbb{K}[x]/(x^2)$ by binary powering and reducing mod x^2 in each step. As before, unrolling recurrence (9), with these initial terms, provides a way to compute $F_N(x)$ in complexity $O(N)$.

2.3 Third method via Creative Telescoping

Writing $F(x, y) = y/(1 - xy - y^2)$ we are interested in a differential equation for the coefficient of y^N in $F(x, y)$. By Cauchy's integral formula, we have for sufficiently small $\epsilon > 0$:

$$F_N(x) = [y^N]F(x, y) = \frac{1}{2\pi i} \oint_{|y|=\epsilon} \frac{y}{(1-xy-y^2)y^{N+1}} \mathrm{d}y.$$

Then the method of creative telescoping can be used to find a differential equation for the integral above. For example, the command

```
DEtools[Zeilberger](1/(1-x*y-y^2)/y^n, x, y, Dx);
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in Maple immediately finds that

$$\left((x^2+4)\partial_x^2 + 3x\partial_x + 1 - n^2\right)\frac{F(x,y)}{y^{n+1}} = \partial_y\left(\frac{F(x,y)}{y^n}C(x,y)\right),$$

where $C(x, y) = \frac{n+1-nxy-(n-1)y^2}{1-xy-y^2}$. By Cauchy's integral theorem, the contour integral of the right-hand side vanishes, and thus (10) follows. Then one can conclude in the same way as in the previous method and compute $F_N(x)$ in complexity $O(N)$.

2.4 Comments on the three approaches

It is natural to ask ourselves what in these approaches was just luck, what was truly specific to the particular example of the Fibonacci polynomials, and what can be extended to the general case.

It is clear that the key for computing $F_N(x)$ in complexity $O(N)$ is the existence of the recurrence (9) (or equivalently the ODE (10)). Even though there is no hope for a closed form solution in general, we shall prove that such a recurrence always exists for polynomial

C-finite sequences. We should, however, definitely be careful and avoid proving tautologic statements. Since $u_N(x)$ is a polynomial, it does satisfy *a priori* a first-order linear differential equation with polynomial coefficients, $u_N(x)y'(x) - u'_N(x)y(x) = 0$, but this one is trivial for our purposes. Indeed, converting this differential equation into a recurrence satisfied by the sequence of coefficients of $u_N(x)$ yields a recurrence of order $\deg(u_N)$, which is obviously useless for computing the coefficients of u_N. Therefore, we need to ensure existence of a recurrence/ODE whose order and degree are independent of N. This is the purpose of the next section. Specifically, in §3.1 we will explain how it can be found by algebraic substitution (generalizing §2.2) and in §3.2 we will show that it can also be found in general via creative telescoping (generalizing §2.3).

3 POLYNOMIAL C-FINITE SEQUENCES

Recall that a polynomial C-finite sequence $(u_n(x))_{n\geq 0}$ is a sequence of polynomials $u_n(x) \in \mathbb{K}[x]$ that satisfies a recurrence

$$u_{n+r}(x) = c_{r-1}(x)u_{n+r-1}(x) + \cdots + c_0(x)u_n(x), \tag{11}$$

of some order $r \in \mathbb{N}$ and polynomial coefficients $c_0(x), \ldots, c_{r-1}(x) \in \mathbb{K}[x]$. The *degree* of (11) is $d = \max_i(\deg c_i(x))$. Eq. (11) defines the sequence $(u_n(x))_{n\geq 0}$ uniquely if r initial terms $u_0(x), \ldots, u_{r-1}(x)$ are prescribed. The characteristic polynomial for (11) is defined as

$$\chi(y) = y^r - c_{r-1}(x)y^{r-1} - \cdots - c_1(x)y - c_0(x) \in \mathbb{K}[x, y].$$

It is well-known and not difficult to see that the generating function $U(x, y) := \sum_{n\geq 0} u_n(x)y^n$ is a rational function and given by

$$U(x, y) = \frac{v_0(x) + \cdots + v_{r-1}(x)y^{r-1}}{y^r\chi(1/y)}, \tag{12}$$

with $v_k(x) := u_k(x) - c_{r-1}(x)u_{k-1}(x) - \cdots - c_{r-k}(x)u_0(x)$.

Let $a_1(x), \ldots, a_k(x) \in \overline{\mathbb{K}(x)}$ be the roots of $\chi(y) = 0$, and $m_1, \ldots, m_k$ be their multiplicities. It follows from the partial fraction decomposition and geometric series that any sequence $(u_n(x))_{n\geq 0}$ satisfying (11) is of the form

$$u_n(x) = q_1(n, x)a_1(x)^n + \cdots + q_k(n, x)a_k(x)^n, \tag{13}$$

where $k \leq r$ and each $q_i(n, x) \in \mathbb{K}(a_1(x), \ldots, a_n(x))[n]$ is a polynomial in n of degree at most $m_i - 1$, for $i = 1, \ldots, k$.

3.1 Computing $u_N(x)$ in $O(N)$

By generalizing the ideas of Section 2.2, it is not difficult to prove that the nth term of a polynomial C-recursive sequence $(u_n(x))_{n\geq 0}$ satisfies a linear ODE whose degree and order are independent of n, and consequently, that there exists a linear recurrence relation for the coefficient sequence of $u_n(x)$ whose order (say s) and degree are again independent of n. Then, for a given $N \in \mathbb{N}$, first computing initial terms by binary powering of the companion matrix in $\mathbb{K}[x]/(x^s)$ and then unrolling this recurrence for $n = N$, we achieve a complexity $O(N)$ for the computation of $u_N(x)$.

Theorem 3.1. *Let $(u_n(x))_{n\geq 0}$ be a polynomial C-recursive sequence. Then there exists $L_n \in \mathbb{K}[n, x]\langle\partial_x\rangle$ with order and degree independent of n, and such that $L_n(u_n(x)) = 0$. Consequently, writing $u_n(x) = \sum_{k\geq 0} c_{n,k}x^k$, there exist, for some $s \in \mathbb{N}$ independent of n, polynomials $p_0(n, x), \ldots, p_s(n, x) \in \mathbb{K}[n, x]$ of degrees independent of n, and such that the sequence $(c_{n,k})_{k\geq 0}$ satisfies the recurrence*

$$p_s(n, k)c_{n,k+s} + \cdots + p_0(n, k)c_{n,k} = 0, \quad k \geq 0. \tag{14}$$

Algorithm 1 SeqTermAS$((u_n)_n, N)$

Input: A polynomial C-finite sequence $(u_n(x))_{n\geq 0}$ given by (11) with initial conditions, and $N \in \mathbb{N}$.
Output: The polynomial $u_N(x)$.

1: $\chi(y) \leftarrow$ the characteristic polynomial of $(u_n(x))_{n\geq 0}$
2: $a_1(x), \ldots, a_k(x) \leftarrow$ the roots of $\chi(y)$
3: Compute minimal polynomials for $q_1(x, n), \ldots, q_k(x, n) \in \mathbb{K}(a_1(x), \ldots, a_k(x))[n]$ such that (13) holds.
4: For each i deduce an ODE $L_{i,n} \in \mathbb{K}[n, x]\langle\partial_x\rangle$ with order and degree independent of n such that $L_{i,n}(q_i(x, n)a_i(x)^n) = 0$
5: $L_n \leftarrow \mathrm{LCLM}(L_{1,n}, \ldots, L_{k,n}) \in \mathbb{K}[n, x]\langle\partial_x\rangle$
6: Compute a recurrence $p_s(n, k)c_{n,k+s} + \cdots + p_0(n, k)c_{n,k} = 0$ satisfied by any solution $f_n(x) = \sum_{k\geq 0} c_{n,k}x^k$ of $L_n y = 0$
7: Using binary powering of the companion matrix of the initial recurrence mod x^s, compute the values $c_{N,0}, \ldots, c_{N,s-1}$
8: Unroll the recurrence from Line 6 for $n = N$ and with initial terms from Line 7
9: **return** $\sum_{k=0}^{Nd} c_{N,k}x^k$, where $d = \deg_x(\chi(y))$

In the theorem above it is crucial that neither the order nor the degree of L_n depend on n. Since each $u_n(x)$ is a polynomial, it is a tautology to say that it satisfies *some* linear differential equation with polynomial coefficients: one may simply take $L = \partial_x^\alpha$, where $\alpha > \deg(u_n(x))$ or $L = u_n(x)\partial_x - u'_n(x)$. However, it is a nontrivial fact that $u_n(x)$ satisfies an ODE of the form

$$p_\ell(n, x)u_n^{(\ell)}(x) + \cdots + p_0(n, x)u_n(x) = 0$$

for some $p_i(n, x) \in \mathbb{K}[n, x]$ (with ℓ and $\deg_x p_i$ independent of n).

