Reallocation with Priorities and Minimal Envy Mechanisms

Julien Combe

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Abstract

We investigate the problem of reallocation with priorities where one has to assign objects or positions to individuals. Agents can have an initial ownership over an object. Each object has a priority ordering over the agents. In this framework, there is no mechanism that is both individually rational (IR) and stable, i.e. has no blocking pairs. Given this impossibility, an alternative approach is to compare mechanisms based on the blocking pairs they generate. A mechanism has minimal envy within a set of mechanisms if there is no other mechanism in the set that always leads to a set of blocking pairs included in the one of the former mechanism. Our main result shows that the modified Deferred Acceptance mechanism (Guillen and Kesten, 2012), is a minimal envy mechanism in the set of IR and strategy-proof mechanisms. We also show that an extension of the Top Trading Cycle (Karakaya, Klaus, and Schlegel, 2019) mechanism is a minimal envy mechanism in the set of IR, strategy-proof and Pareto-efficient mechanisms. These two results extend the existing ones in school choice.

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†Ecole polytechnique, CREST and Institut Polytechnique de Paris. Email: julien.combe@polytechnique.edu.

1 Introduction

Since the seminal work of Gale and Shapley (1962), the study of matching problems without monetary transfers has become increasingly important, and solutions to these problems have been widely used in many practical applications, such as the allocation of interns to hospitals (see, for instance, Roth, 1984), school choice (Abdulkadiroglu and Sonmez, 2003), kidney exchange (Roth, Sonmez, and Unver, 2004) and the reallocation of houses (Shapley and Scarf, 1974). Recently, some authors have investigated the problem of assigning agents with an initial allocation who have preferences over objects and where the objects order the agents according to priorities. In the literature, many applications have been identified for it: the (re)assignment of campus housing (Guillen and Kesten, 2012), the reallocation of workers to positions (Compte and Jehiel, 2008), the assignment of teachers to schools (Pereyra, 2013; Dur and Kesten, 2014; Combe, Tercieux, and Terrier, 2022), intra-district school choice transfer (Hafalir, Kojima, and Yenmez, 2019), re(assignment) of police officers (Sidibe, Ba, Bayer, Rim, and Rivera, 2021) or tuition-exchange programs (Dur and Ünver, 2019).

In a reallocation problem with priorities, there are two types of individuals: *initial owners* who are initially assigned to an object[1] and *initially unassigned individuals* who are looking for their first assignment. A first important constraint, called *individual rationality* (IR), is to ensure that each initial owner obtains an object that she weakly prefers to her initial one. The problem is thus similar to the house allocation problem with existing tenants (Abdulkadiroglu and Sonmez, 1999). In that setting, one has to assign houses to individuals where some of these agents are tenants and initially own one of the houses. In addition in our setting, each object can have multiple copies and has a fixed priority ordering over individuals. Our problem is also related to the standard school choice problem (Abdulkadiroglu and Sonmez, 2003) where students without an initial assignment have to be assigned to schools. In this environment, one would like to design *strategy-proof* (SP) mechanisms that incentivize individuals to report truthfully their preferences and that return a matching with desirable properties. Two main properties, that are incompatible, have been considered:

1. *Stability*: there is no *blocking pairs* i.e. an individual and an object so that either i) the former prefers the latter and she has a higher priority in for that object than some individual that has been assigned to it (*no justified envy*) or ii) there are unassigned copies of that object left (*non wastefulness*). If such a pair exists, for simplicity, we say that the individual *envies* the object.[2]

2. *Pareto-efficiency* (PE): one cannot find another matching where all individuals obtain a weakly preferred object and some a strictly preferred one.

In that case, two well-known mechanisms have been proposed (Abdulkadiroglu and Sonmez, 2003): i) the *Deferred-Acceptance* (DA) mechanism that is SP and stable and the school choice *Top Trading Cycles* (TTC) that is SP and PE. A reallocation problem with priorities can be seen as a hybrid between the school choice and the house allocation problem: if there are only initially unassigned individuals, then it is equivalent to a school choice problem and if there are only initial owners, we are close to a house allocation model where houses have priorities over the agents. A

[1] We generally refer to objects here but one can also think about positions in an institution, e.g. a school, a company and so on.

[2] The term *envy* usually refers to an agent that prefers an object to her current one, independently of her priority ranking in that object.

natural question arises: can we adapt the two most prominent mechanisms of the school choice literature to ensure IR in this environment where there may be initial owners? In practice, priorities of objects over individuals can be decided by law based on critreria independent of the initial ownership. This feature creates an important difference with the school choice problem: there is no matching that is both IR and stable and so DA and TTC are not IR.[3] If IR is considered to be an important desiderata to incentivize initial owners to participate, then one will necessarily end up creating blocking pairs. Thus, a natural question is how to compare different mechanisms in terms of the envy they generate?

A first natural approach followed in the literature (Compte and Jehiel, 2008; Guillen and Kesten, 2012; Pereyra, 2013) is to define a relaxed notion of stability that is compatible with the IR constraint: if an individual prefers an object to her assigned one, then the only individuals that can have a lower priority than her for that object are its initial owners.[4] With this notion, a simple modification of the DA mechanism, introduced by Guillen and Kesten (2012), can be used: i) modify the priorities of the objects so that they all rank their initial owners at the top of their priorities and ii) run the standard DA mechanism using these modified priorities. This mechanism, we refer to it as DA*, is IR, SP and respects the above relaxed notion of stability. However, it is not stable according to the original priorities: blocking pairs remain. This mechanism is used for instance at MIT for the assignment of campus housing to students (Guillen and Kesten, 2012) and also in France to assign teachers to schools (Combe, Tercieux, and Terrier, 2022).

When focusing on SP and PE mechanisms in the presence of initial owners and initially unassigned individuals, one can consider a similar extension of the school choice TTC mechanism: i) modify the priorities of the objects so that they all rank their initial owners at the top of their ranking and ii) run the TTC mechanism using these modified priorities. This mechanism, that we refer to as TTC*, is a simple adaptation of the *You-Request-My-House-I-Get-Your-Turn* (YRMH-IGYT) mechanism proposed by Abdulkadiroglu and Sonmez (1999) for house allocation problems with existing tenants and vacant houses.[5] This mechanism is IR, SP and PE but is not stable.

In this paper, we propose to use an alternative approach to compare mechanisms in this setting in terms of *how much envy they generate*, i.e. their respective sets of blocking pairs. The notion of *minimal envy* has been introduced by Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) to compare PE mechanisms in a school choice setting. If for each preference and priority profile, the set of blocking pairs of a mechanism is always a subset of the one of another mechanism, and this inclusion is strict for some profiles, then the former has strictly less envy than the latter. A mechanism is a *minimal envy mechanism* if there is no other mechanism that has strictly less envy.

Combe, Tercieux, and Terrier (2022) showed that the DA* mechanism is not efficient in a strong sense: it is possible to find assignments such that individuals are better-off and objects (schools in their application) are assigned individuals with higher priority,[6] thus one can improve both sides of the market. Such assignment would result in a set of blocking pairs that is included in the one of DA* while still being IR (Example 1). This observation is of practical importance. Indeed, a very low mobility can lead to important complains from individuals. In their application

[3] We refer to the school choice version of the TTC mechanism introduced by Abdulkadiroglu and Sonmez (2003). At each step, each object points to its highest ranked individual and each individual points to its most preferred object and cycles are implemented. If priorities are arbitrary, the mechanism is trivially not IR.

[4] This notion is also called *fairness* in Guillen and Kesten (2012) but is called μ_0-stability in Compte and Jehiel (2008) and *acceptable matchings* in Pereyra (2013).

[5] In the YRMH-IGYT mechanism, houses are assumed to have a common priority list compared to TTC* where schools can have different priority rankings.

[6] For a definition of the improvement when objects have multiple copies, the reader can refer to the cited article.

of secondary school teacher assignment in France, who are assigned using DA*, they report that more than 60% of the teachers initially assigned to a school and requesting a transfer do not obtain a new assignment. Since we know, from Pereyra (2013) and Compte and Jehiel (2008), that there is no μ_0-stable and SP mechanism that can lead to more mobility than DA*, this tension can lead the policy makers to forgo the μ_0-stability concept. For instance, this is actually what happened for primary school teachers in France. Due to a persistent lack of mobility across regions, an additional phase of exchanges has been added (called ineat/exeat) where teachers can request a transfer which can be granted without any stability constraint. In 2019, more than 20% of the primary school teachers willing to transfer obtain a new assignment through this extra process.[7] For secondary school teachers, a recent report to the Ministry of Education highlights the needs to consider more individuals situations to sometimes go "beyond the priority score" (Inspection générale de l'administration de l'Education nationale et de la recherche, 2015).

In the case where policy makers consider going beyond μ_0-stability, one can still consider the whole set of blocking pairs and the minimal envy requirement. If it was possible to find a SP mechanism that would systematically have less envy than DA*, this would be problematic. Following the positive interpretation of stability (Roth, 1991), one can be worried that mechanisms that are not minimal envy mechanisms, and thus can generate "too much" envy compared to other strategy-proof mechanisms, have a higher risk to be abandoned in practice. Our first result shows that no such mechanism exists: DA* is a minimal envy mechanism in the set of IR and SP mechanisms (Proposition 1).[8] Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) showed that, in a school choice setting where each school has only one seat, the TTC mechanism is a minimal envy mechanism in the set of SP and PE mechanisms. Our second result shows that TTC*, the natural extension of TTC in our reallocation with priorities framework, is also a minimal envy mechanism in the set of IR, SP and PE mechanisms when all schools have only one seat (Proposition 3).

Our contribution is twofold. First, it confirms that the method of modifying the objects' priorities to rank at the top their initial owners prior to running one of the two standard mechanisms of school choice is indeed an effective way to extend some of their properties to account for the IR constraint imposed by the existence of an initial ownership.[9] In the school choice literature, since DA is stable, it is trivially a minimal envy mechanism in the set of SP mechanisms. In the our setting, the result is not trivial since its counterpart, DA*, does generate blocking pairs and, as mentioned earlier, Combe, Tercieux, and Terrier (2022) showed that one can systematically find alternative IR matchings with a set of blocking pairs included in the one of the matching of DA*. Our work also complements the other analysis of DA* that highlighted the properties of the mechanism with respect to a relaxed fairness notion (Compte and Jehiel, 2008; Guillen and Kesten, 2012; Pereyra, 2013). In particular, we show that it extends the characterization of Pereyra (2013) in the set of strategy-proof mechanisms. Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) showed that the TTC mechanism, when objects only have one copy, is a minimal envy mechanism in the set of PE and SP mechanisms. Since the school choice problem is a special case of ours,

[7]The data of primary school teacher assignment in France are not accessible for researchers yet. This figure is taken from the answer of the government to a deputy's question in 2019, see https://questions.assemblee-nationale.fr/q15/15-3033QE.htm (in French)

[8]Coming back to our previous motivation of teacher assignment, the result implies that one cannot find a SP mechanism leading to more mobility than DA* *and* always leading to less envy.

[9]However it is not always possible to extend all of them. Combe, Tercieux, and Terrier (2022) showed for instance that the strong efficiency notion that they consider, that is usually respected by DA in a school choice setting, is not respected by DA*.

our results generalize the ones of the school choice literature concerning minimal envy. This is summarized in Table 1 below. Second, our results also highlight the usefulness of the minimal envy concept in analyzing mechanisms in environments where the primary policy constraint, such as IR for instance, does not guarantee the existence of stable matchings. This "second-best" approach opens the door for interesting further analyses of the mechanisms identified in other similar settings such as the assignment of couples (see the survey of Biró and McDermid, 2014), the matching with constraints (see the recent developments in Kamada and Kojima, 2017a) or the literature on affirmative action policies (Abdulkadiroglu and Sonmez, 2003; Kojima, 2012; Hafalir, Yenmez, and Yildirim, 2013).

Setting	School Choice		Reallocation with priorities	
Mechanism	DA	TTC	DA*	TTC*
Capacities	Multiple seats	One seat	Multiple seats	One seat
Minimal envy in the set	SP	SP & PE	IR & SP	IR & SP & PE

Table 1: Summary of the results

† Notes: The second line states whether the minimal envy result holds in a many-to-one environment (Multiple seats/copies) or a one-to-one environment (One seat/copy). The last line describes the set of mechanisms in which the result holds. SP refers to strategy-proof, PE to Pareto-efficient, IR to individually rational.

To motivate our analysis and concepts, we describe in the next section a real-life applications of the model of reallocation with priorities. We discuss the related literature in Section 2. In Section 3, we formally introduce the setting and the two mechanisms DA* and TTC*. In Section 4, we provide our results alongside with examples. Since the proof of the minimal envy of DA* has intermediary steps of independent interest, we provide it in Section 5. Finally, in Section 6, we discuss possible alternative mechanisms, properties and open questions for future research before concluding in Section 7.

2 Related Literature

School choice and minimal envy. Pareto-efficient mechanisms are not stable. Thus, a natural question is how to compare them in terms of "how much envy" they generate. Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) introduced the concept of *minimal envy* that we use in our analysis (see Section 3 for the definition). In a one-to-one school choice framework where schools only have one seat, they showed that TTC is a minimal envy mechanism in the set of SP and PE mechanisms. Last, one can note that since DA is a stable mechanism, its set of blocking pairs is empty in every preference and priority profile, so it is trivially a minimal envy mechanism in the set of SP mechanisms.[10] In the setting of reallocation with priorities, as already mentioned, there is no IR and stable mechanism. Thus, we use the concept of minimal envy to compare IR mechanisms in that setting. Our result about the TTC* mechanism can be seen as a generalization of theirs to this setting. Recently, a number of papers have investigated and generalized the results of Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) for instance by using blocking triplets and dropping strategy-proofness (Kwon and Shorrer, 2020) or by setting axioms for incomplete orderings comparing mechanisms according to envy (Ehlers et al., 2021).

[10]However, note that DA is not a Pareto-efficient mechanism, so it does not contradict the result concerning TTC since the latter focuses on the set of strategy-proof and Pareto-efficient mechanisms.

