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Georg Alexander Gruetzner

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# Harmonic analysis on the boundary of hyperbolic groups

Georg Grützner

## Abstract

In this paper we show that a Möbius-structure  $\mathcal{M}$  of dimension  $Q$  has a minimal Ahlfors-David constant. This shows that a Möbius space is uniformly  $Q$ -Ahlfors-David regular. In summary, many classical theorems of harmonic analysis on  $\mathbb{R}^n$  admit a Möbius-invariant formulation in the context of Möbius-geometry. We use this observation to show that the Knapp-Stein operator

$$(I_d^\alpha u_d)(x) = \int \frac{u_d(y)}{d(x, y)^{Q-\alpha}} d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

is a continuous operator on the weighted  $L^2$ -space  $L^2((\frac{d'}{d})^\alpha d\mu_d)$ , with a norm independent of  $d$  and  $d'$ .

From here we construct a Sobolev space  $\mathcal{H}_d^{-\alpha}$  on  $s$ -densities for a given  $s$  as a function of  $\alpha$ . We would like to say that the construction is topologically independent of the metric  $d$ . In this paper we prove that the norms on a large class of functions are comparable.

The work is inspired by a paper by Astengo, Cowling, and Di Blasio [1], who construct uniformly bounded representations for simple Lie groups of rank 1. We formulate the problem in a much more general framework of groups acting on Möbius structures. In particular, all hyperbolic groups.

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## 1 Introduction

### 1.1 Representation theory of hyperbolic groups.

**1.1.1 Kazhdan’s Property (T).** A topological group  $G$  has the Kazhdan property (T), if *every affine-isometric action of  $G$  on a real Hilbert space has a fixed point*. The original 1968 definition of Kazhdan, is formulated in terms of ”almost” invariant vectors of representations of  $G$  on complex Hilbert spaces. The equivalence of the two formulations is known as the Delorme-Guichardet Theorem. Kazhdan introduced property (T) as a tool to prove finite generation of nonuniform lattices in simple Lie groups of higher rank. It is a representation-theoretic form of rigidity that has similarities to geometric superrigidity [2]. The property plays a central role in the theory of operator algebra and in geometric group theory. An excellent monograph on property (T) was written by Bekka, de la Harpe and Valette [3].

**Example 1** *For a local field  $\mathbb{K}$ , the groups  $SL_n(\mathbb{K})$ ,  $n \geq 3$ , and  $Sp_{2n}(\mathbb{K})$ ,  $n \geq 2$ , have property (T). More generally, any simple real Lie group of rank at least two has the Kazhdan property (T).*

**Example 2** *The isometry group  $Sp(n, 1)$ ,  $n \geq 2$ , of the hyperbolic space over the quaternionic numbers, has property (T).*

The Kazhdan property (T) shows up *geometrically* in affine isometric actions on real Hilbert spaces, *algebraically* in group cohomology with coefficients in orthogonal representations, and *analytically* as functions of conditional negative type. The connections between the three perspectives are quite straightforward. If  $A$  is an affine isometric action of a topological group  $G$  on a real Hilbert space, then the linear part  $\pi$  is an orthogonal representation of  $G$ , and the translation part  $b$  is a 1-cocycle with coefficients in  $\pi$ .  $b$  defines a class in  $H^1(G, \pi)$  and  $g \mapsto \|b(g)\|^2$  is a conditional negative type function on  $G$ . In this paper we take the geometric point of view on Kazhdan’s property (T).

**1.1.2 Weak a-T-meanability.** It is known that hyperbolic groups can be divided into two classes: those with and those without the Kazhdan property (T).

A strong negation of the Kazhdan property (T) is the Haagerup property, also known as a-T-meanability. A locally compact topological group  $G$  has the *Haagerup property* if there exists a proper affine-isometric action of  $G$  on a real Hilbert space. Morally, such a group is strongly non-rigid.

Hyp. groups with Haagerup's property	Hyp. groups with property (T)
Free groups, groups acting properly on trees, Fuchsian groups, $\mathrm{SO}(n, 1)$ , $\mathrm{SU}(n, 1)$ Random groups at density $d < \frac{1}{6}$ [5]	$\mathrm{Sp}(n, 1)$ , $\mathrm{F}_{4(-20)}$ Random groups at density $\frac{1}{3} < d < \frac{1}{2}$ [4]

Groups having property (T) do not admit unitary representations with cohomology.

Higher rank Lie groups (rank  $\geq 2$ ) satisfy a certain stronger form of Kazhdan's property (T). Namely, higher rank Lie groups do not admit uniformly bounded representations with reduced cohomology.

Conversely, all rank one Lie groups admit uniformly bounded representations with reduced cohomology. This motivates the question whether all hyperbolic groups admit uniformly bounded representations with cohomology. An even stronger conjecture is often quoted and is dedicated to Y. Shalom [6].<sup>1</sup>

**Conjecture 1 (Y. Shalom)** *Let  $G$  be a non-elementary hyperbolic group. There exists a uniformly bounded representation of  $G$  on a Hilbert space, for which  $H^1(G, \pi) \neq 0$  and for which there exists a proper 1-cocycle.*

Shalom's conjecture is known to be true for lattices in  $\mathrm{Sp}(n, 1)$  [7]. It also leads us to a conjectured classification of hyperbolic groups due to Cowling. Namely, if Shalom's conjecture is true, then we can associate to each hyperbolic group  $G$  a number  $\Lambda(G)$  as follows:

$$\Lambda(G) := \inf_{\pi} \sup_{g \in G} \|\pi_g\|, \quad (1)$$

where the infimum is taken over all uniformly bounded representations with cohomology. This invariant measures, in some sense, the degree to which the strong form of the Kazhdan property (T) fails in  $G$ . If there are no uniformly bounded representations with cohomology, we define  $\Lambda(G)$  to be infinite.

$\Lambda(G)$  is presumably related to an invariant defined by Cowling and Haagerup [8] [9]. Cowling believes that  $\Lambda(G)$  is related to the curvature of the hyperbolic group (with the word metric).

Not much is known about  $\Lambda(G)$  in general. The only result in this direction is a spectral gap theorem for automorphism groups of simplicial complexes [10].

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<sup>1</sup>First mentioned in: Mini-Workshop: Geometrization of Kazhdan's Property (T), Mathematisches Forschungsinstitut Oberwolfach, July 7th - July 14th, 2001, Talk: Reduced cohomology of unitary representation, Y. Shalom, (Report No. 29/2001) (see question 14 in open problem session)

## 1.2 Fractional calculus.

At first glance, the theory of fractional calculus has little to do with the abstract representation theory and the questions therein that we addressed above. We shall see, however, that the two topics are closely related.

### 1.2.1 The classical non-compact picture.

Mon idée fondamentale est de généraliser les principes de l'analyse ordinaire, qui ne considère que les différentielles  $dy, d^2y, d^3y, \dots$ , dont les indices sont entiers positifs, et les intégrales  $\int y dx, \int^2 y dx^2, \int^3 y dx^3, \dots$ , que l'on [...] peut regarder comme des différentielles à indices entiers négatifs. Je propose de considérer aussi les différentielles à indices fractionnaires, et même à indices irrationnelles et imaginaires. (J. Liouville, JEP, 1832)

Liouville's 1832 work, [11], marks the starting point for the study of fractional integration and differentiation. For a locally integrable function  $f$  on  $\mathbb{R}$  and a complex number  $\alpha$  in the half-plane  $\Re(\alpha) > 0$ , Liouville introduces the function

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt, \quad (2)$$

where  $\Gamma$  is the gamma function and  $a$  is any base point.

The operator  $I^\alpha$  satisfies

$$\frac{d}{dx} I^{\alpha+1} f(x) = I^\alpha f(x), \quad I^\alpha(I^\beta f) = I^{\alpha+\beta} f. \quad (3)$$

It is therefore useful to define *fractional differentiation* by taking enough ordinary derivatives of  $I^\alpha f$ .

It's worth noting that a formal treatment of the integral  $I^\alpha f$  can also be found in unpublished notes by Riemann from 1847 entitled "Versuch einer allgemeinen Auffassung der Integration und Differentiation" [12]. At that time Riemann was still a student at the University of Göttingen and most likely had no knowledge of Liouville's work. There is a nice article on Riemann's notes by Hermann Weyl from 1917 [13].

A systematic treatment of the properties of "Riemann-Liouville" integrals and derivatives for certain classes of functions ( $L^p(\mathbb{R})$ , Lipschitz, etc.) was given by Hardy and Littlewood in 1928 [14].

In 1949 M. Riesz published a paper which generalized the "Riemann-Liouville" integrals to higher dimensions and gave meaning to the operator  $I^\alpha$  for negative  $\alpha$  by means of analytic continuation [15]. In his honor the operators  $I^\alpha$  on  $\mathbb{R}^n$  are known today as *Riesz-potential operators*.

Today these operators are studied on spaces of *homogeneous type*, i.e., on metric spaces with a doubling measure compatible with the topology (the metric must satisfy the triangle inequality only up to a constant).

**1.2.2 The compact picture.** Let  $\mathbb{S}^1$  be the circle considered as the boundary of the open disk  $\mathbb{D} = \{z : |z| < 1\}$  in the complex plane  $\mathbb{C}$ . We can endow  $\mathbb{S}^1$  with a metric as follows. Given two points  $x, y \in \mathbb{S}^1$ , let  $\theta(x, y) \in [0, \pi]$  be the angle at the origin 0 between

$x$  and  $y$ . Then

$$d_0(x, y) = \sin \frac{1}{2} \theta(x, y), \quad (4)$$

is a metric on  $\mathbb{S}^1$ . We can forget about normalization and naively define fractional integration on  $\mathbb{S}^1$  as

$$I_0^\alpha f(x) = \int_{\mathbb{S}^1} \frac{f(y)}{d_0(x, y)^{1-\alpha}} dy. \quad (0 < \alpha < 1) \quad (5)$$

Here  $dy$  is simply the ordinary Lebesgue measure by identifying  $\mathbb{S}^1$  with  $[0, 1]$ . This definition turns out to be 'natural' in a very precise sense.

We endow  $\mathbb{D}$  with the Poincaré metric

$$d(z_1, z_2) = 2 \tanh^{-1} \left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right|. \quad (6)$$

Given two points  $x, y \in \mathbb{S}^1$  and one point  $z \in \mathbb{D}$ , then the three points  $x, y, z$  form an ideal triangle in  $\mathbb{D}$ . Let  $\theta_z(x, y) \in [0, \pi]$  be the angle at  $z$  of this ideal triangle. Then again

$$d_z(x, y) = \sin \frac{1}{2} \theta_z(x, y), \quad (7)$$

is a metric on  $\mathbb{S}^1$ . We call it the *visual metric* on  $\mathbb{S}^1$  related to the point  $z$ . Let  $\mu_z$  be the harmonic measure on  $\mathbb{S}^1$  with respect to the point  $z$ , then we can define

$$I_z^\alpha f(x) = \int_{\mathbb{S}^1} \frac{f(y)}{d_z(x, y)^{1-\alpha}} d\mu_z(y). \quad (0 < \alpha < 1) \quad (8)$$

This definition agrees with the above when  $z$  is the origin. It turns out that  $I_z^\alpha$  "does not depend on the choice of  $z$ " if we consider  $f$  not as a function but as  $(\frac{1}{2} + \alpha)$ -density, viz., functions  $f_z$  which, when the point  $z$  changes to  $z'$ , turns out to be  $f_{z'} = (\frac{d\mu_z}{d\mu_{z'}})^{\frac{1}{2} + \alpha} f_z$ . One can check that if  $f_z$  is a  $(\frac{1}{2} + \alpha)$ -density, then  $z \mapsto I_z^\alpha f_z$  is a  $(\frac{1}{2} - \alpha)$ -density.

We note that 1-densities are signed measures on  $\mathbb{S}^1$  which are absolutely continuous with respect to harmonic measures. As  $\alpha$  converges to 1, the operator  $I_z^\alpha$  converges to an operator  $I^1$  which takes a signed-measure  $\mu$  to its total mass  $\mu(\mathbb{S}^1)$ .

### 1.3 Principle and complementary series representations.

$s$ -densities and fractional integration and differentiation appear naturally in the context of induced representations. We give a brief overview of this connection in the context of Lie group theory.

Let  $G$  be a connected semisimple Lie group with finite center. Let  $KAN$  be an Iwasawa decomposition of  $G$ . We denote  $M$  and  $M'$  as the centralizer and normalizer, respectively, of  $A$  in  $K$ .

$MAN$  is the minimal standard parabolic subgroup of  $G$ , more compactly we write  $H$  for the subgroup  $MAN$ . For  $s \in \mathbb{R}$ , an  $s$ -density on the smooth manifold  $G/H$  is a tensorial object  $v$  which has only one component in coordinates and which transforms under diffeomorphisms according to the rule:

$$\phi^* v = |\det(d\phi)|^s (v \circ \phi).$$

Densities can be multiplied: The pointwise product of an  $s$  density with an  $s'$  density,

is an  $(s + s')$  density. 1-densities coincide with smooth (absolutely continuous) measures on  $G/H$ . Formally,  $s$ -densities are smooth sections of a  $G$ -homogeneous line bundle  $\mathcal{D}_s(G/H)$ . The action of  $G$  on  $\mathcal{D}_s(G/H)$  is given by  $g \cdot v = (l_g^{-1})^* v$ , where  $l_g$  is the left translation of  $G$  on  $G/H$ .

The irreducible unitary representations of  $MAN$  are all of the form  $(\sigma \otimes \lambda) : man \mapsto a^\lambda \sigma(m)$ , where  $a \mapsto a^\lambda \in \mathbb{C}$  is a unitary character of  $A$  and  $\sigma$  is an irreducible unitary representation of  $M$ .

The characters  $a \mapsto a^\lambda$  of  $A$  are constructed as follows. Given  $\lambda \in \mathfrak{a}_\mathbb{C}^* := \text{Hom}(\mathfrak{a}, \mathbb{C})$ , for  $a \in A$ , we write

$$a^\lambda = e^{\lambda \log a} \in \mathbb{C}.$$

The unitary characters are given by all  $\lambda \in i\mathfrak{a}^*$ .

The category of continuous representations of  $H$  is equivalent to the category of  $G$ -homogeneous vector bundles on  $G/H$ , by the associated vector bundle construction. The  $G$ -homogeneous vector bundle  $\mathcal{D}_s(G/H)$  of  $s$ -densities, is the associated bundle of the character

$$\delta^s : a \mapsto e^{s \text{tr}(\text{ad}(\log a)|_{\mathfrak{n}})}.$$

One checks that  $\delta^s(a) = |\det \text{Ad}_{\mathfrak{g}/\mathfrak{h}}(man)|^{-s}$ , for all  $m \in M$ ,  $n \in N$ , where  $\text{Ad}_{\mathfrak{g}/\mathfrak{h}}$  is the adjoint- representation of  $H$  on  $\mathfrak{g}/\mathfrak{h}$ .

More generally, any continuous representation  $(\sigma, \mathcal{H})$  of  $H$  is associated with a  $G$ -homogeneous vector bundle  $\mathcal{V}$ . By  $\Gamma(\mathcal{V})$  denote the space of continuous sections of  $\mathcal{V}$ . The induced representation of  $G$  from the representation  $(\sigma, \mathcal{H})$  of  $H$ , is the following action of  $G$  on  $\Gamma(\mathcal{V})$ :

$$(\pi(g)s)(x) = g \cdot (s(g^{-1}x)). \quad (s \in \Gamma(\mathcal{V}), x \in G/H, g \in G)$$

This representation  $\pi$  is usually denoted by  $\text{Ind}_H^G(\sigma)$ .

The representation  $\text{Ind}_H^G(\sigma)$  is said to be unitarizable if the space  $\Gamma(\mathcal{V})$  admits such a pre-Hilbert structure that  $\pi$  extends to a unitary representation on the associated Hilbert completion. In general, unitarity of  $(\sigma, \mathcal{H})$  does not imply unitarity of  $\text{Ind}_H^G(\sigma)$ . However, unitarity can be achieved by twisting the representation  $\sigma$  with semidensities.

Indeed, we can tensorize  $\sigma$  with  $\delta^{\frac{1}{2}}$  to obtain a representation  $\sigma \otimes \delta^{\frac{1}{2}}$  of  $H$  in  $\mathcal{H} \otimes \mathcal{D}_{\frac{1}{2}}(\mathfrak{g}/\mathfrak{h})$ . Given  $a, b \in \mathcal{H}$  and  $u, v \in \mathcal{D}_{\frac{1}{2}}(\mathfrak{g}/\mathfrak{h})$ , we define  $(a \otimes u, b \otimes v) = \langle a, b \rangle_{\mathcal{H}} u \bar{v}$ , and extend it to a sesquilinear form  $\mathcal{H} \otimes \mathcal{D}_{\frac{1}{2}}(\mathfrak{g}/\mathfrak{h}) \times \mathcal{H} \otimes \mathcal{D}_{\frac{1}{2}}(\mathfrak{g}/\mathfrak{h}) \rightarrow \mathcal{D}_1(\mathfrak{g}/\mathfrak{h})$ . Thus, for the sections  $A, B \in \Gamma(\mathcal{V} \otimes \mathcal{D}_{\frac{1}{2}}(G/H))$  their product  $(A, B)$  is in  $\Gamma(\mathcal{D}_1(G/H))$ , leading to a measure. We set

$$\langle A, B \rangle = \int_{G/H} (A, B).$$

The action of  $\pi = \text{Ind}_H^G(\sigma \otimes \delta^{\frac{1}{2}})$  in  $\Gamma(\mathcal{V} \otimes \mathcal{D}_{\frac{1}{2}}(G/H))$  is unitary, since  $(\pi(g)A, \pi(g)B) = ((l_g^{-1})^* A, (l_g^{-1})^* B) = (l_g^{-1})^*(A, B)$ , and the integral of the density  $(l_g^{-1})^*(A, B)$  over  $G/H$  agrees with the total mass of  $(A, B)$ .

The *principle series* of unitary representations of  $G$  is parameterized by all  $(\sigma, \lambda)$  such that  $\sigma \otimes \lambda$  is unitary, and is obtained by normalized induction from  $MAN$  to  $G$ . This means

$$\pi_{\sigma, \lambda} = \text{Ind}_H^G((\sigma \otimes \lambda) \otimes \delta^{\frac{1}{2}})$$



The induction of representations on  $G$  works even if  $a \mapsto a^\lambda$  is not a unitary character. Knapp and Stein call the representations  $\pi_{\sigma,\lambda}$  with  $\lambda$  non-unitary, the *non-unitary principal series*. Roughly speaking, the *complementary series* consists of those representations in the nonunitary principle series which can be made unitary by changing the inner product.

From now on let  $G$  be of real rank 1, and  $m' \in M' \setminus M$ . It turns out that a necessary condition for  $\pi_{\sigma,\lambda}$  to be unitarizable is that  $\sigma^{m'}(m) = \sigma(m'mm'^{-1})$  is equivalent to  $\sigma$ , and  $\lambda$  has the form

$$\lambda(a) = \delta^\alpha(a),$$

with  $\alpha \in \mathbb{R}$ . Then

$$\pi_{\sigma,\lambda} = \text{Ind}_{\mathbb{H}}^G((\sigma \otimes \delta^\alpha) \otimes \delta^{\frac{1}{2}}) = \text{Ind}_{\mathbb{H}}^G(\sigma \otimes \delta^{\frac{1}{2}+\alpha}),$$

and the inner product necessarily comes from an operator  $\mathcal{I}^\alpha : \mathcal{D}_{\frac{1}{2}+\alpha}(G/H) \rightarrow \mathcal{D}_{\frac{1}{2}-\alpha}(G/H)$ , which maps  $(\frac{1}{2} + \alpha)$  densities to  $(\frac{1}{2} - \alpha)$  densities [16]. For  $\alpha > 0$  such operators were found by Kunze and Stein in 1964 [17]. These operators amount to fractional integration operators on  $K/M \cong G/H$ , as does the operator (8) on  $\mathbb{S}^1$ . Knapp and Stein observed that the existence of complementary series representations (i.e., the positivity of  $\mathcal{I}^\alpha$ ) is closely related to the analytic continuation of  $\mathcal{I}^\alpha$  to negative values of  $\alpha$  [18].

The following result is essentially due to B. Kostant [19].

**Theorem 1 (Kostant)** *If  $G = \text{SO}(n, 1)$  or  $\text{SU}(n, 1)$ , then  $\mathcal{I}^\alpha$  is positive definite for any  $\alpha \in ]0, \frac{1}{2}[$ . If  $G = \text{Sp}(n, 1)$ , then it is positive definite if and only if  $\alpha < \frac{1}{2} - \frac{1}{2(2n+1)}$ .*

Thus, if  $G = \text{SO}(n, 1)$  or  $\text{SU}(n, 1)$ , then the representations

$$\pi_\alpha = \text{Ind}_{\mathbb{H}}^G(1_H \otimes \delta^{\frac{1}{2}+\alpha}), \quad (0 < \alpha < \frac{1}{2})$$

interpolate between the quasi-regular representation  $\lambda_{G/H} = \text{Ind}_{\mathbb{H}}^G(1_H \otimes \delta^{\frac{1}{2}})$ , which is weakly contained in the regular representation  $\lambda_G$ , and the trivial representation  $1_G$  when  $\alpha$  tends to  $\frac{1}{2}$ .

To understand the latter statement, recall the above case of  $\mathbb{S}^1$ , that  $I_z^\alpha$  tends to an operator  $I^1 : \mu \mapsto \mu(\mathbb{S}^1)$  on signed measures on  $\mathbb{S}^1$  when  $\alpha$  tends to 1 (note that we scaled  $\alpha$  by 2). But the action of  $\text{PSL}(2, \mathbb{R})$  on  $\mathbb{S}^1$  is measure-preserving. Therefore, the diagonal matrix coefficients  $(\pi_\alpha(\cdot)v, v)$  tend to constant functions, showing that  $\pi_\alpha \rightarrow 1_G$  in the Fell-topology.

#### 1.4 Conformal invariance of differential operators.

Densities reappear in connection with the conformal invariance of natural differential operators.

Natural bundles in Riemannian geometry are functors  $T$  from the category of Riemannian manifolds into the category of vector bundles belonging to representations of the general linear group. Natural operators in Riemannian geometry are natural transformations between natural bundles which associate differential operators  $A_g : T(M, g) \rightarrow T'(M, g)$

with Riemannian metrics  $g$  in such a way that if  $\phi$  is a diffeomorphism,

$$A_{\phi^*g} = \phi^*(A_g).$$

Traditionally, such a natural transformation is said to be *conformally invariant* if, in addition, there exist real numbers  $a$  and  $b$ , so that if a Riemannian metric  $g$  is conformally changed to  $g' = e^{2w}g$ , then

$$A_{g'} = e^{-bw} \circ A_g \circ e^{aw}.$$

This means that  $A$  can be transformed into an exactly conformally invariant operator  $A$  from sections of  $D_{a/Q} \otimes T$  to sections of  $D_{b/Q} \otimes T'$ ,  $Q = \text{dimension}$ .

**1.4.1 GJMS-operators** On any flat conformal manifold of dimension  $Q \geq 3$ , for every  $k \in \mathbb{Z}^{>0}$  there is a unique natural conformally invariant operator

$$\mathcal{L}^k : \mathcal{D}_{\frac{1}{2} - \frac{k}{Q}} \rightarrow \mathcal{D}_{\frac{1}{2} + \frac{k}{Q}}, \quad (9)$$

whose leading part is the  $k$ -th power of the Beltrami Klein operator.

For  $k = 1$  this operator is the usual conformal Laplace operator, for  $k = 2$  this operator is commonly known as the Paneitz-operator. If  $Q$  is even, then  $\mathcal{L}^{\frac{Q}{2}}$  exists and is known as the *critical GJMS-operator*. It maps a function to a multiple of the volume form [20]. Note that  $\frac{1}{2} - \frac{k}{Q} + \frac{1}{2} + \frac{k}{Q} = 1$ , so the quadratic form

$$v \mapsto \int v \mathcal{L}^k v \quad (10)$$

is conformally invariant on  $\frac{1}{2} - \frac{k}{Q}$ -densities of Sobolev class  $\mathcal{H}^k$ .