The most direct proof of Theorem 3.1 uses the explicit expression (13) for $u_n(x)$ and the following classical fact about algebraic substitution into D-finite functions. Recall that a function $a(x)$ is called *algebraic* over $\mathbb{K}(x)$ if it satisfies a non-trivial polynomial relation $P(x, a(x)) = 0$ for some $P(x, y) \in \mathbb{K}[x, y]$. Size and complexity bounds on differential equations for algebraic functions, and more generally on algebraic substitution, are given in [5, 26].

Lemma 3.2. *Let $a(x)$ be an algebraic function over $\mathbb{K}(x)$ and let $g(x)$ be D-finite. Then $f(x) = g(a(x))$ is D-finite. In particular, $a(x)^n$ satisfies a linear ODE of order and degree independent of n.*

Proof. The first part is a classical result, see for example [34, Thm. 2.7]. In the proof one shows that the vector space spanned over $\mathbb{K}(x)$ by $(f^{(i)}(x))_{i\geq 0}$ is finite-dimensional over $\mathbb{K}(x, a(x))$ which is itself finite-dimensional over $\mathbb{K}(x)$. For the second part, it is enough to set $g(x) = x^n$ which satisfies $xg'(x) = ng(x)$. □

Example 3.3. Like in Section 2 let $\varphi_\pm(x) = (x \pm \sqrt{x^2 + 4})/2$ be the roots of $y(x)^2 + xy(x) - 1 = 0$. Then $\varphi_\pm(x)^n$ satisfy the ODE

$$(x^2 + 4)y''(x) + xy'(x) - n^2y(x) = 0.$$

Proof of Theorem 3.1. Write $u_n(x)$ as in (13). By Lemma 3.2, each $a_i(x)^n$ satisfies a linear differential equation of order and degree independent of n, hence the same holds for $q_i(n, x)a_i(x)^n$, and finally for $u_n(x)$. It follows that the coefficient sequence of $u_n(x)$ is P-recursive with order and degree independent of n. □

Since all steps in the proofs above are effective and independent of N, this leads to Algorithm 1. Its Lines 1 to 6 can be seen

as "precomputations" since they do not depend on N. As already mentioned, Line 7 has complexity $O(\log N)$ and Line 8 has complexity $O(N)$. Thus, Algorithm 1 solves SeqTerm in complexity $O(N)$, up to a potential issue during the unrolling at Line 8 of the recurrence from Line 6. Indeed, this unrolling may be impossible for some values k, namely those for which $p_s(N,k)$ vanishes. We will explain how to overcome this problem in Section 3.3.

For practical applications, however, computing the polynomials $q_i(x,n)$ in Line 3 as well as the LCLM in Line 5 is algorithmically somewhat cumbersome. Thus, generalizing the approach in Section 2.3, we now propose a variant of Algorithm 1 which replaces Lines 1 to 5 by an algorithm based on creative telescoping.

3.2 Computing L_n with Creative Telescoping

Let $U(x,y) = \sum_{n\geq 0} u_n(x)y^n \in \mathbb{K}[x][\![y]\!]$ be the generating function (12) of $(u_n(x))_{n\geq 0}$. The sequence is C-finite, so $U(x,y)$ is a rational function. Moreover, the Cauchy integral formula implies that

$$u_n(x) = \frac{1}{2\pi i}\oint_{|y|=\epsilon} \frac{U(x,y)}{y^{n+1}}\mathrm{d}y. \tag{15}$$

A *telescoper* of $U(x,y)/y^{n+1}$ is a differential operator

$$L = p_k(x)\partial_x^k + \cdots + p_0(x) \in \mathbb{K}[x]\langle\partial_x\rangle,$$

such that L applied to $U(x,y)/y^{n+1}$ is $\partial_y(C(x,y))$ for some rational function $C(x,y)$ called the *certificate*. By the Cauchy integral theorem, $\oint_{|y|=\epsilon}\partial_y(C(x,y))\mathrm{d}y = 0$, and it follows that $Lu_n(x) = 0$, i.e., L yields a differential equation for $u_n(x)$. In this section we will prove that for $U(x,y)/y^{n+1}$ there exists a telescoper $L_n \in \mathbb{K}[n,x]\langle\partial_x\rangle$ whose order and degree do not depend on n. Our proof relies on reduction-based creative telescoping and repeatedly uses the *Hermite reduction* algorithm [2–4].

We now introduce the necessary definitions and recall the Hermite reduction method. For a more detailed introduction, a full complexity analysis, and applications of reduction-based creative telescoping to integration of bivariate rational functions, we refer to [2]. Let $\mathbb{L} = \mathbb{K}(x)$. For a polynomial $Q(y) \in \mathbb{L}[y]$, let $Q = Q_1Q_2^2\cdots Q_k^k$ be its squarefree factorization and let $Q^* = Q_1\cdots Q_k$ denote the squarefree part of Q. We set $Q^- := Q/Q^*$. Recall that, given $P, Q \in \mathbb{L}[y]$, the *Hermite reduction* algorithm computes two polynomials $A, a \in \mathbb{L}[y]$ with $\deg_y a < \deg_y Q^*$ such that

$$\frac{P}{Q} = \partial_y\left(\frac{A}{Q^-}\right) + \frac{a}{Q^*}.$$

Given a bivariate rational function $H(x,y) = P(y)/Q(y) \in \mathbb{K}(x,y)$, one may compute the Hermite reduction (A_i, a_i) of $\partial_x^i H$ for $i = 0, 1, \ldots$. Since $\deg_y a_i$ is uniformly bounded by $d^* = \deg_y Q^*$ for each i, the d^*+1 functions $\{a_i(x,y) : 0 \leq i \leq d^*\}$ will be linearly dependent over $\mathbb{K}(x)$. Hence one can find $q_0(x), \cdots, q_{d^*}(x) \in \mathbb{K}(x)$ not all zero, such that $\sum_{i=0}^{d^*} q_i(x)a_i(x) = 0$. It follows then that $L = \sum_{i=0}^{d^*} q_i(x)\partial_x^i$ is a telescoper for H.

This procedure cannot be directly applied to $U(x,y)/y^{n+1}$ if n is an indeterminate. At the same time, if $n = N \in \mathbb{N}$ is fixed, it is *a priori* not obvious that $\deg_x q_i(x)$ will be independent of N. Moreover, the complexity of the algorithm will depend on N, which we want to avoid. As we will now explain, to achieve this, one should see $U(x,y)/y^{n+1}$ not as a rational function in x and y with potentially large degree in the numerator, but as a hyperexponential function with the parameter n appearing solely as a coefficient in the logarithmic derivative. Recall that $H(x,y)$ is called *hyperexponential* if $\partial_x H/H$ and $\partial_y H/H$ belong to $\mathbb{K}(x,y)$.

For hyperexponential functions, the Almkvist-Zeilberger algorithm [1] was the first practical method to find telescopers and certificates. Indeed, as we mentioned in Section 2.3, the command

```
DEtools[Zeilberger](1/(1-x*y-y^2)/y^n, x, y, Dx);
```

in Maple immediately finds the differential equation for the nth Fibonacci polynomial for a variable n. Note that if n is specialized to an integer N before the execution of the command above, the implemented algorithm becomes slower as N growths.

