



HAL
open science

Efficient approximation of high-frequency Helmholtz solutions by Gaussian coherent states

Théophile Chaumont-Frelet, Victorita Dolean, Maxime Ingremeau

► **To cite this version:**

Théophile Chaumont-Frelet, Victorita Dolean, Maxime Ingremeau. Efficient approximation of high-frequency Helmholtz solutions by Gaussian coherent states. CMAM 2022, Aug 2022, Vienna, Austria. hal-03940689

HAL Id: hal-03940689

<https://inria.hal.science/hal-03940689>

Submitted on 16 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Efficient approximation of high-frequency Helmholtz solutions by Gaussian coherent states

T. Chaumont-Frelet^{*,†}, V. Dolean^{†,‡} and M. Ingremeau[‡]

CMAM 2022

MS11: Numerical methods for wave propagation problems
TU Wien, August 29

* Inria Université Côte d'Azur, Atlantis project-team

† Laboratoire J.A. Dieudonné

‡ University of Strathclyde, department of mathematics and statistics

Given $f : \mathbb{R}^d \rightarrow \mathbb{C}$, find $u : \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f,$$

where μ and \mathbf{A} are smooth (given) coefficients, and k is the (large) frequency.

Given $f : \mathbb{R}^d \rightarrow \mathbb{C}$, find $u : \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f,$$

where μ and \mathbf{A} are smooth (given) coefficients, and k is the (large) frequency.

A central ingredient of numerical methods is to design a finite-dimensional space W such that

$$\min_{w \in W} \|u - w\|$$

is small in a suitable norm $\|\cdot\|$.

Given $f : \mathbb{R}^d \rightarrow \mathbb{C}$, find $u : \mathbb{R}^d \rightarrow \mathbb{C}$ such that

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f,$$

where μ and \mathbf{A} are smooth (given) coefficients, and k is the (large) frequency.

A central ingredient of numerical methods is to design a finite-dimensional space W such that

$$\min_{w \in W} \|u - w\|$$

is small in a suitable norm $\|\cdot\|$.

Here, we would especially like to design a sequence of spaces $W(k)$ so that

$$\min_{w \in W(k)} \|u - w\| = \text{Cst}$$

uniformly in k while minimizing the dimension of $W(k)$.

Finite element discretization with a mesh of size $h \sim k^{-1}$

$$\dim W(k) \sim k^d, \quad \text{sparse matrix.}$$

Finite element discretization with a mesh of size $h \sim k^{-1}$

$$\dim W(k) \sim k^d, \quad \text{sparse matrix.}$$

Boundary elements with a surface mesh of size $h \sim k^{-1}$

$$\dim W(k) \sim k^{d-1}, \quad \text{dense matrix.}$$

Finite element discretization with a mesh of size $h \sim k^{-1}$

$$\dim W(k) \sim k^d, \quad \text{sparse matrix.}$$

Boundary elements with a surface mesh of size $h \sim k^{-1}$

$$\dim W(k) \sim k^{d-1}, \quad \text{dense matrix.}$$

Our goal is to propose a new family of spaces such that

$$\dim W(k) \sim k^{d-1/2}, \quad \text{sparse matrix.}$$

Quick comparison

	BEM		FEM
Cost	k^{d-1}		k^d
Heterogeneous media	✗		✓
Sparse matrix	✗		✓



Galkowski and Spence, 2022



Melenk and Sauter, 2010

Quick comparison

	BEM	Gaussian states	FEM
Cost	k^{d-1}	$k^{d-1/2}$	k^d
Heterogeneous media	✗	✓	✓
Sparse matrix	✗	✓	✓



Galkowski and Spence, 2022



Melenk and Sauter, 2010



T. Chaumont-Frelet and M. Ingremeau, 2022



T. Chaumont-Frelet, V. Dolean and M. Ingremeau, 2022

- 1 The model scattering problem
- 2 Convergence for general right-hand sides
- 3 Convergence for scattering problems
- 4 Numerical examples
- 5 Concluding remarks

The model scattering problem

The model scattering problem

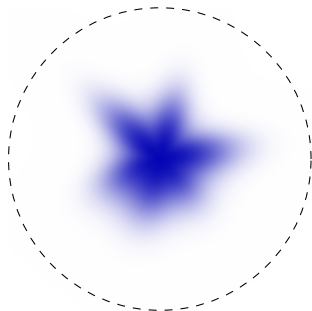
Standard formulation of the scattering problem

The scattering problem by a planewave is usually formulated that way:

$$u^{\text{tot}} = u^{\text{sca}} + e^{ikd \cdot x},$$

$$-k^2 \mu u^{\text{tot}} - \nabla \cdot (A \nabla u^{\text{tot}}) = 0,$$

$$\nabla u^{\text{sca}} \cdot \frac{x}{|x|} - iku^{\text{sca}} \xrightarrow{|x| \rightarrow 0} 0,$$

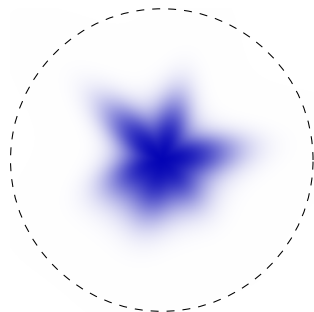


The scattering problem by a planewave is usually formulated that way:

$$\mathbf{u}^{\text{tot}} = \mathbf{u}^{\text{sca}} + e^{ik\mathbf{d}\cdot\mathbf{x}},$$

$$-k^2 \mu \mathbf{u}^{\text{tot}} - \nabla \cdot (\mathbf{A} \nabla \mathbf{u}^{\text{tot}}) = 0,$$

$$\nabla \mathbf{u}^{\text{sca}} \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - ik \mathbf{u}^{\text{sca}} \xrightarrow{|\mathbf{x}| \rightarrow 0} 0,$$



