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Georg Alexander Gruetzner

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# Harmonic analysis on the boundary of hyperbolic groups

Georg Grötzner

## Abstract

In this paper we show that a Möbius-structure  $\mathcal{M}$  of dimension  $Q$  has a minimal Ahlfors-David constant. This shows that a Möbius space is uniformly  $Q$ -Ahlfors-David regular. In summary, many classical theorems of harmonic analysis on  $\mathbb{R}^n$  admit a Möbius-invariant formulation in the context of Möbius-geometry. We use this observation to show that the Knapp-Stein operator

$$(I_d^\alpha u_d)(x) = \int \frac{u_d(y)}{d(x, y)^{Q-\alpha}} d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2})$$

is a continuous operator on the weighted  $L^2$ -space  $L^2((\frac{d'}{d})^\alpha d\mu_d)$ , with a norm independent of  $d$  and  $d'$ .

From here we construct a Sobolev space  $\mathcal{H}_d^{-\alpha}$  on  $s$ -densities for a given  $s$  as a function of  $\alpha$ . We would like to say that the construction is topologically independent of the metric  $d$ . In this paper we prove that the norms on a large class of functions are comparable.

The work is inspired by a paper by Astengo, Cowling, and Di Blasio [1], who construct uniformly bounded representations for simple Lie groups of rank 1. We formulate the problem in a much more general framework of groups acting on Möbius structures. In particular, all hyperbolic groups.

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# 1 Introduction

## 1.1 Haagerup Property vs. Kazhdan’s Property (T)

Kazhdan in 1968 introduced property (T) as a tool to prove finite generation of non uniform lattices in higher-rank simple Lie groups. This property has turned out to play a central role in operator algebra theory and in geometric group theory. An excellent monograph on property (T) was written by Bekka, de la Harpe and Valette.<sup>1</sup>

## 1.2 Weak a-T-meanability

It is known that hyperbolic groups may be divided into two classes: those with and those without Kazhdan’s Property (T). By the Delorme-Guichardet Theorem, those which have Property (T) do not admit unitary representations with cohomology.

Higher rank Lie groups (rank  $\geq 2$ ), satisfy a certain stronger form of Kazhdan’s Property (T). Namely, higher rank Lie groups do not admit uniformly bounded representations with reduced cohomology.<sup>2</sup>

Conversely, all rank one Lie groups admit uniformly bounded representations with reduced cohomology. This motivates the question, if all hyperbolic groups admit uniformly bounded representations with cohomology. An even stronger conjecture is often quoted and devoted to Y. Shalom.<sup>3 4</sup>

**Conjecture 1 (Y. Shalom)** *Let  $G$  be a non-elementary hyperbolic group. There exists a uniformly bounded representation of  $G$  on a Hilbert space, for which  $H^1(G, \pi) \neq 0$  and for which there exists a proper 1-cocycle.*

Shalom’s conjecture is known to be true for lattices in  $\mathrm{Sp}(n, 1)$ .<sup>5</sup> It also leads us to a conjectural classification of hyperbolic groups due to Cowling. Indeed, if Shalom’s conjecture is true, then we may associate a number  $\Lambda(G)$  to every hyperbolic group  $G$  as follows:

$$\Lambda(G) := \inf_{\pi} \sup_{g \in G} \|\pi_g\|, \tag{1}$$

where the infimum is taken over all uniformly bounded representations with cohomology.

<sup>1</sup>Kazhdan’s Property (T), new mathematical monographs: 11, Cambridge, Bekka, de la Harpe, Valette.

<sup>2</sup>Mini-Workshop: Geometrization of Kazhdan’s Property (T), Mathematisches Forschungsinstitut Oberwolfach, July 7th - July 14th, 2001, Talk: Reduced cohomology of unitary representation, Y. Shalom, (Report No. 29/2001)

<sup>3</sup>Group Actions on Banach Spaces, Nowak, 2014, Handbook of Group Actions

<sup>4</sup>First mentioned in: Mini-Workshop: Geometrization of Kazhdan’s Property (T), Mathematisches Forschungsinstitut Oberwolfach, July 7th - July 14th, 2001, Talk: Reduced cohomology of unitary representation, Y. Shalom, (Report No. 29/2001) (see question 14 in open problem session)

<sup>5</sup> $\mathrm{Sp}(n,1)$  ADMITS A PROPER 1-COCYCLE FOR A UNIFORMLY BOUNDED REPRESENTATION, SHINTARO NISHIKAWA

This invariant measures in some sense the degree to which the strong form of Kazhdan's property (T) fails in  $G$ . If there does not exist uniformly bounded representations with cohomology, we define  $\Lambda(G)$  to be infinite.

$\Lambda(G)$  is presumably linked to an invariant produced by Cowling and Haagerup.<sup>6 7</sup> Cowling believes that  $\Lambda(G)$  is related to the curvature of the hyperbolic group (with the word metric).

There is in general not much known about  $\Lambda(G)$ . The only results in this direction is a spectral gap theorem for automorphism groups of simplicial complexes.<sup>8</sup>

## 2 Möbius spaces

### 2.1 Some basics on Möbius spaces

According to Buyalo and Schröder [2], a Möbius structure  $\mathcal{M}$  on a set  $Z$  is a collection of metrics which define the same metric cross-ratios. More precisely, one considers quadruples  $(x, y, z, w)$  of points in  $Z$  for which no entry occurs three or four times. Such an *admissible* quadruple is associated with a *cross-ratio triple*

$$[x, y, z, w]_d = (d(x, y)d(z, w) : d(x, z)d(y, w) : d(x, w)d(y, z)) \in \mathbb{R}P^2. \quad (2)$$

Two metrics  $d$  and  $d'$  on  $Z$  are said to be *Möbius equivalent* if for any admissible quadruple, their respective cross-ratio triple coincides.

In fact the theory makes it necessary to consider *extended* metrics on  $Z$  that allow the existence of an *infinitely remote* point  $\bullet \in X$ , i.e.  $d(x, \bullet) = \infty$  for all  $x \neq \bullet$ . Such a point  $\bullet$  is always assumed to be unique and  $d(\bullet, \bullet) = 0$ . If such a point exists, we will sometimes use the notation  $Z_\bullet$  for the set  $Z \setminus \{\bullet\}$ . If  $\bullet$  occurs once in an admissible quadruple, then its associated cross-ratio triple is  $[x, y, z, \bullet]_d = (d(x, y) : d(x, z) : d(y, z))$ . If it occurs twice, then  $[x, y, \bullet, \bullet]_d = (0 : 1 : 1)$ .

A *Möbius structure* on a set  $Z$  is a class  $\mathcal{M}$  of metrics on  $Z$ , which are pairwise Möbius equivalent.

Two metrics  $d$  and  $d'$  in a Möbius structure are conformal in the strong sense that the limiting ratio  $\frac{d'}{d}(x) := \lim_{y \rightarrow x} \frac{d'(x, y)}{d(x, y)}$  exists for all  $x \in Z \setminus \{\bullet\}$ . Furthermore, for all  $x \neq y \in Z \setminus \{\bullet\}$ ,

$$\frac{d'}{d}(x) \frac{d'}{d}(y) = \left( \frac{d'(x, y)}{d(x, y)} \right)^2. \quad (3)$$

Examples of sets with natural Möbius structures are:

1. The round sphere,
2. Boundaries of rank one symmetric spaces,
3. Boundaries of CAT(-1) spaces [3],
4. Boundaries of strongly hyperbolic metric spaces [4],

---

<sup>6</sup>The Constants of Cowling and Haagerup, Varadharajan Muruganandam, Journal of Lie Theory, Volume 18, (2008)

<sup>7</sup>Completely bounded multipliers of Fourier algebra of a simple Lie group of real rank one, Cowling and Haagerup, Inventiones mathematicae, volume 96, 1989

<sup>8</sup>Automorphism groups of simplicial complexes and rigidity for uniformly bounded representations (J. Koivisto)

5. Boundaries of higher rank symmetric spaces and Euclidean buildings [5], [6],
6. Boundaries of certain CAT(0) cube complexes [7], [8].

A Möbius structure  $\mathcal{M}$  is associated with a family of Hausdorff measures  $\{\mu_d\}_{d \in \mathcal{M}}$  with a uniform Hausdorff-dimension  $Q$ . Given two metrics  $d, d' \in \mathcal{M}$ , the Radon-Nikodým derivative  $\frac{d\mu_d}{d\mu_{d'}}$  is related to the conformal factor  $\frac{d}{d'}$  by

$$\frac{d\mu_d}{d\mu_{d'}}(x) = \frac{d}{d'}(x)^Q \quad \forall x \in Z \setminus \{\bullet\}. \quad (4)$$

In particular, metrics in  $\mathcal{M}$  without remote points, have equivalent measures.

We say that a Möbius space  $(Z, \mathcal{M})$  or Möbius structure  $\mathcal{M}$  is *compact*, if  $(Z, d)$  is compact for all metrics  $d \in \mathcal{M}$ .

We say that a Möbius space  $(Z, \mathcal{M})$  or Möbius structure  $\mathcal{M}$  is *spherical*, if  $\text{diam}(Z, d) \leq 1$  for all  $d \in \mathcal{M}$ .

Strongly hyperbolic metric spaces admit a natural spherical Möbius structure on its boundary. We will discuss this case in detail below. As we will see, it turns out that every hyperbolic group admits strongly hyperbolic left invariant metrics [9].

## 2.2 The metric Cayley transform

Stereographic projection is a conformal bijection from the unit sphere  $\mathbb{S}$  in  $\mathbb{R}^{n+1}$  to  $\mathbb{R}^n \cup \{\infty\}$ . Its generalization to Iwasawa  $N$ -groups associated with rank 1 simple Lie groups is known as the Cayley transform. A purely metric analog of the Cayley transform may be given as follows.

A  $K$ -*semi-metric* is a symmetric function  $\rho(x, y)$  on  $Z$ , such that there exist  $K > 0$  and a metric  $d$  (possibly extended) on  $Z$ , s.t.

$$\frac{1}{K}d(x, y) \leq \rho(x, y) \leq Kd(x, y). \quad (5)$$

A semi-metric space is a set  $Z$  together with a semi-metric  $\rho$ . The semi-metric  $\rho$  and the metric  $d$  induce the same topology on  $Z$ . Furthermore, we may speak of Lipschitz maps, Möbius maps or symmetric maps between semi-metric spaces.

**Definition 1** *Let  $(Z, \rho)$  be a semi-metric space, the Cayley transform w.r.t. a point  $\bullet \in Z$ , is the bijection induced by replacing  $\rho$  on  $Z$  with the semi-metric*

$$\rho_{\bullet}(x, y) = \frac{\rho(x, y)}{\rho(x, \bullet)\rho(\bullet, y)} \quad \text{if } x, y \in Z \setminus \{\bullet\}, \quad (6)$$

$\rho_{\bullet}(x, \bullet) = \infty$  for all  $x \in Z \setminus \{\bullet\}$  and  $\rho_{\bullet}(\bullet, \bullet) = 0$ .

If  $Z$  already has an infinitely remote point  $\bullet'$ , the convention is that  $\rho_{\bullet}(x, \bullet') = \frac{1}{\rho(x, \bullet)}$ .

The Cayley transform is conformal in the sense that for all admissible quadruples  $(x, y, z, w) \in Z$ ,

$$[x, y, z, w]_{\rho} = [x, y, z, w]_{\rho_{\bullet}}. \quad (7)$$

A metric  $d$  is *Ptolemaic* if its Cayley transform  $d_{\bullet}$  with respect to every point  $\bullet \in Z$  is

again a metric. A Möbius space  $(Z, \mathcal{M})$  for which the Cayley transform with respect to every point, maps  $\mathcal{M}$  into itself, is called a *Ptolemy space*.

**Proposition 1 (Bonk, Kleiner)** *The Cayley transform maps  $K$ -semi-metrics to  $4K^3$ -semi-metrics.*

**Proof:** By assumption,

$$\frac{1}{K^3}d_{\bullet}(x, y) \leq \rho_{\bullet}(x, y) \leq K^3d_{\bullet}(x, y). \quad (8)$$

A slight modification of the semi-metric  $d_{\bullet}$  gives a metric by setting

$$d(x, y) = \inf_{x=x_0, x_1, \dots, x_{k-1}, x_k=y} \sum_{j=0}^k d_{\bullet}(x_j, x_{j+1}). \quad (9)$$

One checks (details Bonk and Kleiner) that

$$\frac{1}{4}d_{\bullet}(x, y) \leq d(x, y) \leq d_{\bullet}(x, y).^9 \quad (10)$$

Thus,

$$\frac{1}{4K^3}d(x, y) \leq \rho_{\bullet}(x, y) \leq 4K^3d(x, y). \quad (11)$$

■

The conformal factor between the semi-metrics  $\rho_{\bullet}$  and  $\rho$  is well-defined and given by

$$\frac{\rho_{\bullet}}{\rho}(x) = \frac{1}{\rho(x, \bullet)^2}. \quad \forall x \in Z \setminus \{\bullet\} \quad (12)$$

Therefore, the Hausdorff measure  $\mu_{\rho_{\bullet}}$  of  $\rho_{\bullet}$  can be expressed in terms of the Hausdorff measure  $\mu_{\rho}$  by

$$\mu_{\rho_{\bullet}}(A) = \int_A \frac{1}{\rho(z, \bullet)^{2Q}} d\mu_{\rho}(z). \quad A \subset Z \setminus \{\bullet\} \quad (13)$$

An inverse Cayley transform from an unbounded semi-metric space to a bounded one may be given as follows.

**Definition 2** *Let  $(Z, \rho)$  be a semi-metric space with an infinite remote point  $\bullet \in Z$ . The inverse Cayley transform w.r.t. a center  $c \in Z \setminus \{\bullet\}$  is the map induced by replacing  $\rho$  with the semi-metric*

$$|\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} \quad \text{if } x, y \in Z \setminus \{\bullet\}, \quad (14)$$

$|\rho|(x, \bullet) = |\rho|(\bullet, x) = \frac{1}{(1 + \rho(x, c))}$  for all  $x \in Z \setminus \{\bullet\}$ , and  $|\rho|(\bullet, \bullet) = 0$ .

The inverse Cayley transform is again conformal in the sense that for all admissible quadruples  $(x, y, z, w) \in Z$ ,

$$[x, y, z, w]_{\rho} = [x, y, z, w]_{|\rho|}. \quad (15)$$

---

<sup>9</sup>Rigidity of quasi-möbius group actions, Bonk and Kleiner, lemma 2.2

**Proposition 2 (Bonk, Kleiner)** *The inverse Cayley transform maps  $K$ -semi-metrics to  $3K^3$ -semi-metrics.*

**Proof:** By assumption,

$$\frac{1}{K^3} \frac{d(x, y)}{(1 + d(x, c))(1 + d(y, c))} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} \leq \frac{K^3 d(x, y)}{(1 + d(x, c))(1 + d(y, c))}. \quad (16)$$

As above, a slight modification of the semi-metric  $|d|$  gives a metric by setting

$$d(x, y) = \inf_{x=x_0, x_1, \dots, x_{k-1}, x_k=y} \sum_{j=0}^k |d|(x_j, x_{j+1}). \quad (17)$$

One checks (details see Bonk and Kleiner) that

$$\frac{1}{4} |d|(x, y) \leq d(x, y) \leq |d|(x, y).^{10} \quad (18)$$

Thus

$$\frac{1}{4K^3} d(x, y) \leq |\rho|(x, y) \leq 4K^3 d(x, y). \quad (19)$$

■

The conformal factor between the semi-metric  $|\rho|$  and  $\rho$  is well-defined and given by

$$\frac{|\rho|}{\rho}(x) = \frac{1}{(1 + \rho(x, c))^2}. \quad \forall x \in Z \setminus \{\bullet\} \quad (20)$$

Therefore, the Hausdorff measure  $\mu_{|\rho|}$  of  $|\rho|$  may be expressed in terms of the Hausdorff measure of  $\mu_\rho$ .

$$\mu_{|\rho|}(A) = \int_{A \setminus \{\infty\}} \frac{1}{(1 + \rho(z, c))^{2Q}} d\mu_\rho(z). \quad \forall A \subset Z \quad (21)$$

The following proposition shows that the Cayley transform of a bounded semi-metric space  $(Z, \rho)$ , followed by the inverse Cayley transform, gives back the semi-metric space  $(Z, \rho)$  up to a bi-Lipschitz map. This justifies our naming convention.

**Proposition 3 (Buckley, Herron, Xie)** *Let  $(Z, \rho)$  be a bounded semi-metric space and let  $(Z, \rho_\bullet)$  be its Cayley transform w.r.t. a non-isolated point  $\bullet \in Z$ . The inverse Cayley transform of  $(Z, \rho_\bullet)$  w.r.t. any center in  $Z \setminus \{\bullet\}$ , is bi-Lipschitz equivalent to  $(Z, \rho)$ .*

**Remark 1** *Qualitatively, if we assume that  $\text{diam}(Z) = 1$ , then the proof gives a bi-Lipschitz constant of  $\max\{16, 4K^4\}$ .*

**Proof:** We may assume that  $\text{diam}(Z) = 1$ . Pick  $c \in Z \setminus \{\bullet\}$  with  $\rho(c, \bullet) \geq \frac{1}{2}$ .

We check that for all  $x \in Z_\bullet$ ,

$$\frac{1}{4} \rho(x, \bullet) \leq |\rho_\bullet|(x, \bullet) \leq 2K^2 \rho(x, \bullet). \quad (22)$$

---

<sup>10</sup>Rigidity of quasi-möbius group actions, Bonk and Kleiner, lemma 2.2

The upper bound is calculated as follows,

$$\begin{aligned}
|\rho_\bullet|(x, \bullet) &= \frac{1}{1 + \rho_\bullet(x, c)} \\
&= \frac{\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet)\rho(\bullet, c) + \rho(x, c)} \\
&\leq \frac{2\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet) + 2\rho(x, c)} \leq 2K^2 \rho(x, \bullet).
\end{aligned} \tag{23}$$

For the lower bound,

$$\begin{aligned}
|\rho_\bullet|(x, \bullet) &= \frac{1}{1 + \rho_\bullet(x, c)} \\
&= \frac{\rho(x, \bullet)\rho(\bullet, c)}{\rho(x, \bullet)\rho(\bullet, c) + \rho(x, c)} \\
&\geq \frac{1}{2} \frac{\rho(x, \bullet)}{\rho(x, \bullet) + \rho(x, c)} \geq \frac{1}{4} \rho(x, \bullet).
\end{aligned} \tag{24}$$

If  $x, y \in Z_\bullet$ , then

$$\begin{aligned}
|\rho_\bullet|(x, y) &= \frac{\rho_\bullet(x, y)}{(1 + \rho_\bullet(c, x))(1 + \rho_\bullet(c, y))} \\
&= |\rho_\bullet|(x, \bullet) \rho_\bullet(x, y) |\rho_\bullet|(y, \bullet) \\
&= \frac{|\rho_\bullet|(x, \bullet)|\rho_\bullet|(y, \bullet)}{\rho(x, \bullet)\rho(\bullet, y)} \rho(x, y) \\
&\leq 4K^4 \rho(x, y).
\end{aligned} \tag{25}$$

For the lower bound one obtains,

$$|\rho_\bullet|(x, y) \geq \frac{1}{16} \rho(x, y). \tag{26}$$

■

### 2.3 Uniform Ahlfors-David regularity

We have seen that with every Möbius-structure  $\mathcal{M}$  is associated a family of measures  $\{\mu_d\}_{d \in \mathcal{M}}$ . This family of measures shares regularity properties in the following sense.

**Theorem 1** *Given a Möbius-structure  $\mathcal{M}$ , if there exists  $d \in \mathcal{M}$ , such that  $\mu_d$  is  $Q$ -Ahlfors-David regular, then there exists  $C \geq 1$  such that  $\{\mu_d\}_{d \in \mathcal{M}}$  is uniformly Ahlfors-David regular with constant  $C$ .*

This observation is crucial, because it often allows to treat Möbius spaces as uniformly Ahlfors-David regular. Much of the theory of classical harmonic analysis on Ahlfors-David regular metric spaces admits therefore a Möbius invariant analog on Möbius spaces.

The above theorem essentially follows from the fact, that the Cayley transforms of two metrics  $d, d'$  of a Möbius structure  $\mathcal{M}$  are homothetic.

**Lemma 1 (Buyalo, Schroeder)** *Let  $(Z, \mathcal{M})$  be a Möbius space and  $d, d' \in \mathcal{M}$ , then for any point  $\bullet \in Z$ , the semi-metrics  $d_\bullet$  and  $d'_\bullet$  are homothetic i.e. there exists  $\lambda > 0$ , such that  $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$  for all  $x, y \in Z$ .*

**Proof:** We may assume w.l.o.g. that  $Z_\bullet$  contains two distinct points  $x$  and  $y$ . Pick  $\lambda$ , s.t.  $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$ . If there exists a third point  $z \in Z_\bullet$ , then  $[x, y, z, \bullet]_{d'_\bullet} = [x, y, z, \bullet]_{d_\bullet}$ , i.e.  $(d'_\bullet(x, y) : d'_\bullet(x, z) : d'_\bullet(y, z)) = (d_\bullet(x, y) : d_\bullet(x, z) : d_\bullet(y, z))$ . Since  $d'_\bullet(x, y) = \lambda d_\bullet(x, y)$ , it follows that  $d'_\bullet(x, z) = \lambda d_\bullet(x, z)$  and  $d'_\bullet(y, z) = \lambda d_\bullet(y, z)$ .  $\blacksquare$

Hausdorff measures of homothetic semi-metrics share strong regularity properties.