For  $k = 1$ , this quadratic form is positive on the  $Q$ -sphere,  $Q \geq 3$ , since the scalar curvature is positive.

**Definition 1** *On a  $Q$ -dimensional Riemannian manifold with scalar curvature  $R$ , the conformal Laplacian is the operator acting on smooth functions  $u$  through*

$$Lu = \Delta u + \frac{Q-2}{4(Q-1)}Ru.$$

Graham and Zworski further study natural conformally invariant operators  $\mathcal{L}^\alpha : \mathcal{D}_{\frac{1}{2} - \frac{\alpha}{Q}} \rightarrow \mathcal{D}_{\frac{1}{2} + \frac{\alpha}{Q}}$ ,  $\alpha \in (0, \frac{Q}{2})$  using scattering theory on Poincaré-Einstein-manifolds. They recover the above operators  $\mathcal{L}^k$  when  $\alpha$  is an integer [21].

**1.4.2 sub-GJMS-operators?** On contact manifolds of dimension  $Q = 2n + 1$ , for every  $k \in \{1, \dots, n + 1\}$  there exists a natural conformally invariant operator

$$\mathcal{L}^k : \mathcal{D}_{\frac{1}{2} - \frac{k}{Q+1}} \rightarrow \mathcal{D}_{\frac{1}{2} + \frac{k}{Q+1}}, \quad (11)$$

whose leading part is the  $k$ -th power of the sub-Laplacian [22]. For  $k = 1$  this operator is the CR invariant sub-Laplacian of Jerison-Lee [23].

Roncal and Thangavelu describe natural conformally invariant operators  $\mathcal{L}^\alpha$  on the Heisenberg group for all positive  $\alpha < \frac{n+1}{2}$  [24]. This and Theorem 2.7 from [1] suggest that

fractional GJMS-operators also exist in sub-Riemannian geometry.

If we believe in the existence of natural conformally invariant operators  $\mathcal{L}^k$  on sub-Riemannian manifolds whose leading part is the  $k$ -th power of the sub-Laplacian, then

$$v \mapsto \int v \mathcal{L}^k v \quad (12)$$

gives conformally invariant quadratic forms on all symmetric spaces of rank 1.

### 1.5 From “exact” to “almost” conformal invariance.

Natural conformally invariant operators give us conformally invariant quadratic forms. In general, they are not positive definite. To obtain a pre-Hilbert space structure on  $s$ -densities, we can replace the exact conformally invariant operator  $\mathcal{L}^\alpha$  by the power  $\alpha$  of the conformal Laplacian  $L$ .

Since the conformal Laplacian  $L$  is self-adjoint and positive,  $L^\alpha$  is well-defined. Taking powers breaks the conformal invariance. Nevertheless,  $L^\alpha$  defines for any  $s \in \mathbb{R}$  an *scale invariant* operator on  $s$ -densities, which we denote by

$$\mathcal{L}_s^\alpha : \mathcal{D}_s \rightarrow \mathcal{D}_{s+\frac{2\alpha}{Q}}.$$

Scale invariant means that  $\mathcal{L}_s^\alpha$  is unchanged when the metric is multiplied by a constant factor. We obtain a scale invariant inner product  $Q^\alpha$  on  $\frac{1}{2} - \frac{\alpha}{Q}$ -densities of Sobolev class  $\mathcal{H}^\alpha$ . We can hope that the representation of  $G$  in  $(\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(K/M), Q^\alpha)$  is uniformly bounded.

**1.5.1 The Cayley transform trick.** Let  $N$  denote the unipotent component of the Iwasawa decomposition  $G = KAN$ , and let  $M$  be the centralizer of  $A$  in  $K$ .  $N$  is a simply connected nilpotent Lie group acting simply transitively on the complement of a point of  $K/M$ . The orbit map  $\mathcal{C} : N \rightarrow K/M$  is called the *Cayley transform*.

Since  $N$  is a group of type  $H$ , its Lie algebra  $\mathfrak{n}$  decomposes into a direct product  $\mathfrak{v} \oplus \mathfrak{z}$ , where  $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z}$ . The exponential mapping is a bijection of  $\mathfrak{n}$  onto  $N$ , and for  $X$  in  $\mathfrak{v}$  and  $Z$  in  $\mathfrak{z}$  we can denote by  $(X, Z)$  the element  $\exp(X + Z)$  in  $N$ .

The Korányi norm on  $N$  is given by

$$\mathcal{N}(X, Z) = \left( \frac{|X|^4}{16} + |Z|^2 \right)^{\frac{1}{4}}.$$

Setting  $d_N(n_1, n_2) = \mathcal{N}(n_1^{-1}n_2)$ , gives  $N$  the structure of a Carnot-Carathéory manifold. There exists a metric  $d_{K/M}$  on  $K/M$  such that

$$d_{K/M}(\mathcal{C}((X, Z)), \mathcal{C}((X', Z'))) = \frac{d_N((X, Z), (X', Z'))}{\left( (1 + \frac{|X|^2}{4})^2 + |Z|^2 \right)^{\frac{1}{4}} \left( (1 + \frac{|X'|^2}{4})^2 + |Z'|^2 \right)^{\frac{1}{4}}}$$

The Cayley transform can be viewed as a means of flattening the metric  $d_{K/M}$  on  $K/M$ . In the case of real hyperbolic space, it is a stereographic projection. In all cases it is conformal (Corollary 2.3 in [1]), but far from homothetic.

The Cayley transform conjugates the  $A$ -subgroup in the Iwasawa decomposition with the

1-parameter group of dilations  $(\delta_t)$  of  $N$ , which are homothetic.

$$\delta_t(X, Z) = (tX, t^2Z) \quad \forall (X, Z) \in N.$$

Imagine one can prove that pulling-back densities

$$\mathcal{C}^* : (\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(K/M), Q^\alpha) \rightarrow (\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(N), Q^\alpha)$$

is invertible. Then for each element  $a \in A$  there is  $t \in \mathbb{R}$  such that  $a = \mathcal{C} \circ \delta_t \circ \mathcal{C}^{-1}$ . With respect to the inner product  $Q^\alpha$ ,  $\mathcal{C}$  is invertible and  $\delta_t$  is isometric. Therefore  $a$  is invertible with norm  $\leq \|\mathcal{C}\| \|\mathcal{C}^{-1}\|$  independent of  $a$ .

Let us use the Bruhat decomposition  $G = KAK$ . Then  $K$  acts isometrically on  $(\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(K/M), Q^\alpha)$ . This shows that  $G$  acts uniformly bounded, with norm  $\leq \|\mathcal{C}\| \|\mathcal{C}^{-1}\|$ . Proving the boundedness of the representation of  $G$  in the pre-Hilbert space  $(\mathcal{D}_{\frac{1}{2}-\frac{\alpha}{Q}}(K/M), Q^\alpha)$  amounts to estimating a single operator. This was done by Astengo, Cowling, and Di Blasio in [1].

They obtain Hilbert structures such that  $\{\pi_\alpha = \text{Ind}_{\mathbb{H}}^G(1_{\mathbb{H}} \otimes \delta^{\frac{1}{2}+\frac{\alpha}{Q}})\}_{\alpha \in (-\frac{Q}{2}, \frac{Q}{2})}$  become uniformly bounded representations of  $G$ , which interpolate between the quasi-regular representation  $\lambda_{G/H} = \text{Ind}_{\mathbb{H}}^G(1_{\mathbb{H}} \otimes \delta^{\frac{1}{2}})$  and the trivial representation  $1_G$  of  $G$ .

This fact is an essential step in Julg's program to prove the Baum-Connes conjecture with coefficients for  $\text{Sp}(n, 1)$  and  $F_{4(-20)}$ , i.e., for simple Lie groups of real rank 1 with Kazhdan property (T) [25].

**Remark 1** *The bound obtained by Astengo, Cowling and Di Blasio is not uniform in  $\alpha$  and explodes when  $\alpha$  approaches  $\frac{Q}{2}$ .*

## 1.6 From Lie groups to $Q$ -Ahlfors-David regular spaces.

General Gromov hyperbolic groups act conformally on  $Q$ -Ahlfors-David regular metric spaces. In order to construct uniform bounded representations for all hyperbolic groups that interpolate between the quasi-regular representation and the trivial representation, we would like to apply the ideas of Astengo, Cowling and Di Blasio to this setting. However, we know of no analog of a conformal Laplacian in this context. Therefore, we assume its dual description instead.

Let  $K^\alpha$  be a fundamental solution of a differential operator  $\mathcal{L}^\alpha$ , i.e., a distribution  $K^\alpha$  such that  $\mathcal{L}^\alpha K^\alpha = \delta$ . Then, for any smooth compactly supported function  $u$ ,  $\mathcal{L}^\alpha(K^\alpha \star u) = (\mathcal{L}^\alpha K^\alpha) \star u = \delta \star u = u$ . In other words, the operator  $I^\alpha$ , defined by

$$I^\alpha u = K^\alpha \star u : x \mapsto \int_N K^\alpha(xy^{-1})u(y) dy$$

is  $(\mathcal{L}^\alpha)^{-1}$ . On the Heisenberg group, the natural conformally invariant operator  $\mathcal{L}^\alpha$  of Roncal and Thangavelu has as fundamental solution a constant multiple of  $|x|^{-Q+2\alpha}$  [24]. On Iwasawa  $N$ -groups this distribution is exactly the kernel of the intertwining operators of Knapp and Stein [18].

In this work we work in the category of Möbius-spaces  $(Z, \mathcal{M})$ , i.e. sets  $Z$  together with a collection  $\mathcal{M}$  of metrics defining the same metric cross-ratios. With  $\mathcal{M}$  we associate a

class of  $s$ -densities  $\mathcal{D}_s(\mathcal{M})$  and the Knapp-Stein operator

$$(I_d^\alpha u_d)(x) = \int \frac{u_d(y)}{d(x, y)^{Q-\alpha}} d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

where  $\mu_d$  is the maximum  $Q$ -dimensional Hausdorff measure associated with the metric  $d$ , and  $u_d$  is an  $s$ -density, where  $s$  is a function of  $\alpha$ .

**1.6.1 Results.** We use an analog of the Cayley transform of Astengo, Cowling and Di Blasio on Möbius spaces to prove strong regularity properties of Möbius spaces as measure spaces.

**Theorem 2** *If for a Möbius structure  $\mathcal{M}$  there exists  $d \in \mathcal{M}$  such that  $\mu_d$  is  $Q$ -Ahlfors-David-regular, then there exists  $C \geq 1$  such that  $\{\mu_d\}_{d \in \mathcal{M}}$  is uniformly Ahlfors-David-regular with constant  $C$ .*

We use this observation in the context of strongly hyperbolic metric spaces  $X$  to show in Theorem 6 that powers of conformal factors  $\frac{d'}{d}$  associated to the class of visual metrics  $\mathcal{M}$  on  $\partial X$  are Muckenhoupt weights with uniformly bounded  $A_p$ -constant on  $L^p(\mu_d)$ , independent of  $d$  and  $d'$  in  $\mathcal{M}$ .

From there we prove the following weighted bound on the Knapp-Stein operator.

**Theorem 8** *Let  $X$  be a strongly hyperbolic space,  $\{d_o\}_{o \in X}$  be the natural Möbius structure on  $\partial X$ , and  $0 < \alpha < \frac{Q}{q}$ . Then there is a constant  $C_\alpha > 0$  that depends only on  $\alpha$ , so for all  $d_o, d_{o'} \in \mathcal{M}$  and  $f \in L^p(\mu_o)$ ,*

$$\|(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}} \circ I_o^\alpha \circ (\frac{d_{o'}}{d_o})^{-\frac{\alpha}{2}} f\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}. \quad (13)$$

The Knapp-Stein operator is self-adjoint on  $L^2(d\mu_d)$  and its square has a kernel comparable to  $d(x, y)^{-Q+2\alpha}$ . Thus  $(I_d^\alpha)^2$  is a good candidate for a Laplacian raised to the power  $-\alpha$ . Our guess for the  $\mathcal{H}^{-\alpha}$ -norm on  $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities on compact  $Q$ -Ahlfors regular Möbius spaces is

$$\|u\|_d = \|I_d^\alpha u_d\|_{L^2(d\mu_d)}. \quad (14)$$

By construction it is positive and leads to a Sobolev space  $\mathcal{H}_d^{-\alpha}$  of  $(\frac{1}{2} + \frac{\alpha}{Q})$  densities. We would like to say that the construction is topologically independent of the metric  $d$ . In Proposition 9, we use the previous results from this paper to prove that the norms  $\|\cdot\|_d$  are comparable on a large class of functions.

## 1.7 Organization of the paper

# 2 Möbius spaces

## 2.1 Some basics on Möbius spaces

According to Buyalo and Schröder [26], a Möbius structure  $\mathcal{M}$  on a set  $Z$  is a collection of metrics defining the same metric cross-ratios. More precisely, one considers quadruples  $(x, y, z, w)$  of points in  $Z$  where no entry occurs three or four times. Such a *admissible*

quadruple is associated with a *cross-ratio triple*

$$[x, y, z, w]_d = (d(x, y)d(z, w) : d(x, z)d(y, w) : d(x, w)d(y, z)) \in \mathbb{R}P^2. \quad (15)$$

Two metrics  $d$  and  $d'$  on  $Z$  are called *Möbius equivalent* if for each admissible quadruple their respective cross-ratio triples coincide.

Indeed, the theory makes it necessary to consider *extended* metrics on  $Z$  which allow the existence of an *infinitely distant* point  $\bullet \in X$ , i.e.  $d(x, \bullet) = \infty$  for all  $x \neq \bullet$ . Such a point  $\bullet$  is always assumed to be unique and  $d(\bullet, \bullet) = 0$ . If such a point exists, we sometimes use the notation  $Z_\bullet$  for the set  $Z \setminus \{\bullet\}$ . If  $\bullet$  occurs once in an admissible quadruple, then the associated cross-ratio triple is  $[x, y, z, \bullet]_d = (d(x, y) : d(x, z) : d(y, z))$ . If it occurs twice, then  $[x, y, \bullet, \bullet]_d = (0 : 1 : 1)$ .

A *Möbius structure* on a set  $Z$  is a class  $\mathcal{M}$  of metrics on  $Z$  which are pairwise Möbius equivalent.

Two metrics  $d$  and  $d'$  in a Möbius structure are conformal in the strong sense that the limiting ratio  $\frac{d'}{d}(x) := \lim_{y \rightarrow x} \frac{d'(x, y)}{d(x, y)}$  exists for all  $x \in Z \setminus \{\bullet\}$ . Moreover, for all  $x \neq y \in Z \setminus \{\bullet\}$ ,

$$\frac{d'}{d}(x) \frac{d'}{d}(y) = \left( \frac{d'(x, y)}{d(x, y)} \right)^2. \quad (16)$$

Examples of sets with natural Möbius structures are:

1. The round sphere,
2. Boundaries of symmetric spaces of rank one,
3. Boundaries of CAT(-1) spaces [27],
4. Boundaries of strongly hyperbolic metric spaces [28],
5. Boundaries of symmetric spaces of higher rank and Euclidean buildings [29], [30],
6. Boundaries of certain CAT(0)-cube complexes [30], [31].

A Möbius structure  $\mathcal{M}$  is associated with a family of Hausdorff measures  $\{\mu_d\}_{d \in \mathcal{M}}$  with a uniform Hausdorff dimension  $Q$ . For two metrics  $d, d' \in \mathcal{M}$ , the Radon-Nikodým derivative  $\frac{\mu_d}{\mu_{d'}}$  is connected to the conformal factor  $\frac{d}{d'}$  by

$$\frac{d\mu_d}{d\mu_{d'}}(x) = \frac{d}{d'}(x)^Q \quad \forall x \in Z \setminus \{\bullet\}. \quad (17)$$

In particular, metrics in  $\mathcal{M}$  without distant points, have equivalent measures.

We say that a Möbius space  $(Z, \mathcal{M})$  or a Möbius structure  $\mathcal{M}$  is *compact* if  $(Z, d)$  is compact for all metrics  $d \in \mathcal{M}$ .

We say that a Möbius space  $(Z, \mathcal{M})$  or a Möbius structure  $\mathcal{M}$  is *spherical* if  $\text{diam}(Z, d) \leq 1$  for all  $d \in \mathcal{M}$ .

Strongly hyperbolic metric spaces admit a natural spherical Möbius structure on their boundary. We will discuss this case in detail below. As we will see, it turns out that any hyperbolic group admits strongly hyperbolic left invariant metrics [32].

## 2.2 The metric Cayley transform

The stereographic projection is a conformal bijection from the unit sphere  $\mathbb{S}$  in  $\mathbb{R}^{n+1}$  onto  $\mathbb{R}^n \cup \{\infty\}$ . Its generalization to Iwasawa  $N$ -groups associated to simple Lie groups of rank 1 is known as the Cayley transformation. A pure metric analog of the Cayley transform can be given as follows.

An  $K$ -semi-metric is a symmetric function  $\rho(x, y)$  on  $Z$  such that there exists  $K > 0$  and a (possibly extended) metric  $d$  on  $Z$ , s.t.

$$\frac{1}{K}d(x, y) \leq \rho(x, y) \leq Kd(x, y). \quad (18)$$

A semi-metric space is a set  $Z$  together with a semi-metric  $\rho$ . The semi-metric  $\rho$  and the metric  $d$  induce the same topology on  $Z$ . Furthermore, one can speak of Lipschitz maps, Möbius maps or symmetric maps between semi-metric spaces.

**Definition 2** *Let  $(Z, \rho)$  be a semi-metric space, then the Cayley transformation with respect to a point  $\bullet$  in  $Z$  is the bijection obtained by replacing  $\rho$  on  $Z$  by the semi-metric*

$$\rho_{\bullet}(x, y) = \frac{\rho(x, y)}{\rho(x, \bullet)\rho(\bullet, y)} \quad \text{if } x, y \in Z \setminus \{\bullet\}, \quad (19)$$

$\rho_{\bullet}(x, \bullet) = \infty$  for all  $x \in Z \setminus \{\bullet\}$  and  $\rho_{\bullet}(\bullet, \bullet) = 0$ .

If  $Z$  already has an infinitely distant point  $\bullet'$ , the convention is that  $\rho_{\bullet}(x, \bullet') = \frac{1}{\rho(x, \bullet)}$ .

The Cayley transform is conformal in the sense that for all admissible quadruples  $(x, y, z, w) \in Z$ ,

$$[x, y, z, w]_{\rho} = [x, y, z, w]_{\rho_{\bullet}}. \quad (20)$$

A metric  $d$  is *Ptolemaic* if its Cayley transform  $d_{\bullet}$  with respect to every point  $\bullet$  in  $Z$  is again a metric. A Möbius space  $(Z, \mathcal{M})$  for which the Cayley transform maps into itself with respect to every point  $\mathcal{M}$  is called a *Ptolemaic space*.

**Proposition 1 (Bonk, Kleiner [33])** *The Cayley transform maps  $K$ -semi-metrics to  $4K^3$ -semi-metrics.*

**Proof:** By assumption,

$$\frac{1}{K^3}d_{\bullet}(x, y) \leq \rho_{\bullet}(x, y) \leq K^3d_{\bullet}(x, y). \quad (21)$$

A slight modification of the semi-metric  $d_{\bullet}$  yields a metric by setting

$$d(x, y) = \inf_{x=x_0, x_1, \dots, x_{k-1}, x_k=y} \sum_{j=0}^k d_{\bullet}(x_j, x_{j+1}). \quad (22)$$

One checks (for details see [33] Lemma 2.3.) that

$$\frac{1}{4}d_{\bullet}(x, y) \leq d(x, y) \leq d_{\bullet}(x, y). \quad (23)$$

Thus,

$$\frac{1}{4K^3}d(x, y) \leq \rho_{\bullet}(x, y) \leq 4K^3d(x, y). \quad (24)$$

■

The conformal factor between the semi-metrics  $\rho_\bullet$  and  $\rho$  is well-defined and given by

$$\frac{\rho_\bullet}{\rho}(x) = \frac{1}{\rho(x, \bullet)^2}. \quad \forall x \in Z \setminus \{\bullet\} \quad (25)$$

Therefore, the Hausdorff measure  $\mu_{\rho_\bullet}$  of  $\rho_\bullet$  can be expressed in terms of the Hausdorff measure  $\mu_\rho$  by

$$\mu_{\rho_\bullet}(A) = \int_A \frac{1}{\rho(z, \bullet)^{2Q}} d\mu_\rho(z). \quad A \subset Z \setminus \{\bullet\} \quad (26)$$

An inverse Cayley transform from an unbounded semi-metric space to a bounded one can be given as follows.

**Definition 3** *Let  $(Z, \rho)$  be a semi-metric space with an infinite distant point  $\bullet$  in  $Z$ . The inverse Cayley transform with respect to a center  $c \in Z \setminus \{\bullet\}$  is the mapping obtained by replacing  $\rho$  by the semi-metric*

$$|\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} \quad \text{if } x, y \in Z \setminus \{\bullet\}, \quad (27)$$

$$|\rho|(x, \bullet) = |\rho|(\bullet, x) = \frac{1}{(1 + \rho(x, c))} \quad \text{for all } x \in Z \setminus \{\bullet\}, \quad \text{and } |\rho|(\bullet, \bullet) = 0.$$

The inverse Cayley transform is again conformal in the sense that for all admissible quadruples  $(x, y, z, w) \in Z$ ,

$$[x, y, z, w]_\rho = [x, y, z, w]_{|\rho|}. \quad (28)$$

**Proposition 2 (Bonk, Kleiner [33])** *The inverse Cayley transform maps  $K$ -semi-metrics to  $3K^3$ -semi-metrics.*

**Proof:** By assumption,

$$\frac{1}{K^3} \frac{d(x, y)}{(1 + d(x, c))(1 + d(y, c))} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} \leq \frac{K^3 d(x, y)}{(1 + d(x, c))(1 + d(y, c))}. \quad (29)$$

As above, a slight modification of the semi-metric  $|d|$  yields a metric by setting

$$d(x, y) = \inf_{x=x_0, x_1, \dots, x_{k-1}, x_k=y} \sum_{j=0}^k |d|(x_j, x_{j+1}). \quad (30)$$

One checks (for details see [33] Lemma 2.2.) that

$$\frac{1}{4} |d|(x, y) \leq d(x, y) \leq |d|(x, y). \quad (31)$$

So

$$\frac{1}{4K^3} d(x, y) \leq |\rho|(x, y) \leq 4K^3 d(x, y). \quad (32)$$

■



The conformal factor between the semi-metric  $|\rho|$  and  $\rho$  is well-defined and given by

$$\frac{|\rho|}{\rho}(x) = \frac{1}{(1 + \rho(x, c))^2}. \quad \forall x \in Z \setminus \{\bullet\} \quad (33)$$

Therefore, the Hausdorff measure  $\mu_{|\rho|}$  of  $|\rho|$  can be expressed by the Hausdorff measure of  $\mu_\rho$ .

$$\mu_{|\rho|}(A) = \int_{A \setminus \{\infty\}} \frac{1}{(1 + \rho(z, c))^{2Q}} d\mu_\rho(z). \quad \forall A \subset Z \quad (34)$$

The following proposition shows that the Cayley transform of a bounded semi-metric space  $(Z, \rho)$  followed by the inverse Cayley transform returns the semi-metric space  $(Z, \rho)$  up to a Bi-Lipschitz map. This justifies our naming convention.