It is, however, not clear that the Almkvist-Zeilberger algorithm applied to $U(x,y)/y^{n+1}$ will always find a telescoper whose degree and order are independent of n, even though we know from Section 3.1 that an ODE with this property exists. Therefore, to have a complete algorithm based on creative telescoping, we will invoke the reduction-based method for hyperexponential functions first introduced and analyzed in [3]. Using the implementation of the latter work, the command in Maple

```
HermiteTelescoping(1/(1-x*y-y^2)/y^n, x, y, Dx);
```

also immediately finds the correct ODE for $F_n(x)$. The practical advantage for our purpose of using the reduction based algorithm in comparison to the Almkvist-Zeilberger method is shown in Section 5 (Table 1). The theoretical advantage comes from the following lemma, which guarantees that the algorithm will find a telescoper for $U(x,y)/y^{n+1}$, and consequently an ODE for $u_n(x)$, whose order and degree do not depend on n.

Lemma 3.4. *Let $P(y) \in \mathbb{L}[n,y]$ and $Q(y) \in \mathbb{L}[y]$ with $Q(0) \neq 0$. Set $d_n := \deg_n P(y)$, $d^* := \deg_y Q^*(y)$ and let k be the highest pure power in the square free factorization of $Q(y)$. Then there exist $B(n,y) \in \mathbb{L}(n)[y]$ and $b(n,y) \in \mathbb{L}[n,y]$ with $\deg_y b(n,y) \leq d^*$ and $\deg_n b(n,y) \leq d_n + k$ such that*

$$\frac{P(y)}{Q(y)y^{n+1}} = \partial_y\left(\frac{B(n,y)}{Q^-(y)y^n}\right) + \frac{b(n,y)}{Q^*(y)y^{n+1}}. \tag{16}$$

Proof. We are going prove the statement by induction on $d^- := \deg_y Q^-$. If $d^- = 0$, then $Q^* = Q$ and the Euclidean division gives $P = P_1Q + b_1$ with $\deg_y b_1 < d^*$. Moreover,

$$\frac{P_1(y)}{y^{n+1}} = \partial_y\left(\frac{B_1(n,y)}{y^n}\right),$$

where $B_1(n,y)$ is $P_1(y)$ with the kth coefficient p_k replaced by $p_k/(k-n)$. Setting $B = B_1$ and $b = b_1$ proves the induction basis.

Now assume that $d^- > 0$ and note that

$$\partial_y\left(\frac{B(n,y)}{Q^-(y)y^n}\right) = y^{-n}\partial_y\left(\frac{B(n,y)}{Q^-(y)}\right) - y^{-n-1}n\frac{B(n,y)}{Q^-(y)},$$

so equation (16) is equivalent to

$$\frac{P(y)}{Q(y)y} = \partial_y\left(\frac{B(n,y)}{Q^-(y)}\right) - n\frac{B(n,y)}{Q^-(y)y} + \frac{b(y)}{Q^*(y)y}. \tag{17}$$

The Hermite reduction algorithm applied to $\frac{P(y)}{Q(y)y}$ yields polynomials $A(y), a(y)$ with $\deg_y a(y) \leq d^*$ such that

$$\frac{B(y)}{Q(y)y} = \partial_y\left(\frac{A(y)}{Q^-(y)}\right) + \frac{a(y)}{Q^*(y)y}. \tag{18}$$

Comparing (17) and (18), we now look at

$$H(y) := \frac{a(y)}{Q^*(y)y} + n\frac{A(y)}{Q^-(y)y}.$$

The denominator of $H(y)$ is $R(y)y := \operatorname{lcm}(Q^*, Q^-)y$. Clearly, $R^* = Q^*$ and $\deg_y R^- < d^-$. The highest pure power in the square free factorization of $R(y)$ is at most $k-1$ and the degree of the numerator in n of $H(y)$ is at most $d_n + 1$. Hence, by induction, we may write

$$\frac{a(y)}{Q^*(y)y} + n\frac{A(y)}{Q^-(y)y} = \partial_y\left(\frac{C(n,y)}{Q^-(y)}\right) - n\frac{C(n,y)}{Q^-(y)y} + \frac{c(n,y)}{Q^*(y)y}$$

with $\deg_y c(n,y) < d^*$ and $\deg_n c(n,y) \le d_n + k$. Setting $B(n,y) = A(y) + C(n,y)$ and $b(y) = a(y) + c(n,y)$ finishes the proof. □

The proof of Lemma 3.4 induces an algorithm for the computation of $B(n,y)$ and $b(n,y)$ given $P(y), Q(y) \in \mathbb{K}[x,y]$ such that (16) holds, $\deg_y b \le \deg_y Q^*$ and also $\deg_n b$ bounded in terms of Q. It can be seen as a special case of the procedure HermiteReduction in [3]. The ODE for $u_n(x)$ can be now found as in Algorithm 2.

Algorithm 2 TelescNthTerm($U(x,y)$)

Input: A rational function $U(x,y) \in \mathbb{K}(x,y) \cap \mathbb{K}[\![x,y]\!]$.
Output: A diff. operator in $\mathbb{K}[n,x]\langle\partial_x\rangle$ for $u_n(x) = [y^n]U(x,y)$.
1: Write $U(x,y) = P(x,y)/Q(x,y)$ and let $d^* = \deg_y Q^*(x,y)$
2: For each $i = 0, \dots, d^*$ compute the polynomial $b_i(n,x,y) = b(n,y)$ as in Lemma 3.4 applied to $\partial_x^i U(x,y)/y^{n+1}$
3: Find a linear relation of $\{b_i(n,x,y) : 0 \le i \le d^*\}$ over $\mathbb{K}(n,x)$, that is polynomials $q_0(n,x), \dots, q_{d^*}(n,x)$ not all zero with $\sum_{i=0}^{d^*} q_i(n,x)b_i(n,x,y) = 0$
4: Return the differential operator $\sum_{i=0}^{d^*} q_i(n,x)\partial_x^i \in \mathbb{K}[n,x]\langle\partial_x\rangle$

Note that, as in the usual reduction based creative telecoping, the linear relation at Line 3 exists because $\deg_y(b_i(n,x,y))$ is uniformly bounded by d^*. Writing $U(x,y) = P(x,y)/Q(x,y)$, the operator $L = \sum_{i=0}^{d^*} q_i(n,x)\partial_x^i$ annihilates $u_n(x) = [y^n]U(x,y)$ since

$$2\pi i \cdot L_n u_n(x) = L_n \oint \frac{U(x,y)}{y^{n+1}}\mathrm{d}y = \oint L_n \frac{P(x,y)}{Q(x,y)y^{n+1}}\mathrm{d}y$$
$$= \oint \partial_y \frac{\sum_{i=1}^{d^*} q_i(x,y)B_i(n,y)}{Q^-(x,y)y^n}\mathrm{d}y + \oint \frac{\sum_{i=0}^{d^*} q_i(n,x)b_i(n,x,y)}{Q^*(x,y)y^{n+1}}\mathrm{d}y;$$

the first integral vanishes by Cauchy's integral theorem, and the second integral vanishes by construction of the $q_i(n,x)$.

This provides a variant for Lines 1 to 5 of Algorithm 1, as described in Algorithm 3.

Algorithm 3 SeqTermCT($(u_n)_n, N$)

Input: A polynomial C-finite sequence $(u_n(x))_{n\ge0}$ given by (11) with initial conditions, and $N \in \mathbb{N}$.
Output: The polynomial $u_N(x)$.
1: $U(x,y) \leftarrow$ the rational generating function of $u_n(x)$ in (12)
2: $L_n \leftarrow$ TelescNthTerm($U(x,y)$)
3: ▷ *follow Lines 6 to 9 of Algorithm 1*

As above, a potential issue is that the unrolling step is only possible for $p_s(N,k) \neq 0$. The next section deals with this problem.

3.3 The singular case

In this section we discuss the potential issue of our algorithm that can occur if the sequence for the coefficients of $u_n(x)$ cannot be unrolled due to singularities. We shall first highlight this problem and its solution by means of an example.