**Lemma 2** *Let  $(Z, \mathcal{M})$  be a Möbius space and  $d, d' \in \mathcal{M}$ , then for any point  $\bullet \in Z$ , the Hausdorff measure  $\mu_{d_\bullet}$  is Ahlfors-David regular with a constant  $C \geq 1$  if and only if  $\mu_{d'_\bullet}$  is Ahlfors-David regular with the same constant.*

**Proof:** We assume that  $\mu_{d'_\bullet}$  is Ahlfors-David regular with constant  $C$ .

From the conformality of the Cayley transform follows,

$$\frac{d'_\bullet(x, z)}{d'_\bullet(x, w)} = \frac{d_\bullet(x, z)}{d_\bullet(x, w)}. \quad (x, y, w \in Z_\bullet) \quad (27)$$

Let  $B_{d_\bullet}(x, r)$  be a ball in  $(Z, d_\bullet)$ . Without loss of generality we may assume that  $\partial B_{d_\bullet}(x, r) \neq \emptyset$ . Otherwise  $B_{d_\bullet}(x, r)$  is closed and there exists  $0 \leq r' < r$ , s.t.  $\bar{B}_{d_\bullet}(x, r') = B_{d_\bullet}(x, r)$ ,  $\partial B_{d_\bullet}(x, r') \neq \emptyset$  and we replace  $B(x, r)$  with  $B_{d_\bullet}(x, r')$ .

Let  $w \in Z$ , s.t.  $r = d_\bullet(x, w)$ , then

$$B_{d_\bullet}(x, r) = B_{d'_\bullet}(x, d'_\bullet(x, w)) \quad (28)$$

by the symmetry condition (27). Thus

$$\begin{aligned} \mu_{d_\bullet}(B_{d_\bullet}(x, r)) &= \int_{B_{d_\bullet}(x, r)} \left( \frac{d_\bullet}{d'_\bullet}(y) \right)^Q d\mu_{d'_\bullet}(y) \\ &= \lambda^{-Q} \mu_{d'_\bullet}(B_{d_\bullet}(x, r)) \\ &= \lambda^{-Q} \mu_{d'_\bullet}(B_{d'_\bullet}(x, d'_\bullet(x, w))) \\ &\leq C \lambda^{-Q} d'_\bullet(x, w)^Q \\ &= C \left( \frac{d_\bullet}{d'_\bullet}(x) \right)^Q d'_\bullet(x, w)^Q \\ &= C d_\bullet(x, w)^Q \\ &= C r^Q. \end{aligned} \quad (29)$$

The lower bound is obtained analogously.  $\blacksquare$

To conclude theorem 1 it remains to show that the Cayley transform and the inverse Cayley transform preserve Ahlfors-David regularity.

**Proposition 4 (Li, Shanmugalingam)** *Let  $(Z, \rho)$  be a  $Q$ -Ahlfors-David regular semi-metric space. Then the Cayley transform  $(Z, \rho_\bullet)$  of  $(Z, \rho)$  is  $Q$ -Ahlfors-David regular, with a constant that depends only on the Ahlfors-David constant of  $(Z, \rho)$ .<sup>11</sup>*

**Proof:** W.l.o.g. we may replace  $\rho$  by its true metric approximation  $d$ . Indeed, if  $(Z, \rho)$  is Ahlfors-David regular with constant  $C \geq 1$ , then so is  $(Z, d)$  with constant  $K^{2Q}C$ .

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<sup>11</sup>Li, Shanmugalingam, Preservation of bounded geometry under sphericalization and flattening, Prop. 4.1.

Let  $C_d$  be the Ahlfors-David constant of the measure  $\mu_d$ , and let  $B_{d_\bullet}(x, r)$  be any ball in  $(Z, d_\bullet)$ . We distinguish three cases,  $rd(x, \bullet) \leq \frac{1}{2}$ ,  $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$ , and  $\frac{1}{2} < rd(x, \bullet) < 4C_d^{\frac{2}{Q}}$ .

Case 1 ( $rd(x, \bullet) \leq \frac{1}{2}$ ): We are looking for two comparable balls  $B_d$  and  $B'_d$  s.t.  $B_d \subset B_{d_\bullet}(x, r) \subset B'_d$ .

For  $y \in B_{d_\bullet}(x, r)$ ,

$$d(x, y) < rd(x, \bullet)d(\bullet, y). \quad (30)$$

Thus,  $d(y, \bullet) - d(\bullet, x) \leq d(x, y) < rd(x, \bullet)d(\bullet, y)$  and a rearrangement, together with  $rd(x, \bullet) \leq \frac{1}{2}$ , gives

$$d(y, \bullet) \leq \frac{d(\bullet, x)}{1 - rd(\bullet, x)} < 2d(\bullet, x).$$

By symmetry, we have as well,  $d(x, \bullet) - d(\bullet, y) \leq d(x, y) < rd(x, \bullet)d(\bullet, y)$  and a rearrangement gives

$$d(y, \bullet) \geq \frac{d(\bullet, x)}{1 + rd(\bullet, x)} > \frac{2d(\bullet, x)}{3}.$$

Thus for  $y \in B_{d_\bullet}(x, r)$

$$\frac{2d(\bullet, x)}{3} < d(y, \bullet) < 2d(x, \bullet). \quad (31)$$

The combination of equation (30) and (31) shows that we may take

$$B'_d = B_d(x, 2rd(x, \bullet)^2).$$

Conversely, let  $y \in B_d(x, \frac{2rd(x, \bullet)^2}{3})$ , then

$$d(x, \bullet) - d(\bullet, y) \leq d(x, y) < \frac{2r}{3}d(x, \bullet)^2 \leq \frac{d(x, \bullet)}{3},$$

where for the last inequality we used the assumption  $rd(x, \bullet) \leq \frac{1}{2}$ . Thus,  $\frac{d(x, \bullet)}{d(y, \bullet)} \leq \frac{3}{2}$ , and therefore

$$d_\bullet(x, y) = \frac{d(x, y)}{d(x, \bullet)d(\bullet, y)} < \frac{2r}{3} \frac{d(x, \bullet)}{d(y, \bullet)} < r. \quad (32)$$

Hence, we may set  $B_d = B_d(x, \frac{2rd(x, \bullet)^2}{3})$ .

We estimate the lower bound as follows.

$$\begin{aligned} \mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\geq \int_{B_d} \frac{d\mu_d(z)}{d(z, \bullet)^{2Q}} \\ &\geq \frac{\mu_d(B_d)}{2^{2Q}d(x, \bullet)^{2Q}} \\ &\geq \frac{(\frac{2rd(x, \bullet)^2}{3})^Q}{2^{2Q}C_d} \frac{1}{d(x, \bullet)^{2Q}} \\ &= \frac{r^Q}{6^Q C_d}. \end{aligned} \quad (33)$$

For the upper bound,

$$\begin{aligned}
\mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\leq \int_{B'_d} \frac{d\mu_d(z)}{d(z, \bullet)^{2Q}} \\
&\leq \left(\frac{3}{2}\right)^{2Q} \frac{\mu_d(B'_d)}{d(x, \bullet)^{2Q}} \\
&\leq \left(\frac{3}{2}\right)^{2Q} \frac{C_d (2rd(x, \bullet))^Q}{d(x, \bullet)^{2Q}} \\
&= \left(\frac{9}{2}\right)^Q C_d r^Q.
\end{aligned} \tag{34}$$

Case 2: Assume  $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$ , then in particular  $Z \setminus B_d(\bullet, \frac{4C_d^{\frac{2}{Q}}}{r}) \neq \emptyset$ .

If  $y \in Z \setminus \{B_{d_\bullet}(x, r) \cup \{\bullet\}\}$ , then

$$rd(x, \bullet)d(\bullet, y) \leq d(x, y) \leq d(x, \bullet) + d(\bullet, y). \tag{35}$$

Rearranging and using  $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$  gives

$$\begin{aligned}
d(y, \bullet) &< \frac{d(x, \bullet)}{rd(x, \bullet) - 1} \\
&\leq \frac{1}{r} \frac{4C_d^{\frac{2}{Q}}}{(4C_d^{\frac{2}{Q}} - 1)} < \frac{2}{r}.
\end{aligned} \tag{36}$$

Thus,

$$Z \setminus \bar{B}_d(\bullet, \frac{2}{r}) \subset B_{d_\bullet}(x, r). \tag{37}$$

Conversely, if  $y \in B_{d_\bullet}(x, r)$ , then

$$d(x, \bullet) - d(\bullet, y) \leq d(x, y) \leq rd(x, \bullet)d(\bullet, y). \tag{38}$$

Rearranging and using  $rd(x, \bullet) \geq 4C_d^{\frac{2}{Q}}$ , gives

$$\begin{aligned}
d(y, \bullet) &> \frac{d(x, \bullet)}{rd(x, \bullet) + 1} \\
&\geq \frac{1}{r} \frac{4C_d^{\frac{2}{Q}}}{4C_d^{\frac{2}{Q}} + 1} > \frac{2}{3r}.
\end{aligned} \tag{39}$$

Thus,

$$Z \setminus \bar{B}_d(\bullet, \frac{2}{r}) \subset B_{d_\bullet}(x, r) \subset Z \setminus \bar{B}_d(\bullet, \frac{2}{3r}). \tag{40}$$

We estimate the upper bound as follows,

$$\begin{aligned}
\mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\leq \int_{Z \setminus \bar{B}_d(\bullet, \frac{2}{3r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\leq \sum_{i=1}^{\infty} \int_{\bar{B}_d(\bullet, \frac{2^{i+1}}{3r}) \setminus \bar{B}_d(\bullet, \frac{2^i}{3r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\leq \sum_{i=1}^{\infty} \int_{\bar{B}_d(\bullet, \frac{2^{i+1}}{3r}) \setminus \bar{B}_d(\bullet, \frac{2^i}{3r})} \frac{d\mu_d(z)}{(\frac{2^i}{3r})^{2Q}} \\
&\leq C_d \sum_{i=1}^{\infty} \frac{(\frac{2^{i+1}}{3r})^Q}{(\frac{2^i}{3r})^{2Q}} \\
&= C_d \sum_{i=0}^{\infty} 2^{-iQ} (3r)^Q \\
&= \frac{2^Q}{2^Q - 1} 3^Q C_d r^Q.
\end{aligned} \tag{41}$$

There exists a lower bound as well, since

$$\begin{aligned}
\mu_d(B_d(\bullet, \frac{4C_d^{\frac{2}{Q}}}{r}) \setminus B_d(\bullet, \frac{2}{r})) &\geq \frac{4^Q C_d}{r^Q} - \frac{2^Q C_d}{r^Q} \\
&\geq \frac{2^Q C_d}{r^Q}.
\end{aligned} \tag{42}$$

Therefore, the lower bound can be estimated as follows

$$\begin{aligned}
\mu_{d_\bullet}(B_{d_\bullet}(x, r)) &\geq \int_{Z \setminus \bar{B}_d(\bullet, \frac{2}{r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\geq \int_{B(\bullet, \frac{4C_d^{\frac{2}{Q}}}{r}) \setminus \bar{B}_d(\bullet, \frac{2}{r})} \frac{d\mu_d(z)}{d(\bullet, z)^{2Q}} \\
&\geq \frac{2^Q C_d}{r^Q} \frac{1}{(\frac{4}{r})^{2Q} C_d^4} \\
&= \frac{r^Q}{2^{3Q} C_d^3}
\end{aligned} \tag{43}$$

Case 3: Assume  $\frac{1}{2} < rd(x, \bullet) < 4C_d^{\frac{2}{Q}}$ . Then

$$B_{d_\bullet}(x, \frac{r}{8C_d^{\frac{2}{Q}}}) \subset B_{d_\bullet}(x, r) \subset B_{d_\bullet}(x, 8C_d^{\frac{2}{Q}} r). \tag{44}$$

The interior ball satisfies the hypothesis in case 1, thus

$$\frac{r^Q}{8^Q 6^Q C_d^3} \leq \mu_{d_\bullet}(B_{d_\bullet}(x, \frac{r}{8C_d^{\frac{2}{Q}}})) \leq \mu_{d_\bullet}(B_{d_\bullet}(x, r)). \tag{45}$$

The exterior ball satisfies the hypothesis of case 2, thus

$$\mu_{d_\bullet}(B_{d_\bullet}(x, r)) \leq \mu_{d_\bullet}(B_{d_\bullet}(x, 8C_d^{\frac{2}{Q}} r)) \leq \frac{2^Q}{2^Q - 1} 3^Q 8^Q C_d^3 r^Q. \tag{46}$$

■

**Proposition 5 (Li, Shanmugalingam)** *Let  $(Z, \rho)$  be an unbounded, Ahlfors-David regular, semi-metric space, with constant  $C \geq 1$ . Then its inverse Cayley transform  $(Z, |\rho|)$  is Ahlfors-David regular with a constant at most  $(2K + 1)^{2Q} 20^Q C^2$ , where  $K$  is the semi-metric constant of  $|\rho|$ .<sup>12</sup>*

**Proof:** Let  $\bullet$  be the point at infinity in  $Z$  i.e.  $\rho(x, \bullet) = \infty$  for all  $x \in Z$ . Since  $(Z, |\rho|)$  is compact, it is enough to check Ahlfors-David regularity on balls with  $r \leq \frac{1}{2}$ .

Let  $R = \frac{1}{r} - 1$ , then  $\frac{1}{2r} \leq R < \frac{1}{r}$ . For a ball centered at  $\bullet$ , we have

$$\begin{aligned} B_{|\rho|}(\bullet, r) &= \{x \in Z, \frac{1}{1 + \rho(x, c)} < r\} \cup \{\bullet\} \\ &= \{x \in Z, \rho(x, c) > \frac{1}{r} - 1\} \cup \{\bullet\} \\ &= Z \setminus \bar{B}_\rho(c, R). \end{aligned} \tag{47}$$

Set  $\lambda = (2C_\rho^2)^{\frac{1}{Q}} > 1$ , and define  $B_i = B_\rho(c, \lambda^i R)$  then

$$\begin{aligned} C_\rho(\lambda^{i-1} R)^Q &= (\frac{\lambda^Q}{C_\rho} - C_\rho)(\lambda^{i-1} R)^Q \\ &= \frac{(\lambda^i R)^Q}{C_\rho} - C_\rho(\lambda^{i-1} R)^Q \\ &\leq \mu_\rho(B_i \setminus B_{i-1}), \end{aligned} \tag{48}$$

and

$$B_{|\rho|}(\bullet, r) = \bigcup_{i=1}^{\infty} B_i \setminus \bar{B}_{i-1}. \tag{49}$$

Therefore

$$\mu_{|\rho|}(B_{|\rho|}(\bullet, r)) = \sum_{i=1}^{\infty} \int_{B_i \setminus \bar{B}_{i-1}} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y). \tag{50}$$

Therefore,

$$\sum_{i=1}^{\infty} \frac{\mu_\rho(B_i \setminus \bar{B}_{i-1})}{(1 + \lambda^i R)^{2Q}} \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq \sum_{i=1}^{\infty} \frac{\mu_\rho(B_i \setminus \bar{B}_{i-1})}{(1 + \lambda^{i-1} R)^{2Q}} \tag{51}$$

Using that  $1 \leq \lambda^i R$  and the doubling property of  $\mu_\rho$ ,

$$\frac{C_\rho}{4^Q \lambda^Q R^Q} \sum_{i=0}^{\infty} \lambda^{-iQ} \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq \frac{\lambda^Q C_\rho}{R^Q} \sum_{i=0}^{\infty} \lambda^{-iQ}. \tag{52}$$

Thus

$$\frac{C_\rho}{4^Q} \frac{1}{\lambda^Q - 1} r^Q \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq (2\lambda)^Q C_\rho \frac{\lambda^Q}{\lambda^Q - 1} r^Q. \tag{53}$$

Further simplification yields,

$$\frac{1}{(4\lambda)^Q C_\rho} r^Q \leq \mu_{|\rho|}(B_{|\rho|}(\bullet, r)) \leq (4\lambda)^Q C_\rho r^Q. \tag{54}$$

Let  $x \in Z \setminus \{\bullet\}$ , we will prove this in four cases.

Case 1: Suppose  $|\rho|(x, \bullet) \leq \frac{r}{2K}$ , we reduce this case to the one above.

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<sup>12</sup>Preservation of bounded geometry under sphericalization and flattening, Li, Shanmugalingam, Prop. 3.1.

If  $y \in B_{|\rho|}(x, r)$ , then

$$\begin{aligned} |\rho|(\bullet, y) &\leq K(|\rho|(\bullet, x) + |\rho|(x, y)) \\ &< (K+1)r. \end{aligned} \quad (55)$$

Thus

$$B_\rho^c(x, r) \subset B_\rho^c(\bullet, (K+1)r). \quad (56)$$

If  $y \in B_{|\rho|}(\bullet, \frac{r}{2K})$ , then

$$\begin{aligned} |\rho|(x, y) &\leq K(|\rho|(x, \bullet) + |\rho|(\bullet, y)) \\ &< r. \end{aligned} \quad (57)$$

Thus

$$B_\rho^c(\bullet, \frac{r}{2K}) \subset B_\rho^c(x, r). \quad (58)$$

Therefore

$$\frac{1}{(2K4\lambda)^Q C_\rho} r^Q \leq \mu_{|\rho|}(B_\rho^c(\bullet, \frac{r}{2K})) \leq \mu_{|\rho|}(B_\rho^c(x, r)) \leq \mu_{|\rho|}(B_\rho^c(\bullet, (K+1)r)) \leq ((K+1)4\lambda)^Q C_\rho r^Q. \quad (59)$$

Case 2: Suppose  $\frac{r}{2K} \leq |\rho|(x, \bullet) < 4r$ .

Then as above,

$$B_{|\rho|}(x, r) \subset B_{|\rho|}(\bullet, 5Kr), \quad (60)$$

and therefore

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \leq (5K4\lambda)^Q C_\rho r^Q. \quad (61)$$

For the lower bound, notice that  $\frac{1}{4r} \leq 1 + \rho(x, c) \leq \frac{2K}{r}$ . If  $y \in B_\rho(x, \frac{1}{20r})$ , then

$$1 + \rho(y, c) \geq 1 + \rho(x, c) - \rho(x, y) \geq \frac{1}{4r} - \frac{1}{20r} = \frac{1}{5r}, \quad (62)$$

and therefore

$$|\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < \frac{4r \cdot 5r}{20r} = r. \quad (63)$$

Thus

$$B_\rho(x, \frac{1}{20r}) \subset B_{|\rho|}(x, r). \quad (64)$$

Furthermore, for  $y \in B_\rho(x, \frac{1}{20r})$

$$1 + \rho(y, c) \leq 1 + \rho(x, c) + \rho(x, y) \leq \frac{2K}{r} + \frac{1}{20r} < \frac{2K+1}{r}. \quad (65)$$

Putting all together,

$$\begin{aligned} \mu_{|\rho|}(B_{|\rho|}(x, r)) &\geq \int_{B_\rho(x, \frac{1}{20r})} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \\ &\geq \frac{r^{2Q}}{(2K+1)^{2Q}} \mu_\rho(B_\rho(x, \frac{1}{20r})) \\ &\geq \frac{1}{(2K+1)^{2Q} 20^Q C_\rho} r^Q \end{aligned} \quad (66)$$

Case 3: Suppose  $|\rho|(x, \bullet) \geq 4r$  and  $\rho(x, c) \leq 1$ . Then for all  $y \in B_{|\rho|}(x, r)$ ,

$$\frac{4r\rho(x, y)}{1 + \rho(y, c)} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < r. \quad (67)$$

From this follows that

$$4(1 + \rho(y, c) - 1 - \rho(x, c)) \leq 1 + \rho(y, c). \quad (68)$$

Therefore

$$1 + \rho(y, c) \leq \frac{4}{3}(1 + \rho(x, c)) \leq 3 \quad (69)$$

since  $\rho(x, c) \leq 1$  and hence

$$\frac{1}{6}\rho(x, y) \leq |\rho|(x, y) = \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))}. \quad (70)$$

That is to say  $B_{|\rho|}(x, r) \subset B_\rho(x, 6r)$  and we obtain

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \leq \int_{B_\rho(x, 6r)} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \leq \mu_\rho(B_\rho(x, 6r)) \leq C_\rho 6^Q r^Q. \quad (71)$$

For the lower bound, notice that if  $y \in B_\rho(x, r)$ , then  $|\rho|(x, y) \leq \rho(x, y) < r$  and henceforth  $B_\rho(x, r) \subset B_{|\rho|}(x, r)$ . We therefore obtain

$$\mu_{|\rho|}(B_{|\rho|}(x, r)) \geq \int_{B_\rho(x, r)} \frac{1}{(1 + \rho(y, c))^{2Q}} d\mu_\rho(y) \geq \frac{1}{3^{2Q}} \mu_\rho(B_\rho(x, r)) \geq \frac{1}{9^Q C_\rho} r^Q \quad (72)$$

Case 4: Assume  $|\rho|(x, \bullet) \geq 4r$  and  $\rho(x, c) > 1$ .