**Proposition 3 (Buckley, Herron, Xie [34])** *Let  $(Z, \rho)$  be a bounded semi-metric space and let  $(Z, \rho_\bullet)$  be its Cayley transform with respect to a nonisolated point  $\bullet$  in  $Z$ . The inverse Cayley transform of  $(Z, \rho_\bullet)$  with respect to any center in  $Z \setminus \{\bullet\}$ , is bi-Lipschitz equivalent to  $(Z, \rho)$ .*

**Remark 2** *Qualitatively, if we assume that  $\text{diam}(Z) = 1$ , then the proof yields a bi-Lipschitz constant of  $\max\{16, 4K^4\}$ .*

**Proof:** We can assume that  $\text{diam}(Z) = 1$ . Choose  $c \in Z \setminus \{\bullet\}$  with  $\rho(c, \bullet) \geq \frac{1}{2}$ . We check that for all  $x \in Z_\bullet$ ,

$$\frac{1}{4} \rho(x, \bullet) \leq |\rho_\bullet|(x, \bullet) \leq 2K^2 \rho(x, \bullet). \quad (35)$$

The upper bound is calculated as follows,

$$\begin{aligned} |\rho_\bullet|(x, \bullet) &= \frac{1}{1 + \rho_\bullet(x, c)} \\ &= \frac{\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet)\rho(\bullet, c) + \rho(x, c)} \\ &\leq \frac{2\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet) + 2\rho(x, c)} \leq 2K^2 \rho(x, \bullet). \end{aligned} \quad (36)$$

For the lower bound,

$$\begin{aligned} |\rho_\bullet|(x, \bullet) &= \frac{1}{1 + \rho_\bullet(x, c)} \\ &= \frac{\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet)\rho(\bullet, c) + \rho(x, c)} \\ &\geq \frac{1}{2} \frac{\rho(x, \bullet)}{\rho(x, \bullet) + \rho(x, c)} \geq \frac{1}{4} \rho(x, \bullet). \end{aligned} \quad (37)$$

If  $x, y \in Z_\bullet$ , then

$$\begin{aligned}
|\rho_\bullet|(x, y) &= \frac{\rho_\bullet(x, y)}{(1 + \rho_\bullet(c, x))(1 + \rho_\bullet(c, y))} \\
&= |\rho_\bullet|(x, \bullet) \rho_\bullet(x, y) |\rho_\bullet|(y, \bullet) \\
&= \frac{|\rho_\bullet|(x, \bullet) |\rho_\bullet|(y, \bullet)}{\rho(x, \bullet) \rho(\bullet, y)} \rho(x, y) \\
&\leq 4K^4 \rho(x, y).
\end{aligned} \tag{38}$$

For the lower bound we get,

$$|\rho_\bullet|(x, y) \geq \frac{1}{16} \rho(x, y). \tag{39}$$

■

### 2.3 Uniform Ahlfors-David regularity

We have seen that associated with each Möbius structure  $\mathcal{M}$  is a family of measures  $\{\mu_d\}_{d \in \mathcal{M}}$ . This family of measures has regularity properties in the following sense.

**Theorem 2** *If for a Möbius structure  $\mathcal{M}$  there exists  $d \in \mathcal{M}$  such that  $\mu_d$  is  $Q$ -Ahlfors-David-regular, then there exists  $C \geq 1$  such that  $\{\mu_d\}_{d \in \mathcal{M}}$  is uniformly Ahlfors-David-regular with constant  $C$ .*

This observation is crucial because it often allows us to treat Möbius spaces as uniformly Ahlfors-David regular. Much of the theory of classical harmonic analysis on Ahlfors-David-regular metric spaces therefore admits a Möbius invariant analog on Möbius spaces. The above theorem follows essentially from the fact that the Cayley transformations of two metrics  $d, d'$  of a Möbius structure are  $\mathcal{M}$  are homothetic.

**Lemma 1 (Buyalo, Schroeder [26])** *Let  $(Z, \mathcal{M})$  be a Möbius space and  $d, d' \in \mathcal{M}$ , then for any point  $\bullet \in Z$  the semi-metrics  $d_\bullet$  and  $d'_\bullet$  are homothetic, viz. i.e., there exists  $\lambda > 0$  such that  $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$  for all  $x, y \in Z$ .*

**Proof:** We can assume w.l.o.g. that  $Z_\bullet$  contains two distinct points  $x$  and  $y$ . Choose  $\lambda$ , s.t.  $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$ . If there exists a third point  $z \in Z_\bullet$ , then  $[x, y, z, \bullet]_{d'_\bullet} = [x, y, z, \bullet]_{d_\bullet}$ , i.e.  $(d'_\bullet(x, y) : d'_\bullet(x, z) : d'_\bullet(y, z)) = (d_\bullet(x, y) : d_\bullet(x, z) : d_\bullet(y, z))$ . Since  $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$ , it follows that  $d'_\bullet(x, z) = \lambda d_\bullet(x, z)$  and  $d'_\bullet(y, z) = \lambda d_\bullet(y, z)$ . ■

Hausdorff measures of homothetic semi-metrics have strong regularity properties.

**Lemma 2** *Let  $(Z, \mathcal{M})$  be a Möbius space and  $d, d' \in \mathcal{M}$ , then for any point  $\bullet \in Z$  the Hausdorff measure  $\mu_{d_\bullet}$  is Ahlfors-David regular with a constant  $C \geq 1$  if and only if, if  $\mu_{d'_\bullet}$  is Ahlfors-David-regular with the same constant.*

**Proof:** We assume that  $\mu_{d'_\bullet}$  is Ahlfors-David-regular with constant  $C$ .

From the conformality of the Cayley transform it follows,

$$\frac{d'_\bullet(x, z)}{d'_\bullet(x, w)} = \frac{d_\bullet(x, z)}{d_\bullet(x, w)}. \quad (x, y, w \in Z_\bullet) \tag{40}$$

Let  $B_{d_\bullet}(x, r)$  be a ball in  $(Z, d_\bullet)$ . Without loss of generality, we can assume that  $\partial B_{d_\bullet}(x, r) \neq \emptyset$ . Otherwise  $B_{d_\bullet}(x, r)$  is closed and there exists  $0 \leq r' < r$ , s.t.  $\bar{B}_{d_\bullet}(x, r') = B_{d_\bullet}(x, r)$ ,  $\partial B_{d_\bullet}(x, r') \neq \emptyset$  and we replace  $B(x, r)$  with  $B_{d_\bullet}(x, r')$ .

Let  $w \in Z$ , s.t.  $r = d_\bullet(x, w)$ , then

$$B_{d_\bullet}(x, r) = B_{d'_\bullet}(x, d'_\bullet(x, w)) \quad (41)$$

by the symmetry condition (40). So

$$\begin{aligned} \mu_{d_\bullet}(B_{d_\bullet}(x, r)) &= \int_{B_{d_\bullet}(x, r)} \left( \frac{d_\bullet}{d'_\bullet}(y) \right)^Q d\mu_{d'_\bullet}(y) \\ &= \lambda^{-Q} \mu_{d'_\bullet}(B_{d_\bullet}(x, r)) \\ &= \lambda^{-Q} \mu_{d'_\bullet}(B_{d'_\bullet}(x, d'_\bullet(x, w))) \\ &\leq C \lambda^{-Q} d'_\bullet(x, w)^Q \\ &= C \left( \frac{d_\bullet}{d'_\bullet}(x) \right)^Q d'_\bullet(x, w)^Q \\ &= C d_\bullet(x, w)^Q \\ &= C r^Q. \end{aligned} \quad (42)$$

The lower bound is obtained in an analogous way. ■

To conclude the theorem 2, it remains to show that the Cayley transform and the inverse Cayley transform preserve the Ahlfors-David regularity.

**Proposition 4 (Li, Shanmugalingam [35])** *Let  $(Z, \rho)$  be a  $Q$ -Ahlfors-David regular semi-metric space. Then the Cayley transform  $(Z, \rho_\bullet)$  of  $(Z, \rho)$  is  $Q$ -Ahlfors-David-regular, with a constant depending only on the Ahlfors-David constant of  $(Z, \rho)$ .*

**Proof:** W.l.o.g. we can replace  $\rho$  by its true metric approximation  $d$ . Indeed, if  $(Z, \rho)$  is Ahlfors-David regular with constant  $C \geq 1$ , then so is  $(Z, d)$  with constant  $K^{2Q}C$ .

Let  $C_d$  be the Ahlfors-David constant of measure  $\mu_d$ , and let  $B_{d_\bullet}(x, r)$  be any ball in  $(Z, d_\bullet)$ . We distinguish three cases,  $rd(x, \bullet) \leq \frac{1}{2}$ ,  $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$ , and  $\frac{1}{2} < rd(x, \bullet) < 4C_d^{\frac{2}{Q}}$ .

Case 1 ( $rd(x, \bullet) \leq \frac{1}{2}$ ): We are looking for two comparable balls  $B_d$  and  $B'_d$  s.t.  $B_d \subset B_{d_\bullet}(x, r) \subset B'_d$ .

For  $y \in B_{d_\bullet}(x, r)$ ,

$$d(x, y) < rd(x, \bullet)d(\bullet, y). \quad (43)$$

Thus  $d(y, \bullet) - d(\bullet, x) \leq d(x, y) < rd(x, \bullet)d(\bullet, y)$  and a rearrangement, together with  $rd(x, \bullet) \leq \frac{1}{2}$ , gives

$$d(y, \bullet) \leq \frac{d(\bullet, x)}{1 - rd(\bullet, x)} < 2d(\bullet, x).$$

By symmetry, we also have  $d(x, \bullet) - d(\bullet, y) \leq d(x, y) < rd(x, \bullet)d(\bullet, y)$  and a rearrangement yields

$$d(y, \bullet) \geq \frac{d(\bullet, x)}{1 + rd(\bullet, x)} > \frac{2d(\bullet, x)}{3}.$$

So for  $y \in B_{d_\bullet}(x, r)$

$$\frac{2d(\bullet, x)}{3} < d(y, \bullet) < 2d(x, \bullet). \quad (44)$$

Combining equation (43) and (44) shows that we can take

$$B'_d = B_d(x, 2rd(x, \bullet)^2).$$

Conversely, let  $y \in B_d(x, \frac{2rd(x, \bullet)^2}{3})$ , then

$$d(x, \bullet) - d(\bullet, y) \leq d(x, y) < \frac{2r}{3}d(x, \bullet)^2 \leq \frac{d(x, \bullet)}{3},$$

where for the last inequality we used the assumption  $rd(x, \bullet) \leq \frac{1}{2}$ . Thus  $\frac{d(x, \bullet)}{d(y, \bullet)} \leq \frac{3}{2}$ , and therefore

$$d_\bullet(x, y) = \frac{d(x, y)}{d(x, \bullet)d(\bullet, y)} < \frac{2r}{3} \frac{d(x, \bullet)}{d(y, \bullet)} < r. \quad (45)$$

So we can set  $B_d = B_d(x, \frac{2rd(x, \bullet)^2}{3})$ .

We estimate the lower bound as follows.

$$\begin{aligned} \mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\geq \int_{B_d} \frac{d\mu_d(z)}{d(z, \bullet)^{2Q}} \\ &\geq \frac{\mu_d(B_d)}{2^{2Q}d(x, \bullet)^{2Q}} \\ &\geq \frac{(\frac{2rd(x, \bullet)^2}{3})^Q}{2^{2Q}C_d} \frac{1}{d(x, \bullet)^{2Q}} \\ &= \frac{r^Q}{6^Q C_d}. \end{aligned} \quad (46)$$

For the upper bound,

$$\begin{aligned} \mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\leq \int_{B'_d} \frac{d\mu_d(z)}{d(z, \bullet)^{2Q}} \\ &\leq \left(\frac{3}{2}\right)^{2Q} \frac{\mu_d(B'_d)}{d(x, \bullet)^{2Q}} \\ &\leq \left(\frac{3}{2}\right)^{2Q} C_d \frac{(2rd(x, \bullet)^2)^Q}{d(x, \bullet)^{2Q}} \\ &= \left(\frac{9}{2}\right)^Q C_d r^Q. \end{aligned} \quad (47)$$

Case 2: Suppose  $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$ , then in particular  $Z \setminus B_d(\bullet, \frac{4C_d^{\frac{2}{Q}}}{r}) \neq \emptyset$ .

If  $y \in Z \setminus \{B_{d_\bullet}(x, r) \cup \{\bullet\}\}$ , then

$$rd(x, \bullet)d(\bullet, y) \leq d(x, y) \leq d(x, \bullet) + d(\bullet, y). \quad (48)$$

By rearranging and using  $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$ , we obtain

$$\begin{aligned} d(y, \bullet) &< \frac{d(x, \bullet)}{rd(x, \bullet) - 1} \\ &\leq \frac{1}{r} \frac{4C_d^{\frac{2}{Q}}}{(4C_d^{\frac{2}{Q}} - 1)} < \frac{2}{r}. \end{aligned} \quad (49)$$

Thus,

$$Z \setminus \bar{B}_d(\bullet, \frac{2}{r}) \subset B_{d_\bullet}(x, r). \quad (50)$$

Conversely, if  $y \in B_{d_\bullet}(x, r)$ , then

$$d(x, \bullet) - d(\bullet, y) \leq d(x, y) \leq rd(x, \bullet)d(\bullet, y). \quad (51)$$

If we convert and use  $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$ , we get

$$\begin{aligned} d(y, \bullet) &> \frac{d(x, \bullet)}{rd(x, \bullet) + 1} \\ &\geq \frac{1}{r} \frac{4C_d^{\frac{2}{Q}}}{4C_d^{\frac{2}{Q}} + 1} > \frac{2}{3r}. \end{aligned} \quad (52)$$

Thus,

$$Z \setminus \bar{B}_d(\bullet, \frac{2}{r}) \subset B_{d_\bullet}(x, r) \subset Z \setminus \bar{B}_d(\bullet, \frac{2}{3r}). \quad (53)$$

We estimate the upper bound as follows,

$$\begin{aligned} \mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\leq \int_{Z \setminus \bar{B}_d(\bullet, \frac{2}{3r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\ &\leq \sum_{i=1}^{\infty} \int_{\bar{B}_d(\bullet, \frac{2^{i+1}}{3r}) \setminus \bar{B}_d(\bullet, \frac{2^i}{3r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\ &\leq \sum_{i=1}^{\infty} \int_{\bar{B}_d(\bullet, \frac{2^{i+1}}{3r}) \setminus \bar{B}_d(\bullet, \frac{2^i}{3r})} \frac{d\mu_d(z)}{(\frac{2^i}{3r})^{2Q}} \\ &\leq C_d \sum_{i=1}^{\infty} \frac{(\frac{2^{i+1}}{3r})^Q}{(\frac{2^i}{3r})^{2Q}} \\ &= C_d \sum_{i=0}^{\infty} 2^{-iQ} (3r)^Q \\ &= \frac{2^Q}{2^Q - 1} 3^Q C_d r^Q. \end{aligned} \quad (54)$$

There is also a lower bound, as

$$\begin{aligned} \mu_d(B_d(\bullet, \frac{4C_d^{\frac{2}{Q}}}{r}) \setminus B_d(\bullet, \frac{2}{r})) &\geq \frac{4^Q C_d}{r^Q} - \frac{2^Q C_d}{r^Q} \\ &\geq \frac{2^Q C_d}{r^Q}. \end{aligned} \quad (55)$$

Therefore, the lower bound can be estimated as follows

$$\begin{aligned}
\mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\geq \int_{Z \setminus \bar{B}_d(\bullet, \frac{2}{r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\geq \int_{B(\bullet, \frac{4C_d^{\frac{2}{Q}}}{r}) \setminus \bar{B}_d(\bullet, \frac{2}{r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\geq \frac{2^Q C_d}{r^Q} \frac{1}{(\frac{4}{r})^{2Q} C_d^4} \\
&= \frac{r^Q}{2^{3Q} C_d^3}
\end{aligned} \tag{56}$$

Case 3: Suppose  $\frac{1}{2} < rd(x, \bullet) < 4C_d^{\frac{2}{Q}}$ . Then

$$B_{d_\bullet}(x, \frac{r}{8C_d^{\frac{2}{Q}}}) \subset B_{d_\bullet}(x, r) \subset B_{d_\bullet}(x, 8C_d^{\frac{2}{Q}}r). \tag{57}$$

The inner ball satisfies the hypothesis in case 1, so

$$\frac{r^Q}{8^{2Q} 6^Q C_d^3} \leq \mu_{d_\bullet}(B_{d_\bullet}(x, \frac{r}{8C_d^{\frac{2}{Q}}})) \leq \mu_{d_\bullet}(B_{d_\bullet}(x, r)). \tag{58}$$

The outer ball satisfies the hypothesis of case 2, so

$$\mu_{d_\bullet}(B_{d_\bullet}(x, r)) \leq \mu_{d_\bullet}(B_{d_\bullet}(x, 8C_d^{\frac{2}{Q}}r)) \leq \frac{2^Q}{2^Q - 1} 3^Q 8^Q C_d^3 r^Q. \tag{59}$$

■

**Proposition 5 (Li, Shanmugalingam [35] )** *Let  $(Z, \rho)$  be an unbounded, Ahlfors-David regular semimetric space with constant  $C \geq 1$ . Then its inverse Cayley transform  $(Z, |\rho|)$  is Ahlfors-David regular with constant at most  $(2K + 1)^{2Q} 20^Q C^2$ , where  $K$  is the semimetric constant of  $|\rho|$ .*

**Proof:** Let  $\bullet$  be the point at infinity in  $Z$ , i.e.  $\rho(x, \bullet) = \infty$  for all  $x \in Z_\bullet$ . Since  $(Z, |\rho|)$  is compact, it suffices to check the Ahlfors-David regularity on balls with  $r \leq \frac{1}{2}$ .

Let  $R = \frac{1}{r} - 1$ , then  $\frac{1}{2r} \leq R < \frac{1}{r}$ . For a sphere whose center is  $\bullet$ , we have

$$\begin{aligned}
B_{|\rho|}(\bullet, r) &= \{x \in Z, \frac{1}{1 + \rho(x, c)} < r\} \cup \{\bullet\} \\
&= \{x \in Z, \rho(x, c) > \frac{1}{r} - 1\} \cup \{\bullet\} \\
&= Z \setminus \bar{B}_\rho(c, R).
\end{aligned} \tag{60}$$

Set  $\lambda = (2C_\rho^2)^{\frac{1}{Q}} > 1$ , and define  $B_i = B_\rho(c, \lambda^i R)$  then

$$\begin{aligned}
C_\rho(\lambda^{i-1}R)^Q &= (\frac{\lambda^Q}{C_\rho} - C_\rho)(\lambda^{i-1}R)^Q \\
&= \frac{(\lambda^i R)^Q}{C_\rho} - C_\rho(\lambda^{i-1}R)^Q \\
&\leq \mu_\rho(B_i \setminus B_{i-1}),
\end{aligned} \tag{61}$$

and

$$B_{|\rho|}(\bullet, r) = \bigcup_{i=1}^{\infty} B_i \setminus \bar{B}_{i-1}. \quad (62)$$

Therefore

$$\mu_{|\rho|}(B_{|\rho|}(\bullet, r)) = \sum_{i=1}^{\infty} \int_{B_i \setminus \bar{B}_{i-1}} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_{\rho}(y). \quad (63)$$

It follows,

$$\sum_{i=1}^{\infty} \frac{\mu_{\rho}(B_i \setminus \bar{B}_{i-1})}{(1 + \lambda^i R)^{2Q}} \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq \sum_{i=1}^{\infty} \frac{\mu_{\rho}(B_i \setminus \bar{B}_{i-1})}{(1 + \lambda^{i-1} R)^{2Q}} \quad (64)$$

Using  $1 \leq \lambda^i R$  and the doubling property of  $\mu_{\rho}$ ,

$$\frac{C_{\rho}}{4^Q \lambda^Q R^Q} \sum_{i=0}^{\infty} \lambda^{-iQ} \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq \frac{\lambda^Q C_{\rho}}{R^Q} \sum_{i=0}^{\infty} \lambda^{-iQ}. \quad (65)$$

So

$$\frac{C_{\rho}}{4^Q} \frac{1}{\lambda^Q - 1} r^Q \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq (2\lambda)^Q C_{\rho} \frac{\lambda^Q}{\lambda^Q - 1} r^Q. \quad (66)$$

A further simplification leads to,

$$\frac{1}{(4\lambda)^Q C_{\rho}} r^Q \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq (4\lambda)^Q C_{\rho} r^Q. \quad (67)$$

Let  $x \in Z \setminus \{\bullet\}$ , we will prove this in four cases.

Case 1: Suppose  $|\rho|(x, \bullet) \leq \frac{r}{2K}$ , we reduce this case to the one above.

If  $y \in B_{|\rho|}(x, r)$ , then

$$\begin{aligned} |\rho|(\bullet, y) &\leq K(|\rho|(\bullet, x) + |\rho|(x, y)) \\ &< (K + 1)r. \end{aligned} \quad (68)$$

So

$$B_{\rho}^c(x, r) \subset B_{\rho}^c(\bullet, (K + 1)r). \quad (69)$$

If  $y \in B_{|\rho|}(\bullet, \frac{r}{2K})$ , then

$$\begin{aligned} |\rho|(x, y) &\leq K(|\rho|(x, \bullet) + |\rho|(\bullet, y)) \\ &< r. \end{aligned} \quad (70)$$

So

$$B_{\rho}^c(\bullet, \frac{r}{2K}) \subset B_{\rho}^c(x, r). \quad (71)$$

Therefore

$$\frac{1}{(2K4\lambda)^Q C_{\rho}} r^Q \leq \mu_{|\rho|}(B_{\rho}^c(\bullet, \frac{r}{2K})) \leq \mu_{|\rho|}(B_{\rho}^c(x, r)) \leq \mu_{|\rho|}(B_{\rho}^c(\bullet, (K+1)r)) \leq ((K+1)4\lambda)^Q C_{\rho} r^Q. \quad (72)$$

Case 2: Suppose  $\frac{r}{2K} \leq |\rho|(x, \bullet) < 4r$ .

Then as above,

$$B_{|\rho|}(x, r) \subset B_{|\rho|}(\bullet, 5Kr), \quad (73)$$

and therefore

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \leq (5K4\lambda)^Q C_\rho r^Q. \quad (74)$$

For the lower bound, note that  $\frac{1}{4r} \leq 1 + \rho(x, c) \leq \frac{2K}{r}$ . If  $y \in B_\rho(x, \frac{1}{20r})$ , then

$$1 + \rho(y, c) \geq 1 + \rho(x, c) - \rho(x, y) \geq \frac{1}{4r} - \frac{1}{20r} = \frac{1}{5r}, \quad (75)$$

and therefore

$$|\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < \frac{4r \cdot 5r}{20r} = r. \quad (76)$$

So

$$B_\rho(x, \frac{1}{20r}) \subset B_{|\rho|}(x, r). \quad (77)$$

Moreover, for  $y \in B_\rho(x, \frac{1}{20r})$

$$1 + \rho(y, c) \leq 1 + \rho(x, c) + \rho(x, y) \leq \frac{2K}{r} + \frac{1}{20r} < \frac{2K+1}{r}. \quad (78)$$

All together,

$$\begin{aligned} \mu_{|\rho|}(B_{|\rho|}(x, r)) &\geq \int_{B_\rho(x, \frac{1}{20r})} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \\ &\geq \frac{r^{2Q}}{(2K+1)^{2Q}} \mu_\rho(B_\rho(x, \frac{1}{20r})) \\ &\geq \frac{1}{(2K+1)^{2Q} 20^Q C_\rho} r^Q \end{aligned} \quad (79)$$

Case 3: Suppose  $|\rho|(x, \bullet) \geq 4r$  and  $\rho(x, c) \leq 1$ . Then for all  $y \in B_{|\rho|}(x, r)$ ,

$$\frac{4r\rho(x, y)}{1 + \rho(y, c)} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < r. \quad (80)$$

It follows that

$$4(1 + \rho(y, c) - 1 - \rho(x, c)) \leq 1 + \rho(y, c). \quad (81)$$

Therefore

$$1 + \rho(y, c) \leq \frac{4}{3}(1 + \rho(x, c)) \leq 3 \quad (82)$$

since  $\rho(x, c) \leq 1$  and therefore

$$\frac{1}{6}\rho(x, y) \leq |\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))}. \quad (83)$$

That is,  $B_{|\rho|}(x, r) \subset B_\rho(x, 6r)$  and we obtain

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \leq \int_{B_\rho(x, 6r)} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \leq \mu_\rho(B_\rho(x, 6r)) \leq C_\rho 6^Q r^Q. \quad (84)$$

For the lower bound, if  $y \in B_\rho(x, r)$ , then  $|\rho|(x, y) \leq \rho(x, y) < r$  and henceforth  $B_\rho(x, r) \subset B_{|\rho|}(x, r)$ . So we get

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \geq \int_{B_\rho(x, r)} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \geq \frac{1}{3^{2Q}} \mu_\rho(B_\rho(x, r)) \geq \frac{1}{9^Q C_\rho} r^Q \quad (85)$$



Case 4: Suppose  $|\rho|(x, \bullet) \geq 4r$  and  $\rho(x, c) > 1$ .