Abdulkadiroglu, Pathak, and Roth (2009) showed that there is no mechanism that is SP and always leads to a matching that weakly (and sometimes strictly) Pareto-dominates that of DA. The Pareto-dominance ordering between two mechanisms, similarly to the "less envy than" ordering, defines an incomplete ordering over a set of mechanisms. Similarly to minimal envy mechanisms, one can study the "maximally efficient" mechanisms using this relation. That is exactly what the result of Abdulkadiroglu, Pathak, and Roth (2009) shows for DA in a school choice setting: in the set of SP mechanisms, DA is a "maximally efficient" mechanism. Recently, Kesten and Kurino (2019) extended this result to the case where the preference domain of the students is restricted.[11] They provided the maximal domain of preferences under which it is possible to find an SP mechanism that Pareto-dominates DA. Even if the question is mathematically similar, i.e., finding maximal elements of a partially ordered set of mechanisms, the study of minimal envy mechanisms is conceptually different and their results do not imply ours. Indeed, if between any two matchings μ and μ', the former has a set of blocking pair included in the one of the latter, then one cannot conclude that μ' Pareto dominates μ, so the aforementioned results cannot be used. This impossibility naturally applies to our reallocation with priorities setting. In particular, one can easily construct examples where individuals can be worse off under μ'.[12] However, we show that the analysis of the DA mechanism done in this literature are useful for our question and our work provides some connections between the two problems. Indeed, the use of *weakly underdemanded schools* under DA*, which are schools that all the teachers prefer weakly less than their assignment under DA*, proved to be key for our result. We borrow one technical lemma from Kesten and Kurino (2019) (Lemma 1) concerning these schools to analyze the DA* mechanism.

Last, Alva and Manjunath (2019) proposed a general model of matching where a designer has to select *allocations* but agents can always choose to go for an *outside option* – that can be interpreted as remaining unassigned in our framework. Their general approach allows them to cover many different settings (school choice, object allocation, reallocation with endowments, transferable utilities...). Assuming a richness condition on the ranking of the outside option, they showed that there is at most one strategy-proof mechanism that Pareto-improves upon any individually rational and *participation maximal* mechanism.[13] Their motivation is similar to ours since they investigate the existence and uniqueness of some *maximal* mechanism in the set of strategy-proof mechanisms. However, they focus on the Pareto-dominance ordering over mechanisms which, as mentioned, is conceptually different than the "less envy than" relation that we use. Also, their richness assumption. In our model, similar to housing market models described below, we assume that initial owners prefer their initial school to remaining unassigned, which violates their richness assumption.[14]

Housing market and house allocation. The reallocation with priorities problem shares also similarities with the *housing market* problem introduced by Shapley and Scarf (1974). In this problem, tenants with an initial house would like to exchange it and have preferences over all the houses. This is a one-to-one model so that each house has only one tenant. When reassigning

[11] We refer the reader to their paper for the technical details. Formally, some students do not have access to the possibility of remaining unassigned ($\emptyset$), so the proof in Abdulkadiroglu, Pathak, and Roth (2009), which crucially uses such possibility, cannot be applied.

[12] In a school choice setting, simply compare the student-optimal stable matching to the school-optimal stable matching. Indeed, the two trivially lead to the same set of blocking pairs, the empty set, and the former Pareto dominates the latter.

[13] Participation maximality imposes that one cannot make more agents participate, i.e. not choosing the outside option, without hurting any participating agent. We refer the reader to the formal definition in their article.

[14] This seems to not be a problem in some real life applications such as the assignment of teachers in France.

them, a matching needs to ensure *individual rationality* (IR), i.e. every agent is assigned a weakly preferred house than his initial one. In our setting, the same constraint is imposed on initial owners and their initial object. Shapley and Scarf (1974) introduced the *Top Trading Cycle* mechanism that is IR, SP and PE. Ma (1994) showed that it is the only IR, SP and PE mechanism in this setting. When there are only initial owners, the TTC* mechanism that we consider becomes equivalent to the TTC mechanism of the housing market model.[15]

When some agents do not have any initial house and some houses are possibly vacant, Abdulkadiroglu and Sonmez (1999) proposed an IR, SP and PE mechanism called the *You-Request-My-House-I-Get-Your-Turn* (YRMH-IGYT) algorithm. This setting, called the *house allocation problem with existing tenants* is similar to a teacher assignment problem except that, in the latter, schools, contrary to houses, are imbeded with a priority ordering over the teachers. The TTC* that we study has been proposed by Karakaya, Klaus, and Schlegel (2019) as an extension of the YRMH-IGYT. They showed that it is the only IR, SP, PE, consistent and reallocation-proof mechanism.[16] Their motivation is different than ours since they consider a model with houses so that priorities are not part of the primitives but emerge from their axiomatic analysis. In our setting, the priority profile is fixed and part of the primitives so that we focus on the envy that the TTC* mechanism generates. If houses all have the same priorities then TTC* is equivalent to the YRMH-IGYT mechanism introduced by Abdulkadiroglu and Sonmez (1999) and later characterized by Sonmez and Unver (2010). The question of whether this mechanism is, in a one-to-one setting, a minimal envy mechanism in the set of IR, strategy-proof and Pareto-efficient mechanisms in a teacher assignment problem has not been investigated.

Reallocation with priorities. Guillen and Kesten (2012) showed that the DA* mechanism was used at MIT – the NH4 algorithm – to assign students to off campus housing. They and Pereyra (2013) showed that the matching returned by this mechanism respects a certain form of stability, called *fair matchings* by Guillen and Kesten (2012) or *acceptable matching* by Pereyra (2013).[17] A matching is acceptable if there is no individual and object such that the former prefers the latter to her assignment and has a higher priority than another individual who is assigned to it but was not an initial owner of that object. If such a situation occurs, Pereyra (2013) referred to it as a *justified claim*. Therefore, an acceptable matching is simply a matching with no justified claims. Since DA* returns an acceptable matching, the only possible remaining blocking pairs, as defined in the school choice literature, are those with an individual with a higher priority than an initial owner assigned to his initial object. This type of envy is called *inappropriate claims* by Pereyra (2013). He showed that there is no other acceptable matching that leads to a set of inappropriate claims included in the one of DA* so that the latter *minimizes inappropriate claims among acceptable matchings*. Moreover, he showed that among all the acceptable mechanisms that minimize inappropriate claims, DA* is the most preferred by the individuals. Compte and Jehiel (2008) also showed that no acceptable matching which generates more mobility than DA*.

[15]Since in a housing market, primitives do not include any priority ordering for the houses, the question of minimal envy does not arise. So even if the two mechanisms become equivalent in terms of assignments, the questions that are investigated conceptually differ from each other.

[16]For the latter two properties, we refer the reader to cited article for further details. Intuitively, consistency requires that the mechanism still chooses the same allocation if one removes (consistently) some agents with their assigned house. Reallocation-proofness requires that no two agents can jointly misreport their preferences, exchange their allocation afterward and be better off than when being truthful.

[17]In an independent unpublished work, Compte and Jehiel (2008) also investigated such mechanism. They considered the same relaxed notion of stability that they called μ_0-*stability*.

Our difference from this approach is that we do not differentiate between inappropriate or justified claims but consider the complete set of claims, i.e., all the blocking pairs as defined in the school choice literature, and ask whether it is possible to somehow "minimize it" in the setwise inclusion sense while maintaining strategy-proofness. This process is motivated by the observation that for some preference and priority profiles, it is possible to find a matching that has strictly less envy than the one of DA* (Example 1). Our envy definition is more permissive than the one of Pereyra (2013) since it allows a mechanism to create any type of blocking pair. In that case, there are more possible mechanisms which can potentially lead to less blocking pairs than DA*, not only those returning an acceptable matching like in Pereyra (2013). Since we show that DA* is a minimal envy mechanism, our main result can thus be seen as a strengthening of the one of Pereyra (2013).

Last, Dur and Kesten (2014) studied a model of sequential assignment where one has to assign a set of agents to objects in two-rounds, potentially using two different mechanisms. In their model, agents can also initially own an object. One application that they mention is the assignment of tenured and newly tenured teachers to positions in Turkey. Our goal is different from theirs since we do not focus on two-rounds mechanisms.

Distributional constraints. In a school choice setting, an important literature has been developed concerning *distributional constraints* (see Kamada and Kojima, 2017a for a survey). When assigning students to schools, one has to respect a set of constraints in some schools or group of schools (lower-upper bounds, quotas...). Similarly to the IR constraint in our setting, these assignment constraints can be incompatible with the standard definition of stability. One approach to follow is to define an appropriate notion of stability that is somehow "compatible" with the constraints and to investigate under which conditions on the constraints such stable matching exists.[18] A natural question is whether the IR constraint and the minimal envy analysis that we are studying in our setting can be formulated in terms of a distributional constraint to apply the results of the literature. It turns out not to be the case. Intuitively, the IR constraint imposes the following: if an individual would like to be assigned to her initial object, an IR mechanism has to assign her to it and otherwise, her object can be assigned to someone else. Thus, whether the IR constraint is relevant or not depends on the preferences of the initial owner. This observation makes the IR constraint similar to the *type specific reserves* introduced by Hafalir, Yenmez, and Yildirim (2013) in the context of affirmative action policies. We refer the reader to the discussion in Kamada and Kojima (2017b) who showed that this requirement cannot be embedded into their matching with constraints framework.

3 Setting

Consider the problem of reallocation with priorities where there is a finite set I of N individuals and a set O of M objects.[19] Each individual $i \in I$ has a strict preference ordering P_i over the set of objects and being unassigned, i.e., $O \cup \{\emptyset\}$. Let R_i be the induced weak ordering.[20] Each object

[18]For general constraints, there can be several notions of stability that can be defined and results can vary greatly across them. We refer the reader to the articles of the literature, see for instance Kojima, Tamura, and Yokoo (2018) or Kamada and Kojima (2017b).

[19]Objects can have a different interpretation depending on the application. One can also think about an object as being a school, like in a teacher assignment setting.

[20]It is the ordering s.t. $o' R_i o \Leftrightarrow (o = o') \vee (o' P_i o)$.

$o \in O$ is attached to a priority ordering $\succ_o$ over the individuals and leaving the object unassigned, i.e., $I \cup \{\emptyset\}$. We denote by P the profile of preferences of the individuals, i.e., $P := (P_i)_{i \in I}$, and by $\succ$ the profile of priorities of the objects, i.e., $\succ := (\succ_o)_{o \in O}$. As usual, for an individual $i \in I$, $P_{-i} := (P_{i'})_{i' \neq i}$ and for a set $I' \subseteq I$, $P_{-I} := (P_{i'})_{i' \notin I'}$. For each object o, there can be multiple copies of it: q_o will denote the number of copies of object o, and we let $q := (q_o)_{o \in O}$ be the vector of the number of copies of each object.[21] A matching μ is a mapping from I to $O \cup \{\emptyset\}$ where $\mu(i)$ is the object assigned to individual i where $\mu(i) = \emptyset$ means that individual i remains unassigned. We slightly abuse the notation in letting $\mu(i)$ be the set of individuals who obtain a copy of object o and $\mu(\emptyset)$ be the set of unassigned individuals who do not obtain any object. A matching cannot assign more copies of an object than available, i.e., for all $o \in O$, $|\mu(o)| \leq q_o$. The main difference between the school choice setting proposed by Abdulkadiroglu and Sonmez (2003) is that there is now an **initial matching** μ_0. There are two types of individuals: i) **initially unassigned ones**, who do not have any initial object, i.e., $\mu_0(i) = \emptyset$, and ii) **initial owners**, who are initially assigned to a copy of an object, i.e., $\mu_0(i) \in O$. We let $n \leq N$ be the number of initially unassigned individuals, i.e., $n := |\mu_0(\emptyset)|$. Note that if $n = N$, then we are back to a standard school choice model, and if $n = 0$, then the model becomes a housing market model where houses can have multiple seats and priorities. We also distinguish types of frameworks: i) a **many-to-one** setting if there exists $o \in O$ with $q_o > 1$ and ii) a **one-to-one** setting if for all $o \in O$, $q_o = 1$. Moreover, we make the following assumptions that are relevant in many applications:

A1. Every initial owner prefers his initial object to being unassigned, i.e., if $\mu_0(i) \in I$, then $\mu_0(i) P_i \emptyset$.

A2. Every object's priority ranks all the individuals as acceptable, i.e., for all $i \in I$ and $o \in O$, $i \succ_o \emptyset$.

A3. The number of available copies of all the objects is greater or equal to the number of individuals, i.e., $\sum\limits_{o \in O} q_o \geq N$.

The first assumption is usual in models of matching with an initial assignment (for instance in the housing market model of Shapley and Scarf, 1974). In practice, it can be not possible for an initial owner to participate to the assignment mechanism while declaring that her initial object is unacceptable. For instance, tenured teachers in France will automatically be assigned to their initial school if they do not obtain another one. Thus, the mechanism cannot assign their initial seat to another teacher. The drop out rate remains very low so that it does not seem to be a problem in practice.[22] Concerning the second assumption, it is natural in most public allocation systems to consider all the candidates as acceptable. All our results would hold if one imposes the constraint that object o can never be assigned to an individual i s.t. $\emptyset \succ_o i$ and that every object ranks its initial owners as acceptable, i.e., if $i \in \mu_0(o)$, then $i \succ_o \emptyset$. Finally, the last one is satisfied in most public assignment systems, notably teacher assignment, where the number of copies/seats is calibrated so that all the individuals, especially initially unassigned ones, can obtain an assignment.[23] If there are only initial owners, it is trivially satisfied.

[21] In the application where objects designate schools, copies are thus interpreted as the multiple positions available in a given school.

[22] Whether this assumption is satisfied or not depends on the application that one has in mind. We chose to keep the standard approach of the housing market literature. The reader can refer to the discussion in Alva and Manjunath (2019) on the effect on relaxing this assumption in a housing market setting.

[23] It is usually the case in many public assignment settings, such as in school choice or, for instance, teacher assignment in France.