Define

$$D(x) = \left\{ y \in Z : \frac{4}{5}(1 + \rho(x, c)) < 1 + \rho(y, c) < \frac{4}{3}(1 + \rho(x, c)) \right\}. \quad (73)$$

As previously, if  $y \in B_{|\rho|}(x, r)$ , then

$$\frac{4r\rho(x, y)}{(1 + \rho(y, c))} \leq \frac{\rho(x, y)}{(1 + \rho(x, c))(1 + \rho(y, c))} < r, \quad (74)$$

and

$$4\rho(x, y) < 1 + \rho(y, c). \quad (75)$$

The last equation together with  $4(1 + \rho(y, c) - 1 - \rho(x, c)) < 1 + \rho(y, c)$  and  $4(1 + \rho(x, c) - 1 - \rho(y, c)) < 1 + \rho(y, c)$  gives

$$\frac{4}{5}(1 + \rho(x, c)) < 1 + \rho(y, c) < \frac{4}{3}(1 + \rho(x, c)). \quad (76)$$

Furthermore,

$$\rho(x, y) < r(1 + \rho(x, c))(1 + \rho(y, c)) < \frac{4}{3}(1 + \rho(x, c))^2 r. \quad (77)$$

Therefore,  $B_{|\rho|}(x, r) \subset D(x) \cap B_\rho(x, \frac{4}{3}(1 + \rho(x, c))^2 r)$ .

Therefore,

$$\begin{aligned}
\mu_{|\rho|}(B_{|\rho|}(x, r)) &\leq \int_{D(x) \cap B_\rho(x, \frac{4}{3}(1+\rho(x, c))^2 r)} \frac{1}{(1+\rho(y, c))^{2Q}} d\mu_\rho(y) \\
&\leq \left(\frac{5}{4}\right)^{2Q} \frac{1}{(1+\rho(x, c))^{2Q}} \mu_\rho(B_\rho(x, \frac{4}{3}(1+\rho(x, c))^2 r)) \\
&\leq \left(\frac{5}{4}\right)^{2Q} \left(\frac{4}{3}\right)^Q C_\rho r^Q = \left(\frac{25}{12}\right)^Q C_\rho r^Q.
\end{aligned} \tag{78}$$

If  $y \in B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r)$ , then

$$\begin{aligned}
\frac{4}{5}(1+\rho(x, c)) &= 1+\rho(x, c) - \frac{1}{5}(1+\rho(x, c)) \\
&\leq 1+\rho(x, c) - \frac{4}{5}(1+\rho(x, c))^2 r \\
&\leq 1+\rho(x, c) - \rho(x, y) \\
&\leq 1+\rho(y, c) \\
&\leq 1+\rho(x, c) + \rho(x, y) \\
&< 1+\rho(x, c) + \frac{4r}{5}(1+\rho(x, c))^2 \\
&\leq 1+\rho(x, c) + \frac{1}{5}(1+\rho(x, c)) \\
&< \frac{4}{3}(1+\rho(x, c)).
\end{aligned} \tag{79}$$

Therefore  $B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r) \subset D(x)$ . Furthermore, knowing this inclusion,

$$\begin{aligned}
|\rho|(x, y) &= \frac{\rho(x, y)}{(1+\rho(x, c))(1+\rho(y, c))} \\
&< \frac{4}{5}(1+\rho(x, c))^2 r \frac{5}{4(1+\rho(x, c))^2} \\
&= r.
\end{aligned} \tag{80}$$

Thus  $B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r) \subset B_{|\rho|}(x, r) \subset D(x)$ . It follows that

$$\begin{aligned}
\mu_{|\rho|}(B_{|\rho|}(x, r)) &\geq \int_{B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r)} \frac{1}{(1+\rho(y, c))^{2Q}} d\mu_\rho(y) \\
&\geq \left(\frac{3}{4}\right)^{2Q} \frac{1}{(1+\rho(x, c))^{2Q}} \mu_\rho(B_\rho(x, \frac{4}{5}(1+\rho(x, c))^2 r)) \\
&\geq \left(\frac{3}{4}\right)^{2Q} \left(\frac{4}{5}\right)^Q \frac{1}{C_\rho} r^Q = \left(\frac{9}{20}\right)^Q \frac{1}{C_\rho} r^Q
\end{aligned} \tag{81}$$

■

### 3 The boundary of hyperbolic groups

#### 3.1 $\delta$ -hyperbolic spaces

In the 80's Mikhael Gromov introduced a notion of hyperbolicity for metric spaces that stays invariant under small changes in topology. It generalizes the metric properties of classical hyperbolic geometry and of trees. The basic definitions are remarkably simple. The *Gromov product* of two points  $x$  and  $y$  with respect to an origin  $o$ , is the non-negative

number

$$\langle x, y \rangle_o = \frac{1}{2}(d(x, o) + d(y, o) - d(x, y)). \quad (82)$$

In trees,  $\langle x, y \rangle_o$  is the distance of  $o$  to the branching point of geodesics emanating from  $o$  and passing through  $x$  and  $y$ .

A metric space  $(X, d)$  is said to be *Gromov hyperbolic* or  $\delta$ -*hyperbolic*, if for any triple of points  $x, y, z$ ,

$$\langle x, y \rangle_o \geq \min\{\langle x, z \rangle_o, \langle z, y \rangle_o\} - \delta. \quad (83)$$

This condition is independent of the basepoint in the sense that, if it holds for some  $o$  and  $\delta$ , it holds for all basepoints with  $2\delta$ .

### 3.2 Gromov hyperbolic group

A group  $\Gamma$  is said to be *Gromov hyperbolic*, or *hyperbolic* for short, if it acts by isometries, properly discontinuously and cocompactly on a proper, geodesic  $\delta$ -hyperbolic metric space.

### 3.3 Gromov boundary

Gromov hyperbolic spaces have an interesting geometry at infinity, much like the ideal boundary of classical hyperbolic space.

The construction of the bordification goes as follows. We may say that a sequence of points  $x_n$  in  $X$  'converges at infinity', if  $\langle x_i, x_j \rangle_o \rightarrow +\infty$  as  $i, j$  goes to infinity. Two such sequences  $x_n, y_n$  are considered equivalent, if  $\langle x_i, y_i \rangle_o \rightarrow +\infty$  as  $i$  goes to infinity. The Gromov boundary  $\partial X$  of  $X$  is the set of equivalence classes of such sequences.

The Gromov product can be extended to the boundary by setting

$$\langle \eta, \xi \rangle_o = \inf \liminf_{i, j} \langle x_i, y_j \rangle_o, \quad (84)$$

where the infimum is taken over all representatives  $x_n$  and  $y_n$  of  $\eta$  and  $\xi$  respectively. Although the choice of  $\liminf$  make the above definition non-canonical, it turns out that replacing  $\inf$  with  $\sup$  or  $\limsup$  by  $\liminf$ , all lead to quantities that are boundedly close to each other. The above definition is therefore at least canonical on topological level and defines a topology for convergence at infinity on the bordification  $X \cup \partial X$ .

The topology of convergence at infinity on  $\partial X$ , is in fact metrizable. It turns out, that the metric is for small enough  $\epsilon$  within a constant multiple of  $\exp(-\epsilon \langle \cdot, \cdot \rangle_o)$ . We will see below that this result can be improved.

### 3.4 Roughly geodesic and strongly hyperbolic metric spaces

The classical theory of hyperbolic groups takes place in the world of geodesic metric spaces. However, the extension of the Gromov product to  $\partial X$  requires a non-canonical choice.

By smoothening out the metric  $d$ , we can achieve that the Gromov product extends indeed continuously to the boundary  $\partial X$ . Moreover, the natural topology on  $\partial X$ , will then be induced by an explicitly given metric called the visual metric or the Bourdon metric. The

trade off is that we need to pass from the world of geodesic metric spaces to roughly geodesic metric spaces.

**Definition 3** *A metric space  $(X, d)$  is roughly geodesic, if there exists  $C > 0$  such that between all  $x, y \in X$  there exists a rough geodesic i.e. a map  $\gamma : [a, b] \subset \mathbb{R} \rightarrow X$ , with  $\gamma(a) = x$ ,  $\gamma(b) = y$  and such that for all  $t, s \in [a, b]$ ,  $|t - s| - C \leq d(\gamma(t), \gamma(s)) \leq |t - s| + C$ .*

A rough geodesic ray is a rough geodesic  $r : [0, \infty) \rightarrow X$ . Two rough geodesic rays are considered equivalent if  $\sup_t d(r(t), r'(t)) < \infty$ . An alternative construction of the Gromov boundary of a proper and roughly geodesic space  $X$ , is to consider it as the set of equivalence classes of rough geodesic rays.

The relaxation of the geodesic assumption for hyperbolic spaces to roughly geodesic, was first advocated by Bonk and Schramm.<sup>13</sup> Hyperbolicity is a quasi-isometric invariant for roughly geodesic metric spaces and also the Schwarz-Milnor lemma holds in this context. We may therefore equally well say, that a group is hyperbolic if it acts geometrically on a roughly geodesic, proper hyperbolic metric space.

**Definition 4** *A metric space  $(X, d)$  is  $\epsilon$ -strongly hyperbolic, if for every triple of points  $x, y, z$  and some origin  $o$  in  $X$ ,*

$$\exp(-\epsilon \langle x, y \rangle_o) \leq \exp(\langle x, z \rangle_o) + \exp(\langle z, y \rangle_o).$$

**Theorem 2 (Nica-Spakula)** *Let  $(X, d)$  be an  $\epsilon$ -strongly hyperbolic metric space, then*

1.  $(X, d)$  is  $\log(2)/\epsilon$ -hyperbolic,
2. the Gromov product extends continuously to the bordification  $X \cup \partial X$ ,
3. and the function  $d_{o,\epsilon}(\eta, \xi) = \exp(-\epsilon \langle \eta, \xi \rangle_o)$  is a metric on  $\partial X$ .

**Remark 2** *In the context of roughly geodesic metric spaces, a strongly hyperbolic metric space is strongly bolic in the sense of Lafforgue.<sup>14</sup>*

Examples of strongly hyperbolic metric spaces are  $\text{CAT}(-1)$  spaces.<sup>15</sup> In fact, any hyperbolic group  $\Gamma$ , acts geometrically on a roughly geodesic and strongly hyperbolic metric space. This metric is obtained by smoothening the word metric on  $\Gamma$ .

**Theorem 3 (Nika-Spakula)** *Let  $\Gamma$  be a hyperbolic group. Then the Green metric defined by a symmetric and finitely supported random walk on  $\Gamma$  is strongly hyperbolic.*

### 3.5 A Möbius structure on strongly hyperbolic spaces.

Strongly hyperbolic metric spaces admit a *conformal structure* on its boundary. This stands in opposition to only a *quasi conformal structure* on the boundary of hyperbolic spaces.

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<sup>13</sup>Embedding of Gromov hyperbolic spaces, Bonk and Schramm, 2000

<sup>14</sup>K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes, V. Lafforgue, 2002

<sup>15</sup>Strong Hyperbolicity, Nica Spakula (arXiv 2015)

If  $X$  is a strongly hyperbolic space, the continuity of the Gromov product allows us to define Busemann-functions

$$\beta_\xi(o, o') = 2\langle \xi, o' \rangle_o - d(o, o') = \lim_{x \rightarrow \xi} (d(x, o) - d(x, o')). \quad (o, o' \in X, \xi \in \partial X) \quad (85)$$

For every pair of points  $o, o'$  in an  $\epsilon$ -strongly hyperbolic metric space,

$$d_{o,\epsilon}(\eta, \xi) = \exp\left(\frac{\epsilon}{2}(\beta_\eta(o, o') + \beta_\xi(o, o'))\right) d_{o',\epsilon}(\eta, \xi). \quad (86)$$

This implies, that for any admissible quadruple  $(x, y, z, w)$ , the cross-ratio triple  $[x, y, z, w]_{d_{o,\epsilon}}$  does not depend on  $o$ , and  $\{d_o\}_{o \in X}$  is a Möbius structure on  $\partial X$ .

### 3.6 Petterson-Sullivan measures

We have seen that the ideal boundary of a strongly hyperbolic metric space  $X$ , is a Möbius space when equipped with the set of visual metrics  $\mathcal{M} = \{d_{o,\epsilon}\}_{o \in X}$ . The Hausdorff measure  $\mu_o$  associated with the metric  $d_{o,\epsilon}$  is known as a *Petterson-Sullivan* measure<sup>16</sup>. Every such measure associated with the Möbius-structure  $\mathcal{M}$  has the same Hausdorff dimension  $Q$ .

On strongly hyperbolic spaces, the Radon-Nikodým derivatives  $\frac{d\mu_{o'}}{d\mu_o}$  have a concrete expression in terms of Busemann functions,

$$\frac{d\mu_{o',\epsilon}}{d\mu_{o,\epsilon}}(x) = e^{\epsilon Q \beta_x(o, o')}. \quad (87)$$

Petterson-Sullivan measures are Ahlfors-David regular.<sup>17</sup> As we have seen, in the context of Möbius geometry even more is true. There exists a constant  $C \geq 1$ , such that for every measure  $\mu_o \in \{\mu_o\}_{o \in X}$ ,

$$\frac{1}{C} r^Q \leq \mu_o(B(x, r)) \leq C r^Q. \quad (88)$$

### 3.7 Sullivan's shadow lemma.

Let  $X$  be a hyperbolic group endowed with a left-invariant hyperbolic metric, quasi-isometric to a word metric. We may assume that for any origin  $o$  and any other point  $o' \in X$ , there exists a roughly geodesic ray  $\gamma$  emanating from  $o$ , and that passes within uniform distance to  $o'$ . In terms of the Gromov product, we may assume the existence of a constant  $M > 0$ , such that for all  $o' \in X$ ,

$$\sup_{x \in \partial X} \langle x, o' \rangle_o \geq d(o, o') - M. \quad (89)$$

The constant  $M$  can be chosen independent of the choice of an origin  $o$ .

**Lemma 3 (Garncarek)** *If  $X$  is a  $\delta$ -Gromov-hyperbolic group, then there exists  $M > 0$*

<sup>16</sup>Petterson, The limit set of a Fuchsian group, 1976;

<sup>17</sup>Sullivan, The density at infinity for a discrete group of hyperbolic motions, 1979, Corollary 10

such that for all  $o'$  and all  $o$  in  $X$ ,

$$\sup_{x \in \partial X} \langle x, o' \rangle_o \geq d(o, o') - M. \quad (90)$$

Since there exists for every  $o'$  in  $X$  a roughly geodesic ray  $\gamma$  emanating from  $o$  that passes close to  $o'$ , we may associate to it a forward endpoint  $\bullet'$ . More precisely, let  $\bullet'$  be a point in  $\partial X$ , such that

$$\langle \bullet', o' \rangle_o \geq d(o, o') - M. \quad (91)$$

**Proof:** (of Garncarek's lemma) For all point  $x \in X$ , the Gromov-product satisfies

$$\langle g^{-1}o, x \rangle_o + \langle go, gx \rangle_o = d(o, g^{-1}o).$$

Let  $x \in \partial X$  and  $\{x_i\}$  a representative of  $x$ , then

$$d(o, g^{-1}o) = \liminf_{i \rightarrow \infty} (\langle g^{-1}o, x_i \rangle_o + \langle go, gx_i \rangle_o) \leq \langle g^{-1}o, x \rangle_o + \langle go, gx \rangle_o + 4\delta.$$

Thus

$$\langle go, gx \rangle_o \geq d(o, g^{-1}o) - \langle g^{-1}o, x \rangle_o - 4\delta.$$

and for two distinct points  $y_1, y_2 \in \partial X$ ,

$$\begin{aligned} \sup_{x \in \partial X} \langle x, go \rangle_o &\geq \max_i \langle go, gy_i \rangle_o \\ &\geq d(o, g^{-1}o) - \min\{\langle g^{-1}o, y_1 \rangle_o, \langle g^{-1}o, y_2 \rangle_o\} - 4\delta \\ &\geq d(o, go) - \langle y_1, y_2 \rangle_o - 5\delta. \end{aligned}$$

Set  $M = \inf_{y_1, y_2 \in \partial X} \langle y_1, y_2 \rangle_o + 5\delta$  ■

A ball of radius  $s$  around a point  $o'$  in  $X$ , casts a shadow at infinity seen from  $o$ . Explicitly, we may define the shadow at infinity of a ball of radius  $s$  around  $o'$  as

$$\mathcal{O}_o^{o'}(s) = \{x \in \partial X : \langle x, o' \rangle_o \geq d(o, o') - s\} \quad (92)$$

The shadows at infinity are essentially balls around the endpoint of a rough geodesic ray almost passing through  $o'$ .

**Lemma 4 (Sullivan's shadow lemma)** *Let  $X$  be a strongly-hyperbolic space. Let  $\bullet'$  be a point in the boundary such that*

$$\langle \bullet', o' \rangle_o \geq d(o, o') - M.$$

*Then for all  $s > M + \delta$*

$$\bar{B}_o(\bullet', e^{-\epsilon(d(o, o') - s + \delta)}) \subset \mathcal{O}_o^{o'}(s) \subset \bar{B}_o(\bullet', e^{-\epsilon(d(o, o') - s - \delta)}).$$

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<sup>18</sup>Lukasz Garncarek, Boundary representations of hyperbolic groups

**Proof:** If  $x \in \mathcal{O}'_o(s)$ , then

$$\begin{aligned} \langle \bullet', x \rangle_o &\geq \min\{\langle \bullet', o' \rangle_o, \langle o', x \rangle_o\} - \delta \\ &\geq \min\{d(o, o') - M, d(o, o') - s\} - \delta \\ &= d(o, o') - s - \delta. \end{aligned} \tag{93}$$

Therefore  $d_{o,\epsilon}(\bullet', x) \leq e^{-\epsilon(d(o,o')-s-\delta)}$ . Conversely, if  $x \in \bar{B}_o(\bullet', e^{-\epsilon(d(o,o')-s+\delta)})$ , then

$$\begin{aligned} \langle o', x \rangle_o &\geq \min\{\langle o', \bullet' \rangle_o, \langle \bullet', x \rangle_o\} - \delta \\ &\geq \min\{d(o, o') - M, d(o, o') - s + \delta\} - \delta \\ &= d(o, o') - s. \end{aligned} \tag{94}$$

Equality in the last line follows from the assumption  $s > M + \delta$ . ■

### 3.8 Distances associated to boundary points, the Cayley transform revisited

Let  $X$  denote a strongly hyperbolic metric space, and  $Z = \partial X$  its ideal boundary equipped with its natural Möbius structure, i.e. a family of distances  $d_o$  indexed by points  $o$  of  $X$ :

$$d_{o,\epsilon}(x, y) = e^{-\epsilon \langle x, y \rangle_o}.$$

For points  $p, q \in X$ ,

$$\langle p, q \rangle_o = \frac{1}{2}(d(o, p) + d(o, q) - d(p, q)).$$

One then shows that  $\langle p, q \rangle_o$  has a limit, denoted by  $\langle x, y \rangle_o$ , as  $p$  tends to  $x$  and  $q$  tends to  $y$ .

**Proposition 6** *Let  $x, y \in \partial X$ . As  $p \in X$  tends to a point  $z \in \partial X \setminus \{x, y\}$ ,*

$$e^{d(o,p)} d_p(x, y) \tag{95}$$

*tends to a number, denoted by  $d_{z,o}(x, y)$ , given by*

$$d_{z,o}(x, y) = \frac{d_o(x, y)}{d_o(z, x)d_o(z, y)}. \tag{96}$$

*As  $o$  is changed,  $d_{z,o}$  gets multiplied by a positive constant,*

$$d_{z,o'} = e^{\beta_z(o',o)} d_{z,o}. \tag{97}$$

Indeed, let  $q, q' \in X$  be close to  $x, y \in \partial X$ .

$$\begin{aligned} \langle q, q' \rangle_p - \langle q, q' \rangle_o - d(o, p) &= \frac{1}{2}(d(p, q) + d(p, q') - d(q, q')) \\ &\quad - \frac{1}{2}(d(o, q) + d(o, q') - d(q, q')) - d(o, p) \\ &= \frac{1}{2}(d(p, q) - d(o, q) - d(o, p)) \\ &\quad + \frac{1}{2}(d(p, q') - d(o, q') - d(o, p)) \\ &= -\langle p, q \rangle_o - \langle p, q' \rangle_o, \end{aligned} \tag{98}$$

which tends to  $-\langle z, x \rangle_o - \langle z, y \rangle_o$  as  $p \rightarrow z$ ,  $q \rightarrow x$ ,  $q' \rightarrow y$ . So

$$e^{-d(o,p)} \frac{d_o(x, y)}{d_p(x, y)} \quad (99)$$

tends to  $d_o(z, x)d_o(z, y)$  as  $p$  tends to  $z$ . This provides the formula

$$d_{z,o}(x, y) = \frac{d_o(x, y)}{d_o(z, x)d_o(z, y)}. \quad (100)$$

Furthermore,

$$\begin{aligned} \frac{d_{z,o'}(x, y)}{d_{z,o}(x, y)} &= \lim_{p \rightarrow z} \frac{e^{d(o',p)} d_p(x, y)}{e^{d(o,p)} d_p(x, y)} \\ &= \lim_{p \rightarrow z} e^{d(o',p) - d(o,p)} \\ &= e^{\beta_z(o',o)}. \end{aligned} \quad (101)$$

Although undefined at  $z$ , the semi-metric  $d_{z,o}$  belongs to the Möbius class in an extended sense: with the convention that  $d_{z,o}(z, x) = +\infty$  for all  $x \neq z$ , the semi-metric defines the same cross-ratios.