Define

$$D(x) = \left\{ y \in Z : \frac{4}{5}(1 + \rho(x, c)) < 1 + \rho(y, c) < \frac{4}{3}(1 + \rho(x, c)) \right\}. \quad (86)$$

As before, if  $y \in B_{|\rho|}(x, r)$ , then

$$\frac{4r\rho(x, y)}{(1 + \rho(y, c))} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < r, \quad (87)$$

and

$$4\rho(x, y) < 1 + \rho(y, c). \quad (88)$$

The last equation together with  $4(1 + \rho(y, c) - 1 - \rho(x, c)) < 1 + \rho(y, c)$  and  $4(1 + \rho(x, c) - 1 - \rho(y, c)) < 1 + \rho(y, c)$  results in

$$\frac{4}{5}(1 + \rho(x, c)) < 1 + \rho(y, c) < \frac{4}{3}(1 + \rho(x, c)). \quad (89)$$

Also,

$$\rho(x, y) < r(1 + \rho(x, c))(1 + \rho(y, c)) < \frac{4}{3}(1 + \rho(x, c))^2 r. \quad (90)$$

It follows that,  $B_{|\rho|}(x, r) \subset D(x) \cap B_\rho(x, \frac{4}{3}(1 + \rho(x, c))^2 r)$ .

It follows,

$$\begin{aligned} \mu_{|\rho|}(B_{|\rho|}(x, r)) &\leq \int_{D(x) \cap B_\rho(x, \frac{4}{3}(1 + \rho(x, c))^2 r)} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \\ &\leq \left(\frac{5}{4}\right)^{2Q} \frac{1}{(1 + \rho(x, c))^{2Q}} \mu_\rho(B_\rho(x, \frac{4}{3}(1 + \rho(x, c))^2 r)) \\ &\leq \left(\frac{5}{4}\right)^{2Q} \left(\frac{4}{3}\right)^Q C_\rho r^Q = \left(\frac{25}{12}\right)^Q C_\rho r^Q. \end{aligned} \quad (91)$$

If  $y \in B_\rho(x, \frac{4}{3}(1 + \rho(x, c))^2 r)$ , then

$$\begin{aligned} \frac{4}{5}(1 + \rho(x, c)) &= 1 + \rho(x, c) - \frac{1}{5}(1 + \rho(x, c)) \\ &\leq 1 + \rho(x, c) - \frac{4}{5}(1 + \rho(x, c))^2 r \\ &\leq 1 + \rho(x, c) - \rho(x, y) \\ &\leq 1 + \rho(y, c) \\ &\leq 1 + \rho(x, c) + \rho(x, y) \\ &< 1 + \rho(x, c) + \frac{4r}{5}(1 + \rho(x, c))^2 \\ &\leq 1 + \rho(x, c) + \frac{1}{5}(1 + \rho(x, c)) \\ &< \frac{4}{3}(1 + \rho(x, c)). \end{aligned} \quad (92)$$

Therefore  $B_\rho(x, \frac{4}{5}(1 + \rho(x, c))^2 r) \subset D(x)$ . Moreover, knowing this inclusion,

$$\begin{aligned} |\rho|(x, y) &= \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} \\ &< \frac{4}{5}(1 + \rho(x, c))^2 r \frac{5}{4(1 + \rho(x, c))^2} \\ &= r. \end{aligned} \tag{93}$$

So  $B_\rho(x, \frac{4}{5}(1 + \rho(x, c))^2 r) \subset B_{|\rho|}(x, r) \subset D(x)$ . It follows that

$$\begin{aligned} \mu_{|\rho|}(B_{|\rho|}(x, r)) &\geq \int_{B_\rho(x, \frac{4}{5}(1 + \rho(x, c))^2 r)} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \\ &\geq \left(\frac{3}{4}\right)^{2Q} \frac{1}{(1 + \rho(x, c))^{2Q}} \mu_\rho(B_\rho(x, \frac{4}{5}(1 + \rho(x, c))^2 r)) \\ &\geq \left(\frac{3}{4}\right)^{2Q} \left(\frac{4}{5}\right)^Q \frac{1}{C_\rho} r^Q = \left(\frac{9}{20}\right)^Q \frac{1}{C_\rho} r^Q \end{aligned} \tag{94}$$

■

### 3 The boundary of hyperbolic groups

#### 3.1 $\delta$ -hyperbolic spaces

In the 1980s, Mikhael Gromov introduced a notion of hyperbolicity for metric spaces that remains invariant under small changes in topology. It generalizes the metric properties of classical hyperbolic geometry and of trees. The basic definitions are remarkably simple. The *Gromov product* of two points  $x$  and  $y$  with respect to an origin  $o$ , is the nonnegative number

$$\langle x, y \rangle_o = \frac{1}{2}(d(x, o) + d(y, o) - d(x, y)). \tag{95}$$

In trees,  $\langle x, y \rangle_o$  is the distance from  $o$  to the branch point of geodesics starting from  $o$  and passing through  $x$  and  $y$ .

A metric space  $(X, d)$  is called *Gromov hyperbolic* or  $\delta$ -*hyperbolic* if for any triple of points  $x, y, z$ ,

$$\langle x, y \rangle_o \geq \min\{\langle x, z \rangle_o, \langle z, y \rangle_o\} - \delta. \tag{96}$$

This condition is independent of the base point in the sense that if it holds for some  $o$  and  $\delta$ , it holds for all base points with  $2\delta$ .

#### 3.2 Gromov hyperbolic group

A group  $\Gamma$  is called *Gromov hyperbolic* or *hyperbolic* for short if it acts by isometries, properly discontinuous and cocompact on a proper, geodesic  $\delta$ -hyperbolic metric space.

#### 3.3 Gromov boundary

Gromov hyperbolic spaces have an interesting geometry at infinity, similar to the ideal boundary of classical hyperbolic space.

The construction of the boundary goes as follows. One can say that a sequence of points  $x_n$  in  $X$  'converges at infinity' if  $\langle x_i, x_j \rangle_o \rightarrow +\infty$  as  $i, j$  goes to infinity. Two such

sequences  $x_n, y_n$  are considered equivalent if  $\langle x_i, y_i \rangle_o \rightarrow +\infty$  as  $i$  goes to infinity. The Gromov boundary  $\partial X$  of  $X$  is the set of equivalence classes of such sequences.

The Gromov product can be extended to the boundary by

$$\langle \eta, \xi \rangle_o = \inf_{i,j} \liminf \langle x_i, y_j \rangle_o, \quad (97)$$

where the infimum is taken over all representatives  $x_n$  and  $y_n$  of  $\eta$  and  $\xi$ , respectively. Although the choice of  $\liminf$  does not make the above definition canonical, it turns out that replacing  $\inf$  by  $\sup$  or  $\limsup$  by  $\liminf$  leads to quantities which are all boundedly close. The above definition is therefore canonical, at least at the topological level, and defines a topology for convergence at infinity on the bordification  $X \cup \partial X$ .

The topology of convergence at infinity on  $\partial X$  is indeed metrizable. It turns out that the metric for sufficiently small  $\epsilon$  is within a constant multiple of  $\exp(-\epsilon \langle \cdot, \cdot \rangle_o)$ . We will see below that this result can be improved.

### 3.4 Roughly geodesic and strongly hyperbolic metric spaces

The classical theory of hyperbolic groups takes place in the world of geodesic metric spaces. However, the extension of the Gromov product to  $\partial X$  requires a non-canonical choice.

By smoothing the metric  $d$ , we can make the Gromov product actually extend continuously to the boundary  $\partial X$ . Moreover, the natural topology on  $\partial X$  is then induced by an explicitly given metric, the visual metric or Bourdon metric. The tradeoff is that we must move from the world of geodesic metric spaces to roughly geodesic metric spaces.

**Definition 4** *A metric space  $(X, d)$  is roughly geodesic if there exists  $C > 0$  such that there exists a rough geodesic between all  $x, y \in X$ , i.e., a mapping  $\gamma : [a, b] \subset \mathbb{R} \rightarrow X$ , with  $\gamma(a) = x$ ,  $\gamma(b) = y$  and such that for all  $t, s \in [a, b]$ ,  $|t - s| - C \leq d(\gamma(t), \gamma(s)) \leq |t - s| + C$ .*

A rough geodesic ray is a rough geodesic  $r : [0, \infty) \rightarrow X$ . Two rough geodesic rays are considered equivalent if  $\sup_t d(r(t), r'(t)) < \infty$ . An alternative construction of the Gromov boundary of a proper and roughly geodesic space  $X$  is to consider it as the set of equivalence classes of rough geodesic rays.

The relaxation of the geodesic assumption for hyperbolic spaces to roughly geodesic was first proposed by Bonk and Schramm [36]. Hyperbolicity is a quasi-isometric invariant for roughly geodesic metric spaces and the Schwarz-Milnor lemma also holds in this context. We can therefore just as well say that a group is hyperbolic if it acts geometrically on a roughly geodesic, proper hyperbolic metric space.

**Definition 5** *A metric space  $(X, d)$  is  $\epsilon$ -strongly hyperbolic if for every triple of points  $x, y, z$  and an origin  $o$  in  $X$ ,*

$$\exp(-\epsilon \langle x, y \rangle_o) \leq \exp(\langle x, z \rangle_o) + \exp(\langle z, y \rangle_o).$$

**Theorem 3 (Nica-Spakula)** *Let  $(X, d)$  be a  $\epsilon$ -strongly hyperbolic metric space, then*

1.  $(X, d)$  is  $\log(2)/\epsilon$ -hyperbolic,

2. the Gromov product extends continuously on the boundary  $X \cup \partial X$ ,
3. and the function  $d_{o,\epsilon}(\eta, \xi) = \exp(-\epsilon \langle \eta, \xi \rangle_o)$  is a metric on  $\partial X$ .

**Remark 3** *In the context of roughly geodesic metric spaces, a strongly hyperbolic metric space is strongly bolic in the sense of Lafforgue [37].*

Examples of strongly hyperbolic metric spaces are CAT(−1) spaces [32]. Indeed, any hyperbolic group  $\Gamma$  acts geometrically on a roughly geodesic and strongly hyperbolic metric space. This metric is obtained by smoothing the word metric on  $\Gamma$ .

**Theorem 4 (Nika-Spakula)** *Let  $\Gamma$  be a hyperbolic group. Then the Green metric defined by a symmetric and finitely supported random walk on  $\Gamma$  is strongly hyperbolic.*

### 3.5 A Möbius structure on strongly hyperbolic spaces.

Strongly hyperbolic metric spaces admit a *conformal structure* on their boundary. This is in contrast to only a *quasi-conformal structure* on the boundary of hyperbolic spaces. When  $X$  is a strongly hyperbolic space, the continuity of the Gromov product allows the definition of Busemann functions

$$\beta_\xi(o, o') = 2\langle \xi, o' \rangle_o - d(o, o') = \lim_{x \rightarrow \xi} (d(x, o) - d(x, o')). \quad (o, o' \in X, \xi \in \partial X) \quad (98)$$

For any pair of points  $o, o'$  in a  $\epsilon$ -strongly hyperbolic metric space,

$$d_{o,\epsilon}(\eta, \xi) = \exp\left(\frac{\epsilon}{2}(\beta_\eta(o, o') + \beta_\xi(o, o'))\right) d_{o',\epsilon}(\eta, \xi). \quad (99)$$

This implies that for any admissible quadruple  $(x, y, z, w)$ , the cross-ratio triple  $[x, y, z, w]_{d_{o,\epsilon}}$  does not depend on  $o$ , and  $\{d_o\}_{o \in X}$  is a Möbius structure on  $\partial X$ .

### 3.6 Petterson-Sullivan measures

We have seen that the ideal boundary of a strongly hyperbolic metric space  $X$  is a Möbius space if it is endowed with the set of visual metrics  $\mathcal{M} = \{d_{o,\epsilon}\}_{o \in X}$ . The Hausdorff measure  $\mu_o$  associated with the metric  $d_{o,\epsilon}$  is called the *Petterson-Sullivan* measure [38]. Any such measure associated with the Möbius structure  $\mathcal{M}$  has the same Hausdorff dimension  $Q$ . On strongly hyperbolic spaces, the Radon-Nikodým derivatives  $\frac{d\mu_{o'}}{d\mu_o}$  have a concrete expression in terms of Busemann functions,

$$\frac{d\mu_{o',\epsilon}}{d\mu_{o,\epsilon}}(x) = e^{\epsilon Q \beta_x(o, o')}. \quad (100)$$

Petterson-Sullivan measures are Ahlfors-David regular [39]. As we have seen, even more holds in the context of Möbius geometry. There is a constant  $C \geq 1$  such that for any measure  $\mu_o \in \{\mu_o\}_{o \in X}$ ,

$$\frac{1}{C} r^Q \leq \mu_o(B(x, r)) \leq C r^Q. \quad (101)$$

### 3.7 Sullivan's shadow lemma.

Let  $X$  be a hyperbolic group with a left-invariant hyperbolic metric that is quasi-isometric to a word metric. We can assume that for every origin  $o$  and every other point  $o' \in X$  there is a roughly geodesic ray  $\gamma$  starting from  $o$  and extending at uniform distance from  $o'$ . With respect to the Gromov product, we can assume the existence of a constant  $M > 0$  such that for all  $o' \in X$ ,

$$\sup_{x \in \partial X} \langle x, o' \rangle_o \geq d(o, o') - M. \quad (102)$$

The constant  $M$  can be chosen independently of the choice of an origin  $o$ .

**Lemma 3 (Garncarek [40])** *If  $X$  is a  $\delta$ -Gromov hyperbolic group, then there exists  $M > 0$  such that for all  $o'$  and all  $o$  in  $X$ ,*

$$\sup_{x \in \partial X} \langle x, o' \rangle_o \geq d(o, o') - M. \quad (103)$$

Since for each  $o'$  in  $X$  there is a roughly geodesic ray  $\gamma$  starting from  $o$  and passing near  $o'$ , we can assign it a front endpoint  $\bullet'$ . More precisely, let  $\bullet'$  be a point in  $\partial X$  such that

$$\langle \bullet', o' \rangle_o \geq d(o, o') - M. \quad (104)$$

**Proof:** (of Garncarek's lemma) For any point  $x \in X$ , the Gromov product satisfies

$$\langle g^{-1}o, x \rangle_o + \langle go, gx \rangle_o = d(o, g^{-1}o).$$

Let  $x \in \partial X$  and  $\{x_i\}$  be a representative of  $x$ , then

$$d(o, g^{-1}o) = \liminf_{i \rightarrow \infty} (\langle g^{-1}o, x_i \rangle_o + \langle go, gx_i \rangle_o) \leq \langle g^{-1}o, x \rangle_o + \langle go, gx \rangle_o + 4\delta.$$

So

$$\langle go, gx \rangle_o \geq d(o, g^{-1}o) - \langle g^{-1}o, x \rangle_o - 4\delta.$$

and for two different points  $y_1, y_2 \in \partial X$ ,

$$\begin{aligned} \sup_{x \in \partial X} \langle x, go \rangle_o &\geq \max_i \langle go, gy_i \rangle_o \\ &\geq d(o, g^{-1}o) - \min\{\langle g^{-1}o, y_1 \rangle_o, \langle g^{-1}o, y_2 \rangle_o\} - 4\delta \\ &\geq d(o, go) - \langle y_1, y_2 \rangle_o - 5\delta. \end{aligned}$$

Set  $M = \inf_{y_1, y_2 \in \partial X} \langle y_1, y_2 \rangle_o + 5\delta$  ■

A ball of radius  $s$  around a point  $o'$  in  $X$  casts a shadow at infinity as seen from  $o$ . Explicitly, the shadow at infinity of a ball of radius  $s$  around  $o'$  can be defined as

$$\mathcal{O}_o^{o'}(s) = \{x \in \partial X : \langle x, o' \rangle_o \geq d(o, o') - s\} \quad (105)$$

The shadows at infinity are essentially balls around the endpoint of a rough geodesic ray passing almost through  $o'$ .

**Lemma 4 (Sullivan's shadow lemma)** *Let  $X$  be a strongly hyperbolic space. Let  $\bullet'$  be a point on the boundary for which holds*

$$\langle \bullet', o' \rangle_o \geq d(o, o') - M.$$

Then for all  $s > M + \delta$

$$\bar{B}_o(\bullet', e^{-\epsilon(d(o, o') - s + \delta)}) \subset \mathcal{O}_o^{o'}(s) \subset \bar{B}_o(\bullet', e^{-\epsilon(d(o, o') - s - \delta)}).$$

**Proof:** If  $x \in \mathcal{O}_o^{o'}(s)$ , then

$$\begin{aligned} \langle \bullet', x \rangle_o &\geq \min\{\langle \bullet', o' \rangle_o, \langle o', x \rangle_o\} - \delta \\ &\geq \min\{d(o, o') - M, d(o, o') - s\} - \delta \\ &= d(o, o') - s - \delta. \end{aligned} \tag{106}$$

Therefore  $d_{o, \epsilon}(\bullet', x) \leq e^{-\epsilon(d(o, o') - s - \delta)}$ . Conversely, if  $x \in \bar{B}_o(\bullet', e^{-\epsilon(d(o, o') - s + \delta)})$ , then

$$\begin{aligned} \langle o', x \rangle_o &\geq \min\{\langle o', \bullet' \rangle_o, \langle \bullet', x \rangle_o\} - \delta \\ &\geq \min\{d(o, o') - M, d(o, o') - s + \delta\} - \delta \\ &= d(o, o') - s. \end{aligned} \tag{107}$$

The equality in the last line follows from the assumption  $s > M + \delta$ . ■

### 3.8 Distances associated to boundary points, the Cayley transform revisited

Let  $X$  be a strongly hyperbolic metric space and  $Z = \partial X$  its ideal boundary endowed with its natural Möbius structure, i.e., a family of distances  $d_o$  indexed by points  $o$  of  $X$ :

$$d_{o, \epsilon}(x, y) = e^{-\epsilon \langle x, y \rangle_o}.$$

For points  $p, q \in X$ ,

$$\langle p, q \rangle_o = \frac{1}{2}(d(o, p) + d(o, q) - d(p, q)).$$

One then shows that  $\langle p, q \rangle_o$  has a limit denoted by  $\langle x, y \rangle_o$  when  $p$  tends to  $x$  and  $q$  to  $y$ .

**Proposition 6** *Let  $x, y \in \partial X$ . As  $p \in X$  tends to a point  $z \in \partial X \setminus \{x, y\}$ ,*

$$e^{d(o, p)} d_p(x, y) \tag{108}$$

tends to a number denoted by  $d_{z, o}(x, y)$ , given by

$$d_{z, o}(x, y) = \frac{d_o(x, y)}{d_o(z, x)d_o(z, y)}. \tag{109}$$

As  $o$  is changed,  $d_{z, o}$  is multiplied by a positive constant,

$$d_{z, o'} = e^{\beta_z(o', o)} d_{z, o}. \tag{110}$$

Indeed, let  $q, q' \in X$  be close to  $x, y \in \partial X$ .

$$\begin{aligned}
\langle q, q' \rangle_p - \langle q, q' \rangle_o - d(o, p) &= \frac{1}{2}(d(p, q) + d(p, q') - d(q, q')) \\
&\quad - \frac{1}{2}(d(o, q) + d(o, q') - d(q, q')) - d(o, p) \\
&= \frac{1}{2}(d(p, q) - d(o, q) - d(o, p)) \\
&\quad + \frac{1}{2}(d(p, q') - d(o, q') - d(o, p)) \\
&= -\langle p, q \rangle_o - \langle p, q' \rangle_o,
\end{aligned} \tag{111}$$

which tends to  $-\langle z, x \rangle_o - \langle z, y \rangle_o$  as  $p \rightarrow z, q \rightarrow x, q' \rightarrow y$ . So

$$e^{-d(o,p)} \frac{d_o(x, y)}{d_p(x, y)} \tag{112}$$

tends to  $d_o(z, x)d_o(z, y)$  when  $p$  tends to  $z$ . This gives the formula

$$d_{z,o}(x, y) = \frac{d_o(x, y)}{d_o(z, x)d_o(z, y)}. \tag{113}$$

Furthermore,

$$\begin{aligned}
\frac{d_{z,o'}(x, y)}{d_{z,o}(x, y)} &= \lim_{p \rightarrow z} \frac{e^{d(o',p)} d_p(x, y)}{e^{d(o,p)} d_p(x, y)} \\
&= \lim_{p \rightarrow z} e^{d(o',p) - d(o,p)} \\
&= e^{\beta_z(o',o)}.
\end{aligned} \tag{114}$$

Although the semimetric  $d_{z,o}$  is undefined at  $z$ , it belongs to the Möbius class in an extended sense: by the convention that  $d_{z,o}(z, x) = +\infty$  for all  $x \neq z$ , the semimetric defines the same cross ratios.

The metric associated with an ideal boundary point is defined only up to a multiplicative constant. The same is true for the Hausdorff  $Q$ -measure.

**Corollary 1** *The rescaled Hausdorff  $Q$ -measure  $e^{Q d(o,p)} \mu_p$  converges setwise on  $\partial X \setminus \{z\}$  when  $p \in X$  tends to  $z \in \partial X$ , to the Hausdorff  $Q$ -measure  $\mu_{z,o}$  belonging to  $d_{z,o}$ , which satisfies, for  $x \neq z$ ,*

$$\frac{d\mu_{z,o}}{d\mu_o}(x) = (d_o(z, x))^{-2Q}.$$

As  $o$  is changed,  $\mu_{z,o}$  is multiplied by a positive constant,

$$\mu_{z,o'} = e^{Q \beta_z(o',o)} \mu_{z,o}.$$

Indeed, let  $B$  be such a ball that  $z$  is not a limitpoint of  $B$ . W.l.o.g. suppose the points  $p$  are far enough from  $o$  such that  $B \subset \partial X \setminus \mathcal{O}_o^p(M + 2\delta)$  for all points  $p$  in the sequence, and the endpoints  $\bar{p}$  of rays starting from  $o$  and passing within uniform distance from  $p$  are all  $\epsilon$  close to  $z$ . Then

$$e^{d(o,p)} \frac{d_p}{d_o}(x) \tag{115}$$

is dominated by

$$\frac{e^{2\delta}}{e^{-\langle x, z \rangle_o} - \epsilon}, \quad (116)$$

which is integrable on  $B$ . Thus, by dominated convergence,

$$\lim_{p \rightarrow z} e^{Qd(o,p)} \mu_p(B) = \int_B d_o(z, x)^{-2Q} d\mu_o(x). \quad (117)$$

**Example 3** *Let  $X$  be a symmetric space of rank 1. Then  $d_{z,o}$  is a left-translation invariant dilation-homogeneous distance on the unipotent radical  $N$  of  $\text{Isom}(X)$ . If  $X$  has constant curvature,  $d_{z,o}$  is a Euclidean distance.*

*Indeed,  $d_{z,o}$  is preserved by the stabilizer of  $z$  and the horosphere passing through  $o$  in  $\text{Isom}(X)$  containing  $N$ , and  $N$  acts simply transitively on  $\partial X \setminus \{z\}$ . The dilations of  $N$  are realized by the 1-parameter subgroup  $A$  of  $\text{Isom}(X)$ , whose elements each multiply  $d_{z,o}$  by a constant. When  $X$  has constant curvature, the stabilizer of  $z$  in  $\text{Isom}(X)$  is two-point transitive on  $\partial X \setminus \{z\}$  such that  $d_{z,o}$  is a function of Euclidean distance. By dilaton homogeneity, this function is homogeneous of degree 1, so  $d_{z,o}$  is a constant multiple of the Euclidean distance.*

## 4 Harmonic analysis on the boundary of hyperbolic groups

### 4.1 Introduction

A central tool of classical harmonic analysis on  $\mathbb{R}^n$  are maximal functions. Maximal functions allow to obtain quantitative results on the average of functions.

The best known of all maximal functions is the Hardy-Littlewood maximal function.

**Definition 6** *The Hardy-Littlewood maximal function of  $f$ , is*

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy \quad (118)$$

This operator behaves well on  $L^p$ -spaces.