Consider the polynomial C-recurrent sequence $u_n(x)$ given by

$$u_{n+3}(x) - (x^2+x+2)u_{n+2}(x) + x(x^2+2x+2)u_{n+1}(x) - 2x^3u_n(x) = 0,$$

for all $n \ge 0$ with initial conditions $u_0 = 3, u_1 = x^2+x+2, u_2 = x^4+x^2+4$. The characteristic polynomial of the defining recurrence is easily computed and turns out to factor completely over $\mathbb{K}[x][y]$:

$$\chi(x,y) = (y-2)(y-x)(y-x^2).$$

With the initial conditions and after a partial fraction decomposition it follows that the generating function of $u_n(x)$ is given by

$$U(x,y) = \frac{1}{1-2y} + \frac{1}{1-x^2y} + \frac{1}{1-xy}.$$

Hence, the solution is $u_n(x) = 2^n + x^n + x^{2n}$, and can be even written down in $O(\log(N))$ arithmetic operations for any N. However, as we shall explain now, the direct application of any of the methods described earlier fails.

According to Theorem 3.1, $u_n(x)$ satisfies an ODE whose degree and order are independent of n. Indeed, using creating telescoping one quickly finds an annihilator for $u_n(x) = \oint U(x,y)/y^{n+1}\mathrm{d}x$:

$$\big(x^2\partial_x^3 - 3x(n-1)\partial_x^2 + (2n-1)(n-1)\partial_x\big)u_n(x) = 0.$$

Converting this ODE to a recurrence for the coefficient sequence of $u_n(x) = \sum_{k\ge0} c_{n,k}x^k$ we find

$$(2n-k)(n-k)kc_{n,k} = 0, \quad k \ge 0. \tag{19}$$

In other words, $c_{n,k} = 0$ for all $k \in \mathbb{N}$ except $k \in \{0, n, 2n\}$. In order to "unroll" this recurrence we need to know $c_{n,0}, c_{n,n}$ and $c_{n,2n}$. However, it is not immediately clear how to compute those terms for $n = N$ in $O(N)$ arithmetic operations from the initial input (without using the explicit solution).

We propose the following easily generalizable solution: consider $v_n(x) = u_n(x+1)$. Then the ODE for $v_n(x)$ is given by

$$\big((x+1)^2\partial_x^3 - 3(x+1)(n-1)\partial_x^2 + (2n-1)(n-1)\partial_x\big)v_n(x) = 0,$$

and for the coefficient sequence of $v_n(x) = \sum_{k\ge0} d_{n,k}x^k$ we find

$$(k+1)(k+2)d_{n,k+2} - (k+1)(3N-2k-1)d_{n,k+1} + (2n-k)(n-k)d_{n,k} = 0.$$

Now the leading coefficient of the recurrence is $(k+1)(k+2) \neq 0$, so we can easily unroll it after determining the first two terms, by computing them via binary powering of the corresponding companion matrix mod x^2. Having computed $v_N(x)$, it remains to find $u_N(x) = v_N(x-1)$. Note that expanding the polynomial results in an $O(\mathsf{M}(N))$ algorithm. However, recall from (19) that we only need to compute $c_{N,N}$ and $c_{N,2N}$, or, in other words, the coefficients of x^N and x^{2N} in $v_N(x-1)$. For any i it holds that

$$c_{N,i} = \sum_{k\ge0} d_{N,k}\binom{k}{i}(-1)^{k-i}, \tag{20}$$

and the sum is finite because $v_N(x)$ is a polynomial. Clearly, it can be computed in complexity $O(N)$ for any i.

Generally speaking, an issue with unrolling the recurrence for $(c_{n,k})_{k\ge0}$ occurs if the roots of the leading polynomial are positive

integers that depend on n. Let S be the set of these roots. Note that the size of S is independent of n and S can be non-empty only if the ODE for $u_n(x)$ is singular at 0 (that is, if $q_\ell(x)$ in (4) vanishes at 0). In this case, one can always define $v_n(x) = u_n(x+c)$ for $c \in \mathbb{K}$ a non-singular point of the ODE ($q_\ell(c) \neq 0$). Then the coefficients $d_{n,k}$ of $v_n(x)$ can be computed from $O(1)$ initial conditions via unrolling a recurrence. Using the formula (20) (with $-c$ instead -1) and the fact that $v_n(x)$ is a polynomial, one can compute the coefficients $c_{N,i}$ for $i \in S$. With these, it is then possible to unroll the recurrence for $(c_{N,k})_{k\geq 0}$ and find $u_N(x)$ in complexity $O(N)$.

4 IMPACT ON POLYNOMIAL MATRIX POWER

Here is an algorithm for PolMatPow using SeqTerm. Let $M(x)$ in $\mathbb{K}[x]^{r\times r}_{\leq d}$ and $p_{i,j,n}(x)$ be the (i,j) entry of $M(x)^n$, for $n \geq 0$ and i and j in $\{1,\dots,r\}$. The sequence $(p_{i,j,n}(x))_{n\geq 0}$ is polynomial C-recursive, with a recurrence given by the characteristic polynomial

$$\chi_M(x,y) := \det(y\,I_r - M(x)) = y^r - c_{r-1}(x)y^{r-1} - \cdots - c_0(x).$$

That is, $p_{i,j,n+r}(x) = c_{r-1}(x)p_{i,j,n+r-1}(x) + \cdots + c_0(x)p_{i,j,n}(x)$ for all $n \geq 0$. Thus, to compute $M(x)^N$, it is enough to find $\chi_M(x,y)$ (in $O(1)$, i.e. independent of N), to compute the polynomials $p_{i,j,n}(x)$ for $1 \leq i,j \leq r$ and $0 \leq n < r$ (also in $O(1)$) and to return the entries $p_{i,j,N}(x)$ of $M(x)^N$ via SeqTerm. As such, this approach uses r^2 calls to SeqTerm, with total cost $O(N)$.

This section describes an algorithm for PolMatPow which uses only r such calls, through a direct reduction to BivModPow (see Section 4.2). Our solution for BivModPow, via r calls to SeqTerm, is presented in Section 4.1 and completes the proof of Theorem 1.1.

4.1 Computing bivariate modular powers

Let $\mathbb{L} = \mathbb{K}[x]$ and $P, Q \in \mathbb{L}[y]$. Assume that P, seen as a univariate polynomial in y of degree r, is monic. For $N \in \mathbb{N}$, Euclidean division in $\mathbb{L}[y]$ ensures the existence of unique $S, R \in \mathbb{L}[y]$ such that $\deg_y(R) < r$ and $Q^N = SP + R$. The polynomial R is $Q^N \bmod P$. Assume that P and Q are fixed, and let $d := \deg_x(P)$ (which is thus in $O(1)$). Then, writing $R = \sum_{i=0}^{d-1} r_i(x)y^i$, it holds that $\deg_x r_i(x) = O(N)$. The efficient computation of R when $Q = y$, given $P(x,y)$ and N, is the first step for proving BivModPow in Theorem 1.1.

We shall first illustrate the connection of SeqTerm and BivModPow by means of an example. Let $P(x,y) = y^2 - xy - 1$ and $Q(x,y) = y$, i.e., we are looking for $F_{n-1}(x), F_n(x) \in \mathbb{L}$ such that

$$y^n = S(x,y)(y^2 - xy - 1) + yF_n(x) + F_{n-1}(x), \tag{21}$$

for some polynomial $S(x,y) \in \mathbb{L}[y]$. Replace y by $1/y$ in (21) and then multiply by $y^{n+1}/(1 - xy - y^2)$ to obtain

$$\frac{y}{1-xy-y^2} = Q(x,1/y)y^{n-1} + y^n\frac{F_n(x) + yF_{n-1}(x)}{1-xy-y^2}.$$

Now observe that $\deg_y(Q(x,1/y)y^{n-1}) \leq n-1$, hence by extracting the nth and $(n+1)$st coefficients,

$$F_n(x) = [y^n]\frac{y}{1-xy-y^2} \text{ and}$$
$$F_{n-1}(x) + xF_n(x) = [y^{n+1}]\frac{y}{1-xy-y^2}.$$

We conclude that $F_k(x)$ is the kth Fibonacci polynomial, for $k = n$ and $k = n-1$. In particular, each $F_k(x)$ satisfies a linear recurrence with constant polynomials and can be found in $O(k)$ by SeqTerm.

This strategy, outlined on an example, generalizes in the obvious way. Explicitly, we have the following lemma (see [9, Lem. 2]).