The metric associated to an ideal boundary point is defined only up to a multiplicative constant. The same holds for the Hausdorff  $Q$ -measure.

**Corollary 1** *The rescaled Hausdorff  $Q$ -measure  $e^{Qd(o,p)} \mu_p$  converges setwise on  $\partial X \setminus \{z\}$ , as  $p \in X$  tends to  $z \in \partial X$ , to the Hausdorff  $Q$ -measure  $\mu_{z,o}$  associated to  $d_{z,o}$ , which satisfies, for  $x \neq z$ ,*

$$\frac{d\mu_{z,o}}{d\mu_o}(x) = (d_o(z, x))^{-2Q}.$$

As  $o$  is changed,  $\mu_{z,o}$  gets multiplied by a positive constant,

$$\mu_{z,o'} = e^{Q\beta_z(o',o)} \mu_{z,o}.$$

Indeed, let  $B$  be a ball such that  $z$  is not a limit point of  $B$ . W.l.o.g, assume the points  $p$  to be far enough from  $o$ , such that  $B \subset \partial X \setminus \mathcal{O}_o^p(M + 2\delta)$  for all points  $p$  in the sequence, and the endpoints  $\bar{p}$  of the rays emanating from  $o$  and passing within uniform distance from  $p$ , are all  $\epsilon$  close to  $z$ . Then

$$e^{d(o,p)} \frac{d_p}{d_o}(x) \quad (102)$$

is dominated by

$$\frac{e^{2\delta}}{e^{-\langle x, z \rangle_o} - \epsilon}, \quad (103)$$

which is integrable on  $B$ . Thus, by dominated convergence,

$$\lim_{p \rightarrow z} e^{Qd(o,p)} \mu_p(B) = \int_B d_o(z, x)^{-2Q} d\mu_o(x). \quad (104)$$

**Example 1** *Let  $X$  be a rank one symmetric space. Then  $d_{z,o}$  is a left-translation invariant dilation-homogeneous distance on the unipotent radical  $N$  of  $\text{Isom}(X)$ . If  $X$  has constant curvature,  $d_{z,o}$  is a Euclidean distance.*

*Indeed,  $d_{z,o}$  is preserved by the stabilizer of  $z$  and of the horosphere passing through  $o$  in*

$\text{Isom}(X)$ , which contains  $N$ , and  $N$  acts simply transitively on  $\partial X \setminus \{z\}$ . The dilations of  $N$  are realized by the 1-parameter subgroup  $A$  of  $\text{Isom}(X)$ , each of whose elements multiplies  $d_{z,o}$  by a constant. When  $X$  has constant curvature, the stabilizer of  $z$  in  $\text{Isom}(X)$  is two-point transitive on  $\partial X \setminus \{z\}$ , so  $d_{z,o}$  is a function of Euclidean distance. By dilation-homogeneity, this function is homogeneous of degree 1, therefore  $d_{z,o}$  is a constant multiple of Euclidean distance.

## 4 Harmonic analysis on the boundary of hyperbolic groups

### 4.1 Introduction

A central tool of classical harmonic analysis on  $\mathbb{R}^n$  are maximal functions. Maximal functions allow to get quantitative results on the average of functions.

The most prominent of all maximal functions, is the Hardy-Littlewood maximal function.

**Definition 5** *The Hardy-Littlewood maximal function of  $f$ , is*

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy \quad (105)$$

This operator behaves nicely on  $L^p$  spaces.

**Theorem 4** *The Hardy-Littlewood maximal operator maps  $L^1(\mathbb{R}^n)$  to  $L^{1,\infty}(\mathbb{R}^n)$  with constant at most  $3^n$  and also  $L^p(\mathbb{R}^n)$  to  $L^p(\mathbb{R}^n)$  for  $1 < p < \infty$  with constant at most  $3^{n/p} p(p-1)^{-1}$ .<sup>19</sup>*

**4.1.1 Weighted inequalities.** The study of weighted inequalities for maximal functions appeared as a natural extension to the theory. A principle question is the characterization of all measures  $\mu$  on  $\mathbb{R}^n$  such that

$$\int_{\mathbb{R}^n} M(f)(x)^p d\mu(x) \leq A \int_{\mathbb{R}^n} |f(x)|^p d\mu(x), \quad (106)$$

for some  $p$ ,  $1 < p < \infty$ .

The full solution was given 50 years ago by Muckenhoupt<sup>20</sup>, who established a class of functions  $A_p$  such that (106) and other similar inequalities, hold exactly when  $d\mu(x) = w(x)^p dx$  with  $w \in A_p$ .

Weighted inequalities for singular integrals are another source of examples where Muckenhoupt's  $A_p$  condition plays a central role. If  $T$  is a singular integral operator of convolution type with some regularity assumptions on the kernel, then

$$\int_{\mathbb{R}^n} |Tf(x)|^p d\mu(x) \leq A \int_{\mathbb{R}^n} |f(x)|^p d\mu(x), \quad (107)$$

if and only if  $d\mu(x) = w(x)^p dx$ , with  $w \in A_p$ .

<sup>19</sup>Grafakos, Classical Fourier Analysis, Theorem 2.1.6., p. 80

<sup>20</sup>Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, 1972

## 4.2 Muckenhoupt weights

The class of measures  $\mu$  turns out to consist of exactly those measures, that are absolutely continuous w.r.t. Lebesgue measure,  $d\mu(x) = w(x)^p dx$ , and such that  $w$  satisfies the  $A_p$  inequality

$$\llbracket w \rrbracket_p = \sup_B \left( \frac{1}{|B|} \int_B w^p dx \right)^{\frac{1}{p}} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{p'} dx \right)^{\frac{1}{p'}} < \infty, \quad (108)$$

**Remark 3** *In the literature one often finds a slightly different but equivalent definition of the class  $A_p$ . It amounts to setting  $v = w^p$  and defining  $A_p$  as the class of functions  $v$ , such that  $(\llbracket v^{\frac{1}{p}} \rrbracket_p)^p$  is finite.*

**Example 2** *The function  $|x|^\alpha$  belongs to  $A_p$  on  $\mathbb{R}^n$  if and only if  $-n < \alpha p < n(p-1)$ .*

In the following we list some simple but useful observations on general properties of Muckenhoupt weights.

**4.2.1** Muckenhoupt weights admit an obvious "duality". If  $w \in A_p$ , then  $w^{-1} \in A_{p'}$  with the same Muckenhoupt constant.

**4.2.2** An important property of Muckenhoupt classes is that they are nested up to some renormalization. If  $w \in A_p$ , then  $w^{\frac{p}{q}}$  is in  $A_q$  for every  $q > p$ . Indeed,

$$(\llbracket w^{\frac{p}{q}} \rrbracket_q)^{\frac{q}{p}} = \sup_B \left( \frac{1}{|B|} \int_B w^p dx \right)^{\frac{1}{p}} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{\frac{pq'}{q}} dx \right)^{\frac{q}{pq'}}, \quad (109)$$

is bounded by a multiple of  $\llbracket w \rrbracket_p$ . This follows from Hölder's inequality and the fact that  $q'/q = q' - 1 < p' - 1 = p'/p$ .

**4.2.3** There exists an alternative definition of the class  $A_p$ , that is in notation more closely related to Hardy-Littlewood's maximal function. For any locally integrable function  $f$  we denote by  $f_B$  the average of  $f$  over a ball  $B$ , i.e.  $f_B = \frac{1}{|B|} \int_B f(x) dx$ . We further denote by  $w^p(B)$  the number  $\int_B w(x)^p dx$ . Then  $w$  belongs to the class  $A_p$  if and only if the  $p$ -th power of  $f_B$  is controlled by the mean value of  $f^p$  with respect to the measure  $w(x)^p dx$ . Concretely,

$$(f_B)^p \leq \frac{c}{w^p(B)} \int_B f^p w^p dx \quad (110)$$

holds for all non-negative functions  $f$  and all balls  $B$ . The minimal constant  $c$  for which this inequality is true, agrees with  $(A_p(w))^{\frac{1}{p}}$ .

Assume that  $w \in A_p$ .

$$f_B = \frac{1}{|B|} \int_B f(x) dx = \frac{1}{|B|} \int_B f w w^{-1} dx. \quad (111)$$

We apply Hölder's inequality to the right-hand side with exponent  $p$ .

$$f_B \leq \frac{1}{|B|} \left( \int_B f^p w^p dx \right)^{\frac{1}{p}} \left( \int_B w^{-p'} dx \right)^{\frac{1}{p'}}. \quad (112)$$

Together with Muckenhoupt's inequality (108), the above inequality establishes (110), with minimal constant  $c \leq (A_p(w))^{\frac{1}{p}}$ . Conversely, if (110) holds for  $w$ , we take  $f$  to be

$(w + \epsilon)^{-p'}$ . Then (110) together with  $(1 - p')p = -p'$ , implies

$$\frac{1}{|B|} \left( \int_B (w(x) + \epsilon)^{-p'} dx \right)^{p-1} w^p(B) \leq c. \quad (113)$$

Taking the  $p$ -th root and passing to the limit  $\epsilon \rightarrow 0$ , gives Muckenhoupt's inequality (108).

**4.2.4** Every measure absolutely continuous w.r.t. to Lebesgue's measure,  $d\mu(x) = w(x)^p dx$ , is doubling when  $w \in A_p$ . A measure  $\mu$  is doubling, if there exists a constant  $D > 0$ , s.t.

$$\mu(2B) \leq D\mu(B). \quad (114)$$

If  $w \in A_p$ , then the fact that the measure  $w^p$  is doubling, follows directly from (110) if we set  $B = 2B'$  and  $f = \chi_{B'}$ . The doubling constant is explicitly given by  $c2^{np}$ .

### 4.3 Generalized Muckenhoupt weights.

Muckenhoupt and Wheeden<sup>21</sup> generalized the class  $A_p$  to the operator

$$I^\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad (115)$$

for  $0 < \alpha < n$ . It is often referred to as the fractional integration operator on  $\mathbb{R}^n$ . For  $\alpha = 0$ , it is a Calderon-Zygmund singular integral.

Let  $1 < p < \frac{\alpha}{n}$ , and  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ , then

$$\left( \int_{\mathbb{R}^n} |I^\alpha(f)(x)|^q w(x)^q dx \right)^{\frac{1}{q}} \leq A \left( \int_{\mathbb{R}^n} |f(x)|^p w(x)^p dx \right)^{\frac{1}{p}}, \quad (116)$$

for some non-negative function  $w$ , if and only if,

$$\llbracket w \rrbracket_{q,p} = \sup_B \left( \frac{1}{|B|} \int_B w^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{p'} dx \right)^{\frac{1}{p'}} < \infty. \quad (117)$$

We say that the weight  $w$  is of class  $A_{q,p}$ .

### 4.4 Muckenhoupt weights associated to Möbius structures

Let  $X$  be a strongly hyperbolic metric space with the natural Möbius structure  $\{d_o\}_{o \in X}$  and some fixed origin  $o \in X$ . In this section, we will show, that powers of the conformal factors

$$\frac{d_{o'}}{d_o}(x) = e^{\beta_x(o,o')}, \quad (118)$$

are Muckenhoupt weights w.r.t.  $\mu_o$ , with uniformly bounded  $A_{p,q}$  constant.

In fact, we may allow  $o$  and or  $o'$  to be ideal points. This gives a Muckenhoupt theory on Möbius structures, that is invariant under the Cayley transform and therefore treats the compact and non-compact picture as equivalent with respect to each other.

The definitions and notations introduced above on  $\mathbb{R}^n$  are straight forwardly generalized

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<sup>21</sup>Weighted Norm inequalities for fractional integrals, 1974

to this setting by replacing the Lebesgue measure  $dx$  with the Petterson-Sullivan measure  $\mu_o$ . Indeed,  $\mu_o$  is Ahlfors-David regular and henceforth doubling. The remarks on the properties of  $A_p$  weights on  $\mathbb{R}^n$  made above, generalize to this setting straight forwardly as well, and will not be repeated here.

**Theorem 5** *There exists a constant  $C > 0$ , s.t. for all  $o, o' \in X$ , and  $1 < p \leq q < \infty$ ,  $\alpha < \frac{Q}{q}$ ,*

$$\|(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}}\|_{p,q} = \sup_B \left( \frac{1}{\mu_o(B)} \int_B (\frac{d_{o'}}{d_o})^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B (\frac{d_{o'}}{d_o})^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C < \infty. \quad (119)$$

Clearly, for every pair of points  $o, o'$ , and  $1 < p \leq q < \infty$ , the weight  $(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}}$  is a Muckenhoupt weight of class  $A_{p,q}$ , since the conformal factors  $\frac{d_{o'}}{d_o}$  are bounded above and away from 0. The significance comes from the fact that the bound on the  $A_{p,q}$  constant, is independent of  $o$  and  $o'$ .

We denote by  $A(\mathcal{M})_{p,q}^\alpha$  or simply  $A(\mathcal{M})$  or  $A$ , the minimal  $C > 0$ , s.t. the above inequality is true for all  $o, o' \in X$ .

The proof reveals, that the constant  $A(\mathcal{M})_{p,q}^\alpha$  may not be finite at  $\alpha = \frac{Q}{q}$ . This will become relevant again, when we discuss bounded representations and the limit when  $\alpha$  tends to  $\frac{Q}{q}$ .

Theorem 5 in particular says that  $(\frac{d_{o'}}{d_o})^{\frac{\alpha}{2}}$  is of class  $A_q$  if  $\alpha < \frac{Q}{2}$ , and by the duality property of Muckenhoupt weights,  $(\frac{d_{o'}}{d_o})^{-\frac{\alpha}{2}}$  is of class  $A_{p'}$ . The following corollaries follow from the doubling property of Muckenhoupt weights.

**Corollary 2** *For all  $d_o \in \mathcal{M}$ ,  $\alpha < \frac{Q}{q}$ , the measure*

$$U \mapsto \int_U (\frac{d_{o'}}{d_o})^{\frac{\alpha q}{2}} d\mu_o$$

*is doubling with a doubling constant  $DA^{\frac{1}{q}}$  where  $D$  is the doubling constant of measures associated with  $\mathcal{M}$ .*

**Corollary 3** *For all  $d_o \in \mathcal{M}$ ,  $\alpha < \frac{Q}{2p}$ , the measure*

$$U \mapsto \int_U (\frac{d_{o'}}{d_o})^{-\frac{\alpha p'}{2}} d\mu_o$$

*is doubling with a doubling constant  $DA^{\frac{1}{p'}}$  where  $D$  is the doubling constant of measures associated with  $\mathcal{M}$ .*

**Proof:** (of theorem 5) Let  $B$  be a ball in  $Z$  with center  $z$  and radius  $r$ , and  $o$  and  $o'$  two points in  $X$ .

Let  $\infty$  be a forward endpoint in  $Z$  of a geodesic emanating from  $o$  and passing through  $o'$ . We denote by  $|x|_o$ , the distance of a point  $x$  in  $Z$  to  $\infty$ , w.r.t. the metric  $d_o$ .

We denote by  $B^s = B(\infty, e^\delta e^{-(d(o,o')-s)})$ , and by  $r(s)$  the radius of  $B^s$ . Sullivan's shadow lemma says that if  $s \geq M + 2\delta$ , the shadow  $\mathcal{O}'(s)$  is contained in  $B^s$ .

We will investigate 3 cases separately. Pick a positive integer  $n \geq 2$ ,

1. Case 1:  $|z| \geq (1 + \frac{1}{n})r$ ,
2. Case 2:  $|z| \leq (1 - \frac{1}{n})r$ ,
3. Case 3:  $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$ .

Case 1: For technical reasons, we distinguish two sub-cases:

1. Case 1A:  $|z| \geq (1 + \frac{1}{n})r$  and  $B \cap \mathcal{O}'_o(M + 2\delta) = \emptyset$ ,
2. Case 1B:  $|z| \geq (1 + \frac{1}{n})r$  and  $B \cap \mathcal{O}'_o(M + 2\delta) \neq \emptyset$ .

In case 1A, we have the nice advantage, that we may replace  $e^{-\langle x, o' \rangle_o}$  with the true distance  $|x|_o$ . Indeed, for all  $x$  in  $B$ ,

$$e^{-2\delta\alpha q} |x|^{-\alpha q} \leq e^{\alpha q \langle x, o' \rangle_o} \leq e^{2\delta\alpha q} |x|^{-\alpha q}, \quad (120)$$

and

$$e^{-2\delta\alpha p'} |x|^{\alpha p'} \leq e^{-\alpha p' \langle x, o' \rangle_o} \leq e^{2\delta\alpha p'} |x|^{\alpha p'}. \quad (121)$$

This follows from the following.  $B \cap \mathcal{O}'_o(M + 2\delta) = \emptyset$  is equivalent to

$$\forall x \in B, \quad \langle x, o' \rangle_o < d(o, o') - M - 2\delta. \quad (122)$$

Thus

$$\begin{aligned} \langle x, \bullet \rangle_o &\geq \min\{\langle x, o' \rangle_o, \langle o', \bullet \rangle_o\} - 2\delta \\ &\geq \min\{\langle x, o' \rangle_o, d(o, o') - M\} - 2\delta \\ &\geq \langle x, o' \rangle_o - 2\delta. \end{aligned} \quad (123)$$

The converse follows from the fact that  $B$  does not intersect the ball  $B(\bullet, e^{-d(o, o') - M})$  as well. So

$$\forall x \in B, \quad \langle x, \bullet \rangle_o \leq d(o, o') - M. \quad (124)$$

And therefore

$$\begin{aligned} \langle x, o' \rangle_o &\geq \min\{\langle x, \bullet \rangle_o, \langle \bullet, o' \rangle_o\} - 2\delta \\ &\geq \min\{\langle x, \bullet \rangle_o, d(o, o') - M\} - 2\delta \\ &\geq \langle x, \bullet \rangle_o - 2\delta. \end{aligned} \quad (125)$$

This proves the comparison claim.

This allows us to use the triangle inequality to obtain upper bounds. Since  $|z| \geq (1 + \frac{1}{n})r$ ,

$$\forall x \in B: \quad |x| \geq |z| - 2r \geq |z| - \frac{n}{n+1}|z| = \frac{|z|}{n+1}, \quad (126)$$

and

$$\forall x \in B, \quad |x| \leq |z| + r \leq |z| + \frac{n}{1+n}|z| = \frac{2n+1}{n+1}|z|. \quad (127)$$

Thus,

$$\begin{aligned}
\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o &= e^{-\frac{\alpha q}{2}d(o,o')} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o \\
&\leq e^{-\frac{\alpha q}{2}d(o,o')} e^{2\delta \alpha q} \int_B |x|^{-\alpha q} d\mu_o \\
&\leq e^{-\frac{\alpha q}{2}d(o,o')} e^{2\delta \alpha q} (n+1)^{\alpha q} |z|^{-\alpha q} \mu_o(B),
\end{aligned} \tag{128}$$

and

$$\begin{aligned}
\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o,o')} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\
&\leq e^{\frac{\alpha p'}{2}d(o,o')} e^{2\delta \alpha p'} \int_{2B} |x|^{\alpha p'} d\mu_o \\
&\leq e^{\frac{\alpha p'}{2}d(o,o')} e^{2\delta \alpha p'} \frac{(2n+1)^{\alpha p'}}{(n+1)^{\alpha p'}} |z|^{\alpha p'} \mu_o(B).
\end{aligned} \tag{129}$$

This implies,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{4\delta \alpha} (2n+1)^\alpha \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}, \tag{130}$$

which proves case 1A.

In case 1B,  $B \cap \mathcal{O}_o'(M+2\delta) \neq \emptyset$ . The shadow  $\mathcal{O}_o'(M+2\delta)$  is contained in the ball  $B^{M+2\delta}$ , with  $r_{(M+2\delta)} = e^{-(d(o,o')-M-3\delta)}$ . That  $B$  intersects the shadow  $\mathcal{O}_o'(M+2\delta)$ , implies that the radius  $r$  is comparable to  $r_{(M+2\delta)}$ . Precisely, since  $|z| \geq (1 + \frac{1}{n})r$ ,

$$r \leq n(|z| - r) \leq n r_{(M+2\delta)}. \tag{131}$$

This implies that,

$$B \subset B(\infty, r_{(M+2\delta)} + 2r) \subset B(\infty, (1+2n)r_{(M+2\delta)}) \subset \mathcal{O}_o'(M + \log(1+2n) + 4\delta). \tag{132}$$

Hence,

$$\forall x \in B: \quad \langle x, o' \rangle_{o'} \geq d(o, o') - (M + \log(1+2n) + 4\delta). \tag{133}$$

We use this inequality for the upper bound of the  $p'$ -measure.

$$\begin{aligned}
\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o,o')} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\
&\leq e^{-\frac{\alpha p'}{2}d(o,o')} e^{\alpha p'(M + \log(1+2n) + 4\delta)} \mu_o(B)
\end{aligned} \tag{134}$$

The appropriate upper bound for the  $q$ -measure is straightforward.