We say that a matching μ is:

- **Individually rational** (IR): for all $i \in I$, $\mu(i) R_i \mu_0(i)$. An IR matching is s.t. all the individuals obtain an assignment that they weakly prefer to their initial one.[24]

- **Stable**: $\nexists (i,s) \in I \times O$, s.t. $o P_i \mu(i)$ and either i) $|\mu(o)| < q_o$ and $t \succ_o \emptyset$ or ii) $|\mu(o)| = q_o$ and there exists $i' \in \mu(o)$ s.t. $i' \succ_o i$. Such a pair is called a blocking pair. We also say, in the second case, that individual i has a justified envy toward individual i' at object o. With a slight abuse of language, we say that a matching with the existence of such pairs generates **envy**.[25]

- **Pareto-efficient** (PE): $\nexists \mu'$ s.t. for all $i \in I$, $\mu'(i) R_i \mu(i)$ and the preference is strict for at least one individual.

The set of PE and stable matchings can be disjoint so that any PE matching leads to potential blocking pairs. In a setting with some initial owners ($n < N$), it is possible that no matching is both IR and stable such that any IR matching will have some remaining blocking pairs. For a given profile $(P, \succ)$ and matching μ, we let $B_\mu(P, \succ)$ be the set of blocking pairs of μ under the profile $(P, \succ)$, i.e., the set of pairs that respect the conditions in the above definition of stability. When the context is clear, we refer to $B_\mu(P, \succ)$ as simply B_μ.

A mechanism φ maps each profile $(P, \succ)$ to a matching $\varphi(P, \succ)$.[26] A mechanism is:

- IR if for all $(P, \succ)$, $\varphi(P, \succ)$ is an IR matching.

- Strategy-proof (SP) if for all $i \in I$, for all $\succ$, for all P_i, P'_i, P_{-t}, $\varphi_i(P_i, P_{-i}, \succ) R_i \varphi_i(P'_i, P_{-i}, \succ)$.

- Stable if for all $(P, \succ)$, $\varphi(P, \succ)$ is a stable matching.

- Pareto-efficient (PE) if for all $(P, \succ)$, $\varphi(P, \succ)$ is a Pareto-efficient matching.

We consider concepts introduced by Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020).[27] We say that a matching μ' has **less envy** (resp. **strictly less envy**) than a matching μ if $B_{\mu'} \subseteq B_\mu$ (resp. $B_{\mu'} \subset B_\mu$). A mechanism φ' has **less envy** than a mechanism φ if for all $(P, \succ)$, $\varphi'(P, \succ)$ has less envy than $\varphi(P, \succ)$. A mechanism φ' has **strictly less envy** than a mechanism φ if it has less envy than φ and there exists $(P, \succ)$ s.t. $\varphi'(P, \succ)$ has strictly less envy than $\varphi(P, \succ)$. We say that φ is a **minimal envy mechanism** if there is no mechanism φ' that has strictly less envy than φ.[28]

We now define the two mechanisms important for our purpose: DA* and TTC*. We start by recalling the definition of the well-known DA mechanism:

[24] Note that assumption A1 implies that for any individual $i \in I$, $\mu(i) R_i \mu_0(i) R_i \emptyset$ so that no individual can be assigned to an object that he finds unacceptable.

[25] In the literature, the exact term is justified envy. For simplicity, we shorten the terminology without any risk of confusion.

[26] Formally, a mechanism maps each tuple $(P, \succ, \mu_0)$ to a matching. Since we fixed the initial matching, we omit it here. In particular, an IR mechanism must always return an IR matching for any possible initial assignment.

[27] Though different, it is also closely related to the relation "more stable than" introduced by Chen and Kesten (2017).

[28] Formally, the relation "less envy than" defines an incomplete ordering $\triangleright$ over a set X of mechanisms so that $(X, \triangleright)$ forms a partially ordered set (poset). In a poset $(X, \triangleright)$, $x \in X$ is a *maximal element* if $\nexists y \in X$ such that $y \triangleright x$. When $\triangleright$ is the "less envy than" relation, the minimal envy mechanisms in X are the maximal elements of $(X, \triangleright)$.

Step 1. All the individuals apply to their first-ranked object. If object o receives more than one application, it selects the higher q_o candidates according to $\succ_o$ and rejects the others. If there is no rejection, then the algorithm stops. If there is at least one rejection, then move to Step 2.

Step $k \geq 2$. All the individuals rejected in step $k-1$ apply to their favorite object among those that have not rejected them yet. An object o considers both its previously accepted individuals, if any, and the candidates applying to it and selects again the best q_o candidates according to $\succ_o$. If there is no rejection, then the algorithm stops. If there is at least one rejection, then move to Step $k+1$.

If there are some initial owners ($n < N$), because the DA mechanism is stable, it is not IR. Indeed, at any given step, it is possible for an object to reject one of its initial owners if another individual with a higher priority applies. The mechanism DA* is simple modification of DA that restores the IR property:

1. Let $\tilde{\succ}$ be a modified priority profile such that for all $o \in O$ and for all $i, i' \in I$, if $i \in \mu_0(o)$ and $i' \notin \mu_0(o)$, then $i \tilde{\succ}_o i'$. If $\{i, i'\} \subseteq \mu_0(o)$ or $\{i, i'\} \cap \mu_0(o) = \emptyset$, then $i \tilde{\succ}_o i' \Leftrightarrow i \succ_o i'$. In words, $\tilde{\succ}_o$ moves all the initial owners of object o to the top of the priority ordering but keeps the relative rankings between them and the individuals not initially assigned to it.

2. Let $\text{DA}^*(P, \succ) = \text{DA}(P, \tilde{\succ})$.

By construction, DA* is IR. Indeed, since any individual is ranked first by her initial object, she would be sure to be accepted in the case she is applying to it. Since the modification of the priority profile does not depend on the reporting preferences of the individuals, it is also trivially SP.

As before, we start by recalling the definition of the TTC mechanism introduced by Abdulkadiroglu and Sonmez (2003) in the school choice setting. It is an SP and PE mechanism. Since our result will hold only in a one-to-one framework, we consider only a one-to-one setting when defining the mechanism, so for all $o \in O$, $q_o = 1$.

Step 1. Build the graph where the nodes are the individuals and objects. Every individual points to her favorite object, if she prefers staying unassigned, let him/her point to him/herself. Every object points to its highest-ranked individual. There will be a cycle in this graph, and all cycles will involve different nodes,[29] implement one in letting the individuals be assigned to the object they are pointing to. Delete these individuals and their assigned objects.[30]

Step $k \geq 2$. Build the graph where the nodes are the remaining individuals and the remaining objects. Repeat the same operation as in Step 1 to implement a cycle in this graph.

The process continues until all the individuals obtain an assignment. As before, if there are some initial owners, the above mechanism is not IR. If a school does not rank its initial owner first, it can point to another individual and be assigned before its initial owner such that it does not ensure the latter of obtaining a weakly preferred object compared to her initial one. Similarly to

[29] A cycle is a set of nodes $\{i_1, o_1, \ldots, i_K, o_K\}$ such that for $k = 1, \ldots, K-1$, i_k points to o_k, o_k points to i_{k+1}, and o_K points to i_1. The existence of a cycle and the fact that they are disjoint is a standard result for graphs where nodes have only one outgoing edge. The order in which the cycles are implemented does not influence the outcome of the algorithm.

[30] In a many-to-one setting, one would keep the objects that have additional copies available.

the construction of DA*, one can apply the same technique to define an IR version of the above mechanism, which we call TTC*:

1. Let $\tilde{\succ}$ be a modified priority profile such that for all $o \in O$ and for all $i, i' \in I$, if $i \in \mu_0(o)$ and $i' \notin \mu_0(o)$, then $i \tilde{\succ}_o i'$. If $\{i, i'\} \subseteq \mu_0(o)$ or $\{i, i'\} \cap \mu_0(o) = \emptyset$, then $i \tilde{\succ}_o i' \Leftrightarrow i \succ_o i'$. In words, $\tilde{\succ}_o$ moves all the initial owners of object o to the top of the priority ordering but keeps the relative rankings them and the individuals not initially assigned to it.

2. Let $\text{TTC}^*(P, \succ) = \text{TTC}(P, \tilde{\succ})$.

It can be easily verified that the above mechanism is IR and SP. Additionally, since we only modified the priorities of the schools, the TTC* mechanism is also Pareto-efficient.

4 Results

We now turn to the analysis of DA* and TTC* in terms of minimal envy. In a school choice setting ($n = N$), DA is stable and so generates no blocking pairs. So trivially, it is a minimal envy mechanism and also has trivially less envy than TTC. In our setting, with some initial owners ($n < N$), the answer is non-trivial: there are instances where TTC*, a IR and SP mechanism, can generate strictly less envy than DA*.

Example 1. *There are 3 individuals, i_1, i_2, and i_3, all initially assigned to, respectively, o_1, o_2, and o_3. The preferences and priorities are given by:*

$$\begin{array}{llll} P_{i_1}: & o_2 & o_1 & o_3 \\ P_{i_2}: & o_1 & o_2 & o_3 \\ P_{i_3}: & o_1 & o_2 & o_3 \end{array} \qquad \begin{array}{llll} \succ_{o_1}: & i_3 & i_2 & i_1 \\ \succ_{o_2}: & i_3 & i_1 & i_2 \\ \succ_{o_3}: & i_3 & i_1 & i_2 \end{array}$$

Under the above profile $(P, \succ)$, we have:

$$DA^*(P, \succ) = \begin{pmatrix} i_1 & i_2 & i_3 \\ o_1 & o_2 & o_3 \end{pmatrix}$$
$$TTC^*(P, \succ) = \begin{pmatrix} i_1 & i_2 & i_3 \\ o_2 & o_1 & o_3 \end{pmatrix}$$

Note that $B_{DA^*(P,\succ)} = \{(i_1, o_2), (i_2, o_1), (i_3, o_1), (i_3, o_2)\}$ and $B_{\text{TTC}^*(P,\succ)} = \{(i_3, o_1), (i_3, o_2))\}$, so that TTC∗, an IR and SP mechanism, generates strictly less envy than DA at $(P, \succ)$.*[31]

However, the next example shows that DA* and TTC* cannot be compared in terms of the less envy criterion.

Example 2. *There are 3 individuals, i_1, i_2, and i_3, all initially assigned to, respectively, o_1, o_2, and o_3. The preferences and priorities are given by:*

$$\begin{array}{llll} P_{i_1}: & o_2 & o_3 & o_1 \\ P_{i_2}: & o_1 & o_2 & o_3 \\ P_{i_3}: & o_1 & o_2 & o_3 \end{array} \qquad \begin{array}{llll} \succ_{o_1}: & i_3 & i_2 & i_1 \\ \succ_{o_2}: & i_3 & i_1 & i_2 \\ \succ_{o_3}: & i_1 & i_3 & i_2 \end{array}$$

[31] The example is inspired by the one of Combe, Tercieux, and Terrier (2022) in the context of teacher assignment. Their proposed mechanism, the Teacher-Optimal Block-Exchange (TO-BE), generates the same outcome as TTC* in this example. In addition, it is designed to shrink the set of blocking pairs at each step. We discuss it in Section 6.

Under the above profile $(P, \succ)$*, we have:*

$$DA^*(P,\succ) = \begin{pmatrix} i_1 & i_2 & i_3 \\ o_3 & o_2 & o_1 \end{pmatrix}$$
$$TTC^*(P,\succ) = \begin{pmatrix} i_1 & i_2 & i_3 \\ o_2 & o_1 & o_3 \end{pmatrix}$$

Note that $B_{DA^*(P,\succ)} = \{(i_1, o_2)\}$ *and* $B_{\text{TTC}^*(P,\succ)} = \{(i_3, o_1), (i_3, o_2)\}$*, so the two sets do not intersect.*

Note that Example 1 raises an important question: TTC* generates more "mobility" than DA* since more individuals are allocated an object different than their initial one. As discussed, this feature can be of significant importance in practice since a lack of mobility can trigger complaints by individuals. In that case, one can be tempted to look for mechanism generating less envy than DA* with a better mobility performance.[32]

The next proposition shows that it is not possible: in the set of IR and SP mechanisms, DA* is a minimal envy mechanism.

Proposition 1. *DA* is a minimal envy mechanism in the set of IR and SP mechanisms.*

The proof of Proposition 1 shows that there is no IR and SP mechanism that has strictly less envy than DA*. The structure of the matchings that have less envy than that of DA* must be understood, so several lemmas are dedicated to this study. The lemmas provide interesting results, so we relegate their formal treatment to Section 5. To provide intuition about the proof and its main difficulties, consider the following simple example: there are five individuals $i_1, \ldots, i_5$ and five objects $o_1, \ldots, o_5$ with one copy each. Individual i_k, for $k = 1, 2, 3, 4$, is the initial owner of object o_k. Individual i_5 is initially unassigned, so object o_5 has no initial owner. Preferences and priorities are given by:

$$\begin{array}{llllllllll} P_{i_1}: & o_2 & o_1 & o_3 & \ldots & \succ_{o_1}: & i_3 & i_2 & i_1 & \ldots \\ P_{i_2}: & o_1 & o_2 & \ldots & & \succ_{o_2}: & i_3 & i_1 & i_2 & \ldots \\ P_{i_3}: & o_1 & o_2 & o_3 & \ldots & \succ_{o_3}: & i_3 & i_1 & i_2 & \ldots \\ P_{i_4}: & o_5 & o_4 & \ldots & & \succ_{o_4}: & i_4 & i_5 & \ldots & \\ P_{i_5}: & o_4 & o_5 & \ldots & & \succ_{o_5}: & i_5 & i_4 & \ldots & \end{array}$$

Dots indicate that preferences and priorities can be arbitrary. Let $(P, \succ)$ be the above profile of preferences and priorities. Under this profile, one can check that the matching given by DA* is:

$$\text{DA}^*(P,\succ) = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 & o_5 \\ i_1 & i_2 & i_3 & i_5 & i_4 \end{pmatrix}$$

Note that the set of blocking pairs is $B_{DA^*(P,\succ)} = \{(i_1, o_2), (i_2, o_1), (i_3, o_1), (i_3, o_2)\}$. Now, consider the following matching:

$$\mu' = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 & o_5 \\ i_2 & i_1 & i_3 & i_4 & i_5 \end{pmatrix}$$

[32] As mentioned, Pereyra (2013) showed that there is no *acceptable matching* that generates more mobility than the one of DA*. However, the matching of TTC* in Example 1 is not an acceptable matching in the sense of Pereyra (2013) since we consider the whole set of blocking pairs. It shows that our main result is disconnected from his.