Lemma 4.1. *Let $P \in \mathbb{K}[x,y]$ and $r := \deg_y(P)$, with $P(x,0) \neq 0$ and reversal $\bar{P}(x,y) := y^r P(x, \frac{1}{y})$. Write $\frac{1}{\bar{P}(x,y)} =: \sum_{k\geq 0} u_k(x)y^k$. Finally, let $v(x,y) = (u_{N-r+1}(x) + \cdots + u_N(x)y^{r-1})\bar{P}(x,y) \bmod y^r$. Then $y^N \bmod P(x,y) = v(1/y)y^{r-1}$.*

The sequence $(u_k(x))_{k\geq 0}$ in Lemma 4.1 is C-recursive because its generating function is rational. Hence, using SeqTerm, the $r = O(1)$ many terms $u_{N-r+1}(x), \dots, u_N(x)$ can be computed in complexity $O(N)$. It follows that the case $Q(x,y) = y$ of BivModPow can be solved in $O(N)$ steps as well.

Finally, the computation of $Q(x,y)^N \bmod P(x,y)$ can be reduced to $y^N \bmod P(x,y)$ with a resultant precomputation (see Lemma 4.2). This leads to Algorithm 4, which solves BivModPow in $O(N)$.

Lemma 4.2. *Let $P(y), Q(y) \in \mathbb{L}[y]$. Define $A(t), B(t) \in \mathbb{L}[t]$ by $A(t) = \mathrm{Res}_y(P(y), t - Q(y))$ and $B(t) = t^N \bmod A(t)$. Then*

$$Q(y)^N \bmod P(y) = B(Q(y)) \bmod P(y).$$

Proof. By the definition of the resultant, $A(t) = \prod_i (t - Q(a_i))$ where $a_i \in \overline{\mathbb{L}}$ are the solutions of $P(y) = 0$. Hence, $P(y)$ divides $A(Q(y))$, which, by construction, divides $B(Q(y)) - Q(y)^N$. □

Algorithm 4 BivModPow$(P(x,y), Q(x,y), N)$

Input: $P(x,y), Q(x,y) \in \mathbb{L}[y]$ with $P(x,y)$ monic in y, and $N \in \mathbb{N}$.
Output: $Q(x,y)^N \bmod P(x,y)$.

1: $A(x,t) \leftarrow \mathrm{Res}_y(P(x,y), t - Q(x,y))$
2: $\bar{A}(x,t) \leftarrow t^r A(x,1/t)$, where $r := \deg_t A(x,t)$
3: **for** $i = N-r+1, \dots, N$ **do**
4: $\quad u_i(x) \leftarrow [t^i]\frac{1}{\bar{A}(x,t)}$ using SeqTerm
5: $u(x,t) \leftarrow u_{N-r+1}(x) + \cdots + u_N(x)t^{r-1}$
6: $v(x,t) \leftarrow u(x,t)\bar{A}(x,t) \bmod t^r$; $B(x,t) \leftarrow v(1/t)t^{r-1}$
7: **return** $B(x, Q(x,y)) \bmod P(x,y)$

4.2 Computing polynomial matrix powers

Let $M \in \mathbb{K}[x]^{r\times r}$ be an $r \times r$ polynomial matrix of degree at most d. Its power M^N has degree at most $Nd \in O(N)$. Let $P(x,y)$ be the characteristic polynomial of M. Since $P(x,M) = 0$ by the Cayley-Hamilton theorem, we get $M^N = R(x,M)$ where $R(x,y) = y^N \bmod P(x,y)$. The polynomial R can be computed in $O(N)$ via BivModPow. Then evaluating $R(x,y)$ at $y = M(x)$ has cost $O(N)$ because $\deg_x(R) \in O(N)$, $\deg_y(R) = r \in O(1)$ and $\deg_x(M) = d \in O(1)$. Hence Algorithm 5 is correct and has complexity $O(N)$.

Algorithm 5 PolMatPow(M, N)

Input: matrix $M(x) \in \mathbb{K}[x]^{r\times r}$, integer $N \in \mathbb{N}$.
Output: $M(x)^N \in \mathbb{K}[x]^{r\times r}$.

1: $P(x,y) \leftarrow$ the characteristic polynomial of $M(x)$
2: $R(x,y) \leftarrow y^N \bmod P(x,y)$ ▷ *instance of BivModPow*
3: **return** $R(x, M(x))$

5 EXPERIMENTS

The main precomputation step for all our algorithms consists in starting with a rational function $U \in \mathbb{K}(x,y) \cap \mathbb{K}[\![x,y]\!]$ and in finding a differential operator L_n that annihilates $u_n(x) = [y^n]U(x,y)$ and whose degree and order are independent of n. For this task, in practice, we may either use the method described in §3.1, or creative telescoping algorithms for hyperexponential functions. Table 1 summarizes timings for a variety of implementations.

The table reveals that, among these implementations, the fastest one for computing a telescoper of $U(x,y)/y^{n+1}$ is the reduction-based creative telescoping in Maple. More specifically, `redct` is the fastest, followed by HT. The implementation in *ore_algebra* [25] in SageMath competes best with reduction-based methods.

Table 2 gives timings of an efficient implementation of the remaining stages after precomputations: computing initial terms (IT), and unrolling (UR). We observe that IT takes negligible time compared to UR, except for extreme parameter ranges where, simultaneously, r and d are large and N is small; this is expected since these ranges correspond to cases where the order of the recurrence to be unrolled is close to N. We also see that binary powering is always slower, often by a factor more than 5, than the addition of IT and UR. The speed-up factor is summarized in Figs. 1 to 3; as expected it grows when N grows, with r and d fixed.

For large N, in most of the reported cases, performing both the precomputation and IT+UR is much faster than using binary powering. Still, this is not always true, e.g. for $r = 5$. One has to keep in mind that `redct` is not implemented in low-level Maple, and targets rational coefficients: for a more meaningful assessment of the precomputation part, it would be interesting to have an implementation of creative telescoping which is fully optimized and specialized to coefficients in a word-size prime field.

6 CONCLUSION AND FUTURE WORK

We have shown that it is possible to beat, both in theory and in practice, the very basic and powerful binary powering method for computing: (i) powers of polynomial matrices, (ii) terms in polynomial C-finite sequences and (iii) modular exponentiation for bivariate polynomials. We describe below several lines of work, including possible optimizations and generalizations, that we leave for future investigations.

More detailed complexity analysis. The first and most natural direction for future work is to analyze and improve the complexity of the algorithms in Theorem 1.1 with respect to the parameters r and d. For simplicity, these parameters were assumed to be $O(1)$ in this work. For the Nth power of a polynomial matrix $M(x)$ of size r and degree d, binary splitting has arithmetic complexity $O(\mathsf{M}(Nd)r^2 + Ndr^\omega)$, where $\omega \in [2,3]$ is a feasible exponent of matrix multiplication over $\mathbb{K}$. With our approach, it is legitimate to target a differential equation satisfied by the entries of $M(x)^N$ of order r with coefficients in x of degree $O(dr^3)$, yielding a recurrence of order $O(dr^3)$ and coefficients in n of degree at most r. For large N, the complexity of the new algorithm would thus be $O(Ndr^2\mathsf{M}(r))$. Using different ODEs, of order $O(r)$ and coefficients of degree $O(dr^2)$ could even lead to a complexity $O(Ndr\mathsf{M}(r))$.

The Kth coefficient of the Nth term. For some (large) integers $N, K \in \mathbb{N}$, one might be interested in computing the single coefficient $[x^K y^N]U(x,y)$ of a rational function $U \in \mathbb{K}(x,y) \cap \mathbb{K}[\![x,y]\!]$. Equivalently it is natural to wonder: How fast can one compute the Kth coefficient of the Nth term of a C-recursive sequence $(u_n(x))_{n\geq 0}$? Using our method, a recurrence with initial conditions for the coefficients of $u_N(x)$ can be deduced in $O(\log N)$ arithmetic operations. Then (assuming that the recurrence is non-singular) the Kth coefficient can be found in $O(\mathsf{M}(\sqrt{K}))$ operations by using baby-steps/giant-steps [7, 12]. We expect that, at least under a genericity assumption, this problem can be solved in arithmetic complexity $O(\log(N) + \mathsf{M}(\sqrt{K}))$ which is a big improvement compared to the previous best $O(N+K)$ by [30].