**Theorem 5** *The Hardy-Littlewood maximal operator maps  $L^1(\mathbb{R}^n)$  to  $L^{1,\infty}(\mathbb{R}^n)$  with constant at most  $3^n$  and also  $L^p(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$  for  $1 < p < \infty$  with constant at most  $3^{n/p} p(p-1)^{-1}$  [41].*

**4.1.1 Weighted inequalities** The study of weighted inequalities for maximal functions appeared as a natural extension of the theory. A main question is the characterization of all measures  $\mu$  on  $\mathbb{R}^n$  such that

$$\int_{\mathbb{R}^n} M(f)(x)^p d\mu(x) \leq A \int_{\mathbb{R}^n} |f(x)|^p d\mu(x), \quad (119)$$

for any  $p$ ,  $1 < p < \infty$ .

The complete solution was given 50 years ago by Muckenhoupt [42], who set up a class of functions  $A_p$  such that (119) and other similar inequalities, hold exactly when  $d\mu(x) = w(x)^p dx$  with  $w \in A_p$ .



Weighted inequalities for singular integrals are another source of examples where Muckenhoupt's  $A_p$  condition plays a central role. If  $T$  is a singular integral operator of convolution type with some regularity assumptions for the kernel, then

$$\int_{\mathbb{R}^n} |Tf(x)|^p d\mu(x) \leq A \int_{\mathbb{R}^n} |f(x)|^p d\mu(x), \quad (120)$$

if and only if  $d\mu(x) = w(x)^p dx$ , with  $w \in A_p$ .

## 4.2 Muckenhoupt weights

It turns out that the class of measures  $\mu$  consists exactly of those measures which are absolutely continuous with respect to the Lebesgue measure  $d\mu(x) = w(x)^p dx$ , and that  $w$  satisfies the  $A_p$  inequality

$$\llbracket w \rrbracket_p = \sup_B \left( \frac{1}{|B|} \int_B w^p dx \right)^{\frac{1}{p}} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{p'} dx \right)^{\frac{1}{p'}} < \infty, \quad (121)$$

**Remark 4** *In the literature, one often finds a slightly different but equivalent definition of the class  $A_p$ . It amounts to setting  $v = w^p$  and defining  $A_p$  as the class of functions  $v$  such that  $(\llbracket v^{\frac{1}{p}} \rrbracket_p)^p$  is finite.*

**Example 4** *The function  $|x|^\alpha$  belongs to  $A_p$  on  $\mathbb{R}^n$ , if and only if  $-n < \alpha p < n(p-1)$ .*

Below we list some simple but useful observations on general properties of Muckenhoupt weights.

**4.2.1** Muckenhoupt weights admit an obvious "duality". If  $w \in A_p$ , then  $w^{-1} \in A_{p'}$  with the same Muckenhoupt constant.

**4.2.2** An important property of Muckenhoupt classes is that they are nested up to some renormalization. If  $w \in A_p$ , then  $w^{\frac{p}{q}}$  is in  $A_q$  for any  $q > p$ . Indeed,

$$(\llbracket w^{\frac{p}{q}} \rrbracket_q)^{\frac{q}{p}} = \sup_B \left( \frac{1}{|B|} \int_B w^p dx \right)^{\frac{1}{p}} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{\frac{pq'}{q}} dx \right)^{\frac{q}{pq'}}, \quad (122)$$

is bounded by a multiple of  $\llbracket w \rrbracket_p$ . This follows from Hölder's inequality and the fact that  $q'/q = q' - 1 < p' - 1 = p'/p$ .

**4.2.3** There is an alternative definition of the class  $A_p$  which is more closely related in notation to Hardy-Littlewood's maximal function. For any locally integrable function  $f$ , we denote by  $f_B$  the average of  $f$  over a ball  $B$ , i.e.,  $f_B = \frac{1}{|B|} \int_B f(x) dx$ . We further denote by  $w^p(B)$  the number  $\int_B w(x)^p dx$ . Then  $w$  belongs to the class  $A_p$  if and only if the  $p$ -th power of  $f_B$  is controlled by the mean of  $f^p$  with respect to the measure  $w(x)^p dx$ . Specifically, this means,

$$(f_B)^p \leq \frac{c}{w^p(B)} \int_B f^p w^p dx \quad (123)$$

holds for all nonnegative functions  $f$  and all balls  $B$ . The minimal constant  $c$  for which this inequality is true coincides with  $(A_p(w))^{\frac{1}{p}}$ .

Suppose  $w \in A_p$ .

$$f_B = \frac{1}{|B|} \int_B f(x) dx = \frac{1}{|B|} \int_B f w w^{-1} dx. \quad (124)$$

We apply Hölder's inequality to the right-hand side with exponent  $p$ .

$$f_B \leq \frac{1}{|B|} \left( \int_B f^p w^p dx \right)^{\frac{1}{p}} \left( \int_B w^{-p'} dx \right)^{\frac{1}{p'}}. \quad (125)$$

Together with Muckenhoupt's inequality (121) establishes the above inequality (123), with minimal constant  $c \leq (A_p(w))^{\frac{1}{p}}$ . Conversely, if (123) holds for  $w$ , we take  $f$  to be  $(w + \epsilon)^{-p'}$ . Then (123) together with  $(1 - p')p = -p'$  implies that

$$\frac{1}{|B|} \left( \int_B (w(x) + \epsilon)^{-p'} dx \right)^{p-1} w^p(B) \leq c. \quad (126)$$

Taking the  $p$ -th root and passing to the limit  $\epsilon \rightarrow 0$ , we obtain Muckenhoupt's inequality (121).

**4.2.4** Any absolutely continuous measure with respect to Lebesgue's measure,  $d\mu(x) = w(x)^p dx$ , is doubling if  $w \in A_p$ . A measure  $\mu$  is doubling if there is a constant  $D > 0$ , s.t.

$$\mu(2B) \leq D\mu(B). \quad (127)$$

If  $w \in A_p$ , then the fact that the measure  $w^p$  is doubling, follows directly from (123) if we set  $B = 2B'$  and  $f = \chi_{B'}$ . The doubling constant is explicitly given by  $c2^{np}$ .

### 4.3 Generalized Muckenhoupt weights.

Muckenhoupt and Wheeden [43] generalized the class  $A_p$  to the operator

$$I^\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy, \quad (128)$$

for  $0 < \alpha < n$ . It is often called a fractional integration operator on  $\mathbb{R}^n$ . For  $\alpha = 0$  it is a Calderon-Zygmund singular integral.

Let  $1 < p < \frac{n}{\alpha}$ , and  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ , then

$$\left( \int_{\mathbb{R}^n} |I^\alpha(f)(x)|^q w(x)^q dx \right)^{\frac{1}{q}} \leq A \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)^p dx \right)^{\frac{1}{p}}, \quad (129)$$

for a nonnegative function  $w$  if and only if,

$$\llbracket w \rrbracket_{q,p} = \sup_B \left( \frac{1}{|B|} \int_B w^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{p'} dx \right)^{\frac{1}{p'}} < \infty. \quad (130)$$

We say that the weight  $w$  belongs to the class  $A_{q,p}$ .

#### 4.4 Muckenhoupt weights associated with Möbius structures

Let  $X$  be a strongly hyperbolic metric space with natural Möbius structure  $\{d_o\}_{o \in X}$  and a fixed origin  $o \in X$ . In this section we will show that powers of the conformal factors

$$\frac{d_{o'}}{d_o}(x) = e^{\beta_x(o,o')}, \quad (131)$$

are Muckenhoupt weights with respect to  $\mu_o$ , with uniformly bounded  $A_{p,q}$  constant.

Indeed, we can admit  $o$  and or  $o'$  as ideal points. This gives a Muckenhoupt theory on Möbius structures which is invariant under the Cayley transform and therefore treats the compact and noncompact picture as equivalent to each other.

The definitions and notations introduced above for  $\mathbb{R}^n$  can be readily generalized to this situation by replacing the Lebesgue measure  $dx$  by the Petterson-Sullivan measure  $\mu_o$ . In fact,  $\mu_o$  is Ahlfors-David regular and thus doubling. The remarks made above about the properties of  $A_p$ -weights on  $\mathbb{R}^n$  generalize easily to this setting as well and will not be repeated here.

**Theorem 6** *There exists a constant  $C > 0$ , s.t. for all  $o, o' \in X$ , and  $1 < p \leq q < \infty$ ,  $\alpha < \frac{Q}{q}$ ,*

$$\left\| \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \right\|_{p,q} = \sup_B \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C < \infty. \quad (132)$$

Clearly, for any pair of points  $o, o'$  and  $1 < p \leq q < \infty$ , the weight  $\left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}}$  is a Muckenhoupt weight of class  $A_{p,q}$  since the conformal factors  $\frac{d_{o'}}{d_o}$  are bounded above and away from 0. The significance comes from the fact that the bound on the  $A_{p,q}$  constant is independent of  $o$  and  $o'$ .

We denote by  $A(\mathcal{M})_{p,q}^\alpha$  or simply  $A(\mathcal{M})$  or  $A$  the minimal  $C > 0$ , i.e. the above inequality is true for all  $o, o' \in X$ .

The proof shows that the constant  $A(\mathcal{M})_{p,q}^\alpha$  cannot be finite for  $\alpha = \frac{Q}{q}$ . This will become relevant again when we discuss bounded representations and the limit when  $\alpha$  tends to  $\frac{Q}{q}$ .

Theorem 6 states in particular that  $\left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}}$  is of class  $A_q$ , if  $\alpha < \frac{Q}{2}$ , and by the duality property of the Muckenhoupt weights,  $\left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha}{2}}$  is of class  $A_{p'}$ . The following corollaries follow from the doubling property of the Muckenhoupt weights.

**Corollary 2** *For all  $d_o \in \mathcal{M}$ ,  $\alpha < \frac{Q}{q}$ , the measure*

$$U \mapsto \int_U \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o$$

*is doubling with a doubling constant  $DA^{\frac{1}{q}}$ , where  $D$  is the doubling constant of the measures belonging to  $\mathcal{M}$ .*

**Corollary 3** *For all  $d_o \in \mathcal{M}$ ,  $\alpha < \frac{Q}{2p}$ , the measure*

$$U \mapsto \int_U \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o$$

is doubling with a doubling constant  $DA^{\frac{1}{p}}$ , where  $D$  is the doubling constant of the measures belonging to  $\mathcal{M}$ .

**Proof:** (of the theorem 6) Let  $B$  be a ball in  $Z$  with center  $z$  and radius  $r$ , and  $o$  and  $o'$  two points in  $X$ .

Let  $\infty$  be a front endpoint in  $Z$  of a geodesic starting from  $o$  and passing through  $o'$ . We denote by  $|x|_o$  the distance of a point  $x$  in  $Z$  to  $\infty$ , with respect to the metric  $d_o$ .

We denote by  $B^s = B(\infty, e^\delta e^{-(d(o,o')-s)})$ , and by  $r(s)$  the radius of  $B^s$ . Sullivan's shadow lemma states that the shadow  $\mathcal{O}'_o(s)$  is contained in  $B^s$  if  $s \geq M + 2\delta$ .

We will examine the 3 cases separately. Choose a positive integer  $n \geq 2$ ,

1. Case 1:  $|z| \geq (1 + \frac{1}{n})r$ ,
2. Case 2:  $|z| \leq (1 - \frac{1}{n})r$ ,
3. Case 3:  $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$ .

Case 1: For technical reasons, we distinguish two subcases:

1. Case 1A:  $|z| \geq (1 + \frac{1}{n})r$  and  $B \cap \mathcal{O}'_o(M + 2\delta) = \emptyset$ ,
2. Case 1B:  $|z| \geq (1 + \frac{1}{n})r$  and  $B \cap \mathcal{O}'_o(M + 2\delta) \neq \emptyset$ .

In case 1A, we have the nice advantage of being able to replace  $e^{-\langle x, o' \rangle_o}$  with the true distance  $|x|_o$ . Indeed, for all  $x$  in  $B$ ,

$$e^{-2\delta\alpha q} |x|^{-\alpha q} \leq e^{\alpha q \langle x, o' \rangle_o} \leq e^{2\delta\alpha q} |x|^{-\alpha q}, \quad (133)$$

and

$$e^{-2\delta\alpha p'} |x|^{\alpha p'} \leq e^{-\alpha p' \langle x, o' \rangle_o} \leq e^{2\delta\alpha p'} |x|^{\alpha p'}. \quad (134)$$

This follows from the following.  $B \cap \mathcal{O}'_o(M + 2\delta) = \emptyset$  is equivalent to

$$\forall x \in B, \quad \langle x, o' \rangle_o < d(o, o') - M - 2\delta. \quad (135)$$

So

$$\begin{aligned} \langle x, \bullet \rangle_o &\geq \min\{\langle x, o' \rangle_o, \langle o', \bullet \rangle_o\} - 2\delta \\ &\geq \min\{\langle x, o' \rangle_o, d(o, o') - M\} - 2\delta \\ &\geq \langle x, o' \rangle_o - 2\delta. \end{aligned} \quad (136)$$

The converse follows from the fact that  $B$  does not intersect the ball  $B(\bullet, e^{-d(o,o')-M})$  either. So

$$\forall x \in B, \quad \langle x, \bullet \rangle_o \leq d(o, o') - M. \quad (137)$$

And therefore

$$\begin{aligned} \langle x, o' \rangle_o &\geq \min\{\langle x, \bullet \rangle_o, \langle \bullet, o' \rangle_o\} - 2\delta \\ &\geq \min\{\langle x, \bullet \rangle_o, d(o, o') - M\} - 2\delta \\ &\geq \langle x, \bullet \rangle_o - 2\delta. \end{aligned} \quad (138)$$

This proves the comparison claim.

This allows us to use the triangle inequality to obtain upper bounds. Since  $|z| \geq (1 + \frac{1}{n})r$ ,

$$\forall x \in B : \quad |x| \geq |z| - 2r \geq |z| - \frac{n}{n+1}|z| = \frac{|z|}{n+1}, \quad (139)$$

and

$$\forall x \in B, \quad |x| \leq |z| + r \leq |z| + \frac{n}{1+n}|z| = \frac{2n+1}{n+1}|z|. \quad (140)$$

Thus,

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o &= e^{-\frac{\alpha q}{2}d(o,o')} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{-\frac{\alpha q}{2}d(o,o')} e^{2\delta \alpha q} \int_B |x|^{-\alpha q} d\mu_o \\ &\leq e^{-\frac{\alpha q}{2}d(o,o')} e^{2\delta \alpha q} (n+1)^{\alpha q} |z|^{-\alpha q} \mu_o(B), \end{aligned} \quad (141)$$

and

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o,o')} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{\frac{\alpha p'}{2}d(o,o')} e^{2\delta \alpha p'} \int_{2B} |x|^{\alpha p'} d\mu_o \\ &\leq e^{\frac{\alpha p'}{2}d(o,o')} e^{2\delta \alpha p'} \frac{(2n+1)^{\alpha p'}}{(n+1)^{\alpha p'}} |z|^{\alpha p'} \mu_o(B). \end{aligned} \quad (142)$$

This implies,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{4\delta \alpha} (2n+1)^\alpha \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}, \quad (143)$$

which proves the case 1A.

In case 1B,  $B \cap \mathcal{O}'_o(M+2\delta) \neq \emptyset$ . The shadow  $\mathcal{O}'_o(M+2\delta)$  is contained in the ball  $B^{M+2\delta}$ , with  $r_{(M+2\delta)} = e^{-(d(o,o')-M-3\delta)}$ . That  $B$  intersects the shadow  $\mathcal{O}'_o(M+2\delta)$  implies that the radius  $r$  is comparable to  $r_{(M+2\delta)}$ . More precisely, since  $|z| \geq (1 + \frac{1}{n})r$ ,

$$r \leq n(|z| - r) \leq n r_{(M+2\delta)}. \quad (144)$$

This implies that,

$$B \subset B(\infty, r_{(M+2\delta)} + 2r) \subset B(\infty, (1+2n)r_{(M+2\delta)}) \subset \mathcal{O}'_o(M + \log(1+2n) + 4\delta). \quad (145)$$

From this follows,

$$\forall x \in B : \quad \langle x, o' \rangle_{o'} \geq d(o, o') - (M + \log(1+2n) + 4\delta). \quad (146)$$

We use this inequality for the upper bound of the  $p'$  measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o,o')} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{-\frac{\alpha p'}{2}d(o,o')} e^{\alpha p'(M + \log(1+2n) + 4\delta)} \mu_o(B) \end{aligned} \quad (147)$$

The corresponding upper bound for the  $q$ -measure is straightforward.

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B). \quad (148)$$

This implies,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{\alpha(M+4\delta)} (1+2n)^\alpha \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}, \quad (149)$$

which completely proves the case 1.

Case 2: Let  $S_+$  be the set of all  $s \in \mathbb{R}^+$ , such that  $B \subset \mathcal{O}_o'(s)$ . Let  $s_+^*$  be the infimum of  $S_+$ , and

$$s_+ := d(o, o') - \delta + \log(|z| + r). \quad (150)$$

Let  $S_-$  be the set of all  $s \in \mathbb{R}^+$ , such that  $\mathcal{O}_o'(s) \subset B$ . Note that if  $s \in S_-$ , then for all  $s' \leq s$ ,  $s' \in S_-$ . Let  $s_-^*$  be the supremum of  $S_-$ .

Recall that we denoted by  $B^s$  the ball  $B(\infty, e^\delta e^{-(d(o,o')-s)})$  and by  $r(s)$  its radius.

For technical reasons, we again distinguish two subcases:

1. Case 2A:  $|z| \leq (1 - \frac{1}{n})r$  and  $\mathcal{O}_o'(M+2\delta) \subset B$ ,
2. Case 2B:  $|z| \leq (1 - \frac{1}{n})r$  and  $\mathcal{O}_o'(M+2\delta) \not\subset B$ .

In case 2A,  $s_+ > M+2\delta$  and Sullivan's shadow lemma implies  $s_+ \in S_+$ . Thus,

$$M+2\delta \leq s_+^* \leq s_+. \quad (151)$$

For the upper bound of  $\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o$ , we first note that

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq \int_{\mathcal{O}_o'(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o. \quad (152)$$

The right-hand side has an explicit decomposition by integration by parts.

$$\begin{aligned} \int_{\mathcal{O}_o'(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o &= e^{\frac{\alpha q}{2}d(o,o')} \int_0^\infty \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}_o'(s) \cap \mathcal{O}_o'(s_+)) ds \\ &= e^{\frac{\alpha q}{2}d(o,o')} \int_0^{M+2\delta} \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}_o'(s)) ds \\ &\quad + e^{\frac{\alpha q}{2}d(o,o')} \int_{M+2\delta}^{s_+} \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}_o'(s)) ds \\ &\quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(\mathcal{O}_o'(s_+)) \end{aligned} \quad (153)$$

Further calculations yield,

$$\begin{aligned}
\int_{\mathcal{O}'_o(s_+)} \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o &\leq e^{\frac{\alpha q}{2}d(o,o')} \int_0^{M+2\delta} \alpha q e^{-\alpha q s} \mu_o(B^{M+2\delta}) ds \\
&\quad + e^{\frac{\alpha q}{2}d(o,o')} \int_{M+2\delta}^{s_+} \alpha q e^{-\alpha q s} \mu_o(B^s) ds \\
&\quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\
&\leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B^{M+2\delta}) \\
&\quad + C e^{\delta Q} e^{-Qd(o,o')} e^{\frac{\alpha q}{2}d(o,o')} \int_{M+2\delta}^{s_+} \alpha q e^{(Q-\alpha q)s} ds \\
&\quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\
&\leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B^{M+2\delta}) \\
&\quad + C e^{\delta Q} e^{-Qd(o,o')} e^{\frac{\alpha q}{2}d(o,o')} \left(\frac{\alpha q}{Q-\alpha q}\right) e^{(Q-\alpha q)s_+} \\
&\quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\
&= e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} e^{-\alpha q(M+2\delta)} r_{(M+2\delta)}^{-\alpha q} \mu_o(B^{M+2\delta}) \\
&\quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q} \\
&\quad + e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(s_+)}^{-\alpha q} \mu_o(B^{s_+}) \\
&\leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(M+2\delta)}^{Q-\alpha q} \\
&\quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q} \\
&\quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(s_+)}^{Q-\alpha q} \\
&\leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q-\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q}
\end{aligned} \tag{154}$$

Thus,

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q-\alpha q}{Q-\alpha q}\right) (|z|+r)^{Q-\alpha q}, \tag{155}$$

and using the assumption  $|z| \leq (1 - \frac{1}{n})r$ ,

$$\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q-\alpha q}{Q-\alpha q}\right) \left(\frac{2n+1}{n}\right)^{Q-\alpha q} r^{Q-\alpha q}. \tag{156}$$

The upper bound for the  $p'$  measure is much simpler.

$$\begin{aligned}
\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{-\frac{\alpha p'}{2}d(o,o')} \int_{-\infty}^{d(o,o')} \alpha p' e^{\alpha p' s} \mu_o(B \setminus \mathcal{O}'_o(s)) ds \\
&= e^{-\frac{\alpha p'}{2}d(o,o')} \int_{-\infty}^{s_+^*} \alpha p' e^{\alpha p' s} \mu_o(B \setminus \mathcal{O}'_o(s)) ds \\
&\leq e^{-\frac{\alpha p'}{2}d(o,o')} e^{\alpha p' s_+} \mu_o(B) \\
&= e^{\frac{\alpha p'}{2}d(o,o')} e^{-\alpha p' \delta} (|z|+2r)^{\alpha p'} \mu_o(B) \\
&\leq e^{\frac{\alpha p'}{2}d(o,o')} e^{-\alpha p' \delta} \left(\frac{3n-1}{n}\right)^{\alpha p'} r^{\alpha p'} \mu_o(B).
\end{aligned} \tag{157}$$

For the last inequality we used the assumption  $|z| \leq (1 - \frac{1}{n})r$ .

Both of the above bounds imply,

$$\begin{aligned}
& \left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \\
& \leq C^{\frac{1}{q} + \frac{1}{p'}} \left( \frac{2Q - \alpha q}{Q - \alpha q} \right)^{\frac{1}{q}} \left( \frac{2n + 1}{n} \right)^{\frac{Q}{q} - \alpha} \left( \frac{3n - 1}{n} \right)^{\alpha} r^{Q(\frac{1}{q} + \frac{1}{p'})} \\
& \leq C^{2(\frac{1}{q} + \frac{1}{p'})} \left( \frac{2Q - \alpha q}{Q - \alpha q} \right)^{\frac{1}{q}} \left( \frac{2n + 1}{n} \right)^{\frac{Q}{q} - \alpha} \left( \frac{3n - 1}{n} \right)^{\alpha} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}},
\end{aligned} \tag{158}$$

which proves the case 2A.

Case 2B: We recall that the shadow  $\mathcal{O}'_o(M + 2\delta)$  is contained in the ball  $B^{M+2\delta}$ , with  $r_{(M+2\delta)} = e^{-(d(o,o') - M - 3\delta)}$ . This, together with the assumption  $|z| \leq (1 - \frac{1}{n})r$ , implies that

$$\frac{r}{n} \leq r - |z| \leq r_{(M+2\delta)}. \tag{159}$$

Thus,

$$B \subset B(\infty, |z| + r) \subset B(\infty, (2n - 1)r_{(M+2\delta)}) \subset \mathcal{O}'_o(M + \log(2n - 1) + 4\delta), \tag{160}$$

where the last inclusion again follows from the shadow lemma. This implies

$$\forall x \in B: \quad \langle x, o' \rangle_{o'} \geq d(o, o') - (M + \log(2n - 1) + 4\delta). \tag{161}$$

This inequality provides the following upper bound for the  $p'$  measure.

$$\begin{aligned}
\int_B \left( \frac{d_{o'}}{d_o}(x) \right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2} d(o, o')} \int_B e^{-\alpha p' \langle x, o' \rangle_{o'}} d\mu_o \\
&\leq e^{-\frac{\alpha p'}{2} d(o, o')} e^{\alpha p' (M + 4\delta)} (2n - 1)^{\alpha p'} \mu_o(B).
\end{aligned} \tag{162}$$

The upper bound of the  $q$ -measure is easy to determine.

$$\int_B \left( \frac{d_{o'}}{d_o}(x) \right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2} d(o, o')} \mu_o(B). \tag{163}$$

Both bounds together imply,

$$\left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq e^{\alpha(M+4\delta)} (2n - 1)^{\alpha} \mu_o(B)^{1 - \frac{\alpha}{Q}}, \tag{164}$$

which completely proves case 2.