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B). \tag{135}$$

This implies,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{\alpha(M+4\delta)} (1+2n)^\alpha \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}, \tag{136}$$

which proves case 1 completely.

Case 2: Let  $S_+$  be the set of all  $s \in \mathbb{R}^+$ , such that  $B \subset \mathcal{O}'_o(s)$ . Let  $s_+^*$  be the infimum of  $S_+$ , and

$$s_+ := d(o, o') - \delta + \log(|z| + r). \quad (137)$$

Let  $S_-$  be the set of all  $s \in \mathbb{R}^+$ , such that  $\mathcal{O}'_o(s) \subset B$ . Notice that if  $s \in S_-$ , then for all  $s' \leq s$ ,  $s' \in S_-$ . Let  $s_-^*$  be the supremum of  $S_-$ .

Recall that by  $B^s$  we denoted the ball  $B(\infty, e^\delta e^{-(d(o, o') - s)})$ , and by  $r(s)$  its radius.

For technical reasons we again distinguish two sub-cases:

1. Case 2A:  $|z| \leq (1 - \frac{1}{n})r$  and  $\mathcal{O}'_o(M + 2\delta) \subset B$ ,
2. Case 2B:  $|z| \leq (1 - \frac{1}{n})r$  and  $\mathcal{O}'_o(M + 2\delta) \not\subset B$ .

In case 2A,  $s_+ > M + 2\delta$  and the Sullivan's shadow lemma implies  $s_+ \in S_+$ . Thus,

$$M + 2\delta \leq s_+^* \leq s_+. \quad (138)$$

For the upper bound of  $\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o$ , we first notice that

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq \int_{\mathcal{O}'_o(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o. \quad (139)$$

The right-hand side has an explicit decomposition by integration by parts.

$$\begin{aligned} \int_{\mathcal{O}'_o(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o &= e^{\frac{\alpha q}{2}d(o, o')} \int_0^\infty \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}'_o(s) \cap \mathcal{O}'_o(s_+)) ds \\ &= e^{\frac{\alpha q}{2}d(o, o')} \int_0^{M+2\delta} \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}'_o(s)) ds \\ &\quad + e^{\frac{\alpha q}{2}d(o, o')} \int_{M+2\delta}^{s_+} \alpha q e^{-\alpha q s} \mu_o(\mathcal{O}'_o(s)) ds \\ &\quad + e^{\frac{\alpha q}{2}d(o, o')} e^{-\alpha q s_+} \mu_o(\mathcal{O}'_o(s_+)) \end{aligned} \quad (140)$$

Further calculations give,

$$\begin{aligned}
& \int_{\mathcal{O}'_o(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2}d(o,o')} \int_0^{M+2\delta} \alpha q e^{-\alpha q s} \mu_o(B^{M+2\delta}) ds \\
& \quad + e^{\frac{\alpha q}{2}d(o,o')} \int_{M+2\delta}^{s_+} \alpha q e^{-\alpha q s} \mu_o(B^s) ds \\
& \quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\
& \leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B^{M+2\delta}) \\
& \quad + C e^{\delta Q} e^{-Qd(o,o')} e^{\frac{\alpha q}{2}d(o,o')} \int_{M+2\delta}^{s_+} \alpha q e^{(Q-\alpha q)s} ds \\
& \quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\
& \leq e^{\frac{\alpha q}{2}d(o,o')} \mu_o(B^{M+2\delta}) \\
& \quad + C e^{\delta Q} e^{-Qd(o,o')} e^{\frac{\alpha q}{2}d(o,o')} \left(\frac{\alpha q}{Q-\alpha q}\right) e^{(Q-\alpha q)s_+} \\
& \quad + e^{\frac{\alpha q}{2}d(o,o')} e^{-\alpha q s_+} \mu_o(B^{s_+}) \\
& = e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} e^{-\alpha q(M+2\delta)} r_{(M+2\delta)}^{-\alpha q} \mu_o(B^{M+2\delta}) \\
& \quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q} \\
& \quad + e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(s_+)}^{-\alpha q} \mu_o(B^{s_+}) \\
& \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(M+2\delta)}^{Q-\alpha q} \\
& \quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q} \\
& \quad + C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} r_{(s_+)}^{Q-\alpha q} \\
& \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q-\alpha q}{Q-\alpha q}\right) r_{(s_+)}^{Q-\alpha q}
\end{aligned} \tag{141}$$

Thus,

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q-\alpha q}{Q-\alpha q}\right) (|z|+r)^{Q-\alpha q}, \tag{142}$$

and using the assumption  $|z| \leq (1 - \frac{1}{n})r$ ,

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o,o')} e^{\alpha q \delta} \left(\frac{2Q-\alpha q}{Q-\alpha q}\right) \left(\frac{2n+1}{n}\right)^{Q-\alpha q} r^{Q-\alpha q}. \tag{143}$$

The upper bound for the  $p'$ -measure is significantly simpler.

$$\begin{aligned}
& \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o = e^{-\frac{\alpha p'}{2}d(o,o')} \int_{-\infty}^{d(o,o')} \alpha p' e^{\alpha p' s} \mu_o(B \setminus \mathcal{O}'_o(s)) ds \\
& = e^{-\frac{\alpha p'}{2}d(o,o')} \int_{-\infty}^{s_+^*} \alpha p' e^{\alpha p' s} \mu_o(B \setminus \mathcal{O}'_o(s)) ds \\
& \leq e^{-\frac{\alpha p'}{2}d(o,o')} e^{\alpha p' s_+} \mu_o(B) \\
& = e^{\frac{\alpha p'}{2}d(o,o')} e^{-\alpha p' \delta} (|z|+2r)^{\alpha p'} \mu_o(B) \\
& \leq e^{\frac{\alpha p'}{2}d(o,o')} e^{-\alpha p' \delta} \left(\frac{3n-1}{n}\right)^{\alpha p'} r^{\alpha p'} \mu_o(B).
\end{aligned} \tag{144}$$

For the last inequality, we made use of the assumption  $|z| \leq (1 - \frac{1}{n})r$ .

Both bounds above imply,

$$\begin{aligned}
& \left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \\
& \leq C^{\frac{1}{q} + \frac{1}{p'}} \left( \frac{2Q - \alpha q}{Q - \alpha q} \right)^{\frac{1}{q}} \left( \frac{2n + 1}{n} \right)^{\frac{Q}{q} - \alpha} \left( \frac{3n - 1}{n} \right)^{\alpha} r^{Q(\frac{1}{q} + \frac{1}{p'})} \\
& \leq C^{2(\frac{1}{q} + \frac{1}{p'})} \left( \frac{2Q - \alpha q}{Q - \alpha q} \right)^{\frac{1}{q}} \left( \frac{2n + 1}{n} \right)^{\frac{Q}{q} - \alpha} \left( \frac{3n - 1}{n} \right)^{\alpha} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}},
\end{aligned} \tag{145}$$

which proves case 2A.

Case 2B: Recall, that the shadow  $\mathcal{O}'_o(M + 2\delta)$  is contained in the ball  $B^{M+2\delta}$ , with  $r_{(M+2\delta)} = e^{-(d(o,o')-M-3\delta)}$ . This implies together with the assumption  $|z| \leq (1 - \frac{1}{n})r$ , that

$$\frac{r}{n} \leq r - |z| \leq r_{(M+2\delta)}. \tag{146}$$

Thus,

$$B \subset B(\infty, |z| + r) \subset B(\infty, (2n - 1)r_{(M+2\delta)}) \subset \mathcal{O}'_o(M + \log(2n - 1) + 4\delta), \tag{147}$$

where the last inclusion follows again from the shadow lemma. This implies

$$\forall x \in B: \quad \langle x, o' \rangle_{o'} \geq d(o, o') - (M + \log(2n - 1) + 4\delta). \tag{148}$$

This inequality gives the following upper bound for the  $p'$ -measure.

$$\begin{aligned}
\int_B \left( \frac{d_{o'}}{d_o}(x) \right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2} d(o, o')} \int_B e^{-\alpha p' \langle x, o' \rangle_{o'}} d\mu_o \\
&\leq e^{-\frac{\alpha p'}{2} d(o, o')} e^{\alpha p' (M + 4\delta)} (2n - 1)^{\alpha p'} \mu_o(B).
\end{aligned} \tag{149}$$

The upper bound of the  $q$ -measure is straightforward.

$$\int_B \left( \frac{d_{o'}}{d_o}(x) \right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2} d(o, o')} \mu_o(B). \tag{150}$$

Both bounds together imply,

$$\left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq e^{\alpha(M+4\delta)} (2n - 1)^{\alpha} \mu_o(B)^{1 - \frac{\alpha}{Q}}, \tag{151}$$

which proves the case 2 completely.

Case 3: As above, let  $S_+$  be the set of all  $s \in \mathbb{R}^+$ , such that  $B \subset \mathcal{O}'_o(s)$ . Let  $s_+^*$  be the infimum of  $S_+$ .

Again, we distinguish two sub-cases:

1. Case 3A:  $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$  and  $s_+^* \geq M + 2\delta$ ,
2. Case 3B:  $(1 - \frac{1}{n})r \leq |z| \leq (1 + \frac{1}{n})r$  and  $s_+^* < M + 2\delta$ .

In case 3A, we can apply the shadow lemma to

$$s_+ := d(o, o') - \delta + \log(|z| + r), \tag{152}$$

to conclude  $s_+ \in S_+$ . Thus,

$$\forall x \in B: \quad \langle x, o' \rangle_o \geq d(o, o') - s_+ = -\log(|z| + r) + \delta \geq -\log\left(\frac{2n+1}{n}r\right) + \delta, \quad (153)$$

where in the last inequality we used the assumption  $|z| \leq (1 + \frac{1}{n})r$ .

We use this inequality to get the upper bound for the  $p'$ -measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o, o')} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{\frac{\alpha p'}{2}d(o, o')} \left(\frac{2n+1}{n}\right)^{\alpha p'} e^{-\alpha p' \delta} r^{\alpha p'} \mu_o(B). \end{aligned} \quad (154)$$

For the upper bound of  $\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o$ , we again bound above by the shadow  $\mathcal{O}'_o(s_+)$ .

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq \int_{\mathcal{O}'_o(s_+)} \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o. \quad (155)$$

The right-hand side is bounded as in case 2A. Thus,

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o, o')} e^{\alpha q \delta} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right) (|z| + r)^{Q - \alpha q}, \quad (156)$$

and using the assumption  $|z| \leq (1 + \frac{1}{n})r$ ,

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq C e^{-\frac{\alpha q}{2}d(o, o')} e^{\alpha q \delta} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right) \left(\frac{2n+1}{n}\right)^{Q - \alpha q} r^{Q - \alpha q}. \quad (157)$$

Both bounds together imply,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq C^{2(\frac{1}{q} + \frac{1}{p'})} \left(\frac{2Q - \alpha q}{Q - \alpha q}\right)^{\frac{1}{q}} \left(\frac{2n+1}{n}\right)^{\frac{Q}{q}} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}, \quad (158)$$

which proves the case 3A.

Case 3B: In this case  $B \subset \mathcal{O}'_o(M + 2\delta)$ . Thus,

$$\forall x \in B, \quad \langle x, o' \rangle_{o'} \geq d(o, o') - (M + 2\delta). \quad (159)$$

This inequality gives the following upper bound for the  $p'$ -measure.

$$\begin{aligned} \int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{-\frac{\alpha p'}{2}} d\mu_o &= e^{\frac{\alpha p'}{2}d(o, o')} \int_{2B} e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o \\ &\leq e^{-\frac{\alpha p'}{2}d(o, o')} e^{\alpha p'(M+2\delta)} \mu_o(B). \end{aligned} \quad (160)$$

The appropriate upper bound for the  $q$ -measure straightforward.

$$\int_B \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}} d\mu_o \leq e^{\frac{\alpha q}{2}d(o, o')} \mu_o(B). \quad (161)$$

Both bounds together imply,

$$\left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o\right)^{\frac{1}{q}} \left(\int_B \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o\right)^{\frac{1}{p'}} \leq e^{\alpha(M+2\delta)} \mu_o(B)^{\frac{1}{q} + \frac{1}{p'}}. \quad (162)$$

This proves case 3 completely and finishes the proof.  $\blacksquare$

Theorem 5 is stated with the compact picture in mind. However, the theorem can easily be extended to treat the compact and non-compact picture alike.

**Theorem 6** *There exists a constant  $C > 0$ , s.t. for all  $o, o' \in X \cup \partial X$ , and  $1 < p \leq q < \infty$ ,  $\alpha < \frac{Q}{q}$ ,*

$$\| \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \|_{p,q} = \sup_B \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C < \infty, \quad (163)$$

where the supremum is taken over balls in  $\partial X \setminus \{o\}$ .

**Proof:** Let  $o, o' \in X$ . By multiplying both integrands by  $e^{\frac{\alpha}{2}d(o,o')}$  and  $e^{-\frac{\alpha}{2}d(o,o')}$  respectively,

$$\left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \quad (164)$$

equals

$$\left( \frac{1}{\mu_o(B)} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x) \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B e^{-\alpha p' \langle x, o' \rangle_o} d\mu_o(x) \right)^{\frac{1}{p'}}. \quad (165)$$

We are tempted to take  $o'$  to the boundary. Since  $X$  is strongly hyperbolic, the integrand converges pointwise as  $o' \rightarrow z \in \partial X$ . Careful treatment requires the case when  $x \in \mathcal{O}_o'(M + 2\delta)$ . In favor of readability, we set  $\mathcal{O}_o' = \mathcal{O}_o'(M + 2\delta)$ . Each such integral decomposes as

$$\int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o = \int_B \chi_{B \setminus \mathcal{O}_o'}(x) e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x) + \int_{B \cap \mathcal{O}_o'} e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x). \quad (166)$$

The function  $\chi_{B \setminus \mathcal{O}_o'}(x) e^{\alpha q \langle x, o' \rangle_o}$  is dominated by the integrable function  $e^{2\delta \alpha q} d_o(x, z)^{-\alpha q}$ , and converges pointwise on  $B \setminus \{z\}$  to  $e^{\alpha q \langle x, z \rangle_o}$ . The second integral is bounded by

$$e^{\alpha q d(o, o')} \mu_o(\mathcal{O}_o') \leq e^{Q(M+3\delta)} e^{(\alpha q - Q)d(o, o')}. \quad (167)$$

Thus, the integral  $\int_{B \cap \mathcal{O}_o'} e^{\alpha q \langle x, o' \rangle_o} d\mu_o(x)$  vanishes as  $o' \rightarrow z \in \partial X$ . By dominated convergence

$$\lim_{o' \rightarrow z} \int_B e^{\alpha q \langle x, o' \rangle_o} d\mu_o = \int_B e^{\alpha q \langle x, z \rangle_o} d\mu_o(x) = \int_B \left( \frac{d_{z,o}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o. \quad (168)$$

An equivalent argument works for the  $p'$ -part. This implies,

$$\| \left( \frac{d_{z,o}}{d_o} \right)^{\frac{\alpha}{2}} \|_{p,q} = \sup_B \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{z,o}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)^{\frac{1}{q}} \left( \frac{1}{\mu_o(B)} \int_B \left( \frac{d_{z,o}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right)^{\frac{1}{p'}} \leq C. \quad (169)$$

Since  $d_{z,o'} = e^{\beta z \langle o, o' \rangle} d_{z,o}$ , the expression does not depend on the choice of  $o$  in  $d_{z,o}$ .

Passing from the compact to the non-compact picture, works similarly. Let  $z \in \partial X$ , and let  $B$  be a ball that does not include  $z$  as a limit point. Let  $p$  be a sequence of points,  $p \rightarrow z$ . W.l.o.g. we assume all points to be close enough to  $z$ , s.t.  $B \subset \partial X \setminus \mathcal{O}_o^p(M + 2\delta)$  for all points  $p$ . Further we assume that all endpoints  $\bar{p}$  of rays emanating from  $o$  and passing within uniform distance from  $p$ , are  $\epsilon$  close to  $z$ . Then,

$$e^{\langle x, p \rangle_o} \quad (170)$$

is dominated by

$$\frac{e^{2\delta}}{e^{-\langle x, z \rangle_o} - \epsilon}, \quad (171)$$

which is integrable on  $B$ . By multiplying both integrands by  $e^{\frac{\alpha}{2}d(o,p)}$  and  $e^{-\frac{\alpha}{2}d(o,p)}$  respectively,

$$\left( \frac{1}{\mu_p(B)} \int_B \left( \frac{d_o}{d_p} \right)^{\frac{\alpha q}{2}} d\mu_p \right)^{\frac{1}{q}} \left( \frac{1}{\mu_p(B)} \int_B \left( \frac{d_o}{d_p} \right)^{-\frac{\alpha p'}{2}} d\mu_p \right)^{\frac{1}{p'}} \quad (172)$$

equals

$$\left( \frac{1}{\mu_p(B)} \int_B e^{-\alpha q \langle x, p \rangle_o} d\mu_p(x) \right)^{\frac{1}{q}} \left( \frac{1}{\mu_p(B)} \int_B e^{\alpha p' \langle x, p \rangle_o} d\mu_p(x) \right)^{\frac{1}{p'}}. \quad (173)$$

We may express the integrals w.r.t.  $o$  as

$$\frac{1}{e^{Qd(o,p)} \mu_p(B)} \int_B e^{(Q-\alpha q) \langle x, p \rangle_o} d\mu_o(x). \quad (174)$$

The integrand is dominated as we discussed above. By dominated convergence

$$\begin{aligned} \lim_{p \rightarrow z} \frac{1}{e^{Qd(o,p)} \mu_p(B)} \int_B e^{(Q-\alpha q) \langle x, p \rangle_o} d\mu_o(x) &= \frac{1}{\mu_{z,o}(B)} \int_B e^{(Q-\alpha q) \langle x, z \rangle_o} d\mu_o(x) \\ &= \frac{1}{\mu_{z,o}(B)} \int_B \left( \frac{d_o}{d_{z,o}} \right)^{\frac{\alpha q}{2}} d\mu_{z,o} \end{aligned} \quad (175)$$

An equivalent argument works for the  $p'$ -integral. ■

#### 4.5 Some classical theorems revisited.

The above developed theory allows us to revisit some classical theorems of harmonic analysis and state them in a Möbius invariant manner. All statements make use of Calderon-Zygmund type lemmas on spaces of homogeneous type. To ensure uniformity, careful attention has to be given to the constants however. In this paper we will only prove the statement for the fractional integration operator explicitly and in detail. We postpone this proof until later.

##### 4.5.1 The Hardy-Littlewood maximal function

$$M_o(f)(x) = \sup_{B \ni x} \frac{1}{\mu_o(B)} \int_B |f(y)| d\mu_o, \quad (176)$$

satisfies,

$$\int_Z M_o(f)(y)^p \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha p}{2}} d\mu_o(y) \leq C \int_Z |f(y)|^p \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha p}{2}} d\mu_o(y), \quad (177)$$

for all  $f \in L^p\left(\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha p}{2}} d\mu_o\right)$ , and with a constant  $C$  independent of  $o$  and  $o'$  (including ideal points).

##### 4.5.2 The singular integral

$$T_o(f)(x) = \int_Z \frac{f(y)}{d_o(x,y)^Q} d\mu_o(y) \quad (178)$$

satisfies

$$\int_Z |T_o(f)(y)|^p \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha p}{2}} d\mu_o(y) \leq C \int_Z |f(y)|^p \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha p}{2}} d\mu_o(y), \quad (179)$$

for all  $f \in L^p\left(\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha p}{2}} d\mu_o\right)$ , and with a constant  $C$  independent of  $o$  and  $o'$  (including ideal points).

#### 4.5.3 The fractional integration operator

$$I_o^\alpha(f)(x) = \int_Z \frac{f(y)}{d_o(x, y)^{Q-\alpha}} d\mu_o(y) \quad (180)$$

satisfies

$$\left( \int_Z |I_o^\alpha(f)(y)|^q \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(y) \right)^{\frac{1}{q}} \leq C \left( \int_Z |f(y)|^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha p}{2}} d\mu_o(y) \right)^{\frac{1}{p}}, \quad (181)$$

for  $0 < \alpha < \frac{Q}{q}$ ,  $1 < p < \infty$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ ,  $f \in L^p\left(\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha p}{2}} d\mu_o\right)$ , and with a constant  $C$  independent of  $o$  and  $o'$  (including ideal points).

#### 4.6 Calderon-Zygmund type lemmas on Möbius spaces.

In this section we will revisit some classical Calderon-Zygmund type lemmas. The purpose is to highlight the Möbius invariance of involved constants.

Given a Möbius space  $(Z, \mathcal{M})$ , a ball in  $(Z, \mathcal{M})$  is a metric-ball for some  $d \in \mathcal{M}$ . A family of balls in  $(Z, \mathcal{M})$  is a family of metric-balls for some  $d \in \mathcal{M}$ .