Polynomial P-recursive sequences. A somewhat related task is to study the analogous problem to SeqTerm for polynomial P-recursive sequences, that is for $(u_n(x))_n \in \mathbb{K}[x]^{\mathbb{N}}$ satisfying

$$p_r(x,n)u_{n+r}(x) + \cdots + p_0(x,n)u_n(x) = 0,$$

for $p_i(x,n) \in \mathbb{K}[x,n]$. We expect that, at least under a genericity assumption, a generalization of Lemma 3.4 (based on results in [4]) should exist, implying in particular that $u_N(x)$ satisfies a linear ODE of order and degree independent of N. Generalizing this even further, one might study the Creative Telescoping problem for rational functions of the form $H(\mathbf{x}) = \frac{P(x_1,\ldots,x_s)}{Q(x_1,\ldots,x_s)R(x_1,\ldots,x_s)^n}$. We expect that (at least generically) the minimal telecoper for $H(\mathbf{x})$ has order and degree independent of n and can be found via a Griffiths-Dwork reduction type approach, based on ideas from [8].

Connection to the Jordan–Chevalley decomposition. A different approach for computing powers of matrices uses the *Jordan–Chevalley decomposition* (also called *SN decomposition*), see e.g. [15, 17, 23, 32]. It ensures that any polynomial matrix $M \in \mathbb{K}[x]^{r\times r}$ can be written as $M = S + Z$, where $S \in \mathbb{K}(x)^{r\times r}$ is diagonalizable over $\overline{\mathbb{K}(x)}$, $Z \in \mathbb{K}(x)^{r\times r}$ is nilpotent, and $SZ = ZS$. From this decomposition it follows that $M^N = \sum_{i=0}^{r-1} \binom{N}{i} S^{N-i}Z^i$. After a change of basis, this reduces to computing a power of a diagonal matrix with algebraic functions coefficients. Using Lemma 3.2 this can be performed efficiently in $O(N)$ operations. It would be certainly interesting to compare this approach with the other methods.

A PDE approach for SeqTerm. There is yet another method to deduce recurrence (9). The starting point is that the generating function $F(x,y) = y/(1-xy-y^2)$ of $F_n(x)$ satisfies the linear PDE

$$(x^2+4)\frac{\partial^2 F}{\partial x^2} + 3x\frac{\partial F}{\partial x} - y^2\frac{\partial^2 F}{\partial y^2} - y\frac{\partial F}{\partial y} + F = 0, \qquad (22)$$

and extracting the coefficient of $x^k y^n$ in (22) immediately gives (9). More generally, such a PDE translates into a recurrence if it is linear with polynomial coefficients in x and y and if additionally all terms of the form $x^i y^\ell \frac{\partial^k F}{\partial x^k}\frac{\partial^j F}{\partial y^j}$ have $\ell = j$. A dimension counting argument in the spirit of [29, Lem. 3] proves that such a PDE exists for *any* rational function $F(x,y)$. The existence proof is effective and amounts to linear algebra. A natural question is whether it is possible to compute such a PDE via Creative Telescoping (either Almkvist-Zeilberger [1] or reduction based [2]), and how the corresponding method compares to the aforementioned ones.

Table 1: Timings in seconds for creative telescoping to find a telescoper L_n of $P(x,y)/(y^{n+1}Q(x,y))$. Here $P(x,y)/Q(x,y)$ is the generating function for the sequence of the top-right entry of the powers of a randomly chosen matrix in $\mathbb{F}_p[x]^{r\times r}$ of degree d, for a 50-bit prime p, with $Q(x,y)$ the y-reversal of the characteristic polynomial of this matrix. The order of L_n is ℓ, its degree in n is d_n, and $\mathrm{d}_x = \deg_x(L_n)$. A blank space means that the computation took more than 1000 seconds. We observe empirically that the degree in x is $dr(r+1)(2r-1)/2 - r(r-1)$ while its degree in n is $(r-1)(r+2)/2$; this is expected asymptotically by [2, Thm. 25] and Lemma 3.4, because $\deg_y Q(x,y) = r$ and $\deg_x Q(x,y) = dr$. The tested implementations are

- in Maple: `redct` [4]; `HermiteTelescoping` (HT) [8]; `Zeilberger` (ZB) [1] in DEtools; `creative_telescoping` (c_t) [13];
- in SageMath: creative telescoping (`ct`) from the *ore_algebra* package [25];
- In Mathematica: `FindCreativeTelescoping` (FCT), `CreativeTelescoping` (CT), and `HermiteTelescoping` (HCT), see [28].

		Maple				Sage	Mathematica			ℓ	d_n	d_x
r	d	redct	HT	ZB	c_t	ct	FCT	CT	HCT			
2	2	0.0	0.1	0.0	0.1	0.5	0.2	0.2	0.2	2	2	16
	4	0.0	0.0	0.0	0.1	0.6	0.4	0.4	0.3	2	2	34
	6	0.0	0.0	0.0	0.1	0.6	0.7	0.5	0.5	2	2	52
	8	0.0	0.0	0.0	0.1	0.8	1.0	0.7	0.7	2	2	70
3	1	0.0	0.2	0.0	0.5	2.0	2.0	1.3	1.3	3	5	24
	2	0.0	0.1	0.8	3.4	3.1	4.0	2.6	2.5	3	5	54
	3	0.1	0.2	0.8	9.3	5.6	10	5.7	5.4	3	5	84
	4	0.1	0.5	18	19	8.2	17	9.4	8.9	3	5	114
	5	0.2	1.1	5.1	32	12	25	14	14	3	5	144
	6	0.5	1.7	9.8	49	17	35	19	20	3	5	174
4	1	0.4	2.9	23	117	20	31	25	25	4	9	58
	2	1.7	17	410	749	45	101	96	95	4	9	128
	3	4.4	43			89	295	376	373	4	9	198
	4	12	82			172	388	752	693	4	9	268
	5	18	128			280	635			4	9	338
5	1	11	34	538		163	847	780		5	14	115
	2	64	183			515				5	14	250
	3	159	526							5	14	385
	4	345								5	14	520

Table 2: Timings in seconds, using the C++ library NTL [33] and PML [24], for computing the top-right entry of the Nth power of a randomly chosen matrix in $\mathbb{F}_p[x]^{r\times r}$ of degree d, for a 50-bit prime p. The first tested method is to directly apply binary powering (BP); in the present context, the polynomial matrix multiplication of PML is based on evaluation-interpolation and 3-prime FFT. The second tested method uses Algorithm 3 and we do not count "precomputations" (already showed in Table 1), i.e. we only report timings for the two non-negligible steps that depend on N, namely Line 8 (UR, unrolling) and Line 7 (IT, initial terms) from Algorithm 1.