To go from $\text{DA}^*(P, \succ)$ to μ', i_1 and i_2 exchange their initial objects, as do i_4 and i_5. Note that $B_{\mu'} = \{(i_3, o_1), (i_3, o_2)\} \subset B_{DA^*(P,\succ)}$, so μ' has strictly less envy than $\text{DA}^*(P, \succ)$ but μ' does not Pareto dominate $\text{DA}^*(P, \succ)$: i_4 and i_5 are strictly worse off.[33] However, we note a particular structure: individuals who exchange their initial objects from DA^* to μ' are either all strictly better off or all strictly worse off. Indeed, individuals i_1 and i_2 exchange their assignments and are both strictly better off under μ' than under $\text{DA}^*(P, \succ)$. Individuals i_4 and i_5 also exchange their assignments and are both strictly worse off. Lemma 3 in Section 5 shows that this is a general property: when considering a matching with less envy than DA^*, all individuals involved in an exchange must either all be strictly better off or all strictly worse off. Note that in our example, $B_{\mu'}$ is a strict subset of $B_{DA^*(P,\succ)}$ and that at least one exchange involves individuals who are all strictly better off. Lemma 4 shows that it is generally the case that if a matching has strictly less envy than that of DA^*, at least one exchange must make all its individuals strictly better-off.[34] With this result in hands, we show that these individuals can misreport their preferences at some other preference profiles. It builds upon a technical lemma of Kesten and Kurino (2019) studying the existence and use of some specific objects, called *underdemanded objects*, under DA.

Proposition 1 shows that no IR and SP mechanism has strictly less envy than DA^*. One can ask whether it is possible to find an IR and SP mechanism φ different from DA^* and that generates the same envy, i.e. for all $(P, \succ)$, $B_{\varphi(P,\succ)} = B_{\varphi(P,\succ)}$. Example 3 shows that it is indeed possible.

Example 3. *Consider a setting with 3 individuals, i_1, i_2, and i_3, who are the initial owners of, respectively, o_1, o_2, and o_3, each having only one copy. Let φ be the following mechanism. If the priority profile $\succ$ is:*

$$\begin{array}{cccc} \succ_{o_1} & i_1 & i_3 & i_2 \\ \succ_{o_2} & i_2 & i_3 & i_1 \\ \succ_{o_3} & i_3 & i_1 & i_2 \end{array}$$

Then for all P, let $\varphi(P, \succ) = \mu_0$, i.e. the trivial mechanism where individuals keep their initial object. Otherwise, if the priority profile or the number of individuals differ, let $\varphi(P, \succ) = DA^*(P, \succ)$.

It is clear that φ is an IR mechanism. It is also trivially strategy-proof. Indeed, individuals cannot influence the priority profile of the objects and both the trivial mechanism, if the priority profile is the above one, and DA^* otherwise, are SP mechanisms. Last, note that, when the priority profile is the above one, since all the objects rank their initial owner first, then in the Step 1 of DA^*, there is no modification of the priority profile so $DA^*(P, \succ) = DA(P, \succ)$ and the resulting matching is stable at any preference profile P, i.e. $B_{DA^*(P,\succ)} = \emptyset$. Since $\varphi(P, \succ) = \mu_0$ for this priority profile, then each object is assigned its first ranked individual so that $B_{\varphi(P,\succ)} = \emptyset = B_{DA^*(P,\succ)}$. Since when the priority profile differs from the above one, we have $\varphi(P, \succ) = DA^*(P, \succ)$, then trivially $B_{\varphi(P,\succ)} = B_{DA^*(P,\succ)}$ and we conclude that φ and DA^* generate the same envy even though they differ from each other.

We now turn to the analysis of TTC^*. We start by showing thatn contrary to DA^*, TTC^* is not a minimal envy mechanism in the set of IR and SP mechanisms:

[33]This point is the reason why one cannot use the results of Abdulkadiroglu, Pathak, and Roth (2009) and Kesten and Kurino (2019) in this setting since Pareto-dominance and "less envy than" relations are conceptually different.

[34]Similarly to the remark in Footnote 12, it is possible to find two different matchings with the same set of blocking pairs but where one is Pareto dominated by the other. In our example, simply consider the matching that exchanges the assignments of i_4 and i_5.

Proposition 2. *There are mechanisms that are IR, SP and have strictly less envy than TTC**.

Proof. Consider a setting with 3 individuals, i_1, i_2, and i_3, who are the initial owners of, respectively, o_1, o_2, and o_3, each having only one copy. Now, let φ be the following mechanism: if the priority profile $\succ$ is

$$\begin{array}{cccc} \succ_{o_1} & i_1 & i_3 & i_2 \\ \succ_{o_2} & i_2 & i_3 & i_1 \\ \succ_{o_3} & i_3 & i_1 & i_2 \end{array}$$

Then, let $\varphi(P,\succ) = \text{DA}^*(P,\succ)$. Otherwise, if $\succ$ differs from the above profile or if the number of individuals differ, let $\varphi(P,\succ) = \text{TTC}^*(P,\succ)$. This mechanism is trivially IR, and it is easy to see that it is also SP because the individuals cannot influence the priority profile of the objects and for a fixed profile, the mechanism used is an SP mechanism. Since $\varphi = \text{TTC}^*$, when $\succ$ differs from the above profile or when the number of individuals differs, then the two mechanisms have trivially the same set of blocking pairs. Note that when the priority profile is the above profile, since all the objects rank their initial owner first, then in the Step 1 of DA*, there is no modification of the priority profile so $\text{DA}^*(P,\succ) = \text{DA}(P,\succ)$ and the resulting matching is stable at any preference profile P. Therefore, $B_{\varphi(P,\succ)} = \emptyset$ and is trivially included in $B_{TTC^*(P,\succ)}$. However, note that when i_1 ranks o_2 first and i_2 ranks o_1 first, then:

$$\text{TTC}^*(P,\succ) = \begin{pmatrix} i_1 & i_2 & i_3 \\ o_2 & o_1 & o_3 \end{pmatrix}$$

And $B_{\varphi(P,\succ)} = \emptyset \subset B_{TTC^*(P,\succ)} = \{(i_3,o_1),(i_3,o_2)\}$, so that φ has indeed strictly less envy than TTC*. □

At that point, one can wonder whether DA* is the only minimal envy mechanism in the set of IR and SP mechanisms. Since the "less envy than" ordering is incomplete, one can easily show that it is indeed not the case. Suppose that when there are 3 individuals and $n = 0$, as in Example 2, one uses the TTC* mechanism and if the number of individuals differs, uses DA*. When the profile $(P,\succ)$ is the same as in Example 2, we have seen that $B_{DA^*(P,\succ)} = \{(i_1,o_2)\}$ and $B_{\text{TTC}^*(P,\succ)} = \{(i_3,o_1),(i_3,o_2)\}$. As we will see, TTC* is not a minimal envy mechanism in the set of IR and SP mechanisms. Let φ be a minimal envy, IR and SP mechanism with strictly less envy than TTC*. Note that in Example 2, at profile $(P,\succ)$, there is no IR and stable matching, i.e., there is no IR matching μ such that $B_\mu = \emptyset$. Specifically, $\emptyset \neq B_{\varphi(P,\succ)} \subset B_{\text{TTC}^*(P,\succ)} = \{(i_3,o_1),(i_3,o_2)\}$ and so φ and DA* cannot be compared in terms of blocking pairs.

This negative result for TTC* parallels the standard result in a school choice setting concerning the standard TTC: if one considers only the set of strategy-proof mechanisms, then TTC is not a minimal envy mechanism in that set since, trivially, the DA algorithm always leads to less envy, i.e., no blocking pairs. However, as mentioned, Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) showed that the TTC algorithm in a one-to-one setting is a minimal envy mechanism in the set of SP and PE mechanisms. Our next proposition extends their result to our setting:

Proposition 3. *In a one-to-one setting, TTC** *is a minimal envy mechanism in the set of IR, SP and PE mechanisms.*

The proof of the proposition extends that of Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) and is relegated to Section A of the Appendix. The proof shows that any IR, SP and PE mechanism φ with (weakly) less envy than TTC* must be equal to it. This is achieved by induction over the steps of TTC*. For instance, in the first step of TTC$^*(P, \succ)$, suppose that individual i was assigned to object o' and that object o was pointing to i during cycle C that assigned him/her at that step. If $\varphi(P, \succ)$ assigns i to a object $o'' \neq o'$, then $o' P_i o''$ since i was pointing to her favorite object at the first step of TTC* and o'' was still present at that step. In a school choice setting ($n = N$), if i reports profile $P'_i : o', o, \emptyset$, then the matching of TTC* remains the same and Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) showed that $\varphi_i(P'_i, P_{-t}, \succ) = o$. One can then iterate the argument to reach a profile P' where TTC$^*(P', \succ) =$ TTC$^*(P, \succ)$ and all the individuals in cycle C are assigned under $\varphi(P', \succ)$ to the object that was pointing to them under C. This case contradicts that φ is PE since we know that, because of the cycle C, all these individuals would be better off in exchanging their assignments. In our setting where there are initial owners ($n < N$), our proof uses a similar argument, except now, one has to consider whether object o is the object initially owned by individual i. If not, a similar argument as that in the school choice setting applies. If $\mu_0(i) = o$, then o is not necessarily the highest-ranked individual in the priority profile of o, but one can use the fact that φ is IR to show that $\varphi_i(P'_i, P_{-t}, \succ) = o$. The proof shows that this distinction leads to a conclusion similar to that in the school choice setting.

Similarly to the result of Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020), the one-to-one assumption is necessary for Proposition 3 to hold. This setting is already interesting for many applications, such as campus housing. In a many-to-one setting, using similar arguments, one can exhibit an IR, SP and PE mechanism that has strictly less envy than TTC*. One can also consider whether TTC* is the unique minimal envy mechanism in the set of IR, SP and PE mechanisms. Remember that when there are no initially unassigned individuals ($n = 0$), the model becomes a housing market problem and TTC* becomes the TTC mechanism. We know by Ma (1994) that the latter mechanism is the unique IR, SP and PE mechanism. When there are only individuals who are not intial owners, i.e., $n = N$, Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) provide an example of an SP, PE and minimal envy mechanism that is different from the school choice TTC mechanism. Since in that setting, TTC* becomes equivalent to the latter, it is easy to see that their example can be extended to our setting to show that TTC* is not the only minimal envy mechanism in the set of IR, SP and PE mechanisms.

5 Proof of Proposition 1

We give an overview of the proof strategy which requires several intermediate Lemmas that are proven below. To proof is by contradiction: we assume first that DA* is not minimally stable so that another IR and SP mechanism, say φ, generates less envy than DA* and strictly less for at least one profile $(P, \succ)$. We will show that, at that profile $(P, \succ)$ there exists an individual i who can use his report P_i to successfully manipulate φ at another profile P'_i, i.e. $\varphi_i(P_i, P_{-i}, \succ) P'_i \varphi(P'_i, P_{-i}, \succ)$, contradicting that φ is SP. To do so, we need to understand the properties of matchings generating strictly less envy than the one of DA*. This is what Lemma 2 and 3 do. They allow us to identify such individual i who will be able to manipulate φ: she is the individual who is strictly better-off at $\varphi(P, \succ)$ compared to DA$^*(P, \succ)$. The existence of such agent is guaranteed because $(P, \succ)$ is a profile where φ generates strictly less envy than DA* (Lemma 3). The profile P'_i that individual i will manipulate from is a truncation from the initial profile P_i. This truncation is done using an underdemanded object in the sense of Kesten and Kurino (2019) (Lemma 5 and 6). Underdemanded objects are objects which, at a preference profile, are not strictly preferred by anyone to their current assignment.[35] These objects are important because, under DA*, one can be guaranteed to be matched to them in case she ranks them above their DA* assignment: this is what the profile P_i' does compared to P_i.

We elaborate several lemmas that help to understand the structure of matchings that lead to less envy than DA*. First, we reformulate the definition of μ_0-stability from Guillen and Kesten (2012) or Pereyra (2013) in our framework: if an individual is blocking with an object under the matching of DA*, then he can only have a higher priority than some initial owners of that object who remained assigned to it.

Lemma 1. *For a profile $(P, \succ)$, if $(i, s) \in B_{DA^*(P,\succ)}$, then $|DA_o^*(P, \succ)| = q_o$ and for all $i' \in DA_o^*(P, \succ)$ such that $i \succ_o i'$, $i' \in \mu_0(o)$.*

To prove our result, one has to understand the properties of a matching, say μ, leading to a set of a blocking pairs that is a subset of that of DA*. To study these properties, we want to consider how each copy of an object can be exchanged when one wants to go from the matching of DA* to the matching μ. To do so, for any two IR matchings μ and μ', we define a directed graph that we call the **μ-μ'-exchange graph** representing the exchanges of object copies needed to go from μ to μ'. The μ-μ'-exchange graph is composed of a set of nodes N and a set of edges E. The nodes in N are partitioned into three types:

1. Nodes representing individuals and their assignment under μ which can be either a copy of an object or $\emptyset$ if the individual is unassigned under μ. We call this set N_1 so that $N_1 = \left(\bigcup_{i \in I} \{(i, \mu(i))\}\right)$. So $(i, \mu(i))$ represents individual i and his assignment under μ.

2. Nodes representing copies of objects that are unassigned under μ. Since these copies are unassigned, we denote their assignment under μ as $\emptyset$ and index these copies by integers to differentiate them. We call this set N_2 so that $N_2 = \left(\bigcup_{o:|\mu(o)|<q_o} \bigcup_{k=1}^{q_o - |\mu(o)|} \{(\emptyset_k, o)\}\right)$. So $(\emptyset_k, o)$ represents the k-th unassigned copy of object o under μ.

3. A node that will be used to represent the situation where an individual is assigned an object under μ but not under μ'. This "dummy" node is denoted $(\emptyset, \emptyset)$.