Case 3: As above, let  $S_+$  be the set of all  $s \in \mathbb{R}^+$ , such that  $B \subset \mathcal{O}'_o(s)$ . Let  $s_+^*$  be the infimum of  $S_+$ .

Again, we distinguish two subcases:

1. Case 3A:  $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$  and  $s_+^* \geq M + 2\delta$ ,
2. Case 3B:  $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$  and  $s_+^* < M + 2\delta$ .

In case 3A we can apply the shadow lemma to

$$s_+ := d(o, o') - \delta + \log(|z| + r), \tag{165}$$



to conclude that  $s_+ \in S_+$ . Thus,

$$\forall x \in B: \quad \langle x, o' \rangle_o \geq d(o, o') - s_+ = -\log(|z| + r) + \delta \geq -\log\left(\frac{2n+1}{n}r\right) + \delta, \quad (166)$$

where in the last inequality we used the assumption  $|z| \leq (1 + \frac{1}{n})r$ .

We use this inequality to obtain the upper bound for the  $p'$  measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o, o')} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{\frac{\alpha p'}{2}d(o, o')} \left(\frac{2n+1}{n}\right)^{\alpha p'} e^{-\alpha p' \delta} r^{\alpha p'} \mu_o(B). \end{aligned} \quad (167)$$

For the upper bound of  $\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o$ , we constrain above again by the shadow  $\mathcal{O}'_o(s_+)$ .

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq \int_{\mathcal{O}'_o(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o. \quad (168)$$

The right-hand side is bounded as in case 2A. Thus,

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o, o')} e^{\alpha q \delta} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right) (|z| + r)^{Q - \alpha q}, \quad (169)$$

and using the assumption  $|z| \leq (1 + \frac{1}{n})r$ ,

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o, o')} e^{\alpha q \delta} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right) \left(\frac{2n+1}{n}\right)^{Q - \alpha q} r^{Q - \alpha q}. \quad (170)$$

Both bounds together imply,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq C^{2(\frac{1}{q} + \frac{1}{p'})} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right)^{\frac{1}{q}} \left(\frac{2n+1}{n}\right)^{\frac{Q}{q}} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}, \quad (171)$$

which proves the case 3A.

Case 3B: In this case  $B \subset \mathcal{O}'_o(M + 2\delta)$ . It follows that,

$$\forall x \in B, \quad \langle x, o' \rangle_{o'} \geq d(o, o') - (M + 2\delta). \quad (172)$$

This inequality yields the following upper bound for the  $p'$  measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o, o')} \int_{2B} e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{-\frac{\alpha p'}{2}d(o, o')} e^{\alpha p'(M + 2\delta)} \mu_o(B). \end{aligned} \quad (173)$$

The corresponding upper bound for the  $q$ -measure is simple.

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2}d(o, o')} \mu_o(B). \quad (174)$$

Both bounds together imply,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{\alpha(M + 2\delta)} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}. \quad (175)$$

This completely proves case 3 and ends the proof. ■

Theorem 6 is formulated with the compact picture in mind. However, the theorem can easily be extended to treat the compact and the non-compact picture.

**Theorem 7** *There exists a constant  $C > 0$ , s.t. for all  $o, o' \in X \cup \partial X$ , and  $1 < p \leq q < \infty$ ,  $\alpha < \frac{Q}{q}$ ,*

$$\left\| \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \right\|_{p,q} = \sup_B \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C < \infty, \quad (176)$$

where the supremum is taken over balls in  $\partial X \setminus \{o\}$ .

**Proof:** Let  $o, o' \in X$ . By multiplying both integrands by  $e^{\frac{\alpha}{2}d(o,o')}$  and  $e^{-\frac{\alpha}{2}d(o,o')}$ , respectively,

$$\left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \quad (177)$$

is equal to

$$\left( \frac{1}{\mu_o(B)} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x) \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o(x) \right)^{\frac{1}{p'}}. \quad (178)$$

We are tempted to take  $o'$  to the boundary. Since  $X$  is strongly hyperbolic, the integrand converges pointwise as  $o' \rightarrow z \in \partial X$ . Careful treatment requires the case where  $x \in \mathcal{O}_o'(M + 2\delta)$ . For the sake of readability, we set  $\mathcal{O}_o' = \mathcal{O}_o'(M + 2\delta)$ . Each of these integrals decomposes as

$$\int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o = \int_B \chi_{B \setminus \mathcal{O}_o'}(x) e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x) + \int_{B \cap \mathcal{O}_o'} e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x). \quad (179)$$

The function  $\chi_{B \setminus \mathcal{O}_o'}(x) e^{\alpha q \langle x, o' \rangle_o}$  is dominated by the integrable function  $e^{2\delta \alpha q} d_o(x, z)^{-\alpha q}$  and converges pointwise on  $B \setminus \{z\}$  to  $e^{\alpha q \langle x, z \rangle_o}$ . The second integral is bounded by

$$e^{\alpha q d(o, o')} \mu_o(\mathcal{O}_o') \leq e^{Q(M+3\delta)} e^{(\alpha q - Q)d(o, o')}. \quad (180)$$

Thus, the integral  $\int_{B \cap \mathcal{O}_o'} e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x)$  vanishes as  $o' \rightarrow z \in \partial X$ . By dominant convergence

$$\lim_{o' \rightarrow z} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o = \int_B e^{\alpha q \langle x, z \rangle_o} d\mu_o(x) = \int_B \left( \frac{d_{z,o}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o. \quad (181)$$

An equivalent argument holds for the  $p'$ -part. This implies,

$$\left\| \left( \frac{d_{z,o}}{d_o} \right)^{\frac{\alpha}{2}} \right\|_{p,q} = \sup_B \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{z,o}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{z,o}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C. \quad (182)$$

Since  $d_{z,o'} = e^{\beta_z(o, o')} d_{z,o}$ , the expression does not depend on the choice of  $o$  in  $d_{z,o}$ .

The transition from compact to non-compact picture, works in a similar way. Let  $z \in \partial X$ , and let  $B$  be a ball which does not contain  $z$  as a limitpoint. Let  $p$  be a sequence of points that is  $p \rightarrow z$ . We assume that all points are close enough to  $z$ , i.e.,  $B \subset \partial X \setminus \mathcal{O}_o^p(M + 2\delta)$  for all points  $p$ . Further, suppose that all endpoints  $\bar{p}$  of rays starting from  $o$  and extending

at uniform distance from  $p$  are  $\epsilon$  close to  $z$ . Then,

$$e^{\langle x,p \rangle_o} \quad (183)$$

is dominated by

$$\frac{e^{2\delta}}{e^{-\langle x,z \rangle_o} - \epsilon}, \quad (184)$$

which is integrable on  $B$ . By multiplying both integrands by  $e^{\frac{\alpha}{2}d(o,p)}$  and  $e^{-\frac{\alpha}{2}d(o,p)}$ , respectively,

$$\left( \frac{1}{\mu_p(B)} \int_B \left( \frac{d_o}{d_p} \right)^{\frac{\alpha q}{2}} d\mu_p \right)^{\frac{1}{q}} \left( \frac{1}{\mu_p(B)} \int_B \left( \frac{d_o}{d_p} \right)^{-\frac{\alpha p'}{2}} d\mu_p \right)^{\frac{1}{p'}} \quad (185)$$

is equal to

$$\left( \frac{1}{\mu_p(B)} \int_B e^{-\alpha q \langle x,p \rangle_o} d\mu_p(x) \right)^{\frac{1}{q}} \left( \frac{1}{\mu_p(B)} \int_B e^{\alpha p' \langle x,p \rangle_o} d\mu_p(x) \right)^{\frac{1}{p'}}. \quad (186)$$

We can express the integrals with respect to  $o$  as

$$\frac{1}{e^{Qd(o,p)} \mu_p(B)} \int_B e^{(Q-\alpha q) \langle x,p \rangle_o} d\mu_o(x). \quad (187)$$

The integrand is dominated, as we discussed above. By dominate convergence

$$\begin{aligned} \lim_{p \rightarrow z} \frac{1}{e^{Qd(o,p)} \mu_p(B)} \int_B e^{(Q-\alpha q) \langle x,p \rangle_o} d\mu_o(x) &= \frac{1}{\mu_{z,o}(B)} \int_B e^{(Q-\alpha q) \langle x,z \rangle_o} d\mu_o(x) \\ &= \frac{1}{\mu_{z,o}(B)} \int_B \left( \frac{d_o}{d_{z,o}} \right)^{\frac{\alpha q}{2}} d\mu_{z,o} \end{aligned} \quad (188)$$

An equivalent argument holds for the  $p'$  integral. ■

#### 4.5 Some classical theorems revisited.

The theory developed above allows us to revisit some classical theorems of harmonic analysis and present them in a Möbius invariant way. All statements make use of lemmas of Calderon-Zygmund type on spaces of homogeneous type. However, in order to maintain uniformity, attention must be paid to the constants. In this paper we will prove only the statement for the fractional integration operator explicitly and in detail. We postpone this proof to a later time.

##### 4.5.1 The Hardy-Littlewood maximal function

$$M_o(f)(x) = \sup_{B \ni x} \frac{1}{\mu_o(B)} \int_B |f(y)| d\mu_o, \quad (189)$$

fulfills ,

$$\int_Z M_o(f)(y)^p \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha p}{2}} d\mu_o(y) \leq C \int_Z |f(y)|^p \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha p}{2}} d\mu_o(y), \quad (190)$$

for all  $f \in L^p\left(\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha p}{2}} d\mu_o\right)$ , and with a constant  $C$  (including ideal points) independent of  $o$  and  $o'$ .

#### 4.5.2 The singular integral

$$T_o(f)(x) = \int_Z \frac{f(y)}{d_o(x, y)^Q} d\mu_o(y) \quad (191)$$

fullfills

$$\int_Z |T_o(f)(y)|^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{op}{2}} d\mu_o(y) \leq C \int_Z |f(y)|^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{op}{2}} d\mu_o(y), \quad (192)$$

for all  $f \in L^p\left(\left(\frac{d_{o'}}{d_o}\right)^{\frac{op}{2}} d\mu_o\right)$ , and with a constant  $C$  (including ideal points) independent of  $o$  and  $o'$ .

#### 4.5.3 The fractional integration operator

$$I_o^\alpha(f)(x) = \int_Z \frac{f(y)}{d_o(x, y)^{Q-\alpha}} d\mu_o(y) \quad (193)$$

fullfills

$$\left( \int_Z |I_o^\alpha(f)(y)|^q \left(\frac{d_{o'}}{d_o}\right)^{\frac{oq}{2}} d\mu_o(y) \right)^{\frac{1}{q}} \leq C \left( \int_Z |f(y)|^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{op}{2}} d\mu_o(y) \right)^{\frac{1}{p}}, \quad (194)$$

for  $0 < \alpha < \frac{Q}{q}$ ,  $1 < p < \infty$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ ,  $f \in L^p\left(\left(\frac{d_{o'}}{d_o}\right)^{\frac{op}{2}} d\mu_o\right)$ , and with a constant  $C$  (including ideal points) independent of  $o$  and  $o'$ .

### 4.6 Calderon-Zygmund type lemmas on Möbius spaces.

In this section we will revisit some classical lemmas of the Calderon-Zygmund type. The goal is to highlight the Möbius invariance of the constants involved.

In a Möbius space  $(Z, \mathcal{M})$ , a ball in  $(Z, \mathcal{M})$  is a metric ball for some  $d \in \mathcal{M}$ . A family of balls in  $(Z, \mathcal{M})$  is a family of metric balls for some  $d \in \mathcal{M}$ .

**Lemma 5** *Let  $\{B_i\}_{i \in \mathcal{I}}$  be a family of balls in  $(Z, \mathcal{M})$  contained in a fixed ball  $B$ . Then there is a countable subset  $\{B_i\}_{i \in I}$ ,  $I \subset \mathcal{I}$ , such that*

1.  $B_i \cap B_j = \emptyset$  if  $i \neq j$ ,
2. Every  $B_i$  is contained in some  $5B_i$ ,
3. For  $d \in \mathcal{M}$  such that  $\{B_i\}_{i \in \mathcal{I}}$  is a family of balls in  $(Z, d)$ ,  $\mu_d(\cup_{i \in \mathcal{I}} B_i) \leq C_Q \sum_{i \in I} \mu_d(B_i)$ .

and where  $C_Q = 5^Q C^2$  is a constant depending only on the Ahlfors-David constant  $C$  and the dimension  $Q$  of  $\mathcal{M}$ .

**Proof:** Vitali's selection procedure. ■

The classical theory on the bounds of fractional integration in Euclidean space is based on the decomposition of Calderon and Zygmund by dyadic cubes. We adapt this idea to our setting as follows.

For each  $k$  in  $\mathbb{Z}$ , let  $\{\hat{B}_j^k\}_j$  be a sequence of  $d$ -balls of radius  $3^{k-1}$ , maximal with respect to the property that  $\hat{B}_i^k \cap \hat{B}_j^k = \emptyset$  for  $i \neq j$ . Set  $B_j^k = 3\hat{B}_j^k$ , we call  $\{B_j^k\}_j$  *dyadic  $d$ -balls of order  $k$*  in  $Z$ . We call  $\{B_j^k\}_j$  *dyadic balls of order  $k$*  in  $(Z, \mathcal{M})$  if  $\{B_j^k\}_j$  are dyadic  $d$ -balls of order  $k$  for any  $d \in \mathcal{M}$ . An *dyadic ball* in  $(Z, \mathcal{M})$  is a dyadic  $d$ -ball of order  $k$  for some metric  $d \in \mathcal{M}$ .

**Lemma 6** *If  $\{B_j^k\}_j$  is a family of dyadic balls of order  $k$  in  $(Z, \mathcal{M})$ , then*

1.  $\frac{1}{3}B_i^k \cap \frac{1}{3}B_j^k = \emptyset$  for  $i \neq j$ ,  $k \in \mathbb{Z}$ .
2.  $\sum_j \chi_{B_j^k} \leq M$ , for all  $k$  in  $\mathbb{Z}$ , where  $M$  is a constant that depends only on the Ahlfors-David constant  $C$  of  $\mathcal{M}$ .
3. Let  $d \in \mathcal{M}$  be such that  $\{B_j^k\}_j$  are dyadic  $d$ -balls, then every  $d$ -ball of radius  $3^{k-1}$  is contained in at least one of the  $d$ -balls  $B_j^k$ .

**Proof:** Let  $d \in \mathcal{M}$  be such that  $\{B_j^k\}_j$  is a family of dyadic  $d$ -balls. Suppose  $x \in B_j^k$ ,  $1 \leq k \leq N$ , then  $B_j^k \subset B(x, 2 \times 3^k)$  for  $1 \leq j \leq N$ . According to the Ahlfors-David regularity  $\mu_d(B(x, 2 \times 3^k)) \leq C2 \times 3^k \leq 6C^2\mu_d(\hat{B}_j^k)$  and therefore  $N\mu_d(B(x, 2 \times 3^k)) \leq 6C^2 \sum_{j=1}^N \mu_d(\hat{B}_j^k) = 6C^2\mu_d(\cup_{j=1}^N \hat{B}_j^k) \leq 6C^2\mu_d(B(x, 2 \times 3^k))$ . So  $N \leq 6C^2$  and we can set  $M = 6C^2$ .

For the third property, we choose  $x \in Z$ , the  $d$ -ball  $B(x, 3^{k-1})$  intersects at least one of the  $\hat{B}_j^k$  by the maximality property of  $\{\hat{B}_j^k\}_j$ . But then  $B(x, 3^{k-1})$  is contained in  $B_j^k$  since for each  $w \in B(x, 3^{k-1})$  and  $z \in B(x, 3^{k-1}) \cap \hat{B}_j^k$ ,

$$\begin{aligned} d(x_j^k, w) &\leq d(x_j^k, z) + d(z, x) + d(x, w) \\ &< 3^{k-1} + 3^{k-1} + 3^{k-1} \\ &= 3^k. \end{aligned} \tag{195}$$

■

**Lemma 7** *Let  $\{B_\iota\}_{\iota \in \mathcal{I}}$  be a family of dyadic balls in  $(Z, \mathcal{M})$ . If  $\{B_j\}_{j \in J}$  is a collection of maximal balls with respect to the inclusion in  $\{B_\iota\}_{\iota \in \mathcal{I}}$ , then the balls  $\{\frac{1}{3}B_j\}_{j \in J}$  are pairwise disjoint.*

**Proof:** Note that due to the construction  $\frac{1}{3}B_i^k \cap \frac{1}{3}B_j^k = \emptyset$  for dyadic balls of the same degree when  $i \neq j$ . Suppose  $z \in \frac{1}{3}B_i^k \cap \frac{1}{3}B_j^l$  with  $l < k$ . Then  $B_j^l \subset B_i^k$ , because if  $y \in B_j^l$ ,

$$\begin{aligned} d(x_i^k, y) &\leq d(x_i^k, z) + d(z, x_j^l) + d(x_j^l, y) \\ &< 3^{k-1} + 3^{l-1} + 3^l \\ &\leq 3^k. \end{aligned} \tag{196}$$

From this and from the maximality of  $\{B_j\}_{j \in J}$  follows the lemma. ■

#### 4.7 Fractional integration on the boundary of strongly hyperbolic spaces.

**Theorem 8** *Let  $X$  be a strongly hyperbolic space,  $\{d_o\}_{o \in X}$  the natural Möbius structure on  $\partial X$ , and  $0 < \alpha < \frac{Q}{q}$ . Then there is a constant  $C_\alpha > 0$  that depends only on  $\alpha$ , so for all  $d_o, d_{o'} \in \mathcal{M}$  and  $f \in L^p(\mu_o)$ ,*

$$\|(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}} \circ I_o^\alpha \circ (\frac{d_o}{d_{o'}})^{-\frac{\alpha}{2}} f\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}, \tag{197}$$

where  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ .

**Proof:** (Of Theorem 8) Showing

$$\|(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}} \circ I_o^\alpha \circ (\frac{d_{o'}}{d_o})^{-\frac{\alpha}{2}} f\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}, \quad (198)$$

is equivalent to

$$\left( \int_{\partial X} [I_o^\alpha(f)(x)]^q \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \leq C_\alpha \left( \int_{\partial X} f(x)^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha p}{2}} d\mu_o(x) \right)^{\frac{1}{p}}. \quad (199)$$

This is further equivalent to

$$\left( \int_{\partial X} [I_o^\alpha \circ (\frac{d_{o'}}{d_o})^{-\frac{\alpha p'}{2}}(f)(x)]^q \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \leq C_\alpha \left( \int_{\partial X} f(x)^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha(1-p')p}{2}} d\mu_o(x) \right)^{\frac{1}{p}}. \quad (200)$$

So by duality we can prove

$$\begin{aligned} & \int_{\partial X} I_o^\alpha \circ (\frac{d_{o'}}{d_o})^{-\frac{\alpha p'}{2}}(f)(x) g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \\ & \leq C_\alpha \left( \int_{\partial X} f(x)^p \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o(x) \right)^{\frac{1}{p}} \left( \int_{\partial X} g(x)^{q'} \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q'}}. \end{aligned} \quad (201)$$

for all  $f, g \geq 0$ .

We bound the operator  $I_o^\alpha$  by its dyadic approximation

$$I_o^{\alpha, \text{dy}} f(x) := \sum_{B: x \in B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \int_B f d\mu_o. \quad (202)$$

In fact,

$$I_o^\alpha f(x) = \int_{\partial X} \frac{f(y)}{d_o(x, y)^{Q-\alpha}} d\mu_o(y) \leq C_\alpha \sum_{B: x \in B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \int_B f d\mu_o, \quad (203)$$

where  $C_\alpha = 9^{(Q-\alpha)} C^{1-\frac{\alpha}{Q}}$  with  $C$  the Ahlfors-David constant and  $Q$  the dimension of  $\mathcal{M}$ . This follows from the fact that if  $3^{k-1} \leq d_o(x, y) \leq 3^k$ , then both  $x$  and  $y$  are in  $B(x, 3^k)$ , which by Lemma 6.3 is contained in some  $B_j^{k+1}$ . So  $\mu_o(B_j^{k+1})^{1-\frac{\alpha}{Q}} \leq C^{1-\frac{\alpha}{Q}} 3^{(Q-\alpha)(k+1)} \leq 9^{(Q-\alpha)} C^{1-\frac{\alpha}{Q}} d_o(x, y)^{Q-\alpha}$  and thus

$$d_o(x, y)^{\alpha-Q} \leq C_\alpha \mu_o(B_j^{k+1})^{\frac{\alpha}{Q}-1} \chi_{B_j^{k+1}}(x) \chi_{B_j^{k+1}}(y) \leq C_\alpha \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \chi_B(x) \chi_B(y).$$

The bound (203) follows by multiplying both sides by  $f(y)$  and integrating with respect to  $y$ . Again multiplying both sides by  $g(x) \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}}$  and integrating with respect to  $x$  yields

$$\begin{aligned} & \int_{\partial X} I_o^\alpha \circ (\frac{d_{o'}}{d_o})^{-\frac{\alpha p'}{2}}(f)(x) g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \\ & \leq C_\alpha \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \left( \int_B f(x) \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_B g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right). \end{aligned} \quad (204)$$

Let  $\{Q_j^k\}_j$  be a collection of maximal dyadic  $d_o$ -balls over which the average of  $g$  exceeds

$\gamma^k$  with  $\gamma = 3^Q C^2$ , where  $C$  is the Ahlfors-David constant of  $\mathcal{M}$ . Then

$$\gamma^k < \frac{1}{\mu_o(Q_j^k)} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_d \leq \gamma^{k+1}, \quad (205)$$

since every element  $Q_j^k$  has radius  $3^l$  for any  $l \in \partial\mathbb{X}$ , and hence by lemma 6.3 is contained in some  $B_i^{l+1}$ . By Ahlfors-David regularity,  $\mu_o(B_i^{l+1}) \leq C 3^{Q(l+1)} \leq \gamma \mu_o(Q_j^k)$  and by the maximality property of  $\{Q_j^k\}_j$ ,  $\frac{1}{\mu_o(B_i^{l+1})} \int_{B_i^{l+1}} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^k$ , thus

$$\frac{1}{\mu_o(Q_j^k)} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \frac{\mu_o(B_i^{l+1})}{\mu_o(Q_j^k)} \left[ \frac{1}{\mu_o(B_i^{l+1})} \int_{B_i^{l+1}} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right] \leq \gamma \gamma^k = \gamma^{k+1}. \quad (206)$$

The collection  $\{Q_j^k\}_j$  has the following two important properties, which are necessary towards the end of this proof for the application of Lemma 8.

$$\sum_{i:Q_i^l \subset Q_j^k} \mu_o(Q_i^l) \leq C_Q \gamma^{1-(l-k)} \mu_o(Q_j^k), \quad \forall k, j, l, \quad (207)$$

and

$$Q_i^l \subsetneq Q_j^k \implies l > k. \quad (208)$$

The first property (207) is proved as follows.

$$\begin{aligned} \sum_{i:Q_i^l \subset Q_j^k} \mu_o(Q_i^l) &\leq 3^Q C^2 \sum_{i:Q_i^l \subset Q_j^k} \mu_o\left(\frac{1}{3}Q_i^l\right) \\ &\leq 3^Q C^2 \mu_o(\cup_{i:Q_i^l \subset Q_j^k} Q_i^l) \\ &\leq 15^Q C^4 \sum_{i \in \Gamma} \mu_o(Q_i^l) \end{aligned} \quad (209)$$

where the first inequality follows from the fact that the collection  $\{\frac{1}{3}Q_j^k\}_j$  is pairwise disjoint by lemma 7 and the second inequality holds by lemma 5 for a pairwise disjoint subcollection  $\{Q_i^l\}_{i \in \Gamma}$ . We chose  $\{Q_j^k\}_j$  such that  $\mu_o(Q_i^l)^{-1} \int_{Q_i^l} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o > \gamma^l$  and  $\mu_o(Q_j^k)^{-1} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^{k+1}$ . From this follows

$$\sum_{i \in \Gamma} \mu_o(Q_i^l) \leq \sum_{i \in \Gamma} \gamma^{-l} \int_{Q_i^l} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^{-l} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^{1-(l-k)} \mu_o(Q_j^k)$$

and together with (209) follows the property (207) with  $C_Q = 15^Q C^4$ .