		$N=2^{10}$			$N=2^{12}$			$N=2^{14}$			$N=2^{16}$			$N=2^{18}$			$N=2^{20}$			$N=2^{22}$		
r	d	BP	UR	IT	BP	UR	IT	BP	UR	IT	BP	UR	IT	BP	UR	IT	BP	UR	IT	BP	UR	IT
2	2	1.2e-3	5.7e-4	3.7e-5	5.3e-3	2.4e-3	4.3e-5	2.5e-2	9.7e-3	4.9e-5	1.1e-1	3.9e-2	5.5e-5	5.3e-1	1.5e-1	6.2e-5	3.3e+0	6.2e-1	6.7e-5	1.5e+1	2.5e+0	7.5e-5
	4	2.6e-3	1.3e-3	7.8e-5	1.2e-2	5.2e-3	9.4e-5	5.2e-2	2.1e-2	1.1e-4	2.4e-1	8.4e-2	1.3e-4	1.4e+0	3.4e-1	1.4e-4	7.2e+0	1.4e+0	1.6e-4	3.1e+1	5.4e+0	1.8e-4
	6	3.8e-3	2.1e-3	1.2e-4	1.7e-2	8.7e-3	1.5e-4	7.9e-2	3.5e-2	1.8e-4	3.6e-1	1.4e-1	2.1e-4	2.3e+0	5.5e-1	2.4e-4	1.0e+1	2.2e+0	2.7e-4	4.6e+1	8.9e+0	3.0e-4
	8	5.3e-3	3.1e-3	1.9e-4	2.4e-2	1.2e-2	2.4e-4	1.1e-1	5.0e-2	2.8e-4	5.3e-1	2.0e-1	3.3e-4	3.3e+0	8.0e-1	3.8e-4	1.5e+1	3.2e+0	4.3e-4	7.0e+1	1.2e+1	4.9e-4
3	1	1.4e-3	3.0e-4	1.3e-4	6.0e-3	1.3e-3	1.7e-4	2.6e-2	5.5e-3	2.1e-4	1.2e-1	2.2e-2	2.4e-4	5.8e-1	8.8e-2	2.8e-4	3.4e+0	3.5e-1	3.1e-4	1.6e+1	1.4e+0	3.5e-4
	2	2.9e-3	7.8e-4	4.0e-4	1.2e-2	3.2e-3	5.3e-4	5.6e-2	1.3e-2	6.5e-4	2.6e-1	5.2e-2	7.8e-4	1.5e+0	2.1e-1	9.1e-4	7.6e+0	8.4e-1	1.0e-3	3.4e+1	3.3e+0	1.2e-3
	3	4.3e-3	1.4e-3	7.4e-4	1.9e-2	5.8e-3	9.9e-4	8.4e-2	2.3e-2	1.2e-3	3.9e-1	9.3e-2	1.5e-3	2.2e+0	3.7e-1	1.7e-3	1.1e+1	1.5e+0	2.0e-3	4.9e+1	6.0e+0	2.2e-3
	4	6.0e-3	2.1e-3	8.0e-4	2.6e-2	8.8e-3	1.0e-3	1.2e-1	3.5e-2	1.3e-3	5.8e-1	1.4e-1	1.5e-3	3.5e+0	5.7e-1	1.8e-3	1.7e+1	2.3e+0	2.0e-3	7.1e+1	9.1e+0	2.3e-3
	5	7.4e-3	3.0e-3	1.0e-3	3.3e-2	1.2e-2	1.3e-3	1.5e-1	5.0e-2	1.7e-3	7.2e-1	2.0e-1	2.0e-3	4.3e+0	7.9e-1	2.3e-3	2.0e+1	3.2e+0	2.6e-3	8.8e+1	1.3e+1	2.9e-3
	6	9.1e-3	4.0e-3	1.2e-3	4.0e-2	1.6e-2	1.6e-3	1.8e-1	6.6e-2	1.9e-3	8.2e-1	2.7e-1	2.3e-3	5.3e+0	1.1e+0	2.7e-3	2.3e+1	4.2e+0	3.1e-3	1.1e+2	1.7e+1	3.4e-3
4	1	2.7e-3	4.2e-4	7.8e-4	1.1e-2	1.8e-3	1.1e-3	4.9e-2	7.5e-3	1.4e-3	2.2e-1	3.0e-2	1.7e-3	1.1e+0	1.2e-1	2.0e-3	6.2e+0	4.8e-1	2.3e-3	2.9e+1	1.9e+0	2.6e-3
	2	5.5e-3	1.2e-3	1.3e-3	2.4e-2	5.2e-3	1.8e-3	1.1e-1	2.1e-2	2.3e-3	4.9e-1	8.6e-2	2.8e-3	2.8e+0	3.4e-1	3.2e-3	1.4e+1	1.4e+0	3.7e-3	6.2e+1	5.5e+0	4.2e-3
	3	8.2e-3	2.4e-3	2.1e-3	3.6e-2	1.0e-2	2.9e-3	1.6e-1	4.2e-2	3.7e-3	7.3e-1	1.7e-1	4.5e-3	4.4e+0	6.7e-1	5.3e-3	2.1e+1	2.7e+0	6.1e-3	9.3e+1	1.1e+1	6.9e-3
	4	1.1e-2	4.1e-3	3.0e-3	5.0e-2	1.7e-2	4.1e-3	2.3e-1	6.9e-2	5.2e-3	1.1e+0	2.8e-1	6.4e-3	6.6e+0	1.1e+0	7.5e-3	3.1e+1	4.5e+0	8.6e-3	1.3e+2	1.8e+1	9.7e-3
	5	1.4e-2	6.0e-3	3.7e-3	6.3e-2	2.5e-2	5.2e-3	2.8e-1	1.0e-1	6.6e-3	1.3e+0	4.1e-1	8.0e-3	7.7e+0	1.6e+0	9.4e-3	3.6e+1	6.5e+0	1.1e-2	1.6e+2	2.6e+1	1.2e-2
5	1	4.4e-3	6.0e-4	1.8e-3	1.8e-2	2.7e-3	2.5e-3	8.2e-2	1.1e-2	3.3e-3	3.7e-1	4.5e-2	4.1e-3	1.7e+0	1.8e-1	4.9e-3	1.0e+1	7.3e-1	5.7e-3	4.7e+1	2.9e+0	6.5e-3
	2	9.1e-3	2.0e-3	3.5e-3	3.9e-2	9.1e-3	5.1e-3	1.8e-1	3.7e-2	6.7e-3	8.1e-1	1.5e-1	8.3e-3	4.6e+0	6.0e-1	9.9e-3	2.3e+1	2.4e+0	1.2e-2	1.0e+2	9.6e+0	1.3e-2
	3	1.3e-2	4.3e-3	5.7e-3	5.8e-2	1.9e-2	8.3e-3	2.6e-1	7.8e-2	1.1e-2	1.2e+0	3.2e-1	1.4e-2	7.1e+0	1.3e+0	1.6e-2	3.4e+1	5.1e+0	1.9e-2	1.5e+2	2.0e+1	2.2e-2
	4	1.8e-2	7.4e-3	7.8e-3	8.0e-2	3.3e-2	1.2e-2	3.8e-1	1.3e-1	1.5e-2	1.8e+0	5.4e-1	1.9e-2	1.1e+1	2.2e+0	2.3e-2	4.9e+1	8.7e+0	2.6e-2	2.1e+2	3.5e+1	3.0e-2

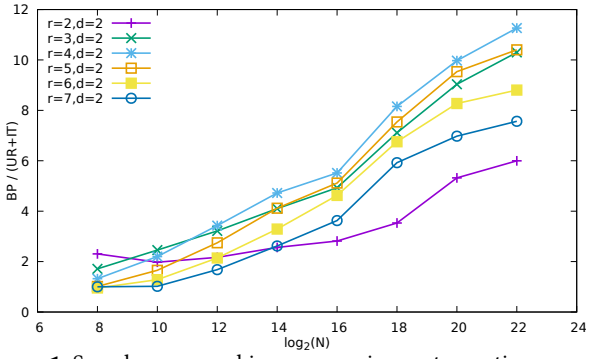

Figure 1: Speed-up versus binary powering, not counting precomputations, for $r = 2 \ldots 7$, $N = 2^8, 2^{10}, \ldots, 2^{22}$, and fixed $d = 2$.

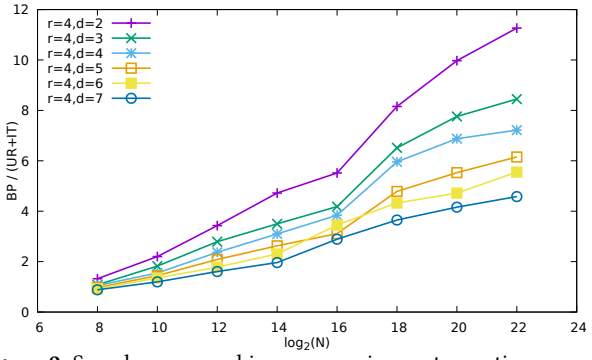

Figure 2: Speed-up versus binary powering, not counting precomputations, for $d = 2 \ldots 7$, $N = 2^8, 2^{10}, \ldots, 2^{22}$, and fixed $r = 4$.

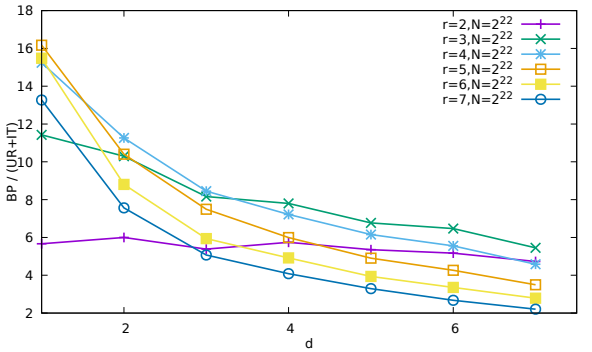

Figure 3: Speed-up versus binary powering, not counting precomputations, for $r = 2 \ldots 7$, $d = 1 \ldots 7$, and fixed $N = 2^{22}$.