Now, we want to define the set E of edges of the **μ-μ'-exchange graph**. An edge will simply represent how an individual switches assignment from μ to μ'. The possible cases are the following:

1. An individual i is assigned a copy of object o under μ and is assigned a copy of object $o' \neq o$ under μ'. In that case, the node $(i, o) \in N_1$ will point to all the nodes representing copies of object o'. These nodes can be assigned copies of o' under μ (so belong to N_1). In that case, if agent i' has a copy of o' under μ, then $[(i, o), (i', o')] \in E$. Or there can also be copies of o' that are unassigned under μ (so belong to N_1) and so $[(i, o), (\emptyset_k, o')] \in E$ for all $k = 1, \ldots, q_o - |\mu(o)|$.[36]

2. An individual i is assigned a copy of object o under μ and is unassigned under μ'. In that case, the node (i, o) will point to the dummy node $(\emptyset, \emptyset)$, i.e. $[(i, o), (\emptyset, \emptyset)] \in N$.

[35] In school choice setting, this truncation is usually done using the outside option of remaining unassigned. In our setting, this is not possible for initial owners so that underdemanded objects are needed.

[36] Remember that the index k is just used to distinguish different unassigned copies of o.

3. An individual i is unassigned under μ but assigned under μ'. In that case, the pointing is the same as in Case 1 above by just replacing o by $\emptyset$.

4. An individual does not change his assignment between μ and μ'. In that case, we just let this node points to itself, i.e. $[(i,\mu(i)),(i,\mu(i)] \in N$. We refer to this case as a self-cycle.[37]

To illustrate how to build the **μ-μ'-exchange graph**, consider the example after Proposition 1 which has five individuals and five objects with one copy each. Let $\mu = \mathrm{DA}^*(P,\succ)$ and μ' be the matching given in the example which has a set of blocking pairs strictly included in the one of $\mathrm{DA}^*(P,\succ)$. Remember that the two matchings are given by:

$$\mu = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 & o_5 \\ i_1 & i_2 & i_3 & i_5 & i_4 \end{pmatrix}$$

$$\mu' = \begin{pmatrix} o_1 & o_2 & o_3 & o_4 & o_5 \\ i_2 & i_1 & i_3 & i_4 & i_5 \end{pmatrix}$$

Figure 1 gives an illustration of the μ-μ'-exchange graph. As we see, to go from the matching μ to μ', the nodes can be partitioned into cycles: let i_1 and i_2 as well as i_4 and i_5 exchange their assigned object under μ. As i_3 is not changing its assignment between μ and μ' he can be thought as exchanging with himself (self-cycle).[38]

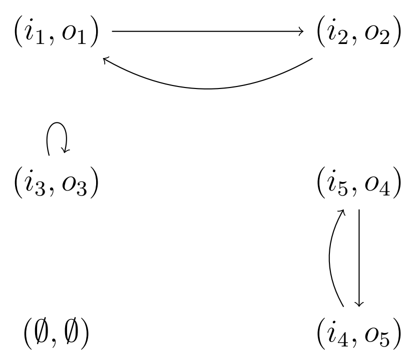

Figure 1: Example of the μ-μ'-exchange graph.

The following lemma formally proves the previous observation that in the μ-μ'-exchange graph, one can decompose the move from μ to μ' as a collection of cycles and chains:

Lemma 2. *Take any two IR matchings μ and μ'. Then, in the μ-μ'-exchange graph (N,E), there exists a partition $C_1,\ldots,C_K$ of $N_1 \cup N_2$ such that for all $k = 1,\ldots K$, $C_k := \{(i_1^k, o_1^k),\ldots,(i_{n_k}^k, o_{n_k}^k)\}$ and for all $\ell = 1,\ldots,n_k - 1$, $[(i_\ell^k, o_\ell^k),(i_{\ell+1}^k, o_{\ell+1}^k)] \in E$ so that $o_{\ell+1}^k = \mu'(i_\ell^k) \neq \emptyset$ and either:*

1. *$i_{n_k}^k \in I$ and $o_1^k = \mu'(i_{n_k}^k) \in O$ so that $[(i_{n_k}^k, o_{n_k}^k),(i_1^k, o_1^k)] \in E$. In that case, C_k is called a cycle.*

[37] Note that i can be either assigned or unassigned under μ.

[38] In the example, because no individual moves from being assigned under μ to unassigned under μ', then the node $(\emptyset,\emptyset)$ is not used. For instance, if μ' would rather say unmatched individual i_5, then the node (i_5, o_4) would point to $(\emptyset,\emptyset)$ and, in the decomposition as defined in Lemma 2, we would have a chain starting from (i_4, o_5) and ending at node $(\emptyset,\emptyset)$.

2. *Or* $(i^k_{n_k}, o^k_{n_k}) \in N_2 \cup \{(\emptyset, \emptyset)\}$*. In that case,* C_k *is called a chain.*

We refer to these subsets as the decomposition (of the μ*-*μ'*-exchange graph) from* μ *to* μ'*.*[39]

Proof. The goal of the proof is to sequentially build the sets $C_1, \dots, C_k$ in the claim of the lemma. Remember that we start from two different matchings μ and μ'. Let $\tilde{I} := \{i \in I : \mu(i) \neq \mu'(i)\}$ and let $\tilde{n} := |\tilde{I}|$.

If individual i does not change her allocation from μ to μ', i.e., $i \notin \tilde{I}$, then $\mu(i) = \mu'(i) := o$ (with possibly $o = \emptyset$) and trivially, in the graph defined in the lemma, the node (i, o) points to itself. Let $C_1 := \{(i, o)\}$ such that this set trivially respects condition (1) of the lemma. There are $N - \tilde{n}$ such individuals, so one can create $C_1, \dots, C_{N-\tilde{n}}$ singleton sets representing self-cycles of individuals who do not move between μ and μ'.

Now, consider the subgraph $(\tilde{N}_1, \tilde{E}_1)$, where $\tilde{N}_1$ deletes any nodes representing self-cycles from the previous step and $\tilde{E}_1$ deletes any edge ending at one of the nodes of an above individual, i.e., $\tilde{N}_1 = N \backslash \bigcup_{k=1}^{N-\tilde{n}} C_k$ and $\tilde{E}_1 := E \backslash \{[(i, o), (i', o')] \in E : i' \notin \tilde{I}\}$. We will define the remaining sets of the decomposition by using "a path of nodes" in $(\tilde{N}_1, \tilde{E}_1)$. Take any individual from $\tilde{I}$ and call him i_1. This individual is part of a node in the μ-μ'-exchange graph, call this node (i_1, o_1). Take any of the outgoing edges starting at node (i_1, o_1), call this outgoing node (i_2, o_2). There are two cases possible:

1. Either this node points back to (i_1, o_1).[40] In that case, we found a new set in our decomposition, $C_{N-\tilde{n}+1} := \{(i_1, o_1), (i_2, o_2)\}$, which respects condition 1 in the lemma: it represents a cycle.
2. Either it has no outgoing edge, so that $(i_2, o_2) = (\emptyset, o_2)$ with either $o_2 = \emptyset$ or $o_2 \in O$ if $(\emptyset, o_2) \in N_2$. In that case, we found the end of a chain (condition 2 of the lemma). Two cases are possible:
   (a) Either (i_1, o_1) has no incoming edge. So we found the entire chain and we can set $C_{N-\tilde{n}+1} := \{(i_1, o_1), (\emptyset, o_2)\}$.
   (b) Or (i_1, o_1) has an incoming edge from another node, call it (i, o) so that $[(i, o), (i_1, o_1)] \in E$. In that case, we can iterate the process we just had for case 2 with the node (i, o). By finiteness of the graph, it must stop at an edge with no incoming cycle (case 2a above) so that we can set $C_{N-\tilde{n}+1}$ as being the set of all the nodes we found in this process.
3. Either it has an outgoing edge to a node (i, o), i.e. $[(i_2, o_2), (i, o)] \in E$. In that case, we can apply the same process we used for node (i_1, o_1) to the new node (i, o). By finiteness, this process must stop either with Case 1 (a cycle was found) or Case 2 (a chain was found).

The end of the above process allows us to define the set $C_{N-\tilde{n}+1}$ in the decomposition of the lemma. If there are still nodes left in the μ-μ'-exchange graph, then one can just iterate this

[39] In a many-to-one framework, this decomposition is not unique, but our results apply to any such decomposition. For instance, in our previous example, if o_4 was actually another copy of object o_1, then, in addition to the edges in Figure 1, (i_2, o_2) would also point to $(i_5, o_4$ and (i_4, o_5) to (i_1, o_1). In that case, several decompositions can be found.

[40] Note that we already removed the self-cycles before so that the node (i_2, o_2) cannot point to itself.

process with the subgraph $(\tilde{N}_2, \tilde{E}_2)$ such that $\tilde{N}_2 := \tilde{N}_1 \backslash C_{N-\tilde{n}+1}$ and $\tilde{E}_2 := \tilde{E}_1 \backslash \{(n,n') \in \tilde{E}_1 : n' \in C_{N-\tilde{n}+1}\}$. The same steps as before can be performed to define the next set $C_{N-\tilde{n}+2}$. By finiteness, all the nodes of the μ-μ'-exchange graph end up being partitioned by this procedure. By construction, each of the sets of the decomposition respects the conditions of the lemma. □

The following lemma exhibits the property of decomposition from the matching $\mathrm{DA}^*(P,\succ)$ to a matching μ with $B_\mu \subseteq B_{DA^*(P,\succ)}$. It states that each cycle or chain in the decomposition from $\mathrm{DA}^*(P,\succ)$ to μ that involves more than one individual must either make all its individuals strictly better off or all of them strictly worse off.

Lemma 3. *Fix an IR matching μ such that $B_\mu \subseteq B_{DA^*(P,\succ)}$ and a decomposition $C_1, \ldots, C_K$ from $DA^*(P,\succ)$ to μ, as defined in Lemma 2. Fix any $k = 1, \ldots, K$ such that $n_k > 1$, then either:*

1. *for all $\ell = 1, \ldots, n_k$: $\mu(i^k_\ell) = o^k_{\ell+1} P_{i^k_\ell} o^k_\ell = DA^*_{i^k_\ell}(P,\succ)$.*
2. *Or for all $\ell = 1, \ldots, n_k$: $DA^*_{i^k_\ell}(P,\succ) = o^k_\ell P_{i^k_\ell} o^k_{\ell+1} = \mu(i^k_\ell)$.*

In particular, in case 1, the set C_k must respects condition 1 of Lemma 2, i.e. represents a cycle.

Proof. Fix a set $C_k := \{(i_1, o_1), \ldots, (i_{n_k}, o_{n_k})\}$ with $n_k > 1$. W.l.o.g. fix individual i_1 and assume that $\mu(i_1) = o_2 P_{i_1} \mathrm{DA}^*_{i_1}(P,\succ)$ so that, by IR, $o_2 \neq \emptyset$. First, note that we cannot have $|\mathrm{DA}^*_{o_2}(P,\succ)| < q_{o_2}$. Indeed, if this was the case, then i_1 would block the matching of $\mathrm{DA}^*(P,\succ)$ with the object whose copies are not all assigned, a contradiction to Lemma 1. Therefore, $|\mathrm{DA}^*_{o_2}(P,\succ)| = q_{o_2}$ and $i_2 \neq \emptyset$.

Case 1. $i_2 \in \mathrm{DA}^*_{o_2}(P,\succ) \backslash \mu_0(o_2)$.

First note that trivially, since i_2 is assigned to o_2 under $\mathrm{DA}^*(P,\succ)$, we have that $(i_2, o_2) \notin B_{DA^*(P,\succ)}$. Using Lemma 1, since $i_2 \in \mathrm{DA}^*_{o_2}(P,\succ)$, we have that $i_2 \succ_{o_2} i_1$; otherwise, i_1 would have a higher priority than an individual in $\mathrm{DA}^*_{o_2}(P,\succ)$ who is not the initial owner of o_2, a contradiction of Lemma 1. Therefore, if $\mathrm{DA}^*_{i_2}(P,\succ) = o_2 P_{i_2} \mu(i_2)$, then we would have $(i_2, o_2) \in B_\mu$, a contradiction to $B_\mu \subseteq B_{DA^*(P,\succ)}$.

Case 2. $i_2 \in \mathrm{DA}^*_{o_2}(P,\succ) \cap \mu_0(o_2)$.

In this case, if $\mathrm{DA}^*_{i_2}(P,\succ) = \mu_0(i_2) = o_2 P_{i_2} \mu(i_2)$, then we would have the contradiction that μ is IR.

We can repeat the argument to show that for all $\ell = 1, \ldots, n_k$, we must have $\mu(i_\ell) = o_{\ell+1} P_{i_\ell} o_\ell = \mathrm{DA}^*_{i_\ell}(P,\succ)$ (with $n_k + 1 := 1$). Note that this result necessarily implies that C_k is a cycle and not a chain.

Now, fix a set $C_k := \{(i_1, o_1), \ldots, (i_{n_k}, o_{n_k})\}$.

First, assume that C_k is a cycle. In that case, all the nodes involve an individual and an object. W.l.o.g, fix individual i_1 and assume that $o_1 = \mathrm{DA}^*_{i_1}(P,\succ) P_{i_1} o_2 = \mu(i_1)$. Note that, since $\mathrm{DA}^*(P,\succ)$ and μ are IR, we have $o_1 \neq \mu_0(i_1)$. Consider individual i_{n_k} and assume that $o_1 = \mu(i_{n_k}) P_{i_{n_k}} \mathrm{DA}^*_{i_{n_k}}(P,\succ) = o_{n_k}$. Since i_1 is assigned to o_1 under $\mathrm{DA}^*(P,\succ)$, we trivially have that $(i_1, o_1) \notin B_{DA^*(P,\succ)}$. Since $i_1 \in \mathrm{DA}^*_{o_1}(P,\succ) \backslash \mu_0(o_1)$, Lemma 1 implies that $i_1 \succ_{o_1} i_{n_k}$; otherwise, i_{n_k} would have a higher priority in o_1 than an individual assigned to it who was not an inintial owner of o_1, contradicting Lemma 1. Therefore, $(i_1, o_1) \in B_\mu$, a contradiction to $B_\mu \subseteq B_{DA^*(P,\succ)}$.

Assume now that C_k is a chain that ends with the empty node, i.e., $(i_{n_k}, o_{n_k}) := (\emptyset, \emptyset)$. By construction, if $i := i_{n_k - 1}$, $\mu(i) = \emptyset \neq \mathrm{DA}^*_i(P,\succ)$. Since DA^* is IR, we have that

$\mathrm{DA}^*_i(P,\succ)P_i\emptyset = \mu(i)$. We can apply the same argument as before to show that i_{n_k-2} must also be worse off and continue to show that all the individuals in the chain must be worse off under μ.