The second property (208) follows from  $\gamma^k < \mu_o(Q_j^k)^{-1} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^l$ , where the last inequality follows from the maximality property of  $Q_i^l$  and  $Q_i^l \subsetneq Q_j^k$ .

For each  $k \in \mathbb{Z}$  define,

$$C^k = \left\{ B \text{ dyadic} : \gamma^k < \frac{1}{\mu_o(B)} \int_B g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^{k+1} \right\} \quad (210)$$

Every dyadic  $d_o$ -ball over which the average of  $g(x) \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}}$  does not vanish is contained

in exactly one of the  $\mathcal{C}^k$ . Moreover,  $Q_j^k \in \mathcal{C}^k$  and therefore

$$\frac{1}{\mu_o(B)} \int_B g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma \frac{1}{\mu_o(Q_j^k)} \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \quad \forall B \in \mathcal{C}^k, \forall j. \quad (211)$$

If  $B \in \mathcal{C}^k$ , then  $B$  is contained in some  $Q_j^k$  by the maximality property, since  $\frac{1}{\mu_o(B)} \int_B g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o > \gamma^k$ . The sum  $\sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \left( \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_B g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)$  on the right side of the equation (204) can be bounded as follows.

$$\begin{aligned} & \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \left( \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_B g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\ &= \sum_k \sum_{B \in \mathcal{C}^k} \mu_o(B) \mu_o(B)^{\frac{\alpha}{Q}-1} \left( \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \frac{1}{\mu_o(B)} \int_B g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\ &\leq \gamma \sum_k \sum_j \left[ \sum_{B \subset Q_j^k} \mu_o(B)^{\frac{\alpha}{Q}} \left( \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \right] \left( \frac{1}{\mu_o(Q_j^k)} \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\ &\leq MC_\alpha \gamma \sum_{k,j} \mu_o(Q_j^k)^{\frac{\alpha}{Q}-1} \left( \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \end{aligned}$$

where the last inequality follows from

$$\begin{aligned} \sum_{B \subset Q_j^k} \mu_o(B)^{\frac{\alpha}{Q}} \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o &\leq C^{\frac{2\alpha}{Q}} \mu_o(Q_j^k)^{\frac{\alpha}{Q}} \sum_{B \subset Q_j^k} \left( \frac{r(B)}{r(Q_j^k)} \right)^\alpha \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \\ &\leq C^{\frac{2\alpha}{Q}} \mu_o(Q_j^k)^{\frac{\alpha}{Q}} \sum_{l=0}^{\infty} \sum_{B \subset Q_j^k, r(B)=\frac{1}{3^l} r(Q_j^k)} \left( \frac{r(B)}{r(Q_j^k)} \right)^\alpha \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \\ &\leq C^{\frac{2\alpha}{Q}} \mu_o(Q_j^k)^{\frac{\alpha}{Q}} \left( \sum_{l=0}^{\infty} 3^{-l\alpha} \right) M \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \\ &= MC_\alpha \mu_o(Q_j^k)^{\frac{\alpha}{Q}} \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \end{aligned}$$

where  $C_\alpha = C^{\frac{2\alpha}{Q}} \left( \sum_{l=0}^{\infty} 3^{-l\alpha} \right)$ . Substituting the above bounds into equation (204), we get

$$\begin{aligned} & \int_{\partial X} I_o^\alpha \circ \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} (f)(x) g(x) d\mu_o \\ &\leq MC_\alpha \gamma \sum_{k,j} \mu_o(Q_j^k)^{\frac{\alpha}{Q}-1} \left( \int_{Q_j^k} f(x) \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_{Q_j^k} g(x) \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \end{aligned} \quad (212)$$

where  $C_\alpha = \frac{3}{2} 9^{(Q-\alpha)} C^{1+\frac{\alpha}{Q}}$ .

By Theorem 6,

$$\left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o(x) \right)^{\frac{1}{p'}} \leq C_\alpha \mu_o(B)^{1-\frac{\alpha}{Q}}. \quad (213)$$

Define the measures

$$M_q(B) := \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o, \quad (214)$$

and

$$M_{p'}(B) := \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o. \quad (215)$$



Therefore, the sum on the right side of the equation (212) can be written as follows.

$$\begin{aligned}
& \sum_{k,j} \mu_o(Q_j^k)^{\frac{\alpha}{Q}-1} \left( \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
& \leq C_\alpha \sum_{k,j} M_{p'}(Q_j^k)^{-\frac{1}{p'}} M_q(Q_j^k)^{-\frac{1}{q}} \left( \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
& = C_\alpha \sum_{k,j} M_{p'}(Q_j^k)^{\frac{1}{p}} \left( \frac{1}{M_{p'}(Q_j^k)} \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) M_q(Q_j^k)^{\frac{1}{q'}} \left( \frac{1}{M_q(Q_j^k)} \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
& \leq C_\alpha \left( \sum_{k,j} M_{p'}(Q_j^k) \left( \frac{1}{M_{p'}(Q_j^k)} \int_{Q_j^k} f dM_{p'} \right)^p \right)^{\frac{1}{p}} \left( \sum_{k,j} M_q(Q_j^k)^{\frac{p'}{q'}} \left( \frac{1}{M_q(Q_j^k)} \int_{Q_j^k} g dM_q \right)^{p'} \right)^{\frac{1}{p'}} \\
& \leq C_\alpha \left( \sum_{k,j} M_{p'}(Q_j^k) \left( \frac{1}{M_{p'}(Q_j^k)} \int_{Q_j^k} f dM_{p'} \right)^p \right)^{\frac{1}{p}} \left( \sum_{k,j} M_q(Q_j^k) \left( \frac{1}{M_q(Q_j^k)} \int_{Q_j^k} g dM_q \right)^{q'} \right)^{\frac{1}{q'}}
\end{aligned} \tag{216}$$

where the first inequality follows from Hölder's inequality and the second from  $q' \leq p'$ . It remains only to show that

$$\left( \sum_{k,j} M_{p'}(Q_j^k) \left( \frac{1}{M_{p'}(Q_j^k)} \int_{Q_j^k} f dM_{p'} \right)^p \right)^{\frac{1}{p}} \leq C_p \left( \int_{\partial X} f^p dM_{p'} \right)^{\frac{1}{p}} \tag{217}$$

and

$$\left( \sum_{k,j} M_q(Q_j^k) \left( \frac{1}{M_q(Q_j^k)} \int_{Q_j^k} g dM_q \right)^{q'} \right)^{\frac{1}{q'}} \leq C_{q'} \left( \int_{\partial X} g^{q'} dM_q \right)^{\frac{1}{q'}} \tag{218}$$

for some  $C_p$  and  $C_{q'}$  depending only on  $p$ ,  $q'$  and the Ahlfors-David constant  $C$ . This follows from the following lemma with  $\Gamma = \{Q_j^k\}$ .

Note that thanks to the doubling property of  $M_q$  proved in Lemma 2,

$$\begin{aligned}
\sum_{l,i:Q_i^l \subset Q_j^k} M_q(Q_i^l) & \leq C_{M_q}^2 \sum_{l,i:Q_i^l \subset Q_j^k} M_q\left(\frac{1}{3}Q_i^l\right) \\
& \leq C_{M_q}^2 M_q(Q_j^k),
\end{aligned} \tag{219}$$

where  $C_{M_q}$  is the doubling constant of  $M_q$ . An analogous inequality holds for  $M_{p'}$  thanks to the doubling property proved in Lemma 3.

Marcinkiewicz's interpolation theorem yields the explicit constants

$$C_p = 2 \left( \frac{p}{p-1} \right)^{\frac{1}{p}} C_{M_{p'}}^{\frac{7}{p}}, \tag{220}$$

and

$$C_p = 2 \left( \frac{q'}{q'-1} \right)^{\frac{1}{q'}} C_{M_q}^{\frac{7}{q'}}. \tag{221}$$

■

**Lemma 8** *Let  $M$  be a doubling measure on a metric space  $Z$  with doubling constant  $C_M \geq 1$ ,  $\beta \geq 1$  and  $\Gamma$  a collection of dyadic balls in  $X$  s.t.*

$$\sum_{B \in \Gamma, B \subset B'} M(B)^\beta \leq K M(B')^\beta, \quad \forall B' \in \Gamma,$$

for some  $K \geq 1$ . Then

$$\left( \sum_{B \in \Gamma} M(B)^\beta \left( \frac{1}{M(B)} \int_B f dM \right)^p \right)^{\frac{1}{p}} \leq C_p \left( \int_Z f^p dM \right)^{\frac{1}{p}},$$

for all  $f \geq 0$ ,  $1 < p < \infty$  and where  $C_p$  depends only on  $p$ ,  $K$  and the doubling constant  $C_M$ .

**Proof:** We show that the map  $f \mapsto \left( \frac{1}{M(B)} \int_B f dM \right)_{B \in \Gamma}$  takes  $L^\infty(Z, dM)$  to  $l^\infty(\Gamma, M(B)^\beta)$  and  $L^1(Z, dM)$  to  $l^{1,\infty}(\Gamma, M(B)^\beta)$ . The lemma then follows by applying Marcinkiewicz's interpolation theorem with  $1 < p < \infty$  [44].

The  $(\infty, \infty)$ -boundedness is clear with constant 1.

We show that the mapping from  $L^1(Z, dM)$  to  $l^{1,\infty}(\Gamma, M(B)^\beta)$  is bounded. Assume w.l.o.g. that  $f$  is bounded with compact support and fix  $\lambda > 0$ . Let  $\{Q_j\}_{j \in J}$  be the maximal dyadic balls  $B$  in  $\Gamma$  such that  $\frac{1}{M(B)} \int_B |f| dM > \lambda$ . Then

$$\begin{aligned} \sum_{B \in \Gamma: \frac{1}{M(B)} \int_B |f| dM > \lambda} M(B)^\beta &\leq \sum_{j \in J} \sum_{B \subset Q_j} M(B)^\beta \\ &\leq K \sum_{j \in J} M(Q_j)^\beta \\ &\leq K C_M^2 \left( \sum_{j \in J} M\left(\frac{1}{3}Q_j\right) \right)^\beta \end{aligned} \quad (222)$$

where in the second inequality we used the summation assumption over  $\Gamma$  and in the last inequality we used the doubling property of  $M$  and the fact that  $\beta \geq 1$ .

By Lemma 5, there exists a pairwise disjoint subcollection  $\{Q_i\}_{i \in I}$  of dyadic balls  $\{Q_j\}_{j \in J}$  with the property that every  $Q_j$ ,  $j \in J$ , is contained in a  $5Q_i$ ,  $i \in I$ . Note also that the collection  $\{\frac{1}{3}Q_j\}$  is pairwise disjoint by lemma 7. Therefore

$$\begin{aligned} \sum_{j \in J} M\left(\frac{1}{3}Q_j\right) &\leq \sum_{i \in I} \sum_{j \in J: Q_j \subset 5Q_i} M\left(\frac{1}{3}Q_j\right) \\ &\leq \sum_{i \in I} M(5Q_i) \\ &\leq C_M^3 \sum_{i \in I} M(Q_i) \\ &\leq \frac{C_M^3}{\lambda} \sum_{i \in I} \int_{Q_i} |f| dM \\ &\leq \frac{C_M^3}{\lambda} \int_X |f| dM. \end{aligned} \quad (223)$$

Combining the above, it can be seen that

$$\left( \sum_{B \in \Gamma: \frac{1}{M(B)} \int_B |f| dM > \lambda} M(B)^\beta \right)^{\frac{1}{\beta}} \leq \frac{K^{\frac{1}{\beta}} C_M^{\frac{2}{\beta}+3}}{\lambda} \int_X |f| dM, \quad (224)$$

which was to be proved.

Marcinkiewicz's interpolation theorem provides an explicit constant

$$C_p = 2 \left( \frac{p}{p-1} K C_M^5 \right)^{\frac{1}{p}}. \quad (225)$$

■

## 5 A quadratic form on Möbius spaces

### 5.1 Densities on Möbius spaces

**Definition 7** Let  $(Z, \mathcal{M})$  be a Möbius space of Hausdorff dimension  $Q$ . An  $s$ -density on  $(Z, \mathcal{M})$  is a mapping  $d \mapsto u_d$  which assigns to each metric  $d$  in  $\mathcal{M}$  a function  $u_d : Z \rightarrow \mathbb{C}$  such that for all metrics  $d, d' \in \mathcal{M}$  and all  $x \in Z$ ,  $u_d(x) = \left(\frac{d'}{d}(x)\right)^{sQ} u_{d'}(x)$ .

For simplicity, we denote  $s$ -densities compactly by  $u$  or  $u_d (d\mu_d)^s$ , where  $\mu_d$  is the Hausdorff  $Q$ -dimensional measure on  $Z$ . The space of  $s$ -densities  $\mathcal{D}_s(\mathcal{M})$  on  $Z$ , forms a vector space under pointwise addition.

Densities can be multiplied. The pointwise product of an  $s$ -density with an  $s'$ -density is an  $(s + s')$ -density.  $\frac{1}{2}$ -densities have a natural Möbius invariant inner product, multiply them and integrate the resulting measure. This works only for  $s = \frac{1}{2}$ . To generate invariant inner products for other values of  $s$ , we need more structure.

**Example 5** A function  $f$  on  $Z$  is a 0 density.

**Example 6** A signed measure  $\mu$  on  $Z$  that is absolutely continuous with respect to a Hausdorff measure of  $Q$  dimension defines a 1 density. In fact, for every metric  $d'$  in the class there is a non-negative measurable function  $u_{d'}$  such that  $d\mu = u_{d'} d\mathcal{H}_{d'}^Q$ . Since

$$\frac{d\mathcal{H}_{d'}^Q}{d\mathcal{H}_d^Q} = \left(\frac{d'}{d}\right)^Q,$$

$$u_d d\mathcal{H}_d^Q = d\mu = u_{d'} d\mathcal{H}_{d'}^Q = u_{d'} \left(\frac{d'}{d}\right)^Q d\mathcal{H}_d^Q,$$

thus for almost any  $x \in Z$ ,

$$u_d(x) = \left(\frac{d'}{d}(x)\right)^Q u_{d'}(x).$$

This motivates to consider  $s$ -densities as fractional measures.

**Example 7** The distance  $(x, y) \mapsto d(x, y)$  as a 2 point function is a  $(-\frac{1}{2Q}, -\frac{1}{2Q})$  bi-density. Indeed, going from one distance  $d'$  to another  $d$  in the Möbius class,

$$\frac{d(x, y)}{d'(x, y)} = \left(\frac{d'}{d}(x)\right)^{-1/2} \left(\frac{d'}{d}(y)\right)^{-1/2}.$$

**Example 8** On the ideal boundary of strongly hyperbolic spaces, we consider the densities associated with the Möbius structure  $\{d_{\epsilon, o}\}_{o \in X}$  as maps  $o \mapsto u_o$ . These maps transform under a change of origin as

$$u_o(x) = e^{\epsilon \beta_x(o, o')} u_{o'}(x). \quad (226)$$

## 5.2 Densities viewed from an ideal origin

On the boundary of a strongly hyperbolic space  $X$  we can extend the definition of densities  $o \mapsto u_o$  to the boundary  $\partial X$ . Recall that the sequence of measures  $\{e^{Qd(o,p)}\mu_p\}$  converges to a measure  $\mu_{z,o}$  if  $p \rightarrow z \in \partial X$ .

**Proposition 7** *Let  $u$  be an  $s$ -density on  $\partial X$ , and let  $x \in \partial X$ . Since  $p \in X$  tends to a point  $z \in \partial X \setminus \{x\}$ ,*

$$e^{-Qs d(o,p)} u_p(x)$$

*tends to a number denoted by  $u_{z,o}(x)$  and given by*

$$u_{z,o}(x) = u_o(x) d_o(z, x)^{2Qs}. \quad (227)$$

*As  $o$  is changed,  $u_{z,o}$  is multiplied by a positive constant,*

$$u_{z,o'} = e^{Qs \beta_z(o', o)} u_{z,o}.$$

Indeed, by definition,

$$\frac{u_p(x)}{u_o(x)} = \left( \frac{d\mu_o}{d\mu_p}(x) \right)^s. \quad (228)$$

Since  $e^{Qd(o,p)}\mu_p$  converges to a  $Q$ -dimensional Hausdorff measure  $\mu_{z,o}$  when  $p \rightarrow o$ ,

$$e^{-Qs d(o,p)} \frac{u_p(x)}{u_o(x)} = \left( e^{Qd(o,p)} \frac{d\mu_p}{d\mu_o}(x) \right)^{-s} \quad (229)$$

converges to

$$\left( \frac{d\mu_{z,o}}{d\mu_o} \right)^{-s} = d_o(z, x)^{2Qs}. \quad (230)$$

Furthermore,

$$\begin{aligned} \frac{u_{z,o'}(x)}{u_{z,o}(x)} &= \lim_{p \rightarrow z} \frac{e^{-Qs d(o',p)} u_p(x)}{e^{-Qs d(o,p)} u_p(x)} \\ &= \lim_{p \rightarrow z} e^{-Qs(d(o',p) - d(o,p))} \\ &= e^{-Qs \beta_z(o', o)}. \end{aligned} \quad (231)$$

In particular, the notation  $u = u_{z,o}(d\mu_{z,o})^s$  is valid.

## 5.3 Fractional integration revisited

Let  $(Z, \mathcal{M})$  be a Möbius space of dimension  $Q$ . The Riesz-Markov-Kakutani theorem states that the dual space of continuous compactly supported functions can be identified with the space of signed measures (with minor regularity assumptions). This establishes a duality between the spaces of 0-densities and 1-densities.

The Riesz-Markov-Kakutani duality is transferable to  $s$ -densities. Given a  $(1-s)$ -density  $v$ , then

$$u \mapsto \int uv, \quad (u \in \mathcal{D}_s(\mathcal{M})) \quad (232)$$

is an element of  $\mathcal{D}_s^*$ . Morally,  $\mathcal{D}_{1-s}$  and  $\mathcal{D}_s$  are dual to each other. The space  $\mathcal{D}_{\frac{1}{2}}$  is dual

to itself, it carries the inner product  $(u, v) = \int_Z uv$ .

P. Julg [45] noted that the operator

$$\mathcal{I}^s(u)(x) = \int_Z \frac{u_d(y)}{d(x, y)^{2Q(1-s)}} \mu_d(y), \quad \left(\frac{1}{2} < s < 1\right) \quad (233)$$

maps  $s$ -densities to  $(1 - s)$ -densities. Indeed,

$$\int_Z \frac{u_d(y)}{d(x, y)^{2Q(1-s)}} d\mu_d(y) = \left(\frac{d'}{d}(x)\right)^{(1-s)Q} \int_Z \frac{u_{d'}(y)}{d'(x, y)^{2Q(1-s)}} d\mu_{d'}(y). \quad (234)$$

$\mathcal{I}^s(u)$  thus transforms as  $(1 - s)$ -density.

Replacing  $s$  by  $\frac{1}{2} + \frac{\alpha}{Q}$ , we get the fractional integration operator

$$I_d^{2\alpha}(f)(x) = \int_Z \frac{f(y)}{d(x, y)^{Q-2\alpha}} d\mu_d(y) \quad \left(0 < \alpha < \frac{Q}{2}\right) \quad (235)$$

defined on functions.

The operator  $\mathcal{I}^s$  yields a Möbius invariant quadratic form,

$$Q^s(u) = \int_Z u \mathcal{I}^s(u). \quad (u \in \mathcal{D}_s) \quad (236)$$

The operator  $I_d^s$  naturally appears in the theory of  $s$ -Poisson transformations. We will illustrate this fact with the instructive example of trees.

#### 5.4 The case of regular trees

On a tree  $T$  of degree  $q + 1$ , the harmonic measure of a vertex  $o$  is the unique (obvious) probability measure  $\mu_o$  on  $Z = \partial T$  which is invariant under all automorphisms fixing  $o$ . Since it coincides with the Patterson-Sullivan measure, the Radon-Nikodým derivatives are exponentials of Busemann functions,

$$\frac{d\mu_{o'}}{d\mu_o}(\zeta) := P(o, o', \zeta) = q^{\beta_\zeta(o, o')} \quad (237)$$

for all  $\zeta \in Z$  and vertices  $x, y \in T$ . Here  $\beta_\zeta(o, o')$  tends to  $+\infty$  when  $o'$  tends to  $\zeta$ .

Fix an origin  $o \in T$ , and let  $s \in \mathbb{C}$ . Consider the  $s$ -Poisson transform defined for functions  $u : Z \rightarrow \mathbb{C}$  by

$$\mathcal{P}^s(u)(o') = \int_Z P(o, o', \zeta)^{1-s} u(\zeta) d\mu_o(\zeta). \quad (238)$$

For  $s = 0$ , the change of variable formula (237) shows that  $\mathcal{P}^0(u)$  does not depend on the choice of  $o$ ,  $\mathcal{P}^0$  is the usual Poisson transform yielding harmonic functions on  $T$ . This invariance is preserved for  $s \neq 0$ , provided one considers functions on  $Z$  as  $s$ -densities.

**5.4.1 Densities on the ideal boundary of a regular tree** We give a definition related to the natural Möbius structure of visual metrics associated with vertices in  $T$ .

**Definition 8** An  $s$ -density on  $Z$  is a map  $v : o \mapsto u_o$  that associates a function  $u_o : Z \rightarrow \mathbb{C}$  with each vertex of  $T$  such that for all vertices  $o, o' \in T$  and  $\zeta \in Z$ ,

$$u_{o'}(\zeta) = P(o, o', \zeta)^s u_o(\zeta). \quad (239)$$

The image of  $\mathcal{P}^s$  consists of eigenfunctions of the Laplacian on  $T$ .

**Theorem 9 ([46] page 37)** *Let  $u$  be an  $s$ -density on  $Z$ , then let  $f = \mathcal{P}^s(u)$ . Then  $f$  does not depend on the choice of the origin  $o$ , and*

$$\Delta f = \rho(s)f, \quad \text{where } \rho(s) = 1 - \frac{q^s + q^{1-s}}{1+q}. \quad (240)$$

*Conversely, any  $\rho(s)$ -eigenfunction of the Laplacian on  $T$  is the image by  $\mathcal{P}^s$  of a unique finitely additive measure (viewed as  $s$ -density) on  $Z$ .*

Since  $\rho(1-s) = \rho(s)$ , the operator

$$\mathcal{I}^s = (\mathcal{P}^s)^{-1} \circ \mathcal{P}^{1-s} \quad (241)$$

is well-defined. It maps  $(1-s)$ -densities to  $s$ -densities, i.e. the quadratic form

$$Q^s : v \mapsto \int_Z v \mathcal{I}^{1-s}(v) \quad (242)$$

is well-defined on  $\mathcal{D}_s$ . If  $\Re(s) = \frac{1}{2}$ , the Hermitian form is

$$H^s : v \mapsto \int_Z \bar{v} \mathcal{I}^{1-s}(v) \quad (243)$$

is well-defined.

**Lemma 9 ([46] page 45)** *1. For all  $s \in \mathbb{C}$  for which  $\Re(s) = \frac{1}{2}$  holds,  $H^s$  is positive definite.*

*2.  $Q^s$  is real iff  $\Im(s) \in \frac{\pi}{\log q} \mathbb{Z}$ . In these cases,  $Q^s$  is positive definite iff  $0 < \Re(s) < 1$ . Otherwise, it has exactly 1 positive direction.*

The proof amounts to an explicit computation of eigenvectors and eigenvalues of  $\mathcal{I}^s$  (acting on functions, i.e., without considering densities). The spectrum of  $\mathcal{I}^{1-s}$  is the set

$$\text{sp}(\mathcal{I}^{1-s}) = \{1\} \cup \frac{q^{-s} - q^s}{q^{1-s} - q^{s-1}} q^{(2s-1)\mathbb{N}^*}. \quad (244)$$

Both cases yield families of unitary representations of  $\text{Aut}(T)$ . The first family is called *principal series*, the second *complementary series*.