**Lemma 5** *Let  $\{B_i\}_{i \in \mathcal{I}}$  be a family of balls in  $(Z, \mathcal{M})$  contained in some fixed ball  $B$ . Then there is a countable sub-collection  $\{B_i\}_{i \in I}$ ,  $I \subset \mathcal{I}$  such that*

1.  $B_i \cap B_j = \emptyset$  if  $i \neq j$ ,
2. Every  $B_i$  is contained in some  $5B_i$ ,
3. For  $d \in \mathcal{M}$  such that  $\{B_i\}_{i \in \mathcal{I}}$  is a family of balls in  $(Z, d)$ ,  $\mu_d(\cup_{i \in \mathcal{I}} B_i) \leq C_Q \sum_{i \in I} \mu_d(B_i)$ .

and where  $C_Q = 5^Q C^2$  is a constant depending only on Ahlfors-David constant  $C$  and the dimension  $Q$  of  $\mathcal{M}$ .

**Proof:** Selection process of Vitali. ■

The classical theory on bounds of fractional integration in Euclidean space is based on Calderon's and Zygmund's decomposition using dyadic cubes. We adapt this idea to our setting as follows.

For each  $k$  in  $\mathbb{Z}$ , let  $\{\hat{B}_j^k\}_j$  be a sequence of  $d$ -balls of radius  $3^{k-1}$ , maximal with respect to the property that  $\hat{B}_i^k \cap \hat{B}_j^k = \emptyset$  for  $i \neq j$ . Set  $B_j^k = 3\hat{B}_j^k$ , we call  $\{B_j^k\}_j$  *dyadic  $d$ -balls of order  $k$  in  $Z$* . We call  $\{B_j^k\}_j$  *dyadic balls of order  $k$  in  $(Z, \mathcal{M})$*  if  $\{B_j^k\}_j$  are dyadic  $d$ -balls of order  $k$  for some  $d \in \mathcal{M}$ . A *dyadic ball* in  $(Z, \mathcal{M})$  is a dyadic  $d$ -ball of some order  $k$  for some metric  $d \in \mathcal{M}$ .

**Lemma 6** *If  $\{B_j^k\}_j$  is a family of dyadic balls of order  $k$  in  $(Z, \mathcal{M})$ , then*

1.  $\frac{1}{3}B_i^k \cap \frac{1}{3}B_j^k = \emptyset$  for  $i \neq j$ ,  $k \in \mathbb{Z}$ .
2.  $\sum_j \chi_{B_j^k} \leq M$ , for all  $k$  in  $\mathbb{Z}$ , where  $M$  is a constant depending only on the Ahlfors-David constant  $C$  of  $\mathcal{M}$ .

3. Let  $d \in \mathcal{M}$  such that  $\{B_j^k\}_j$  are dyadic  $d$ -balls, then every  $d$ -ball of radius  $3^{k-1}$  is contained in at least one of the  $d$ -balls  $B_j^k$ .

**Proof:** Let  $d \in \mathcal{M}$  such that  $\{B_j^k\}_j$  is a family of dyadic  $d$ -balls. Suppose  $x \in B_j^k$ ,  $1 \leq k \leq N$ , then  $B_j^k \subset B(x, 2 \times 3^k)$  for  $1 \leq j \leq N$ . By Ahlfors-David regularity  $\mu_d(B(x, 2 \times 3^k)) \leq C 2 \times 3^k \leq 6C^2 \mu_d(\hat{B}_j^k)$  and therefore  $N \mu_d(B(x, 2 \times 3^k)) \leq 6C^2 \sum_{j=1}^N \mu_d(\hat{B}_j^k) = 6C^2 \mu_d(\cup_{j=1}^N \hat{B}_j^k) \leq 6C^2 \mu_d(B(x, 2 \times 3^k))$ . Thus  $N \leq 6C^2$  and we may set  $M = 6C^2$ .

For the third property, pick  $x \in Z$ , the  $d$ -ball  $B(x, 3^{k-1})$  intersects at least one of the  $\hat{B}_j^k$  by the maximality property of  $\{\hat{B}_j^k\}_j$ . But then  $B(x, 3^{k-1})$  is contained in  $B_j^k$  since for any  $w \in B(x, 3^{k-1})$  and  $z \in B(x, 3^{k-1}) \cap \hat{B}_j^k$ ,

$$\begin{aligned} d(x_j^k, w) &\leq d(x_j^k, z) + d(z, x) + d(x, w) \\ &< 3^{k-1} + 3^{k-1} + 3^{k-1} \\ &= 3^k. \end{aligned} \tag{182}$$

■

**Lemma 7** Let  $\{B_\iota\}_{\iota \in \mathcal{I}}$  be a family of dyadic balls in  $(Z, \mathcal{M})$ . If  $\{B_j\}_{j \in J}$  is a collection of maximal balls with respect to inclusion in  $\{B_\iota\}_{\iota \in \mathcal{I}}$ , then the balls  $\{\frac{1}{3}B_j\}_{j \in J}$  are pairwise disjoint.

**Proof:** Notice that by construction,  $\frac{1}{3}B_i^k \cap \frac{1}{3}B_j^k = \emptyset$  for dyadic balls of the same degree if  $i \neq j$ . Suppose  $z \in \frac{1}{3}B_i^k \cap \frac{1}{3}B_j^l$  where  $l < k$ . Then  $B_j^l \subset B_i^k$  since if  $y \in B_j^l$ ,

$$\begin{aligned} d(x_i^k, y) &\leq d(x_i^k, z) + d(z, x_j^l) + d(x_j^l, y) \\ &< 3^{k-1} + 3^{l-1} + 3^l \\ &\leq 3^k. \end{aligned} \tag{183}$$

From this and the maximality of  $\{B_j\}_{j \in J}$ , the lemma follows. ■

#### 4.7 Fractional integration on the boundary of strongly hyperbolic spaces.

**Theorem 7** Let  $X$  be a strongly hyperbolic space,  $\{d_o\}_{o \in X}$  the natural Möbius structure on  $\partial X$ , and  $0 < \alpha < \frac{Q}{q}$ . Then there exists a constant  $C_\alpha > 0$  depending only on  $\alpha$ , such that for all  $d_o, d_{o'} \in \mathcal{M}$  and  $f \in L^p(\mu_o)$ ,

$$\left\| \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha}{2}} f \right\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}. \tag{184}$$

**Proof:** (Of Theorem 7) Showing

$$\left\| \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha}{2}} f \right\|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)}, \tag{185}$$

is equivalent to

$$\left( \int_{\partial X} [I_o^\alpha(f)(x)]^q \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \leq C_\alpha \left( \int_{\partial X} f(x)^p \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha p}{2}} d\mu_o(x) \right)^{\frac{1}{p}}. \tag{186}$$

This is further equivalent to

$$\left( \int_{\partial X} [I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}}(f)(x)]^q \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \leq C_\alpha \left( \int_{\partial X} f(x)^p \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha(1-p')p}{2}} d\mu_o(x) \right)^{\frac{1}{p}}. \quad (187)$$

By duality we may therefore prove

$$\begin{aligned} & \int_{\partial X} I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}}(f)(x) g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \\ & \leq C_\alpha \left( \int_{\partial X} f(x)^p \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o(x) \right)^{\frac{1}{p}} \left( \int_{\partial X} g(x)^q \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}}. \end{aligned} \quad (188)$$

for all  $f, g \geq 0$ .

We bound the operator  $I_o^\alpha$  by its dyadic approximation

$$I_o^{\alpha, \text{dy.}} f(x) := \sum_{B: x \in B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \int_B f d\mu_o. \quad (189)$$

Indeed,

$$I_o^\alpha f(x) = \int_{\partial X} \frac{f(y)}{d_o(x, y)^{Q-\alpha}} d\mu_o(y) \leq C_\alpha \sum_{B: x \in B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \int_B f d\mu_o, \quad (190)$$

where  $C_\alpha = 9^{(Q-\alpha)} C^{1-\frac{\alpha}{Q}}$  with  $C$  the Ahlfors-David constant and  $Q$  the dimension of  $\mathcal{M}$ . This follows from the fact that if  $3^{k-1} \leq d_o(x, y) \leq 3^k$ , then both  $x$  and  $y$  are in  $B(x, 3^k)$ , which is contained in some  $B_j^{k+1}$  by lemma 6.3. Thus  $\mu_o(B_j^{k+1})^{1-\frac{\alpha}{Q}} \leq C^{1-\frac{\alpha}{Q}} 3^{(Q-\alpha)(k+1)} \leq 9^{(Q-\alpha)} C^{1-\frac{\alpha}{Q}} d_o(x, y)^{Q-\alpha}$  and therefore

$$d_o(x, y)^{\alpha-Q} \leq C_\alpha \mu_o(B_j^{k+1})^{\frac{\alpha}{Q}-1} \chi_{B_j^{k+1}}(x) \chi_{B_j^{k+1}}(y) \leq C_\alpha \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \chi_B(x) \chi_B(y).$$

The bound (190) follows by multiplying both sides by  $f(y)$  and integrating with respect to  $y$ . Multiplying again both sides by  $g(x) \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}}$  and integrating with respect to  $x$  gives

$$\begin{aligned} & \int_{\partial X} I_o^\alpha \circ \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}}(f)(x) g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \\ & \leq C_\alpha \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \left( \int_B f(x) \left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_B g(x) \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right). \end{aligned} \quad (191)$$

Let  $\{Q_j^k\}_j$  be a collection of maximal dyadic  $d_o$ -balls over which the average of  $g$  exceeds  $\gamma^k$  with  $\gamma = 3^Q C^2$  where  $C$  is the Ahlfors-David constant of  $\mathcal{M}$ . Then

$$\gamma^k < \frac{1}{\mu_o(Q_j^k)} \int_{Q_j^k} g \left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_d \leq \gamma^{k+1}, \quad (192)$$

since any element  $Q_j^k$  has radius  $3^l$  for some  $l \in \partial \mathbb{X}$  and is therefore contained in some  $B_i^{l+1}$  by lemma 6.3. By Ahlfors-David regularity,  $\mu_o(B_i^{l+1}) \leq C 3^{Q(l+1)} \leq \gamma \mu_o(Q_j^k)$  and

by the maximality property of  $\{Q_j^k\}_j$ ,  $\frac{1}{\mu_o(B_i^{l+1})} \int_{B_i^{l+1}} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^k$ , therefore

$$\frac{1}{\mu_o(Q_j^k)} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \frac{\mu_o(B_i^{l+1})}{\mu_o(Q_j^k)} \left[ \frac{1}{\mu_o(B_i^{l+1})} \int_{B_i^{l+1}} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right] \leq \gamma \gamma^k = \gamma^{k+1}. \quad (193)$$

The collection  $\{Q_j^k\}_j$  has the following two important properties that are essential towards the end of this proof to apply lemma ?.

$$\sum_{i:Q_i^l \subset Q_j^k} \mu_o(Q_i^l) \leq C_Q \gamma^{1-(l-k)} \mu_o(Q_j^k), \quad \forall k, j, l, \quad (194)$$

and

$$Q_i^l \subsetneq Q_j^k \implies l > k. \quad (195)$$

The first property (194) is proven as follows.

$$\begin{aligned} \sum_{i:Q_i^l \subset Q_j^k} \mu_o(Q_i^l) &\leq 3^Q C^2 \sum_{i:Q_i^l \subset Q_j^k} \mu_o\left(\frac{1}{3}Q_i^l\right) \\ &\leq 3^Q C^2 \mu_o\left(\cup_{i:Q_i^l \subset Q_j^k} Q_i^l\right) \\ &\leq 15^Q C^4 \sum_{i \in \Gamma} \mu_o(Q_i^l) \end{aligned} \quad (196)$$

where the first inequality follows from the fact that the collection  $\{\frac{1}{3}Q_j^k\}_j$  is pairwise disjoint by lemma 7 and the second inequality holds for some pairwise disjoint sub-collection  $\{Q_i^l\}_{i \in \Gamma}$  by lemma 5. We have chosen  $\{Q_j^k\}_j$  such that  $\mu_o(Q_i^l)^{-1} \int_{Q_i^l} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o > \gamma^l$  and  $\mu_o(Q_j^k)^{-1} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^{k+1}$ . Hence

$$\sum_{i \in \Gamma} \mu_o(Q_i^l) \leq \sum_{i \in \Gamma} \gamma^{-l} \int_{Q_i^l} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^{-l} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^{1-(l-k)} \mu_o(Q_j^k)$$

and together with (196), property (194) follows with  $C_Q = 15^Q C^4$ .

The second property (195) follows from  $\gamma^k < \mu_o(Q_j^k)^{-1} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^l$ , where the last inequality follows from the maximality property of  $Q_i^l$  and  $Q_i^l \subsetneq Q_j^k$ .

For each  $k \in \mathbb{Z}$  define,

$$C^k = \left\{ B \text{ dyadic} : \gamma^k < \frac{1}{\mu_o(B)} \int_B g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma^{k+1} \right\}. \quad (197)$$

Every dyadic  $d_o$ -ball over which the average of  $g(x) \left(\frac{d_{o'}}{d_o}(x)\right)^{\frac{\alpha q}{2}}$  does not vanish, is contained in exactly one of the  $C^k$ . Furthermore,  $Q_j^k \in C^k$  and therefore

$$\frac{1}{\mu_o(B)} \int_B g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \leq \gamma \frac{1}{\mu_o(Q_j^k)} \int_{Q_j^k} g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \quad \forall B \in C^k, \forall j. \quad (198)$$

If  $B \in C^k$  then since  $\frac{1}{\mu_o(B)} \int_B g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o > \gamma^k$ ,  $B$  is contained in some  $Q_j^k$  by the maximality property. The sum  $\sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \left( \int_B f\left(\frac{d_{o'}}{d_o}\right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_B g\left(\frac{d_{o'}}{d_o}\right)^{\frac{\alpha q}{2}} d\mu_o \right)$  on

the right-hand side of equation (191) can be bounded as follows.

$$\begin{aligned}
& \sum_{B \text{ dyadic}} \mu_o(B)^{\frac{\alpha}{Q}-1} \left( \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_B g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
&= \sum_k \sum_{B \in \mathcal{C}^k} \mu_o(B) \mu_o(B)^{\frac{\alpha}{Q}-1} \left( \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \frac{1}{\mu_o(B)} \int_B g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
&\leq \gamma \sum_k \sum_j \left[ \sum_{B \subset Q_j^k} \mu_o(B)^{\frac{\alpha}{Q}} \left( \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \right] \left( \frac{1}{\mu_o(Q_j^k)} \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
&\leq MC_\alpha \gamma \sum_{k,j} \mu_o(Q_j^k)^{\frac{\alpha}{Q}-1} \left( \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right)
\end{aligned}$$

where the last inequality follows from

$$\begin{aligned}
\sum_{B \subset Q_j^k} \mu_o(B)^{\frac{\alpha}{Q}} \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o &\leq C^{\frac{2\alpha}{Q}} \mu_o(Q_j^k)^{\frac{\alpha}{Q}} \sum_{B \subset Q_j^k} \left( \frac{r(B)}{r(Q_j^k)} \right)^\alpha \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \\
&\leq C^{\frac{2\alpha}{Q}} \mu_o(Q_j^k)^{\frac{\alpha}{Q}} \sum_{l=0}^{\infty} \sum_{B \subset Q_j^k, r(B)=\frac{1}{3^l} r(Q_j^k)} \left( \frac{r(B)}{r(Q_j^k)} \right)^\alpha \int_B f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \\
&\leq C^{\frac{2\alpha}{Q}} \mu_o(Q_j^k)^{\frac{\alpha}{Q}} \left( \sum_{l=0}^{\infty} 3^{-l\alpha} \right) M \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \\
&= MC_\alpha \mu_o(Q_j^k)^{\frac{\alpha}{Q}} \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o
\end{aligned}$$

with  $C_\alpha = C^{\frac{2\alpha}{Q}} (\sum_{l=0}^{\infty} 3^{-l\alpha})$ . Plugging the above bounds into equation (191), gives

$$\begin{aligned}
& \int_{\partial X} I_o^\alpha \circ \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} (f)(x) g(x) d\mu_o \\
&\leq MC_\alpha \gamma \sum_{k,j} \mu_o(Q_j^k)^{\frac{\alpha}{Q}-1} \left( \int_{Q_j^k} f(x) \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_{Q_j^k} g(x) \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \quad (199)
\end{aligned}$$

with  $C_\alpha = \frac{3}{2} g^{(Q-\alpha)} C^{1+\frac{\alpha}{Q}}$ .

By lemma 5,

$$\left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o(x) \right)^{\frac{1}{q}} \left( \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o(x) \right)^{\frac{1}{p'}} \leq C_\alpha \mu_o(B)^{1-\frac{\alpha}{Q}}. \quad (200)$$

Define the measures

$$M_q(B) := \int_B \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o, \quad (201)$$

and

$$M_{p'}(B) := \int_B \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o. \quad (202)$$

The sum on the right hand side of equation (199) can therefore be written as follows.

$$\begin{aligned}
& \sum_{k,j} \mu_o(Q_j^k)^{\frac{\alpha}{q}-1} \left( \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
& \leq C_\alpha \sum_{k,j} M_{p'}(Q_j^k)^{-\frac{1}{p'}} M_q(Q_j^k)^{-\frac{1}{q}} \left( \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) \left( \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
& = C_\alpha \sum_{k,j} M_{p'}(Q_j^k)^{\frac{1}{p}} \left( \frac{1}{M_{p'}(Q_j^k)} \int_{Q_j^k} f \left( \frac{d_{o'}}{d_o} \right)^{-\frac{\alpha p'}{2}} d\mu_o \right) M_q(Q_j^k)^{\frac{1}{q'}} \left( \frac{1}{M_q(Q_j^k)} \int_{Q_j^k} g \left( \frac{d_{o'}}{d_o} \right)^{\frac{\alpha q}{2}} d\mu_o \right) \\
& \leq C_\alpha \left( \sum_{k,j} M_{p'}(Q_j^k) \left( \frac{1}{M_{p'}(Q_j^k)} \int_{Q_j^k} f dM_{p'} \right)^p \right)^{\frac{1}{p}} \left( \sum_{k,j} M_q(Q_j^k)^{\frac{p'}{q'}} \left( \frac{1}{M_q(Q_j^k)} \int_{Q_j^k} g dM_q \right)^{p'} \right)^{\frac{1}{p'}} \\
& \leq C_\alpha \left( \sum_{k,j} M_{p'}(Q_j^k) \left( \frac{1}{M_{p'}(Q_j^k)} \int_{Q_j^k} f dM_{p'} \right)^p \right)^{\frac{1}{p}} \left( \sum_{k,j} M_q(Q_j^k) \left( \frac{1}{M_q(Q_j^k)} \int_{Q_j^k} g dM_q \right)^{q'} \right)^{\frac{1}{q'}}
\end{aligned} \tag{203}$$

where the first inequality follows from Hölder's inequality and the second from  $q' \leq p'$ . It only remains to show that

$$\left( \sum_{k,j} M_{p'}(Q_j^k) \left( \frac{1}{M_{p'}(Q_j^k)} \int_{Q_j^k} f dM_{p'} \right)^p \right)^{\frac{1}{p}} \leq C_p \left( \int_{\partial X} f^p dM_{p'} \right)^{\frac{1}{p}} \tag{204}$$

and

$$\left( \sum_{k,j} M_q(Q_j^k) \left( \frac{1}{M_q(Q_j^k)} \int_{Q_j^k} g dM_q \right)^{q'} \right)^{\frac{1}{q'}} \leq C_{q'} \left( \int_{\partial X} g^{q'} dM_q \right)^{\frac{1}{q'}} \tag{205}$$

for some  $C_p$  and  $C_{q'}$  depending only on  $p$ ,  $q'$  and the Ahlfors-David constant  $C$ . This follows from the lemma below with  $\Gamma = \{Q_j^k\}$ .

Notice that thanks to the doubling property of  $M_q$  proven in Lemma 2,

$$\begin{aligned}
\sum_{l,i:Q_i^l \subset Q_j^k} M_q(Q_i^l) & \leq C_{M_q}^2 \sum_{l,i:Q_i^l \subset Q_j^k} M_q\left(\frac{1}{3}Q_i^l\right) \\
& \leq C_{M_q}^2 M_q(Q_j^k),
\end{aligned} \tag{206}$$

where  $C_{M_q}$  is the doubling constant of  $M_q$ . An analogous inequality holds for  $M_{p'}$  thanks to its doubling property proven in Lemma 3.