Finally, assume that C_k is a chain that ends with a non-empty node, i.e., $(i_{n_k}, o_{n_k}) = (\emptyset_l, o)$ for some integer l and $o \in O$. Take individual $i := i_{n_k-1}$. Note that it cannot be the case that $o_{n_k} = oP_i\mathrm{DA}^*_i(P,\succ)$. Indeed, this implies that $|\mathrm{DA}^*_o(P,\succ)| < q_o$, which would imply that i is blocking $\mathrm{DA}^*(P,\succ)$ with object which has some copies left unassigned, a contradiction to Lemma 1. Therefore, we must have that $\mathrm{DA}^*_i(P,\succ)P_i\mu(i) = o$. The same argument as before applies, so we conclude that all the individuals in C_k must be worse off under μ. □

Note that the previous proof implies that if the individuals in the set C_k are all strictly better off under μ, then this set must be a cycle. The next lemma tells us that if an IR matching μ has strictly less envy than $\mathrm{DA}^*(P,\succ)$, i.e., $B_\mu \subset B_{DA^*(P,\succ)}$, then there must be a cycle in the decomposition from $DA^*(P,\succ)$ to μ that makes all the individuals strictly better off:

Lemma 4. *Assume that a IR matching μ has strictly less envy than $DA^*(P,\succ)$, i.e., $B_\mu \subset B_{DA^*(P,\succ)}$. Then, in any decomposition given in Lemma 2, there must exist a cycle C_k that respects the condition 1 of Lemma 3, i.e., that makes all its individuals strictly better-off.*

Proof. From Lemma 3, we know that each cycle of the decomposition from $\mathrm{DA}^*(P,\succ)$ to μ either makes all of its individuals strictly better-off or all of them strictly worse-off. In addition, every chain of the decomposition necessarily makes all of its individuals strictly worse-off. So if no cycle in the decomposition makes all of its individuals strictly better-off (condition 2 of Lemma 3) then all the cycles and chains make all their individuals strictly worse off, i.e., for all $i \in I, \mathrm{DA}^*_i(P,\succ)P_i\mu(i)$. Now, fix any pair $(i,o) \in B_{DA^*(P,\succ)}$, Lemma 1 implies that $|DA^*_o(P,\succ)| = q_o$ and there exists $i' \in \mathrm{DA}^*_o(P,\succ) \cap \mu_0(o)$ such that $i \succ_o i'$. If $\mu(i') \neq \mathrm{DA}^*_{i'}(P,\succ)$, then our assumption implies that $\mu_0(i') = \mathrm{DA}^*_{i'}(P,\succ)P_{i'}\mu(i')$, contradicting that μ is IR. Therefore, $i' \in \mathrm{DA}^*_o(P,\succ) \cap \mu(o)$ such that $(i,o) \in B_\mu$. We conclude that $B_{DA^*(P,\succ)} \subseteq B_\mu$, contradicting that $B_\mu \subset B_{DA^*(P,\succ)}$. □

The next terminologies and lemmas are borrowed from Kesten and Kurino (2019). For these results to hold, assumption A3 is needed. For a given profile $(P,\succ)$ and a matching μ, we say that a object o is:

- **Overdemanded at P under** μ: there exists $i \in I$ such that $oP_i\mu(i)$.
- **Weakly-underdemanded at P under** μ: for all $i \in I$: $\mu(i)R_io$.

Lemma 5 (Lemma 1, Kesten and Kurino, 2019)**.** *For any profile $(P,\succ)$, there exists a weakly underdemanded object at P under $DA^*(P,\succ)$.*[41]

The following terminology is also defined in Kesten and Kurino (2019): fix a preference profile P_i for an individual i and two objects o, o' such that oP_io'. We say that a profile P'_i **upgrades** o' **above** o **in** P_i if i) $o'P'_io$, ii) there is no $o'' \in O$ such that $o'P_io''P_io$ and iii) the relative ranking of all the other objects remains the same between P'_i and P_i. Our proof will use the following technical lemma adapted to our setting:

Lemma 6 (Lemma 1, Kesten and Kurino, 2019)**.** *Suppose that under $DA^*(P,\succ)$, individual i is assigned to object o that is overdemanded at P under $DA^*(P,\succ)$. Then, there exist P'_i and a weakly underdemanded object o' at P under $DA^*(P,\succ)$ such that P'_i upgrades o' above o in P_i, $DA^*_i(P'_i, P_{-i}, \succ) = o'$, and o' is weakly underdemanded at (P'_i, P_{-i}) under $DA^*(P'_i, P_{-i}, \succ)$.*

[41] Kesten and Kurino (2019) study a school choice model so that they use the terminology "school" instead of "object".

It is easy to show that the lemma naturally applies to DA*. Indeed, the definitions of overdemanded and weakly-underdemanded objects do not depend on the priority profile of the objects; DA* just modifies the priority profile of the objects and runs a standard DA over it. Thus, with assumption A3, both lemmas hold in our setting. The following lemma elicits an important property of an underdemanded object:

Lemma 7. *If μ is an IR matching such that $B_\mu \subseteq B_{DA^*(P,\succ)}$ and $o := DA^*_i(P,\succ)$ is a weakly-underdemanded object at P under $DA^*(P,\succ)$, then $DA^*_i(P,\succ) R_i \mu(i)$.*

Proof. By contradiction, assume that $\mu(i) P_i \mathrm{DA}^*_i(P,\succ)$. Since μ is IR and $\mu(i) \neq \mathrm{DA}^*_i(P,\succ)$, Lemma 2 tells us that for any decomposition from $\mathrm{DA}^*(P,\succ)$ to μ, there exists a cycle $C_k := \{(i_1,o_1),\ldots,(i_{n_k},o_{n_k})\}$ and $\ell \in \{1,\ldots,n_k\}$ such that $i_\ell = i$. Since $\mu(i) P_i \mathrm{DA}^*_i(P,\succ)$, Lemma 3 tells us that all the individuals in cycle C_k must be strictly better off under μ than under $\mathrm{DA}^*(P,\succ)$. In particular, for individual $i_{\ell-1}$, we have that $o = \mathrm{DA}^*_{i_\ell}(P,\succ) = \mu(i_{\ell-1}) P_{i_{\ell-1}} \mathrm{DA}^*_{i_{\ell-1}}(P,\succ)$, contradicting that o was weakly-underdemanded at P under $\mathrm{DA}^*(P,\succ)$ □

We now have all the elements to prove Proposition 1:

Proof of Proposition 1. Assume that there exists an IR mechanism φ that has strictly less envy than DA*. We will show that this mechanism cannot be strategy-proof.

If φ has strictly less envy than DA*, there must exists a profile $(P,\succ)$ such that $B_{\varphi(P,\succ)} \subset B_{DA^*(P,\succ)}$. By Lemma 2, there exists a decomposition $C_1,\ldots,C_K$ from $\mathrm{DA}^*(P,\succ)$ to $\varphi(P,\succ)$. By Lemma 4, there exists one of these sets that is a cycle. W.l.o.g., say C_1 such that all the individuals involved in it are strictly better-off under $\varphi(P,\succ)$ than under $\mathrm{DA}^*(P,\succ)$. Fix any individual i who is part of cycle C_1 and let $o := \mathrm{DA}^*_i(P,\succ)$, $o' := \varphi_i(P,\succ)$ and i' be the individual of the node that points to (i,o) so that $\varphi_{i'}(P,\succ) = o$. Since all the individuals in cycle C_1 are strictly better off,$o P_{i'} \mathrm{DA}^*_{i'}(P,\succ)$ and o is overdemanded at P under $\mathrm{DA}^*(P,\succ)$. Using Lemma 6, there exists a weakly-underdemanded object o'' at P under $\mathrm{DA}^*(P,\succ)$ and a preference profile P'_i that upgrades o'' above o in P_i such that i) $\mathrm{DA}^*_i(P'_i,P_{-i},\succ) = o''$ and ii) o'' is weakly-underdemanded at (P'_i,P_{-i}) under $\mathrm{DA}^*(P'_i,P_{-i},\succ)$.

According to ii), since o'' is weakly-underdemanded at (P'_i,P_{-i}) under $\mathrm{DA}^*(P'_i,P_{-i},\succ)$ and by the assumption $B_{\varphi(P'_i,P_{-i},\succ)} \subseteq B_{DA^*(P'_i,P_{-i},\succ)}$, Lemma 7 implies that:

$$o'' := \mathrm{DA}^*_i(P'_i,P_{-i},\succ) R'_i \varphi_i(P'_i,P_{-i},\succ)$$

However, since $o' = \varphi_i(P,\succ) P_i \mathrm{DA}^*_i(P,\succ) = o$ and P'_i upgrades o'' above o, then o'' is ranked immediately before o and o' is still strictly preferred to o'' under P'_i:

$$o' = \varphi_i(P_i,P_{-i},\succ) P'_i o'' := \mathrm{DA}^*_i(P'_i,P_{-i},\succ) R'_i \varphi_i(P'_i,P_{-i},\succ)$$

Therefore, we conclude that φ is not strategy-proof. □

6 Discussions

In the previous section, we showed that there are other mechanisms than DA* (resp. TTC*) that are minimal envy mechanisms in the set of IR and SP (resp. IR, SP and PE) mechanisms. However, as our examples showed, their constructions are quite specific and select a particular mechanism for a particular priority profile. A legitimate question is whether one can find other "natural" mechanisms than DA* or TTC* that are minimal envy mechanisms in some well defined sets.

In considering the assignment of teachers to schools, Combe, Tercieux, and Terrier (2022) studied another mechanism: the Teacher-Optimal Block Exchange algorithm (TO-BE). They considered a many-to-one environment with only initial owners, i.e., $n = 0$. To study minimal envy, because their mechanism shares similar features as those of the TTC mechanism, consider a one-to-one setting. Their motivation is twofold:

1. They note that the DA* mechanism is not efficient in a strong sense: one can reassign teachers to schools such that both **teachers and schools** obtain an assignment that they rank higher. They refer to a matching/mechanism not suffering from this problem as **two-sided Pareto-efficient** (2-PE).[42]

2. Even if schools are not strategic entities and their priorities are fixed by law, they argue that those priorities incorporate some welfare objectives that the policy maker would like to achieve. For instance, in France, those priorities incorporate points regarding the experience of the teachers or their ability to move closer to their partner. Thus, they consider a 2-sided notion of individual rationality: **2-Individual Rationality** (2-IR). A matching μ is 2-IR if for all $i \in I$, $\mu(i) R_i \mu_0(i)$ and for all $o \in O$, $\mu(o) \succeq_o \mu_0(o)$, so individuals and objects obtain a weakly better assignment.

The TO-BE mechanism is a 2-IR, 2-PE and SP mechanism. Using data on the assignment of teachers in France, they show that this mechanism performs better than the DA* algorithm in terms of the movement and welfare of teachers. They also note that the 2-IR property allows the achievement of better policy objectives with respect to the distribution of teachers compared to those of other IR (but not 2-IR) mechanisms, such as TTC.[43] The TO-BE mechanism works as follows: first, for $O' \subseteq O$, a matching μ and individual i, define the *opportunity set* of individual i as $\text{Opp}(i, O') := \{o \in O' : i \succeq_o \mu(o')\}$, i.e., the objects in O' that give i a weakly higher priority than their initial owner. Then:

Step 0. Let $O(0) = O$, $I(0) = I$ and $\mu(0) = \mu_0$.

Step $k \geq 1$ Build the graph (N, E) such that $N = \{(i, \mu_0(i)) : i \in I(k-1)\}$. For a node $(i, o) \in N$, let (i, o) point to (i', o') if o' is the favorite object of i in $\text{Opp}(i, O(k-1))$.[44]. There will be cycles in this graph, and all the cycles are disjoint. Implement a cycle[45] in matching each individual to the object of the node her node is pointing to. Define $I(k)$ and $O(k)$ in deleting the assigned individuals and objects. If these sets are empty, stop the algorithm; otherwise, go to step $k + 1$.

For some profile $(P, \succ)$, $B_{\text{TO-BE}(P,\succ)} \subset B_{DA^*(P,\succ)}$. However, as for TTC*, DA* and TO-BE are not comparable according to the less envy relation.[46] In addition, one can also easily show that TTC*

[42]Our previous notion of Pareto-efficiency only considered welfare improvement on the individuals' side.

[43]We refer the interested reader to the cited article for further details. In a one-to-one framework as we are considering here, the TO-BE mechanism and the 2S-TTC, proposed by Dur and Ünver (2019) to study the Tuition Exchange Programs such as Erasmus, are equivalent.

[44]Their original definition is slightly more general since they consider a many-to-one setting. Note that, at any step k, for any individual $i \in I(k-1)$, $\mu_0(i) \in \text{Opp}(i, O(k-1))$ so that a node (i, o) can point to itself, forming a (self-)cycle

[45]The order in which the cycles are implemented does not influence the final matching, so if there are many cycles, the selection of which one to implement does not matter.

[46]Note that in Example 2, TTC* and TO-BE lead to the same matching.

and TO-BE are not comparable according to the former relation. Since TO-BE is 2-PE, it is not in the set of IR, SP and PE mechanisms that we considered for TTC*. Note that TO-BE is a 2-IR mechanism. Thus, the natural class of mechanisms to consider is the class of 2-IR and SP mechanisms rather than, as for DA*, the class of IR and SP mechanisms. In the latter, one can easily exhibit mechanisms that have strictly less envy than TO-BE, so the latter is not a minimal envy mechanism in this set:

Proposition 4. *There are mechanisms that are IR, SP and have strictly less envy than TO-BE.*

Proof. Consider the following simple environment with only 2 individuals, i_1 and i_2, initially assigned to, respectively, o_1 and o_2. In this setting, the standard Shapley-Scarf TTC mechanism is IR and SP and trivially leads to fewer blocking pairs than TO-BE since there is only one possible exchange. Under TO-BE, if one of the two individuals has a lower priority in the object of the other one, then TO-BE does not implement the exchange. On can define a mechanism φ that is the Shapley Scarf TTC mechanism when there are only two individuals and is TO-BE otherwise. It can be trivially verified that such a mechanism is IR and SP and has strictly less envy than TO-BE. □

The following proposition shows that, in a one-to-one environment with only initial owners ($n = 0$) and in the class of 2-IR and SP mechanisms, there is indeed no mechanism with strictly less envy than TO-BE, so TO-BE is a minimal envy mechanism in this set.