**Proposition 8** *Be  $s \in \mathbb{C}$  with  $\frac{1}{2} < \Re(s) < 1$ . Fix a vertex  $o \in T$ . Then, for  $u \in \mathcal{D}_s$ ,*

$$\begin{aligned} Q^s(u) &= \frac{1 - q^{-2}}{1 - q^{-2s}} \int \int_{Z \times Z} q^{2(1-s)\langle x, y \rangle_o} u_o(x) u_o(y) d\mu_o(x) d\mu_o(y) \\ &= \frac{1 - q^{-2}}{1 - q^{-2s}} \int \int_{Z \times Z} d_o(x, y)^{-2(1-s) \log q} u_o(x) u_o(y) d\mu_o(x) d\mu_o(y). \end{aligned} \quad (245)$$

Except for one normalization constant, this quadratic form agrees exactly with the previously defined one.

**Proof:** *(of proposition 8)* It relies on Proposition 1.3 on page 40 of [46], which describes the inverse of the  $s$ -Poisson transform. Here is a formula for the finitely additive measure

$m$  whose  $s$ -Poisson transform is the eigenfunction  $f$ . Fix the origin  $o$ . For the edge  $e$ , let  $(o'', o')$  be the endpoints of  $e$ , where  $o''$  lies between  $o$  and  $o'$ . Let  $\mathcal{O}_o^{o'}$  be the set of endpoints of the subtree intersected by  $e$  and facing  $o$ , i.e., the shadow of  $o'$  as seen from  $o$ . Note that  $\mu_o(\mathcal{O}_o^{o'}) = \frac{q}{q+1} q^{-d(o, o')}$ . Then

$$m(\mathcal{O}_o^{o'}) = \frac{1}{q^{(1-s)} - q^{-(1-s)}} q^{-(1-s)d(o, o'')} (f(o') - q^{-(1-s)} f(o'')). \quad (246)$$

Suppose  $f = \mathcal{P}^{1-s}(u)$ , where  $u$  is a  $(1-s)$  density realized at the origin  $o$  by the function  $u_o$ , i.e.

$$f(o') = \int_Z P(o, o', y)^s u_o(y) d\mu_o(y). \quad (247)$$

If  $o'$  is close enough to a point  $x \in Z$ ,  $x \neq y$ , then  $o''$  lies between  $y$  and  $o'$  along a geodesic, so  $\beta_y(o', o'') = -1$ ,  $\beta_y(o, o'') = \beta_y(o, o') - 1$ ,  $P(o, o'', y) = q^{-1} P(o, o', y)$ ,

$$P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s = (1 - q^{-1}) P(o, o', y)^s, \quad (y \notin \mathcal{O}_o^{o'}) \quad (248)$$

On the other hand, if  $y \in \mathcal{O}_o^{o'}$ , then  $P(o, o'', y) = q P(o, o', y)$ ,

$$P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s = (1 - q^{2s-1}) P(o, o', y)^s, \quad (y \in \mathcal{O}_o^{o'}). \quad (249)$$

This gives,

$$\begin{aligned} f(o') - q^{-(1-s)} f(o'') &= \int_Z (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &= \int_{\mathcal{O}_o^{o'}} (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &\quad + \int_{Z \setminus \mathcal{O}_o^{o'}} (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &= (1 - q^{2s-1}) \int_{\mathcal{O}_o^{o'}} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &\quad + (1 - q^{-1}) \int_{Z \setminus \mathcal{O}_o^{o'}} P(o, o', y)^s u_o(y) d\mu_o(y) \end{aligned} \quad (250)$$

Therefore

$$\begin{aligned} m(\mathcal{O}_o^{o'}) &= \frac{1 - q^{2s-1}}{q^{(1-s)} - q^{-(1-s)}} \int_{\mathcal{O}_o^{o'}} q^{-(1-s)d(o, o'')} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &\quad + \frac{1 - q^{-1}}{q^{(1-s)} - q^{-(1-s)}} \int_{Z \setminus \mathcal{O}_o^{o'}} q^{-(1-s)d(o, o'')} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &= \frac{q^{(1-s)}(1 - q^{2s-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{\mathcal{O}_o^{o'}} q^{-(1-s)d(o, o')} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &\quad + \frac{q^{(1-s)}(1 - q^{-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{Z \setminus \mathcal{O}_o^{o'}} q^{-(1-s)d(o, o')} P(o, o', y)^s u_o(y) d\mu_o(y) \end{aligned} \quad (251)$$

Note that

$$q^{d(o, o')} P(o, o', y) = q^{d(o, o') + \beta_y(o, o')}. \quad (252)$$

This gives,

$$\begin{aligned}
m(\mathcal{O}'_o) &= \frac{q^{(1-s)}(1-q^{2s-1})}{q^{(1-s)}-q^{-(1-s)}} \int_{\mathcal{O}'_o} q^{-d(o,o')} q^{s(d(o,o')+\beta_y(o,o'))} u_o(y) d\mu_o(y) \\
&\quad + \frac{q^{(1-s)}(1-q^{-1})}{q^{(1-s)}-q^{-(1-s)}} \int_{Z \setminus \mathcal{O}'_o} q^{-(1-s)d(o,o')} P(o,o',y)^s u_o(y) d\mu_o(y)
\end{aligned} \tag{253}$$

Since  $o'$  tends to  $x \neq y$ ,  $q^{d(o,o')} P(o,o',y) = q^{d(o,o')+\beta_y(o,o')}$  is stationary. In fact, it remains constant once  $d(o,o') \geq \langle x,y \rangle_o$ , hence its limit is  $q^{2\langle x,y \rangle_o}$ . Note also that the first integral becomes negligible only when  $s < \frac{1}{2}$ . So if  $\mathcal{O}'_o$  is small enough, and  $s < \frac{1}{2}$ ,

$$\begin{aligned}
m(\mathcal{O}'_o) &\sim \frac{1-q^{-1}}{1-q^{-2(1-s)}} \int_Z q^{-d(o,o')} q^{2s\langle x,y \rangle_o} u_o(y) d\mu_o(y) \\
&\sim \frac{1-q^{-1}}{1-q^{-2(1-s)}} \frac{1+q}{q} \left( \int_Z q^{2s\langle x,y \rangle_o} u_o(y) d\mu_o(y) \right) \mu_o(\mathcal{O}'_o).
\end{aligned} \tag{254}$$

In other words,  $m$  has density

$$\frac{1-q^{-2}}{1-q^{-2(1-s)}} \int_Z q^{2s\langle x,y \rangle_o} u_o(y) d\mu_o(y) \tag{255}$$

with respect to  $\mu_o$ .

Replace  $s$  with  $1-s$ . By definition,  $Q^s(u)$  amounts to the integration of  $u_o$  with respect to this measure, so

$$Q^s(u) = \frac{1-q^{-2}}{1-q^{-2s}} \int_{Z \times Z} q^{2(1-s)\langle x,y \rangle_o} u_o(y) u_o(x) d\mu_o(y) d\mu_o(x). \tag{256}$$

■

## 6 Uniformly bounded representations

Given a Möbius space  $(Z, \mathcal{M})$  we have seen that the quadratic form

$$Q^s(u) = \int_{Z \times Z} d(x,y)^{-2Q(1-s)} u_d(x) u_d(y) \mu_d(x) \mu_d(y), \quad \left( \frac{1}{2} < s < 1 \right) \tag{257}$$

on  $s$ -densities, is exactly Möbius invariant. In other words,  $Q^s$  does not depend on a choice of  $d \in \mathcal{M}$ . A thorny question remains about the positivity of  $Q^s$ .

In the world of simple Lie groups of real rank 1,  $Q^s$  occurs as an invariant inner product of complementary series representations. The necessary and sufficient conditions for the positivity of  $Q^s$  were given by B. Kostant [19]. For  $(Z, \mathcal{M})$  as a Möbius space on the boundary of  $\mathrm{SO}(n, 1)$  and  $\mathrm{SU}(n, 1)$ ,  $Q^s$  is positive definite on the whole strip  $s \in (\frac{1}{2}, 1)$ . This is in contrast to the case of  $\mathrm{Sp}(n, 1)$ , for which  $Q^s$  is positive definite only if  $s \in (\frac{1}{2}, 1 - \frac{1}{2n+1})$ . The existence of a point  $s^*$  for which  $Q^s$  is not positive definite for all  $s \in [s^*, 1)$  is typical for groups with Kazhdan property  $T$  such as  $\mathrm{Sp}(n, 1)$  or  $F_{4(-20)}$ .<sup>2</sup>

In the limit  $s \rightarrow 1$ , the quadratic form  $Q^s$  degenerates to  $Q^1(u) = \left( \int_Z u \right)^2$  on measures. However, the derivative of  $Q^s$  with respect to  $s = 1$  again yields an invariant quadratic

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<sup>2</sup> $F_{4(-20)}$  is the rank one real form of the simple complex Lie group of type  $F_4$ .



form

$$H(u) = \frac{dQ^s(u)}{ds} \Big|_{s=1} = 2Q \int_{Z \times Z} \log d(x, y) u_d(x) u_d(y) \mu_d(x) \mu_d(y). \quad (258)$$

P. Julg gives a short proof of the fact that  $H$  is positive definite in the case of  $\mathrm{SO}(n, 1)$  and  $\mathrm{SU}(n, 1)$ . This also follows from the positivity of  $Q^s$  on the entire strip  $s \in (\frac{1}{2}, 1)$ . The affine space

$$\mathcal{A} = \{u \in D^1 : H(u) < \infty, \int_Z u = 0\}, \quad (259)$$

is Möbius invariant. The actions of  $\mathrm{SO}(n, 1)$  and  $\mathrm{SU}(n, 1)$  on  $(\mathcal{A}, H)$  are isometric and proper, confirming the well-known fact that these groups have the Haagerup property.

The above discussion shows that for a strongly hyperbolic space  $X$  the positivity of  $Q^s$  on  $\partial X$  fails in general. Overcoming this restriction is the topic of the next section.

## 6.1 A Hilbert space attached to Möbius spaces

The fractional integration operator

$$I_d^\alpha(u) = \int_Z d(x, y)^{-Q+\alpha} u_d(y) d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2}) \quad (260)$$

is self-adjoint on  $L^2(d\mu_d)$  and its square has the kernel

$$K(x, y) = \int_Z d(x, z)^{-Q+\alpha} d(z, y)^{-Q+\alpha} d\mu_d(z). \quad (261)$$

This kernel is comparable to  $d(x, y)^{-Q+2\alpha}$  and  $(I_d^\alpha)^2$  is a good candidate for a Laplacian raised to the power  $-\alpha$ . Our guess for the  $\mathcal{H}^{-\alpha}$ -norm on  $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities on compact  $Q$ -Ahlfors regular Möbius spaces is

$$\|u\|_d = \|I_d^\alpha u_d\|_{L^2(d\mu_d)}. \quad (262)$$

It is positive by construction and leads to a Sobolev space  $\mathcal{H}_d^{-\alpha}$  of  $(\frac{1}{2} + \frac{\alpha}{Q})$  densities.

**6.1.1 Uniform boundedness** Apriori, the construction of  $\mathcal{H}_d^{-\alpha}$  depends on the choice of the metric  $d$  within the Möbius structure. We can ask whether it is canonical, at least at the topological level. That is, for fixed  $0 < \alpha < \frac{Q}{2}$ , there exists a  $C > 0$ , s.t.

$$\frac{1}{C} \|u\|_d \leq \|u\|_{d'} \leq C \|u\|_d, \quad (263)$$

for all  $d, d' \in \mathcal{M}$ .

Topological uniqueness is directly related to uniformly bounded representations of groups. If  $G$  is a group acting on  $Z$ , then  $G$  acts naturally on  $\mathcal{D}_s(\mathcal{M})$  by  $g \cdot u = g \cdot u_d (dg_* \mu_d)^s$ , where  $g \cdot u_d$  is left-translation on functions and  $g_* \mu_d$  denotes the pushforward measure of  $\mu_d$  by  $g$ . If  $G$  acts on  $Z$  through Möbius automorphisms, then  $\|g \cdot u\|_d = \|u\|_{g^* d}$ . Here  $g^* d$  denotes the pullback of  $d$  by  $g$ . In fact,  $\frac{dg_* \mu_d}{d\mu_d}(x) = \left(\frac{(g^{-1})^* d}{d}(x)\right)^Q = \left(\frac{d}{g^* d}(g^{-1}x)\right)^Q$ .

For better readability we set  $\frac{1}{2} + \frac{\alpha}{Q} = s$ .

$$\begin{aligned}
\|g \cdot u\|_d^2 &= \int_{Z \times Z} \frac{u_d(g^{-1}x)u_d(g^{-1}y) \left(\frac{(g^{-1})^*d}{d}(x)\right)^{sQ} \left(\frac{(g^{-1})^*d}{d}(y)\right)^{sQ}}{d(x,y)^{Q-\alpha}} d\mu_d(y)d\mu_d(x) \\
&= \int_{Z \times Z} \frac{u_d(g^{-1}x)u_d(g^{-1}y) \left(\frac{d}{g^*d}(g^{-1}x)\right)^{sQ} \left(\frac{d}{g^*d}(g^{-1}y)\right)^{sQ}}{g^*d(g^{-1}x, g^{-1}y)^{Q-\alpha}} d\mu_d(y)d\mu_d(x) \\
&= \int_{Z \times Z} \frac{u_d(x)u_d(y) \left(\frac{d}{g^*d}(x)\right)^{sQ} \left(\frac{d}{g^*d}(y)\right)^{sQ}}{g^*d(x,y)^{Q-\alpha}} d(g^{-1})_*\mu_d(y)d(g^{-1})_*\mu_d(x) \\
&= \int_{Z \times Z} \frac{u_{g^*d}(x)u_{g^*d}(y)}{g^*d(x,y)^{Q-\alpha}} d\mu_{g^*d}(y)d\mu_{g^*d}(x).
\end{aligned} \tag{264}$$

If  $G$  maps  $\mathcal{M}$  into itself, then (263) implies,

$$\frac{1}{C}\|u\|_d \leq \|g \cdot u\|_d \leq C\|u\|_d. \quad (\forall g \in G) \tag{265}$$

In other words, the action of  $G$  on  $\mathcal{H}^{-\alpha}$  is uniformly bounded.

**Lemma 10** *The space  $\mathcal{H}^{-\alpha}$  is topologically uniquely defined (i.e., independent of the choice of a metric in  $\mathcal{M}$ ) if and only if there exists a  $C > 0$  such that for all  $d, d' \in \mathcal{M}$ ,*

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^2(\mu_d)} \leq C \|I_d^\alpha u_d\|_{L^2(\mu_d)}. \tag{266}$$

**Proof:** In fact, due to the symmetry of the problem, it is only necessary to show the upper bound in (263) for all  $d, d' \in \mathcal{M}$ . Rewriting the norm  $\|u\|_{d'}$  in terms of a fixed background measure  $\mu_d$ , we get

$$\begin{aligned}
\|u\|_{d'}^2 &= \|I_{d'}^\alpha(u_{d'}(d\mu_{d'})^{\frac{1}{2} + \frac{\alpha}{Q}})\|_{L^2(\mu_{d'})}^2 \\
&= \int_Z \left( \int_Z \frac{u_{d'}(y)}{d'(x,y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\
&= \int_Z \left( \int_Z \frac{\left(\frac{d}{d'}(y)\right)^{\frac{Q}{2} + \alpha} u_d(y)}{\left(\frac{d'}{d}(x)\frac{d'}{d}(y)\right)^{\frac{Q}{2} - \frac{\alpha}{2}} d(x,y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\
&= \int_Z \left( \left(\frac{d}{d'}(x)\right)^{\frac{Q}{2} - \frac{\alpha}{2}} \int_Z \frac{\left(\frac{d}{d'}(y)\right)^{Q + \frac{\alpha}{2}} u_d(y)}{d(x,y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\
&= \int_Z \left( \left(\frac{d}{d'}(x)\right)^{-\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d}{d'}(y)\right)^{\frac{\alpha}{2}} u_d(y)}{d(x,y)^{Q-\alpha}} d\mu_d(y) \right)^2 d\mu_d(x) \\
&= \left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^2(\mu_d)}^2.
\end{aligned} \tag{267}$$

■

Directly showing the inequality  $\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^2(\mu_d)} \leq C \|I_d^\alpha u_d\|_{L^2(\mu_d)}$ , proves to be difficult. However, the weaker statement  $\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^2(\mu_d)} \leq C \|u_d\|_{L^2(\mu_d)}$ , follows from Theorem 8, when  $\mathcal{M}$  is a spherical Möbius structure.

Indeed, by Theorem 8,  $\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^q(\mu_d)} \leq C \|u_d\|_{L^p(\mu_d)}$ , with  $1 < p \leq q$ . Let  $q = 2$ , and  $r > 1$  such that  $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ , then by Hölder's inequality

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} f \right\|_{L^2(\mu_d)} \leq C \|f\|_{L^p(\mu_d)} \leq C \|f\|_{L^2(\mu_d)}. \tag{268}$$

This means that the operator norm  $\|I_d^\alpha\|_{L^2((\frac{d'}{d})^\alpha \mu_d)}$  of  $I_d^\alpha$  is uniformly bounded by a constant  $C$ . We will see that for spherical Möbius structures the weaker statement (268) implies the stronger one (266) for a family of special functions.

**6.1.2 Powers of conformal factors.** In a Möbius space  $(Z, \mathcal{M})$ , the conformal factors raised to the power of  $sQ$  are  $s$ -densities on  $Z$ . It is clear that for any metric  $d' \in \mathcal{M}$ , the function

$$d \mapsto \left(\frac{d'}{d}\right)^{sQ} \quad (269)$$

transforms as an  $s$ -density. Morally,  $(\frac{d'}{d})^{sQ}$  is the  $s$ -density, that is, the constant function 1 on the metric  $d'$ . We recall that  $I_d^\alpha$  generates an operator on  $s$ -densities for a given  $s$  as a function of  $\alpha$ . It turns out that  $s = \frac{1}{2} + \frac{\alpha}{2Q}$ . This fact implies strong regularity properties on spherical Möbius spaces.

**Proposition 9** *Let  $(Z, \mathcal{M})$  be a spherical,  $Q$ -Ahlfors-David regular Möbius space,  $0 < \alpha < \frac{Q}{2}$ . If the operator norm  $\|I_d^\alpha\|_{L^2((\frac{d'}{d})^\alpha \mu_d)}$  is uniformly bounded, i.e., bounded by a constant independent of  $d$  and  $d'$ , then there exists  $C > 0$ , uniformly over  $\mathcal{M}$ , s.t. any function  $f$  from the family*

$$\left\{ \left(\frac{d''}{d}\right)^{sQ} \right\}_{d'' \in \mathcal{M}} \quad \left( s = \frac{1}{2} + \frac{\alpha}{2Q} \right) \quad (270)$$

satisfies the inequality  $\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} f\|_{L^2(\mu_d)} \leq C \|I_d^\alpha f\|_{L^2(\mu_d)}$ .

This theorem holds in particular for the natural Möbius structure  $\{d_{\epsilon,o}\}_{o \in X}$  on a strongly hyperbolic space  $X$ .

We set  $s = \frac{1}{2} + \frac{\alpha}{2Q}$ . We denote the dual of  $s$  by  $s' = \frac{1}{2} - \frac{\alpha}{2Q}$ . The conformal invariance of  $I_d^\alpha$  on  $s$  densities, manifests itself explicitly as follows.

$$\begin{aligned} I_d^\alpha \left( \left(\frac{d'}{d}\right)^{sQ} \right)(x) &= \int_Z \frac{(\frac{d'}{d}(y))^{sQ}}{d(x,y)^{Q-\alpha}} d\mu_d(y) \\ &= \int_Z \frac{(\frac{d'}{d}(y))^{sQ}}{(\frac{d}{d'}(x) \frac{d}{d'}(y))^{s'Q} d'(x,y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{s'Q} \int_Z \frac{(\frac{d'}{d}(y))^Q}{d'(x,y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{s'Q} \int_Z \frac{1}{d'(x,y)^{Q-\alpha}} d\mu_{d'}(y). \end{aligned} \quad (271)$$

If  $\mathcal{M}$  is a compact Möbius structure, then  $\int_Z \frac{1}{d'(x,y)^{Q-\alpha}} d\mu_{d'}(y)$  is finite.

**Lemma 11** *If  $\mathcal{M}$  is a spherical,  $Q$ -Ahlfors-David regular Möbius structure, then there exists  $C > 0$  s.t.  $\forall d \in \mathcal{M}$ ,*

$$1 + \frac{Q-\alpha}{\alpha} \frac{1}{C} \leq \int_Z \frac{1}{d(x,y)^{Q-\alpha}} d\mu_d(y) \leq 1 + \frac{Q-\alpha}{\alpha} C. \quad (272)$$

**Proof:** We use integration by parts to integrate over balls.

$$\begin{aligned}
\int_Z \frac{1}{d(x,y)^{Q-\alpha}} d\mu_d(u) &= \int_Z (Q-\alpha) \int_0^\infty t^{-Q+\alpha-1} \chi_{\{t \geq d(x,y)\}} dt d\mu_d(y) \\
&= (Q-\alpha) \int_0^\infty t^{-Q+\alpha-1} \mu_d(B_d(x,t)) dt \\
&= 1 + (Q-\alpha) \int_1^\infty t^{-Q+\alpha-1} \mu_d(B_d(x,t)) dt.
\end{aligned} \tag{273}$$

The last line takes advantage of the fact that  $\mathcal{M}$  is spherical. The uniform Ahlfors-David regularity of  $\mathcal{M}$ , gives a  $C \geq 1$  such that  $\frac{t^Q}{C} \leq \mu_d(B_d(x,t)) \leq Ct^Q$ . Using this above and integrating out, we prove the lemma.  $\blacksquare$

We are now in a position to prove the proposition 9.

**Proof:** (of proposition 9) Recall that we set  $s = \frac{1}{2} + \frac{\alpha}{2Q}$  and  $s' = \frac{1}{2} - \frac{\alpha}{2Q}$ . For any three metrics  $d, d', d'' \in \mathcal{M}$  we have,

$$\begin{aligned}
\left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \left(\left(\frac{d''}{d}\right)^{sQ}\right)(x) &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^{sQ}}{d(x,y)^{Q-\alpha}} d\mu_d(y) \\
&= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^{sQ}}{\left(\frac{d}{d''}(x)\frac{d}{d''}(y)\right)^{s'Q} d''(x,y)^{Q-\alpha}} d\mu_d(y) \\
&= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d''}{d}(x)\right)^{s'Q} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^Q}{d''(x,y)^{Q-\alpha}} d\mu_d(y) \\
&= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d''}{d}(x)\right)^{s'Q} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}}}{d''(x,y)^{Q-\alpha}} d\mu_{d''}(y) \\
&= \left(\frac{d''}{d}(x)\right)^{\frac{Q}{2}} \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d}{d''}(x)\right)^{\frac{\alpha}{2}} I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right)(x) \\
&= \left(\frac{d''}{d}(x)\right)^{\frac{Q}{2}} \left(\frac{d'}{d''}(x)\right)^{\frac{\alpha}{2}} I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right)(x).
\end{aligned} \tag{274}$$

Thus,

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \left(\left(\frac{d''}{d}\right)^{sQ}\right) \right\|_{L^2(d\mu_d)} = \left\| I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right) \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)} \tag{275}$$

Assuming that there is  $C > 0$  s.t

$$\left\| I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right) \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)} \leq C \left\| \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)}. \tag{276}$$

But  $\left\| \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)} = \left\| \left(\frac{d''}{d}\right)^{s'Q} \right\|_{L^2(\mu_d)}$ , and lemma 11 implies for a spherical Möbius structure,

$$\left\| \left(\frac{d''}{d}\right)^{s'Q} \right\|_{L^2(\mu_d)} \sim \left\| I_d^\alpha \left(\left(\frac{d''}{d}\right)^{sQ}\right) \right\|_{L^2(\mu_d)}. \tag{277}$$

Thus  $C > 0$  exists independently of  $d, d', d''$ , s.t.

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \left(\left(\frac{d''}{d}\right)^{sQ}\right) \right\|_{L^2(d\mu_d)} \leq C \left\| I_d^\alpha \left(\left(\frac{d''}{d}\right)^{sQ}\right) \right\|_{L^2(\mu_d)}. \tag{278}$$

$\blacksquare$

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