Marcinkiewicz's interpolation theorem gives the explicit constants

$$C_p = 2 \left( \frac{p}{p-1} \right)^{\frac{1}{p}} C_{M_{p'}}^{\frac{7}{p}}, \tag{207}$$

and

$$C_p = 2 \left( \frac{q'}{q'-1} \right)^{\frac{1}{q'}} C_{M_q}^{\frac{7}{q'}}. \tag{208}$$

In the special case when for every  $d_o \in \mathcal{M}$ ,  $\mu_o$  is a probability measure,  $1 < p \leq q$  implies that there exists  $r > 1$  such that  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$  and by Hölder's inequality

$$\| \left( \frac{d_{o'}}{d} \right)^{\frac{\alpha}{2}} \circ I_o^\alpha \circ \left( \frac{d_{o'}}{d} \right)^{-\frac{\alpha}{2}} f \|_{L^q(\mu_o)} \leq C_\alpha \|f\|_{L^p(\mu_o)} \leq C_\alpha \mu_o(\partial X)^{\frac{1}{r}} \|f\|_{L^q(\mu_o)} = C_\alpha \|f\|_{L^q(\mu_o)}.$$

The second inequality in the theorem follows by setting  $q = 2$ . ■

**Lemma 8** Let  $M$  be a doubling measure on a metric space  $Z$  with doubling constant  $C_M \geq 1$ ,  $\beta \geq 1$ , and  $\Gamma$  a collection of dyadic balls in  $X$  s.t.

$$\sum_{B \in \Gamma, B \subset B'} M(B)^\beta \leq K M(B')^\beta, \quad \forall B' \in \Gamma,$$

for some  $K \geq 1$ . Then

$$\left( \sum_{B \in \Gamma} M(B)^\beta \left( \frac{1}{M(B)} \int_B f dM \right)^p \right)^{\frac{1}{p}} \leq C_p \left( \int_Z f^p dM \right)^{\frac{1}{p}},$$

for all  $f \geq 0$ ,  $1 < p < \infty$  and where  $C_p$  depends only on  $p$ ,  $K$ , and the doubling constant  $C_M$ .

**Proof:** We show that the map  $f \mapsto \left( \frac{1}{M(B)} \int_B f dM \right)_{B \in \Gamma}$  takes  $L^\infty(Z, dM)$  to  $l^\infty(\Gamma, M(B)^\beta)$  and  $L^1(Z, dM)$  to  $l^{1,\infty}(\Gamma, M(B)^\beta)$ . The lemma then follows by applying Marcinkiewicz's interpolation theorem with  $1 < p < \infty$  [10].

The  $(\infty, \infty)$ -boundedness is clear with constant 1.

We show that the map is bounded from  $L^1(Z, dM)$  to  $l^{1,\infty}(\Gamma, M(B)^\beta)$ . Assume w.l.o.g. that  $f$  is bounded with compact support and fix  $\lambda > 0$ . Let  $\{Q_j\}_{j \in J}$  be the maximal dyadic balls  $B$  in  $\Gamma$  such that  $\frac{1}{M(B)} \int_B |f| dM > \lambda$ . Then

$$\begin{aligned} \sum_{B \in \Gamma: \frac{1}{M(B)} \int_B |f| dM > \lambda} M(B)^\beta &\leq \sum_{j \in J} \sum_{B \subset Q_j} M(B)^\beta \\ &\leq K \sum_{j \in J} M(Q_j)^\beta \\ &\leq K C_M^2 \left( \sum_{j \in J} M\left(\frac{1}{3}Q_j\right) \right)^\beta \end{aligned} \tag{209}$$

where in the second inequality we used the summation assumption on  $\Gamma$  and in the last inequality the doubling property of  $M$  and the fact that  $\beta \geq 1$ .

By lemma 5, there is a pairwise disjoint sub-collection  $\{Q_i\}_{i \in I}$  of the dyadic balls  $\{Q_j\}_{j \in J}$  with the property that every  $Q_j$ ,  $j \in J$ , is contained in some  $5Q_i$ ,  $i \in I$ . Notice also that the collection  $\{\frac{1}{3}Q_j\}$  is pairwise disjoint by lemma 7. Thus

$$\begin{aligned} \sum_{j \in J} M\left(\frac{1}{3}Q_j\right) &\leq \sum_{i \in I} \sum_{j \in J: Q_j \subset 5Q_i} M\left(\frac{1}{3}Q_j\right) \\ &\leq \sum_{i \in I} M(5Q_i) \\ &\leq C_M^3 \sum_{i \in I} M(Q_i) \\ &\leq \frac{C_M^3}{\lambda} \sum_{i \in I} \int_{Q_i} |f| dM \\ &\leq \frac{C_M^3}{\lambda} \int_X |f| dM. \end{aligned} \tag{210}$$

Combining the above, shows that

$$\left( \sum_{B \in \Gamma: |\frac{1}{M(B)} \int_B f dM| > \lambda} M(B)^\beta \right)^{\frac{1}{\beta}} \leq \frac{K^{\frac{1}{\beta}} C_M^{\frac{2}{\beta}+3}}{\lambda} \int_X |f| dM, \quad (211)$$

which was to be proven.

The Marcinkiewicz's interpolation theorem gives an explicit constant

$$C_p = 2 \left( \frac{p}{p-1} K C_M^5 \right)^{\frac{1}{p}}. \quad (212)$$

■

## 5 A quadratic form on Möbius spaces

### 5.1 Densities on Möbius spaces

**Definition 6** Let  $(Z, \mathcal{M})$  be a Möbius space of Hausdorff dimension  $Q$ . A  $s$ -density on  $(Z, \mathcal{M})$  is a map  $d \mapsto u_d$  which associates a function  $u_d : Z \rightarrow \mathbb{C}$  with each metric  $d$  in  $\mathcal{M}$ , in such a way that for all metrics  $d, d' \in \mathcal{M}$  and all  $x \in Z$ ,  $u_d(x) = (\frac{d'}{d}(x))^{sQ} u_{d'}(x)$ .

For convenience we denote  $s$ -densities compactly by  $u$  or  $u_d (d\mu_d)^s$ , where  $\mu_d$  is the Hausdorff  $Q$ -dimensional measure on  $Z$ . The space of  $s$ -densities  $\mathcal{D}_s(\mathcal{M})$  on  $Z$ , forms a vector space under point-wise addition.

Densities can be multiplied. The pointwise product of an  $s$ -density with an  $s'$ -density is a  $(s + s')$ -density.  $\frac{1}{2}$ -densities have a natural Möbius invariant inner product, multiply them and integrate the resulting measure. This works only for  $s = \frac{1}{2}$ . To produce invariant inner products for other values of  $s$ , we need more structure.

**Example 3** A function  $f$  on  $Z$  is a 0-density.

**Example 4** A signed measure  $\mu$  on  $Z$  which is absolutely continuous with respect to Hausdorff  $Q$ -dimensional measure defines a 1-density. Indeed, for each metric  $d'$  in the class, there exists a non-negative measurable function  $u_{d'}$  such that  $d\mu = u_{d'} d\mathcal{H}_d^Q$ . Since

$$\frac{d\mathcal{H}_{d'}^Q}{d\mathcal{H}_d^Q} = \left(\frac{d'}{d}\right)^Q,$$

$$u_d d\mathcal{H}_d^Q = d\mu = u_{d'} d\mathcal{H}_{d'}^Q = u_{d'} \left(\frac{d'}{d}\right)^Q d\mathcal{H}_d^Q,$$

hence, for almost every  $x \in Z$ ,

$$u_d(x) = \left(\frac{d'}{d}(x)\right)^Q u_{d'}(x).$$

This motivates to think of  $s$ -densities as fractional measures.

**Example 5** The distance  $(x, y) \mapsto d(x, y)$ , as a 2-point function, is a  $(-\frac{1}{2Q}, -\frac{1}{2Q})$ -bi-density. Indeed, when passing from one distance  $d'$  to another  $d$  in the Möbius class,

$$\frac{d(x, y)}{d'(x, y)} = \left(\frac{d'}{d}(x)\right)^{-1/2} \left(\frac{d'}{d}(y)\right)^{-1/2}.$$

**Example 6** *On the ideal boundary of strongly hyperbolic spaces, we view densities associated with the Möbius structure  $\{d_{\epsilon,o}\}_{o \in X}$  as maps  $o \mapsto u_o$ . These maps transform under a change of origin as*

$$u_o(x) = e^{\epsilon \beta_x(o,o')} u_{o'}(x). \quad (213)$$

## 5.2 Densities viewed from an ideal origin

On the boundary of a strongly hyperbolic space  $X$ , we may extend the definition of densities  $o \mapsto u_o$  up to the boundary  $\partial X$ . Recall that the sequence of measures  $\{e^{Qd(o,p)} \mu_p\}$  converged to a measure  $\mu_{z,o}$ , when  $p \rightarrow z \in \partial X$ .

**Proposition 7** *Let  $u$  be an  $s$ -density on  $\partial X$ , and let  $x \in \partial X$ . As  $p \in X$  tends to a point  $z \in \partial X \setminus \{x\}$ ,*

$$e^{-Qs d(o,p)} u_p(x)$$

*tends to a number, denoted by  $u_{z,o}(x)$ , given by*

$$u_{z,o}(x) = u_o(x) d_o(z, x)^{2Qs}. \quad (214)$$

*As  $o$  is changed,  $u_{z,o}$  gets multiplied by a positive constant,*

$$u_{z,o'} = e^{Qs \beta_z(o',o)} u_{z,o}.$$

Indeed, by definition,

$$\frac{u_p(x)}{u_o(x)} = \left( \frac{d\mu_o}{d\mu_p}(x) \right)^s. \quad (215)$$

Therefore, since  $e^{Qd(o,p)} \mu_p$  converges to a  $Q$ -dimensional Hausdorff measure  $\mu_{z,o}$  as  $p \rightarrow o$ ,

$$e^{-Qs d(o,p)} \frac{u_p(x)}{u_o(x)} = \left( e^{Qd(o,p)} \frac{d\mu_p}{d\mu_o}(x) \right)^{-s} \quad (216)$$

converges to

$$\left( \frac{d\mu_{z,o}}{d\mu_o} \right)^{-s} = d_o(z, x)^{2Qs}. \quad (217)$$

Furthermore,

$$\begin{aligned} \frac{u_{z,o'}(x)}{u_{z,o}(x)} &= \lim_{p \rightarrow z} \frac{e^{-Qs d(o',p)} u_p(x)}{e^{-Qs d(o,p)} u_p(x)} \\ &= \lim_{p \rightarrow z} e^{-Qs(d(o',p) - d(o,p))} \\ &= e^{-Qs \beta_z(o',o)}. \end{aligned} \quad (218)$$

In particular, the notation  $u = u_{z,o}(d\mu_{z,o})^s$  is valid.

## 5.3 Fractional integration revisited

Let  $(Z, \mathcal{M})$  be a Möbius space of dimension  $Q$ . The Riesz-Markov-Kakutani theorem tells us that the dual space of continuous compactly supported functions can be identified with the space of signed measures (with minor regularity assumptions). This establishes a duality between the spaces of 0-densities and 1-densities.

The Riesz-Markov-Kakutani duality carries over to  $s$ -densities. Given an  $(1-s)$ -density  $v$ , then

$$u \mapsto \int uv, \quad (u \in \mathcal{D}_s(\mathcal{M})) \quad (219)$$

is an element of  $\mathcal{D}_s^*$ . Morally,  $\mathcal{D}_{1-s}$  and  $\mathcal{D}_s$  are dual to each other. The space  $\mathcal{D}_{\frac{1}{2}}$  is dual to itself, it carries the inner product  $(u, v) = \int_Z uv$ .

P. Julg [11] remarked, that the operator

$$\mathcal{I}^s(u)(x) = \int_Z \frac{u_d(y)}{d(x, y)^{2Q(1-s)}} \mu_d(y), \quad \left(\frac{1}{2} < s < 1\right) \quad (220)$$

maps  $s$ -densities to  $(1-s)$ -densities. Indeed,

$$\int_Z \frac{u_d(y)}{d(x, y)^{2Q(1-s)}} d\mu_d(y) = \left(\frac{d'}{d}(x)\right)^{(1-s)Q} \int_Z \frac{u_{d'}(y)}{d'(x, y)^{2Q(1-s)}} d\mu_{d'}(y). \quad (221)$$

So  $\mathcal{I}^s(u)$  transforms as a  $(1-s)$ -density.

Replacing  $s$  by  $\frac{1}{2} + \frac{\alpha}{Q}$ , reveals the fractional integration operator

$$I_d^{2\alpha}(f)(x) = \int_Z \frac{f(y)}{d(x, y)^{Q-2\alpha}} d\mu_d(y) \quad \left(0 < \alpha < \frac{Q}{2}\right) \quad (222)$$

defined on functions.

The operator  $\mathcal{I}^s$  yields a Möbius invariant quadratic form,

$$Q^s(u) = \int_Z u \mathcal{I}^s(u). \quad (u \in \mathcal{D}_s) \quad (223)$$

The operator  $I_d^s$  appears naturally within the theory of  $s$ -Poisson transforms. We will highlight this fact within the instructive example of trees.

#### 5.4 The case of regular trees

On a tree  $T$  of degree  $q+1$ , the harmonic measure of a vertex  $o$  is the unique (obvious) probability measure  $\mu_o$  on  $Z = \partial T$  which is invariant under all automorphisms fixing  $o$ . Since it coincides with Patterson-Sullivan's measure, the Radon-Nikodým derivatives are exponentials of Busemann functions,

$$\frac{d\mu_{o'}}{d\mu_o}(\zeta) := P(o, o', \zeta) = q^{\beta_\zeta(o, o')} \quad (224)$$

for all  $\zeta \in Z$  and vertices  $x, y \in T$ . Here,  $\beta_\zeta(o, o')$  tends to  $+\infty$  as  $o'$  tends to  $\zeta$ .

Fix an origin  $o \in T$ , and let  $s \in \mathbb{C}$ . Consider the  $s$ -Poisson transform defined on functions  $u : Z \rightarrow \mathbb{C}$  by

$$\mathcal{P}^s(u)(o') = \int_Z P(o, o', \zeta)^{1-s} u(\zeta) d\mu_o(\zeta). \quad (225)$$

For  $s = 0$ , the change of variable formula (224) shows that  $\mathcal{P}^0(u)$  does not depend on the choice of  $o$ ,  $\mathcal{P}^0$  is the usual Poisson transform, yielding harmonic functions on  $T$ . This invariance persists for  $s \neq 0$ , provided one considers functions on  $Z$  as  $s$ -densities.

**5.4.1 Densities on the ideal boundary of a regular tree** We provide a definition which refers to the natural Möbius structure of visual metrics associated to vertices in  $T$ .

**Definition 7** An  $s$ -density on  $Z$  is a map  $v : o \mapsto u_o$  which associates a function  $u_o : Z \rightarrow \mathbb{C}$  to each vertex of  $T$ , in such a way that for all vertices  $o, o' \in T$  and  $\zeta \in Z$ ,

$$u_{o'}(\zeta) = P(o, o', \zeta)^s u_o(\zeta). \quad (226)$$

The image of  $\mathcal{P}^s$  consists of eigenfunctions of the Laplacian on  $T$ .

**Theorem 8 (FTN page 37)** Let  $u$  be an  $s$ -density on  $Z$ , let  $f = \mathcal{P}^s(u)$ . Then  $f$  does not depend on the choice of origin  $o$ , and

$$\Delta f = \rho(s)f, \quad \text{where } \rho(s) = 1 - \frac{q^s + q^{1-s}}{1+q}. \quad (227)$$

Conversely every  $\rho(s)$ -eigenfunction of the Laplacian on  $T$  is the image by  $\mathcal{P}^s$  of a unique finitely additive measure (viewed as an  $s$ -density) on  $Z$ .

Since  $\rho(1-s) = \rho(s)$ , the operator

$$\mathcal{I}^s = (\mathcal{P}^s)^{-1} \circ \mathcal{P}^{1-s} \quad (228)$$

is well-defined. It maps  $(1-s)$ -densities to  $s$ -densities, hence the quadratic form

$$Q^s : v \mapsto \int_Z v \mathcal{I}^{1-s}(v) \quad (229)$$

is well-defined on  $\mathcal{D}_s$ . If  $\Re(s) = \frac{1}{2}$ , the Hermitian form

$$H^s : v \mapsto \int_Z \bar{v} \mathcal{I}^{1-s}(v) \quad (230)$$

is well defined.

**Lemma 9 (FTN page 45)** 1. For all  $s \in \mathbb{C}$  such that  $\Re(s) = \frac{1}{2}$ ,  $H^s$  is positive definite.

2.  $Q^s$  is real iff  $\Im(s) \in \frac{\pi}{\log q} \mathbb{Z}$ . In these cases,  $Q^s$  is positive definite iff  $0 < \Re(s) < 1$ . Otherwise, it has exactly 1 positive direction.

The proof amounts to an explicit computation of eigenvectors and eigenvalues of  $\mathcal{I}^s$  (acting on functions, i.e. ignoring densities). The spectrum of  $\mathcal{I}^{1-s}$  is the set

$$\text{sp}(\mathcal{I}^{1-s}) = \{1\} \cup \frac{q^{-s} - q^s}{q^{1-s} - q^{s-1}} q^{(2s-1)\mathbb{N}^*}. \quad (231)$$

Both cases give rise to families of unitary representations of  $\text{Aut}(T)$ . The first family is called the *principal series*, the second the *complementary series*.

**Proposition 8** Let  $s \in \mathbb{C}$  with  $\frac{1}{2} < \Re(s) < 1$ . Fix a vertex  $o \in T$ . Then, for  $u \in \mathcal{D}_s$ ,

$$\begin{aligned} Q^s(u) &= \frac{1 - q^{-2}}{1 - q^{-2s}} \int \int_{Z \times Z} q^{2(1-s)\langle x, y \rangle_o} u_o(x) u_o(y) d\mu_o(x) d\mu_o(y) \\ &= \frac{1 - q^{-2}}{1 - q^{-2s}} \int \int_{Z \times Z} d_o(x, y)^{-2(1-s) \log q} u_o(x) u_o(y) d\mu_o(x) d\mu_o(y). \end{aligned} \quad (232)$$

Up to a normalization constant, this quadratic form agrees exactly with the one defined previously.

**Proof:** (of Proposition 8) It relies on Proposition 1.3 on page 40 of [FTN] which describes the inverse of  $s$ -Poisson transform. Here is a formula for the finitely additive measure  $m$  whose  $s$ -Poisson transform is eigenfunction  $f$ . Fix origin  $o$ . Given edge  $e$ , let  $(o'', o')$  denote the endpoints of  $e$ , with  $o''$  between  $o$  and  $o'$ . Let  $\mathcal{O}_o^{o'}$  be the set of endpoints of the sub-tree cut by  $e$  and opposite to  $o$ , i.e. the shadow of  $o'$  seen from  $o$ . Notice that  $\mu_o(\mathcal{O}_o^{o'}) = \frac{q}{q+1} q^{-d(o, o')}$ . Then

$$m(\mathcal{O}_o^{o'}) = \frac{1}{q^{(1-s)} - q^{-(1-s)}} q^{-(1-s)d(o, o'')} (f(o') - q^{-(1-s)} f(o'')). \quad (233)$$

Assume  $f = \mathcal{P}^{1-s}(u)$  where  $u$  is a  $(1-s)$ -density, realized at origin  $o$  by function  $u_o$ , i.e.

$$f(o') = \int_Z P(o, o', y)^s u_o(y) d\mu_o(y). \quad (234)$$

If  $o'$  is close enough to a point  $x \in Z$ ,  $x \neq y$ , then  $o''$  is between  $y$  and  $o'$  along a geodesic, hence  $\beta_y(o', o'') = -1$ ,  $\beta_y(o, o'') = \beta_y(o, o') - 1$ ,  $P(o, o'', y) = q^{-1} P(o, o', y)$ ,

$$P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s = (1 - q^{-1}) P(o, o', y)^s, \quad (y \notin \mathcal{O}_o^{o'}) \quad (235)$$

If  $y \in \mathcal{O}_o^{o'}$  however, then  $P(o, o'', y) = q P(o, o', y)$ ,

$$P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s = (1 - q^{2s-1}) P(o, o', y)^s, \quad (y \in \mathcal{O}_o^{o'}). \quad (236)$$

This gives,

$$\begin{aligned} f(o') - q^{-(1-s)} f(o'') &= \int_Z (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &= \int_{\mathcal{O}_o^{o'}} (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &\quad + \int_{Z \setminus \mathcal{O}_o^{o'}} (P(o, o', y)^s - q^{-(1-s)} P(o, o'', y)^s) u_o(y) d\mu_o(y) \\ &= (1 - q^{2s-1}) \int_{\mathcal{O}_o^{o'}} P(o, o', y)^s u_o(y) d\mu_o(y) \\ &\quad + (1 - q^{-1}) \int_{Z \setminus \mathcal{O}_o^{o'}} P(o, o', y)^s u_o(y) d\mu_o(y) \end{aligned} \quad (237)$$

Therefore

$$\begin{aligned}
m(\mathcal{O}_o^{o'}) &= \frac{1 - q^{2s-1}}{q^{(1-s)} - q^{-(1-s)}} \int_{\mathcal{O}_o^{o'}} q^{-(1-s)d(o,o'')} P(o, o', y)^s u_o(y) d\mu_o(y) \\
&\quad + \frac{1 - q^{-1}}{q^{(1-s)} - q^{-(1-s)}} \int_{Z \setminus \mathcal{O}_o^{o'}} q^{-(1-s)d(o,o'')} P(o, o', y)^s u_o(y) d\mu_o(y) \\
&= \frac{q^{(1-s)}(1 - q^{2s-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{\mathcal{O}_o^{o'}} q^{-(1-s)d(o,o')} P(o, o', y)^s u_o(y) d\mu_o(y) \\
&\quad + \frac{q^{(1-s)}(1 - q^{-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{Z \setminus \mathcal{O}_o^{o'}} q^{-(1-s)d(o,o')} P(o, o', y)^s u_o(y) d\mu_o(y)
\end{aligned} \tag{238}$$

Notice that

$$q^{d(o,o')} P(o, o', y) = q^{d(o,o') + \beta_y(o,o')}. \tag{239}$$

Which gives,

$$\begin{aligned}
m(\mathcal{O}_o^{o'}) &= \frac{q^{(1-s)}(1 - q^{2s-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{\mathcal{O}_o^{o'}} q^{-d(o,o')} q^{s(d(o,o') + \beta_y(o,o'))} u_o(y) d\mu_o(y) \\
&\quad + \frac{q^{(1-s)}(1 - q^{-1})}{q^{(1-s)} - q^{-(1-s)}} \int_{Z \setminus \mathcal{O}_o^{o'}} q^{-(1-s)d(o,o')} P(o, o', y)^s u_o(y) d\mu_o(y)
\end{aligned} \tag{240}$$

As  $o'$  tends to  $x \neq y$ ,  $q^{d(o,o')} P(o, o', y) = q^{d(o,o') + \beta_y(o,o')}$  is stationary. In fact, it stays constant as soon as  $d(o, o') \geq \langle x, y \rangle_o$ , hence its limit is  $q^{2\langle x, y \rangle_o}$ . Notice also that the first integral becomes negligible only if  $s < \frac{1}{2}$ . Thus when  $\mathcal{O}_o^{o'}$  is small enough, and  $s < \frac{1}{2}$ ,

$$\begin{aligned}
m(\mathcal{O}_o^{o'}) &\sim \frac{1 - q^{-1}}{1 - q^{-2(1-s)}} \int_Z q^{-d(o,o')} q^{2s\langle x, y \rangle_o} u_o(y) d\mu_o(y) \\
&\sim \frac{1 - q^{-1}}{1 - q^{-2(1-s)}} \frac{1 + q}{q} \left( \int_Z q^{2s\langle x, y \rangle_o} u_o(y) d\mu_o(y) \right) \mu_o(\mathcal{O}_o^{o'}).
\end{aligned} \tag{241}$$

In other words,  $m$  has density

$$\frac{1 - q^{-2}}{1 - q^{-2(1-s)}} \int_Z q^{2s\langle x, y \rangle_o} u_o(y) d\mu_o(y) \tag{242}$$

with respect to  $\mu_o$ .