Proposition 5. *Consider a one-to-one setting with only initial owners ($n = 0$); then, TO-BE is a minimal envy mechanism in the set of 2-IR and SP mechanisms.*

Proof. The proof is relegated to Section B of the Appendix. □

Combe, Tercieux, and Terrier (2022) showed that, in a one-to-one setting with $n = 0$, TO-BE is the unique 2-IR, 2-PE and SP. However, the result of Proposition 4 is stronger since it considers the set of all the 2-IR and SP mechanisms and not just the set of 2-IR, SP and 2-PE mechanisms. This is different from the TTC* mechanism, which is not a minimal envy mechanism if one considers the set of IR and SP mechanisms, even if there are only initial owners. If one relaxes the 2-PE property, then it is possible to find other minimal envy mechanisms in the set of 2-IR and SP mechanisms. Take the same setting as in Example 2. Define the mechanism φ as follows: if the priority profile is $\succ$, as in Example 2, then for all P, let $\varphi(P, \succ) = \text{TO-BE}(P, \tilde{\succ})$, where $\tilde{\succ} := (\succ_{o_1}, \tilde{\succ}_{o_2}, \succ_{o_3})$ with $\tilde{\succ}_{o_2} : i_3, i_2, i_1$, i.e., we artificially modify o_2's priority profile in ranking i_1 below i_2. If the priority profile differs from $\succ$ or the number of individuals differs, let φ be equal to TO-BE. Note that, in profile $(P, \succ)$ of Example 2, $\varphi(P, \succ) = \text{DA}^*(P, \succ)$ and $\text{TO-BE}(P, \succ) = \text{TTC}^*(P, \succ)$, so $B_{\varphi(P,\succ)} = B_{DA^*(P,\succ)} \neq B_{\text{TO-BE}}(P, \succ)$ and φ and TO-BE cannot be compared using the "strictly less envy" relation. Note that, by construction, φ is 2-IR and also trivially SP.[47] One can also easily verify that it is a minimal envy mechanism in the set of 2-IR and SP mechanisms.

Similarly to TTC*, if one considers a many-to-one setting with only initial owners, then one can find a 2-IR, 2-PE and SP mechanism that has strictly less envy than TO-BE.[48] However, if one considers an one-to-one intermediate setting with $0 < n < N$, then it is still an open question

[47]The modification of the priority ordering of o_2 in profile $(P, \succ)$ does not depend on the reported preferences of the individuals and the mechanism used is TO-BE, so strategy-proofness is satisfied.

[48]Since the statement follows naturally, we omit the example here but can provide it upon request.

whether we can naturally extend the TO-BE mechanism to maintain its minimal envy property in the set of 2-IR, 2-PE and SP mechanisms. The main difficulty is that the 2-IR property imposes that an object with an initial owner cannot end up being unassigned. If one assumes that for all $o \in O$, $\emptyset \succeq_o \mu_0(o)$, then one can naturally extend the TO-BE algorithm in letting objects without initial owner[49] point to the node of their highest-ranked individual in each step. In doing so, as in TTC*, an object with an initial owner can end up unassigned, but the results would still respect the 2-IR definition under the assumption of priorities. One can use a similar proof as that of Proposition 3 and Proposition 4 to show that TO-BE would indeed be a minimal envy mechanism in the set of 2-IR, 2-PE and SP mechanisms.

However, if for all $o \in O$, $\mu_0(o) \succeq_o \emptyset$, then a 2-IR matching cannot leave an object with an initial owner unassigned. Notably, if the highest-ranked individual of an object without an initial owner is initially assigned to another object, then letting the former object point to such an individual, as in TTC*, can leave the latter object unassigned. Therefore, if the objects without an initial owner are no longer pointing to their highest-ranked individual, the techniques used in the proof of Proposition 3 and 4 cannot be used. In that case, the question of whether one can naturally extend the TO-BE algorithm to properly define the pointing behavior for objects without initial owner to maintain the minimal envy property remains open and is left for future research.

7 Conclusion

Our paper showed that, a the setting of reallocation with priorities, two of the main individually rational mechanisms, the DA* and the TTC*, are minimal envy mechanisms. For the former, we showed that it is not possible to find another individually rational and strategy-proof mechanism that always leads to fewer blocking pairs (in the setwise inclusion sense). For TTC*, when objects only have one copy, we showed that one cannot find another individually rational, strategy-proof and Pareto-efficient mechanism that always results in fewer blocking pairs.

A special case of our setting is the standard school choice framework, i.e., all the agents are initially unassigned, and the two mechanisms that we consider, DA* and TTC*, both collapse to the school DA and TTC mechanisms in that setting. Therefore, our results extend two important results of the literature on school choice to a more general model where i) the DA mechanism is, trivially, a minimal envy mechanism in the set SP mechanisms and ii) the TTC algorithm is a minimal envy mechanism in the set of SP and PE mechanisms.

Our results open three possible paths for future research. First, other natural mechanisms of the literature can be investigated in this setting such as the TO-BE mechanism of Combe, Tercieux, and Terrier (2022). The latter mechanism is individually rational for individuals and objects (2-IR), is strategy-proof and respects a stronger form of efficiency, i.e., 2-Pareto-efficiency. In a one-to-one framework, where all the agents are initially assigned, we show that TO-BE is a minimal envy mechanism in the set of 2-IR and SP mechanisms. However, there remains an open question of how to naturally extend the mechanism to incorporate, in a one-to-one setting, possible agents without any initial assignment while maintaining the minimal envy property. Second, the use of the minimal envy property in a setting where no stable matching exists offers an interesting tool to analyze mechanisms. For instance, as mentioned in Section 2, in the literature on distributional constraints in school choice (Kamada and Kojima, 2017a), the constraints imposed on the matchings are

[49] Nodes of the form $(\emptyset, o)$ in the graph of TO-BE if $\mu_0(o) = \emptyset$. Individuals who are not initial owners would appear in nodes of the form $(i, \emptyset)$.

incompatible with stability. Similarly to our setting, the literature has developed alternative notions of stability and studied their existence. To complement the analysis of these mechanisms, knowing whether they are minimal envy mechanisms in their respective constrained sets is of interest and can provide additional properties to compare them. Last, as mentioned in Section 2, recent papers have investigated alternative orders to compare non stable mechanisms in the context of school choice (see for instance, Ehlers et al., 2021 or Kwon and Shorrer, 2020). Applying their new concepts to the setting of reallocation with priorities would also be a promising direction for future research.

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APPENDIX

A Proof of Proposition 3

We will prove a slightly stronger claim: any IR, SP and PE mechanism φ that has less envy than TTC* must be equivalent to the latter. Assume that there exists another IR, SP and PE mechanism φ and a preference profile $(P, \succ)$ such that $\varphi(P, \succ) \neq \text{TTC}^*(P, \succ)$ and $B_{\varphi(P,\succ)} \subseteq B_{TTC^*(P,\succ)}$. We will show that $\varphi(P, \succ) = \text{TTC}^*(P, \succ)$. The proof follows a similar strategy to that of Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) for the TTC mechanisms in a school choice setting and uses an induction over the steps of TTC*.

Consider the first step of TTC$^*(P, \succ)$ and an individual i assigned at that step. We will show that $\text{TTC}^*_i(P, \succ) = \varphi_i(P, \succ)$. Fix the cycle C with which i has been assigned and let $t := i_1$ in $C := \{i_1, o_1, i_2, o_2, \dots, i_K, o_K\}$.

Assume that $s := \varphi_{i_1}(P, \succ) \neq \text{TTC}^*(P, \succ) = o_1$. In this step, i is pointing to her favorite object so that necessarily $o_1 P_{i_1} s$. There are two cases to consider:

Case 1: $o_K = \mu_0(i_1)$. Assume that i_1 reports the profile $P'_{i_1} : o_1, o_K$. Note that, with this new profile, TTC* is the same so that $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ) = \text{TTC}^*(P, \succ)$ and, in particular, $o_1 = \text{TTC}^*_{i_1}(P'_{i_1}, P_{-i_1}, \succ)$. Note that, since φ is SP and $o_1 P_{i_1} s$, then $\varphi(P'_{i_1}, P_{-i_1}, \succ) \neq o_1$. However, since φ is IR and $o_K = \mu_0(i_1)$, it must be the case that $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_K$.

Case 2: $o_K \neq \mu_0(i_1)$. In the first step of TTC*, objects point to their first-ranked individual according to $\tilde{\succ}$, the modified priority profile where objects with an initial owner rank him/her at the top of their profile. Since $o_K \neq \mu_0(i_1)$ but o_K is pointing to i_1, we have that[50] $\mu_0(o_K) = \emptyset$. Therefore, $\tilde{\succ}_{o_K} = \succ_{o_K}$ and individual i_1 is the highest-ranked individual in the priority profile $\succ_{o_K}$. Assume i_1 reports the profile $P'_{i_1} : o_1, o_K, \mu_0(i_1)$. Again, TTC* is the same so $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ) = \text{TTC}^*(P, \succ)$ and, in particular, $o_1 = \text{TTC}^*_{i_1}(P'_{i_1}, P_{-i_1}, \succ)$. Note that since φ is SP and $o_1 P_{i_1} s$, then $\varphi(P'_{i_1}, P_{-i_1}, \succ) \neq o_1$, and since φ is IR, $\varphi(P'_{i_1}, P_{-i_1}, \succ) \in \{o_K, \mu_0(i_1)\}$.[51] Since $o_1 P'_{i_1} o_K$, we trivially have that $(i_1, o_K) \notin B_{TTC^*(P'_{i_1}, P_{-i_1}, \succ)}$. If $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = \mu_0(i_1)$, then since i_1 is the highest-ranked individual in $\succ_{o_K}$, we would have $(i_1, o_K) \in B_{\varphi(P'_{i_1}, P_{-i_1}, \succ)}$, a contradiction to $B_{\varphi(P'_{i_1}, P_{-i_1}, \succ)} \subseteq B_{TTC^*(P'_{i_1}, P_{-i_1}, \succ)}$. Therefore, we conclude that $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_K$.

Let P'_{i_1} be the preference profile of i_1 under one of the two above cases, depending on whether $o_K = \mu_0(i_1)$. In both cases, we have seen that $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_K$. Since we are in a one-to-one framework, $o_K = \text{TTC}^*_{i_K}(P^*_{i_1}, P_{-i_1}, \succ) \neq \varphi_{i_K}(P'_{i_1}, P_{-i_1}, \succ)$. Again, since individuals point to their favorite object in the first step of TTC*, we have:

$$o_K = \text{TTC}^*_{i_K}(P^*_{i_1}, P_{-i_1}, \succ) P_{i_K} \varphi_{i_K}(P'_{i_1}, P_{-i_1}, \succ)$$

Similarly as before, let i_K be profile $P'_{i_K} : o_K, o_{K-1}$ if $o_{K-1} = \mu_0(i_K)$ or $P'_{i_K} : o_K, o_{K-1}, \mu_0(i_K)$ if $o_{K-1} \neq \mu_0(i_K)$. Using a similar argument as that for individual i_1, one can show that:

$$\begin{aligned} \text{TTC}^*(P'_{i_1}, P'_{i_K}, P_{-\{i_1, i_K\}}, \succ) &= TTC^*(P'_{i_1}, P_{-i_1}, \succ) \\ \varphi_{i_K}(P'_{i_1}, P'_{i_K}, P_{-\{i_1, i_K\}}, \succ) &= o_{K-1} \end{aligned}$$

[50] Remember that we are in a one-to-one setting so that objects can only have one initial owner.

[51] Note that it can be the case that $\mu_0(i_1) = \emptyset$.

By repeating the argument recursively for i_{K-1} and other individuals in C, we obtain the following: for all $k = 1, \ldots K$, $\varphi_{i_k}(P', \succ) = o_{k-1}$ (with $0 := K$) with $P'_{i_k} : o_k, o_{k-1}$ if $\mu_0(i_k) = o_{k-1}$ or $P'_{i_k} : o_k, o_{k-1}, \mu_0(i_k)$ if $\mu_0(i_k) \neq o_{k-1}$. Note that this result contradicts that φ is PE since all the individuals in C can now exchange their assignment under $\varphi(P', \succ)$ and be strictly better off. Therefore, we have shown that, for any profile $(P, \succ)$, all the individuals assigned at the first step of $\text{TTC}^*(P, \succ)$ must have the same assignment between $\text{TTC}^*(P, \succ)$ and $\varphi(P, \succ)$.

Now, assume that for any profile $(P, \succ)$, all the individuals assigned up to step $k-1$ of $\text{TTC}^*(P, \succ)$ have the same assignment between $\text{TTC}^*(P, \succ)$ and $\varphi(P, \succ)$.

Consider an individual i assigned in step k of $\text{TTC}^*(P, \succ)$. Let $i_1 := i$ and $C := \{i_1, o_1, i_2, o_2, \ldots, \ldots, i_K, o_K\}$ be the cycle of step k $\text{TTC}^*(P, \succ)$ that assigned individual i so that $\text{TTC}^*(P, \succ)_{i_1} = o_1$. Assume that $o := \varphi(P, \succ)_{i_1} \neq o_1$. Using our induction hypothesis, note that o cannot be an object that, under $\text{TTC}^*(P, \succ)$, has been assigned at a step $k' < k$. Indeed, if this was the case, then the individual assigned to o at that step would still be matched to o under $\varphi(P, \succ)$. Since we are in a one-to-one setting, it would contradict that i_1 was assigned to it. Therefore, in step k of $\text{TTC}^*(P, \succ)$, object o is still in the graph and has not been assigned. Since in that step, individual i_1 points to o_1, her favorite object among those still available, we have that $o_1 P_{i_1} s$. As before, there are two cases to consider:

Case 1: $o_K = \mu_0(i_1)$. Assume that i_1 reports the profile $P'_{i_1} : o_1, o_K$. Note that with this new profile, TTC^* is the same, so $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ) = \text{TTC}^*(P, \succ)$ and, in particular, $o_1 = \text{TTC}^*_{i_1}(P'_{i_1}, P_{-i_1}, \succ)$. Note that since φ is SP and $o_1 P_{i_1} s$, $\varphi(P'_{i_1}, P_{-i_1}, \succ) \neq o_1$. However, since φ is IR and $o_K = \mu_0(i_1)$, it must be the case that $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_K$.