REFERENCES

[1] Gert Almkvist and Doron Zeilberger. 1990. The method of differentiating under the integral sign. *J. Symbolic Comput.* 10, 6 (1990), 571–591. https://doi.org/10.1016/S0747-7171(08)80159-9

[2] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, and Ziming Li. 2010. Complexity of creative telescoping for bivariate rational functions. In *ISSAC'10*. ACM, 203–210. https://doi.org/10.1145/1837934.1837975

[3] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, Ziming Li, and Guoce Xin. 2013. Hermite reduction and creative telescoping for hyperexponential functions. In *ISSAC'13*. ACM, 77–84. https://doi.org/10.1145/2465506.2465946

[4] Alin Bostan, Frédéric Chyzak, Pierre Lairez, and Bruno Salvy. 2018. Generalized Hermite reduction, creative telescoping and definite integration of D-finite functions. In *ISSAC'18*. ACM, 95–102. https://doi.org/10.1145/3208976.3208992

[5] Alin Bostan, Frédéric Chyzak, Grégoire Lecerf, Bruno Salvy, and Éric Schost. 2007. Differential equations for algebraic functions. In *ISSAC'07*. ACM, 25–32. https://doi.org/10.1145/1277548.1277553

[6] Alin Bostan, Frédéric Chyzak, Ziming Li, and Bruno Salvy. 2012. Fast computation of common left multiples of linear ordinary differential operators. In *ISSAC'12*. ACM, 99–106. https://doi.org/10.1145/2442829.2442847

[7] Alin Bostan, Pierrick Gaudry, and Éric Schost. 2007. Linear recurrences with polynomial coefficients and application to integer factorization and Cartier-Manin operator. *SIAM J. Comput.* 36, 6 (2007), 1777–1806. https://doi.org/10.1137/S0097539704443793

[8] Alin Bostan, Pierre Lairez, and Bruno Salvy. 2013. Creative telescoping for rational functions using the Griffiths-Dwork method. In *ISSAC'13*. ACM, 93–100. https://doi.org/10.1145/2465506.2465935

[9] Alin Bostan and Ryuhei Mori. 2021. A Simple and Fast Algorithm for Computing the N-th Term of a Linearly Recurrent Sequence. In *SOSA'21 (Symposium on Simplicity in Algorithms)*. SIAM. https://doi.org/10.1137/1.9781611976496.14

[10] Paul F. Byrd. 1963. Expansion of analytic functions in polynomials associated with Fibonacci numbers. *Fibonacci Quart.* 1, 1 (1963), 16–29.

[11] David G. Cantor and Erich Kaltofen. 1991. On fast multiplication of polynomials over arbitrary algebras. *Acta Inform.* 28, 7 (1991), 693–701. https://doi.org/10.1007/BF01178683

[12] D. V. Chudnovsky and G. V. Chudnovsky. 1988. Approximations and complex multiplication according to Ramanujan. In *Ramanujan revisited (Urbana-Champaign, Ill., 1987)*. Academic Press, Boston, MA, 375–472.

[13] Frédéric Chyzak. 2000. An extension of Zeilberger's fast algorithm to general holonomic functions. *Discrete Math.* 217, 1-3 (2000), 115–134. https://doi.org/10.1016/S0012-365X(99)00259-9

[14] James W. Cooley and John W. Tukey. 1965. An algorithm for the machine calculation of complex Fourier series. *Math. Comp.* 19 (1965), 297–301. https://doi.org/10.2307/2003354

[15] Danielle Couty, Jean Esterle, and Rachid Zarouf. 2011. Décomposition effective de Jordan-Chevalley. *Gaz. Math.* 129 (2011), 29–49.

[16] Sándor Czirbusz. 2012. Comparing the computation of Chebyshev polynomials in computer algebra systems. *Ann. Univ. Sci. Budapest. Sect. Comput.* 36 (2012), 23–39.

[17] Saber N. Elaydi and William A. Harris, Jr. 1998. On the computation of A^n. *SIAM Rev.* 40, 4 (1998), 965–971. https://doi.org/10.1137/S0036144597319235

[18] Charles M. Fiduccia. 1985. An efficient formula for linear recurrences. *SIAM J. Comput.* 14, 1 (1985), 106–112. https://doi.org/10.1137/0214007

[19] Philippe Flajolet and Bruno Salvy. 1997. The SIGSAM challenges: symbolic asymptotics in practice. *ACM SIGSAM Bull.* 31, 4 (1997), 36–47. https://doi.org/10.1145/274888.274890

[20] Henry W. Gould. 1972. *Combinatorial identities.* viii+106 pages. A standardized set of tables listing 500 binomial coefficient summations.

[21] David Harvey and Joris van der Hoeven. 2021. Integer multiplication in time $O(n \log n)$. *Ann. of Math. (2)* 193, 2 (2021), 563–617. https://doi.org/10.4007/annals.2021.193.2.4

[22] David Harvey and Joris van der Hoeven. 2022. Polynomial multiplication over finite fields in time $O(n \log n)$. *J. ACM* 69, 2 (2022), Art. 12, 40. https://doi.org/10.1145/3505584

[23] Po-Fang Hsieh, Mitsuhiko Kohno, and Yasutaka Sibuya. 1996. Construction of a fundamental matrix solution at a singular point of the first kind by means of the SN decomposition of matrices. *Linear Algebra Appl.* 239 (1996), 29–76. https://doi.org/10.1016/S0024-3795(96)90003-8

[24] Seung Gyu Hyun, Vincent Neiger, and Éric Schost. 2019. Implementations of Efficient Univariate Polynomial Matrix Algorithms and Application to Bivariate Resultants. In *ISSAC'19*. ACM, 235–242. https://doi.org/10.1145/3326229.3326272

[25] Manuel Kauers and Marc Mezzarobba. 2019. Multivariate Ore polynomials in SageMath. *ACM Commun. Comput. Algebra* 53, 2 (2019), 57–60. https://doi.org/10.1145/3371991.3371998

[26] Manuel Kauers and Gleb Pogudin. 2017. Bounds for substituting algebraic functions into D-finite functions. In *ISSAC'17*. ACM, 245–252. https://doi.org/10.1145/3087604.3087616

[27] Wolfram Koepf. 1999. Efficient computation of Chebyshev polynomials in computer algebra. In *Computer Algebra Systems: A Practical Guide*. Wiley, 79–99.

[28] Christoph Koutschan. 2010. A fast approach to creative telescoping. *Math. Comput. Sci.* 4, 2-3 (2010), 259–266. https://doi.org/10.1007/s11786-010-0055-0

[29] L. Lipshitz. 1988. The diagonal of a D-finite power series is D-finite. *J. Algebra* 113, 2 (1988), 373–378. https://doi.org/10.1016/0021-8693(88)90166-4

[30] P. Massazza and R. Radicioni. 2005. On computing the coefficients of bivariate holonomic formal series. *Theoret. Comput. Sci.* 346, 2-3 (2005), 418–438. https://doi.org/10.1016/j.tcs.2005.08.011

[31] J. C. P. Miller and D. J. Spencer Brown. 1966. An algorithm for evaluation of remote terms in a linear recurrence sequence. *Comput. J.* 9 (1966), 188–190. https://doi.org/10.1093/comjnl/9.2.188

[32] Dieter Schmidt. 2000. Construction of the Jordan decomposition by means of Newton's method. *Linear Algebra Appl.* 314, 1-3 (2000), 75–89. https://doi.org/10.1016/S0024-3795(00)00111-7

[33] V. Shoup. 2021. NTL: A library for doing number theory, v11.5.1. https://libntl.org.

[34] R. P. Stanley. 1980. Differentiably finite power series. *European J. Combin.* 1, 2 (1980), 175–188. https://doi.org/10.1016/S0195-6698(80)80051-5