Replace  $s$  by  $1 - s$ . By definition,  $Q^s(u)$  consists in integrating  $u_o$  with respect to this measure, hence

$$Q^s(u) = \frac{1 - q^{-2}}{1 - q^{-2s}} \int_{Z \times Z} q^{2(1-s)\langle x, y \rangle_o} u_o(y) u_o(x) d\mu_o(y) d\mu_o(x). \tag{243}$$

■

## 6 Uniformly bounded representations

Given a Möbius space  $(Z, \mathcal{M})$ , we have seen that the quadratic form

$$Q^s(u) = \int_{Z \times Z} d(x, y)^{-2Q(1-s)} u_d(x) u_d(y) \mu_d(x) \mu_d(y), \quad \left(\frac{1}{2} < s < 1\right) \tag{244}$$

on  $s$ -densities, is exactly Möbius invariant. In other words,  $Q^s$  does not depend on a choice of  $d \in \mathcal{M}$ . A subtle question remains the positivity of  $Q^s$ .

In the world of real rank 1 simple Lie groups,  $Q^s$  arises as the invariant inner product of complementary series representations. The necessary and sufficient conditions for positivity of  $Q^s$  were given by B. Kostant.<sup>22</sup> For  $(Z, \mathcal{M})$  being the Möbius space on the boundary of  $\mathrm{SO}(n, 1)$  and  $\mathrm{SU}(n, 1)$ ,  $Q^s$  is positive definite on the whole strip  $s \in (\frac{1}{2}, 1)$ . Contrary to the case of  $\mathrm{Sp}(n, 1)$ , for which  $Q^s$  is positive definite if and only if  $s \in (\frac{1}{2}, \frac{1}{2n+1})$ . The existence of a point  $s^*$ , such that  $Q^s$  fails to be positive definite for all  $s \in [s^*, 1)$ , is typical for groups with Kazhdan's property  $T$  such as  $\mathrm{Sp}(n, 1)$  or  $F_{4(-20)}$ .<sup>23</sup>

In the limit  $s \rightarrow 1$ , the quadratic form  $Q^s$  degenerates to  $Q^1(u) = \left( \int_Z u \right)^2$  on measures. The derivative of  $Q^s$  at  $s = 1$  however, yields again an invariant quadratic form

$$H(u) = \left. \frac{dQ^s(u)}{ds} \right|_{s=1} = 2Q \int_{Z \times Z} \log d(x, y) u_d(x) u_d(y) \mu_d(x) \mu_d(y). \quad (245)$$

P. Julg gives a short proof of the fact that  $H$  is positive definite in the case of  $\mathrm{SO}(n, 1)$  and  $\mathrm{SU}(n, 1)$ . This also follows from the positivity of  $Q^s$  on the whole strip  $s \in (\frac{1}{2}, 1)$ . The affine space

$$\mathcal{A} = \{u \in D^1 : H(u) < \infty, \int_Z u = 0\}, \quad (246)$$

is Möbius invariant. The actions of  $\mathrm{SO}(n, 1)$  and  $\mathrm{SU}(n, 1)$  on  $(\mathcal{A}, H)$  are isometric and proper, confirming the known fact that these groups have Haagerup property.

The above discussion shows that for a strongly hyperbolic space  $X$ , the positivity of  $Q^s$  on  $\partial X$  fails in general. Overcoming this limitation is the subject of the next section.

## 6.1 A Hilbert space attached to Möbius spaces

The fractional integration operator

$$I_d^\alpha(u) = \int_Z d(x, y)^{-Q+\alpha} u_d(y) d\mu_d(y), \quad (0 < \alpha < \frac{Q}{2}) \quad (247)$$

is self-adjoint on  $L^2(d\mu_d)$  and its square has kernel

$$K(x, y) = \int_Z d(x, z)^{-Q+\alpha} d(z, y)^{-Q+\alpha} d\mu_d(z). \quad (248)$$

This kernel is comparable to  $d(x, y)^{-Q+2\alpha}$  and  $(I_d^\alpha)^2$  a good candidate for a Laplacian raised to the power  $-\alpha$ . Our guess for the  $\mathcal{H}^{-\alpha}$  norm on  $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities on compact  $Q$ -Ahlfors regular Möbius spaces, is

$$\|u\|_d = \|I_d^\alpha u_d\|_{L^2(d\mu_d)}. \quad (249)$$

By construction, it is positive and gives rise to a Sobolev space  $\mathcal{H}_d^{-\alpha}$  of  $(\frac{1}{2} + \frac{\alpha}{Q})$ -densities.

**6.1.1 Uniform boundedness.** Apriori, the construction of  $\mathcal{H}_d^{-\alpha}$  depends on a choice of metric  $d$  within the Möbius structure. We may ask, if it is at least canonical on

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<sup>22</sup>B. Kostant. *On the existence and irreducibility of certain series of representations*. Bull. Amer. Math. Soc., **75**:627642, (1969).

<sup>23</sup> $F_{4(-20)}$  is the rank one real form of the simple complex Lie group of type  $F_4$ .

topological level. That means, for fixed  $0 < \alpha < \frac{Q}{2}$ , does there exists a  $C > 0$ , s.t.

$$\frac{1}{C} \|u\|_d \leq \|u\|_{d'} \leq C \|u\|_d, \quad (250)$$

for all  $d, d' \in \mathcal{M}$ ?

Topological uniqueness is directly related to uniformly bounded representations of groups. If  $G$  is a group acting on  $Z$ , then  $G$  acts naturally on  $\mathcal{D}_s(\mathcal{M})$  by  $g \cdot u = g \cdot u_d (dg_* \mu_d)^s$ , where  $g \cdot u_d$  is left-translation on functions and  $g_* \mu_d$  denotes the pushforward measure of  $\mu_d$  by  $g$ . If  $G$  acts by Möbius automorphisms on  $Z$ , then  $\|g \cdot u\|_d = \|u\|_{g^* d}$ . Here,  $g^* d$  denotes the pullback of  $d$  by  $g$ . Indeed,  $\frac{dg_* \mu_d}{d\mu_d}(x) = \left(\frac{(g^{-1})^* d}{d}(x)\right)^Q = \left(\frac{d}{g^* d}(g^{-1}x)\right)^Q$ . For better readability we set  $\frac{1}{2} + \frac{\alpha}{Q} = s$ .

$$\begin{aligned} \|g \cdot u\|_d^2 &= \int_{Z \times Z} \frac{u_d(g^{-1}x) u_d(g^{-1}y) \left(\frac{(g^{-1})^* d}{d}(x)\right)^{sQ} \left(\frac{(g^{-1})^* d}{d}(y)\right)^{sQ}}{d(x, y)^{Q-\alpha}} d\mu_d(y) d\mu_d(x) \\ &= \int_{Z \times Z} \frac{u_d(g^{-1}x) u_d(g^{-1}y) \left(\frac{d}{g^* d}(g^{-1}x)\right)^{sQ} \left(\frac{d}{g^* d}(g^{-1}y)\right)^{sQ}}{g^* d(g^{-1}x, g^{-1}y)^{Q-\alpha}} d\mu_d(y) d\mu_d(x) \\ &= \int_{Z \times Z} \frac{u_d(x) u_d(y) \left(\frac{d}{g^* d}(x)\right)^{sQ} \left(\frac{d}{g^* d}(y)\right)^{sQ}}{g^* d(x, y)^{Q-\alpha}} d(g^{-1})_* \mu_d(y) d(g^{-1})_* \mu_d(x) \\ &= \int_{Z \times Z} \frac{u_{g^* d}(x) u_{g^* d}(y)}{g^* d(x, y)^{Q-\alpha}} d\mu_{g^* d}(y) d\mu_{g^* d}(x). \end{aligned} \quad (251)$$

If  $G$  maps  $\mathcal{M}$  into itself, then (250) implies,

$$\frac{1}{C} \|u\|_d \leq \|g \cdot u\|_d \leq C \|u\|_d. \quad (\forall g \in G) \quad (252)$$

In other words, the action of  $G$  on  $\mathcal{H}^{-\alpha}$  is uniformly bounded.

**Lemma 10** *The space  $\mathcal{H}^{-\alpha}$  is topologically uniquely defined (i.e. independent of the choice of a metric in  $\mathcal{M}$ ), if and only if there exists a  $C > 0$  such that for all  $d, d' \in \mathcal{M}$ ,*

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^2(\mu_d)} \leq C \|I_d^\alpha u_d\|_{L^2(\mu_d)}. \quad (253)$$

**Proof:** Indeed, by the symmetry of the problem, it is only necessary to show the upper bound in (250) for all  $d, d' \in \mathcal{M}$ . Rewriting the norm  $\|u\|_{d'}$  w.r.t. a fixed background measure  $\mu_d$ , yields

$$\begin{aligned} \|u\|_{d'}^2 &= \|I_{d'}^\alpha (u_{d'} (d\mu_{d'})^{\frac{1}{2} + \frac{\alpha}{Q}})\|_{L^2(\mu_{d'})}^2 \\ &= \int_Z \left( \int_Z \frac{u_{d'}(y)}{d'(x, y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\ &= \int_Z \left( \int_Z \frac{\left(\frac{d}{d'}(y)\right)^{\frac{Q}{2} + \alpha} u_d(y)}{\left(\frac{d'}{d}(x) \frac{d'}{d}(y)\right)^{\frac{Q}{2} - \frac{\alpha}{2}} d(x, y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\ &= \int_Z \left( \left(\frac{d}{d'}(x)\right)^{\frac{Q}{2} - \frac{\alpha}{2}} \int_Z \frac{\left(\frac{d}{d'}(y)\right)^{Q + \frac{\alpha}{2}} u_d(y)}{d(x, y)^{Q-\alpha}} d\mu_{d'}(y) \right)^2 d\mu_{d'}(x) \\ &= \int_Z \left( \left(\frac{d}{d'}(x)\right)^{-\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d}{d'}(y)\right)^{\frac{\alpha}{2}} u_d(y)}{d(x, y)^{Q-\alpha}} d\mu_d(y) \right)^2 d\mu_d(x) \\ &= \left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} u_d \right\|_{L^2(\mu_d)}^2. \end{aligned} \quad (254)$$

■

Directly showing the inequality  $\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} u_d\|_{L^2(\mu_d)} \leq C \|I_d^\alpha u_d\|_{L^2(\mu_d)}$ , proves to be challenging. However, the weaker statement  $\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} u_d\|_{L^2(\mu_d)} \leq C \|u_d\|_{L^2(\mu_d)}$ , is equivalent to the weighted norm inequality for  $I_d^\alpha$  that we have seen previously

$$\|I_d^\alpha(u_d)\|_{L^2((\frac{d'}{d})^\alpha \mu_d)} \leq C \|u_d\|_{L^2((\frac{d'}{d})^\alpha \mu_d)}. \quad (\forall d, d' \in \mathcal{M}) \quad (255)$$

That means, the operator norm  $\|I_d^\alpha\|_{L^2((\frac{d'}{d})^\alpha \mu_d)}$  of  $I_d^\alpha$ , is uniformly bounded by a constant  $C$ . We will see that for spherical Möbius structures, the weaker statement (255) implies the stronger (253) on a family of special functions.

**6.1.2 Powers of conformal factors.** Given a Möbius space  $(Z, \mathcal{M})$ , the conformal factors raised to the power of  $sQ$  are  $s$ -densities on  $Z$ . Clearly, for every metric  $d' \in \mathcal{M}$ , the function

$$d \mapsto \left(\frac{d'}{d}\right)^{sQ} \quad (256)$$

transforms as an  $s$ -density. Morally,  $(\frac{d'}{d})^{sQ}$  is the  $s$ -density, that is the constant function 1 at the metric  $d'$ . Recall, that  $I_d^\alpha$  gives rise to an operator on  $s$ -densities for a certain  $s$  depending on  $\alpha$ . It turns out,  $s = \frac{1}{2} + \frac{\alpha}{2Q}$ . This fact implies strong regularity properties on spherical Möbius spaces.

**Proposition 9** *Let  $(Z, \mathcal{M})$  be a spherical,  $Q$ -Ahlfors-David regular Möbius space,  $0 < \alpha < \frac{Q}{2}$ . If the operator norm  $\|I_d^\alpha\|_{L^2((\frac{d'}{d})^\alpha \mu_d)}$  is uniformly bounded, i.e. bounded by a constant independent of  $d$  and  $d'$ , then there exists  $C > 0$ , uniform over  $\mathcal{M}$ , s.t. every function  $f$  out of the family*

$$\left\{ \left(\frac{d''}{d}\right)^{sQ} \right\}_{d'' \in \mathcal{M}} \quad \left( s = \frac{1}{2} + \frac{\alpha}{2Q} \right) \quad (257)$$

satisfies the inequality  $\|(\frac{d'}{d})^{\frac{\alpha}{2}} \circ I_d^\alpha \circ (\frac{d'}{d})^{-\frac{\alpha}{2}} f\|_{L^2(\mu_d)} \leq C \|I_d^\alpha f\|_{L^2(\mu_d)}$ .

This proposition particularly applies to the natural Möbius structure  $\{d_{\epsilon, o}\}_{o \in X}$  on a strongly hyperbolic space  $X$ .

Let us set  $s = \frac{1}{2} + \frac{\alpha}{2Q}$ . The dual of  $s$  we denote by  $s' = \frac{1}{2} - \frac{\alpha}{2Q}$ . Conformal invariance of  $I_d^\alpha$  on  $s$ -densities, manifests itself explicitly as follows.

$$\begin{aligned} I_d^\alpha \left( \left(\frac{d'}{d}\right)^{sQ} \right)(x) &= \int_Z \frac{(\frac{d'}{d}(y))^{sQ}}{d(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \int_Z \frac{(\frac{d'}{d}(y))^{sQ}}{(\frac{d}{d'}(x) \frac{d}{d'}(y))^{s'Q} d'(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{s'Q} \int_Z \frac{(\frac{d'}{d}(y))^Q}{d'(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{s'Q} \int_Z \frac{1}{d'(x, y)^{Q-\alpha}} d\mu_{d'}(y). \end{aligned} \quad (258)$$

If  $\mathcal{M}$  is a compact Möbius structure, then  $\int_Z \frac{1}{d'(x, y)^{Q-\alpha}} d\mu_{d'}(y)$  is finite.

**Lemma 11** *If  $\mathcal{M}$  is a spherical,  $Q$ -Ahlfors-David regular Möbius structure, then there exists  $C > 0$  s.t.  $\forall d \in \mathcal{M}$ ,*

$$1 + \frac{Q - \alpha}{\alpha} \frac{1}{C} \leq \int_Z \frac{1}{d(x, y)^{Q-\alpha}} d\mu_d(y) \leq 1 + \frac{Q - \alpha}{\alpha} C. \quad (259)$$

**Proof:** We use integration by parts, to integrate over balls.

$$\begin{aligned} \int_Z \frac{1}{d(x, y)^{Q-\alpha}} d\mu_d(y) &= \int_Z (Q - \alpha) \int_0^\infty t^{-Q+\alpha-1} \chi_{\{t \geq d(x, y)\}} dt d\mu_d(y) \\ &= (Q - \alpha) \int_0^\infty t^{-Q+\alpha-1} \mu_d(B_d(x, t)) dt \\ &= 1 + (Q - \alpha) \int_1^\infty t^{-Q+\alpha-1} \mu_d(B_d(x, t)) dt. \end{aligned} \quad (260)$$

The last line uses the fact that  $\mathcal{M}$  is spherical. The uniform Ahlfors-David regularity of  $\mathcal{M}$ , gives a  $C \geq 1$ , such that  $\frac{t^Q}{C} \leq \mu_d(B_d(x, t)) \leq Ct^Q$ . Using this above and integrating out, proves the lemma.  $\blacksquare$

We are now in the position to prove proposition 9.

**Proof:** (of proposition 9) Recall that we have set  $s = \frac{1}{2} + \frac{Q}{2}$  and  $s' = \frac{1}{2} - \frac{Q}{2}$ . For any three metrics  $d, d', d'' \in \mathcal{M}$  we have,

$$\begin{aligned} \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \left(\left(\frac{d''}{d}\right)^{sQ}\right)(x) &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^{sQ}}{d(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^{sQ}}{\left(\frac{d'}{d''}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d''}{d''}(y)\right)^{s'Q} d''(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d''}{d}(x)\right)^{s'Q} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}} \left(\frac{d''}{d}(y)\right)^{sQ}}{d''(x, y)^{Q-\alpha}} d\mu_d(y) \\ &= \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d''}{d}(x)\right)^{s'Q} \int_Z \frac{\left(\frac{d'}{d}(y)\right)^{-\frac{\alpha}{2}}}{d''(x, y)^{Q-\alpha}} d\mu_{d''}(y) \\ &= \left(\frac{d''}{d}(x)\right)^{\frac{Q}{2}} \left(\frac{d'}{d}(x)\right)^{\frac{\alpha}{2}} \left(\frac{d''}{d''}(x)\right)^{\frac{\alpha}{2}} I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right)(x) \\ &= \left(\frac{d''}{d}(x)\right)^{\frac{Q}{2}} \left(\frac{d'}{d''}(x)\right)^{\frac{\alpha}{2}} I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right)(x). \end{aligned} \quad (261)$$

Thus,

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \left(\left(\frac{d''}{d}\right)^{sQ}\right) \right\|_{L^2(d\mu_d)} = \left\| I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right) \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)} \quad (262)$$

By assumption there exists  $C > 0$  s.t.

$$\left\| I_{d''}^\alpha \left(\left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}}\right) \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)} \leq C \left\| \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)}. \quad (263)$$

But  $\left\| \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \right\|_{L^2\left(\left(\frac{d'}{d''}\right)^\alpha \mu_{d''}\right)} = \left\| \left(\frac{d''}{d}\right)^{s'Q} \right\|_{L^2(\mu_d)}$ , and lemma 11 implies for a spherical Möbius structure,

$$\left\| \left(\frac{d''}{d}\right)^{s'Q} \right\|_{L^2(\mu_d)} \sim \left\| I_d^\alpha \left(\left(\frac{d''}{d}\right)^{sQ}\right) \right\|_{L^2(\mu_d)}. \quad (264)$$

Thus, there exists  $C > 0$  independent of  $d, d', d''$ , s.t.

$$\left\| \left(\frac{d'}{d}\right)^{\frac{\alpha}{2}} \circ I_d^\alpha \circ \left(\frac{d'}{d}\right)^{-\frac{\alpha}{2}} \left(\left(\frac{d''}{d}\right)^{s_Q}\right) \right\|_{L^2(d\mu_d)} \leq C \left\| I_d^\alpha \left(\left(\frac{d''}{d}\right)^{s_Q}\right) \right\|_{L^2(\mu_d)}. \quad (265)$$

■

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