Case 2: $o_K \neq \mu_0(i_1)$. Assume i_1 reports the profile $P'_{i_1} : o_1, o_K, \mu_0(i_1)$.[52] Again, TTC^* is the same, so $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ) = \text{TTC}^*(P, \succ)$ and, in particular, $o_1 = \text{TTC}^*_{i_1}(P'_{i_1}, P_{-i_1}, \succ)$. Additionally, note that all the individuals assigned before step k of $\text{TTC}^*(P, \succ)$ are the same as those assigned before step k of $\text{TTC}^*(P'_{i_1}, P'_{-i_1}, \succ)$. Since $o_1 P'_{i_1} o_K$, we trivially have that $(i_1, o_K) \notin B_{TTC^*(P'_{i_1}, P_{-i_1}, \succ)}$, and since φ has less envy than TTC^*, it must be the case that $(i_1, o_K) \notin B_{\varphi(P'_{i_1}, P_{-i_1}, \succ)}$. Note that since φ is SP and $o_1 P_{i_1} s$, $\varphi(P'_{i_1}, P_{-i_1}, \succ) \neq o_1$, and since φ is IR, $\varphi(P'_{i_1}, P_{-i_1}, \succ) \in \{o_K, \mu_0(i_1)\}$. By construction of TTC^*, for all the individuals $i' \neq i_1$ who have not been assigned yet at step k of $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ)$ and since o_K points to o_1, we have that $i_1 \tilde{\succ}_{o_K} t'$. If $\mu_0(o_K) = \emptyset$, then the modified priority ordering $\tilde{\succ}_{o_K}$ can be changed into the true $\succ_{o_K}$. If $\mu_0(o_K) := \tilde{i} \in I$, note that since $o_K \neq \mu_0(i_1)$ and o_K was pointing to $i_1 \neq \tilde{i}$ at step k, it must be the case that $\tilde{i}$ has been assigned at a step $k' < k$. Since the priority ordering $\tilde{\succ}_{o_K}$ maintains the same relative ranking as $\succ_{o_K}$ between individuals who are not the initial owners of o_K, we also have that $i_1 \succ_{o_K} i'$ for any individual i' who has not been assigned yet at the beginning of step k of $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ)$. Now, note that individual $i' := \varphi_{o_K}(P'_{i_1}, P_{-i_1}, \succ)$ has not been assigned yet at the beginning of step k of $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ)$. Indeed, if this was not the case, our induction hypothesis would imply that:

$$\varphi_{i'}(P'_{i_1}, P_{-i_1}, \succ) = \text{TTC}^*_{i'}(P'_{i_1}, P_{-i_1}, \succ) = o_K$$

Therefore, $i' = i_K$, implying that i_K was assigned to a step $k' < k$ of $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ)$, a contradiction. i' is still present at the beginning of step k of $\text{TTC}^*(P'_{i_1}, P_{-i_1}, \succ)$and $i_1 \succ_{o_K} i'$. If $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) \neq o_K$ (so $i_1 \neq i'$), then $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = \mu_0(i_1)$, and since $o_K P'_{i_1} \mu_0(i_1)$ and

[52] If $o = \mu_0(i_1)$, it can be the case that $P'_{i_1} = P_{i_1}$. However, the same contradiction as the one shown below will hold and we would still conclude that $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_K$.

$i_1 \succ_{o_K} i' = \varphi_{o_K}(P'_{i_1}, P_{-i_1}, \succ)$, we have that $(i_1, o_K) \in B_{\varphi(P'_{i_1}, P_{-i_1}, \succ)}$, a contradiction. Therefore, we conclude that $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_K$.

Let P'_{i_1}, be one of the two above profiles depending on whether we are in Case 1 or Case 2. In both cases, $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_K$. Since we are in a one-to-one setting, $\text{TTC}^*_{i_K}(P'_{i_1}, P_{-i_1}, \succ) = o_K \neq \varphi_{i_K}(P'_{i_1}, P_{-i_1}, \succ) := s$. Using our induction hypothesis, o cannot be an object that has been assigned before step k of $\text{TTC}^*_{i_K}(P'_{i_1}, P_{-i_1}, \succ)$, so object o must have been available to i_K at the beginning of step k. Since in each step, individuals are pointing to their favorite object among those remaining, we conclude that $o_K P_{i_K} s$. As before, let i_K be the profile $P'_{i_K} : o_K, o_{K-1}$ if $o_{K-1} = \mu_0(i_K)$ or $P'_{i_K} : o_K, o_{K-1}, \mu_0(i_K)$ if $o_{K-1} \neq \mu_0(i_K)$. With similar arguments as those for i_1, we have that:

$$\begin{aligned}
\text{TTC}^*(P'_{i_1}, P'_{i_K}, P_{-\{i_1,i_K\}}, \succ) &= TTC^*(P'_{i_1}, P_{-i_1}, \succ) \\
\varphi_{i_K}(P'_{i_1}, P'_{i_K}, P_{-\{i_1,i_K\}}, \succ) &= o_{K-1}
\end{aligned}$$

By repeating the argument recursively for i_{K-1} and other individuals in C, we conclude that for all $k = 1, \ldots, K$, $\varphi_{i_k}(P', \succ) = o_{k-1}$ with $P'_{i_k} : o_k, o_{k-1}$ if $\mu_0(i_k) = o_{k-1}$ or $P'_{i_k} : o_k, o_{k-1}, \mu_0(i_k)$ if $\mu_0(i_k) \neq o_{k-1}$. Again, this contradicts that φ is PE since at profile P', all the individuals in C can exchange their allocation under φ and be strictly better off.

Therefore, we conclude that if a mechanism is IR, SP and PE and has less envy than TTC*, then the two must be equivalent such that TTC* is indeed a minimal envy mechanism in the set of IR, SP and PE mechanisms.

B Proof of Proposition 5

Let φ be a 2-IR and SP mechanism. Fix $(P, \succ)$ such that $\varphi(P, \succ) \neq \text{TO-BE}(P, \succ)$. The proof is by induction over the steps of TO-BE$(P, \succ)$.

Let $C_1 := \{(i_1, o_1), \ldots, (i_K, o_K)\}$ be the cycle implemented at the first step of TO-BE. Note that, by construction of TO-BE, we have that $i_K \succ_{o_1} i_1$ so that $(i_K, o_1) \in B_{\mu_0}$ and since $o_1 = \text{TO-BE}_{i_K}(P, \succ)$, $(i_K, o_1) \notin B_{\text{TO-BE}(P,\succ)}$. Assume that C_1 is not implemented by φ. There is at least one individual in the cycle C_1, w.l.o.g. say i_1, who is assigned to a different object than the one she is pointing to in the cycle C_1, i.e. $o := \varphi_{i_1}(P, \succ) \neq \text{TO-BE}_{i_1}(P, \succ) = o_2$. Since φ is 2-IR, we must have that $i_1 \succ_o \mu_0(o)$ so that $o \in \text{Opp}(i, o)$. Since, by definition of TO-BE, i_1 is pointing to her favorite object in her opportunity set, we have that $o_2 P_{i_1} o = \varphi_{i_1}(P, \succ)$. Now, assume that i_1 reports the following profile: $P'_{i_1} : o_2, o_1$. First, note that it does not change the assignment of TO-BE so that $\text{TO-BE}(P'_{i_1}, P_{-i_1}, \succ) = \text{TO-BE}(P, \succ)$ and, in particular, $(i_K, o_1) \notin B_{\text{TO-BE}(P'_{i_1}, P_{-i_1}, \succ)}$. But since φ is SP, we must have $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_1$. Again, since φ is 2-IR, let $o' := \varphi_{i_K}(P'_{i_1}, P_{-i_1}, \succ)$, we must have that $i_K \succeq_{o'} \mu_0(o')$ so that $o' \in \text{Opp}(i_K, O(0))$ and since i_K was pointing to her favorite object in this set, $o_1 P_{i_K} o'$ so that $(i_K, o_1) \in B_{\varphi(P'_{i_1}, P_{-i_1}, \succ)}$. A contradiction to $B_{\varphi(P'_{i_1}, P_{-i_1}, \succ)} \subseteq B_{\text{TO-BE}(P'_{i_1}, P_{-i_1}, \succ)}$. So we conclude that all the individuals assigned by the cycle C_1 must have the same assignment between TO-BE$(P, \succ)$ and $\varphi(P, \succ)$.[53]

If one considers any profile P' such that for all $i \in I$ for all $o \in O$, we have $s R'_i \text{TO-BE}_i(P, \succ) \Rightarrow s R_i \text{TO-BE}_i(P, \succ)$, then $\text{TO-BE}(P, \succ) = \text{TO-BE}(P', \succ)$. Indeed, all the individuals rank

[53] Note that if the cycle C_1 is a self-cycle, i.e. $K = 1$. Then there is no 2-IR matching that assigns i_1 to a different object than o_1 so that the same conclusion holds.

weakly higher the object that they obtain under TO-BE$(P, \succ)$ so that the same cycles form under TO-BE$(P', \succ)$. So for the individuals in C_1, which is the cycle of the first step of TO-BE$(P, \succ)$, they can still be assigned with the same cycle C_1 at the step 1 of TO-BE$(P', \succ)$. The same argument can be applied to show that these individuals must have the same assignment between TO-BE$(P', \succ)$ = TO-BE$(P, \succ)$ and $\varphi(P', \succ)$.[54] So our induction hypothesis is the following: assume that all the individuals assigned up to the step $k-1$ of TO-BE$(P, \succ)$ have the same assignment between TO-BE$(P', \succ)$ and $\varphi(P', \succ)$ for any P' such that for all $i \in I$, we have sP'_iTO-BE$_i(P, \succ) \Rightarrow sP_i$TO-BE$_i(P, \succ)$.[55]

Let $C_k := \{(i_1, o_1), \ldots, (i_K, o_K)\}$ be the cycle implemented at the step k of TO-BE$(P, \succ)$. Assume that there is an individual in this cycle, say i_1 who is assigned to a different object than the one she is pointing to under C_k, i.e. $o := \varphi_{i_1}(P, \succ) \neq$ TO-BE$_{i_1}(P, \succ) = o_2$. Since φ is 2-IR, we must have that $i_1 \succ_o \mu_0(o)$. Using our induction hypothesis, note that all objects assigned before the step k of TO-BE are assigned to the same individuals under both $\varphi(P, \succ)$ and TO-BE$(P, \succ)$. Thus, since $o = \varphi_i(P, \succ)$, we must have that $o \in \text{Opp}(i, O(k-1))$. Since, at the step k of TO-BE, individual i is pointing to the node with her favorite object in $\text{Opp}(i, O(k-1))$, we have that $o_2 :=$ TO-BE$_i(P, \succ)P_i o$. Now, assume that i_1 reports the following profile: $P'_{i_1} : o_2, o_1$. First, note that it does not change the assignment of TO-BE so that TO-BE$(P'_{i_1}, P_{-i_1}, \succ) =$ TO-BE$(P, \succ)$ and, in particular, $(i_K, o_1) \notin B_{\text{TO-BE}(P'_{i_1}, P_{-i_1}, \succ)}$. But since φ is SP, we must have $\varphi_{i_1}(P'_{i_1}, P_{-i_1}, \succ) = o_1$. Again, since φ is 2-IR, let $o' := \varphi_{i_K}(P'_{i_1}, P_{-i_1}, \succ) \neq o_1$, we must have that $i_K \succeq_{s'} \mu_0(o')$.

If $o_1 P_{i_K} o'$, then $(i_K, o_1) \in B_{\varphi(P'_{i_1}, P_{-i_1}, \succ)}$. But since $(i_K, o_1) \notin B_{\text{TO-BE}(P, \succ)} = B_{\text{TO-BE}(P'_{i_1}, P_{-i_1}, \succ)}$, it contradicts that $B_{\varphi(P'_{i_1}, -P_{i_1}, \succ)} \subseteq B_{\text{TO-BE}(P'_{i_1}, P_{-i_1}, \succ)}$.

If $o' P_{i_K} o_1$, then it means that o' was assigned at a step $k' < k$ of TO-BE$(P, \succ)$. Now, note that the profile $(P'_{i_1}, P_{-i_1}, \succ)$ trivially respects the condition of our induction hypothesis. Using the latter, all the objects assigned at a step $k' < k$ of TO-BE$(P, \succ)$ have the same assignment between $\varphi(P'_{i_1}, P_{-i_1}, \succ)$ and TO-BE$(P'_{i_1}, P_{-i_1}, \succ)$ = TO-BE$(P, \succ)$, contradicting that o' is the object assigned to i_K under $\varphi(P'_{i_1}, P_{-i_1}, \succ)$.

So we conclude that TO-BE$(P, \succ) = \varphi(P, \succ)$, a contradiction to $\varphi(P, \succ) \neq$ TO-BE$(P, \succ)$.

[54]Note that, under TO-BE$(P', \succ)$, there might be additional new cycles at the step 1 of TO-BE$(P', \succ)$ compared to the step 1 of TO-BE$(P, \succ)$. However, the cycle C_1 will still remain at the step 1 of TO-BE$(P', \succ)$ so that the claim concerning the individuals in C_1 is true.

[55]Once again, we do not claim that a cycle implemented at the step $k' < k$ of TO-BE$(P, \succ)$ appears only at the step k' of TO-BE$(P', \succ)$. However, since all the individuals rank weakly higher their assignment under TO-BE$(P, \succ)$ in the profile P' compared to P, we can always pick the cycles appearing in TO-BE$(P', \succ)$ in an order that will indeed assign at the step k' of TO-BE$(P', \succ)$ the individuals assigned at the step k' of TO-BE$(P, \succ)$. Since the order in which we implement several existing cycles at a given step of the TO-BE algorithm does not influence its final outcome, our induction using the steps of TO-BE$(P, \succ)$ is valid. The only effect of P' is to make cycles appear earlier in TO-BE$(P', \succ)$ compared to TO-BE$(P, \succ)$.