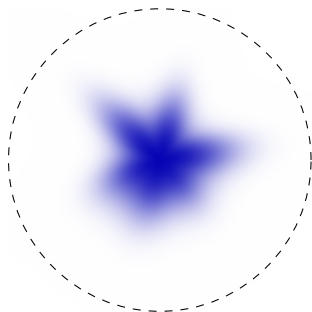
μ and \mathbf{A} are smooth coefficients, with $\mu(\mathbf{x}) = 1$ and $\mathbf{A}(\mathbf{x}) = \mathbf{I}$ if $|\mathbf{x}| \geq R$.

The scattering problem by a planewave is usually formulated that way:

$$\mathbf{u}^{\text{tot}} = \mathbf{u}^{\text{sca}} + e^{ik\mathbf{d}\cdot\mathbf{x}},$$

$$-k^2 \mu \mathbf{u}^{\text{tot}} - \nabla \cdot (\mathbf{A} \nabla \mathbf{u}^{\text{tot}}) = 0,$$

$$\nabla \mathbf{u}^{\text{sca}} \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - ik \mathbf{u}^{\text{sca}} \xrightarrow{|\mathbf{x}| \rightarrow 0} 0,$$



μ and \mathbf{A} are smooth coefficients, with $\mu(\mathbf{x}) = 1$ and $\mathbf{A}(\mathbf{x}) = \mathbf{I}$ if $|\mathbf{x}| \geq R$.

Problem:

we do not have $\mathbf{u}^{\text{sca}} \in H^1(\mathbb{R}^d)$, and the right-hand side is not convenient.

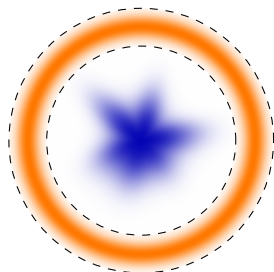
The model scattering problem
Volume right-hand side

Fix a smooth cutoff χ s.t. $\chi(\mathbf{x}) = 1$ if $|\mathbf{x}| \leq R$, and $\chi(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq R'$.

Instead of solving for u^{tot} , we look for $u = \chi u^{\text{tot}} + (1 - \chi)u^{\text{sca}}$. Then

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f,$$

$$\nabla u \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - iku \xrightarrow{|\mathbf{x}| \rightarrow 0} 0,$$



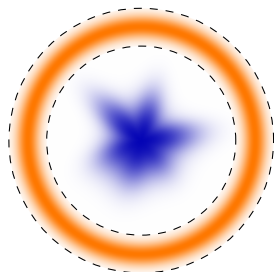
Reformulation with a compactly supported right-hand side

Fix a smooth cutoff χ s.t. $\chi(\mathbf{x}) = 1$ if $|\mathbf{x}| \leq R$, and $\chi(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq R'$.

Instead of solving for u^{tot} , we look for $u = \chi u^{\text{tot}} + (1 - \chi)u^{\text{sca}}$. Then

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f,$$

$$\nabla u \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - iku \xrightarrow{|\mathbf{x}| \rightarrow 0} 0,$$



where the right-hand side has the specific form

$$f := (2ik\mathbf{d} \cdot \nabla \chi + \Delta \chi) e^{ik\mathbf{d} \cdot \mathbf{x}} = \phi e^{ik\mathbf{d} \cdot \mathbf{x}}$$

with ϕ smooth and compactly supported.

The model scattering problem
Perfectly matched layers

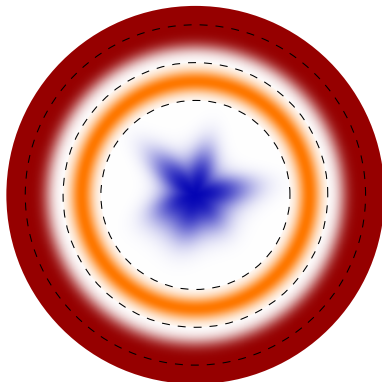
The scattering problem

Finally, we employ a perfectly matched layer (PML).

In essence, we artificially modify the coefficients for $|\mathbf{x}| > R'$ to add absorption. This approach is standard, and do not modify the value of u for $|\mathbf{x}| < R'$.

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f$$

$$u \in H^1(\mathbb{R}^d)$$



We end up with a problem set over $H^1(\mathbb{R}^d)$ with smooth coefficients, no boundary conditions, and $f \in L^2(\mathbb{R}^d)$ compactly supported.

The model scattering problem

Key assumptions

Our model problem is to find $u \in H^1(\mathbb{R}^d)$ such that

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f$$

where

$f \in L^2(\mathbb{R}^d)$ is compactly supported,

μ and \mathbf{A} are $C^\infty(\mathbb{R}^d, \mathbb{C})$ and bounded along with all their derivatives.

Our model problem is to find $u \in H^1(\mathbb{R}^d)$ such that

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f$$

where

$f \in L^2(\mathbb{R}^d)$ is compactly supported,

μ and \mathbf{A} are $C^\infty(\mathbb{R}^d, \mathbb{C})$ and bounded along with all their derivatives.

In the second part of the talk, we will also specifically use the assumption that

$$f = \phi e^{ikd \cdot x}$$

for a smooth compactly supported function ϕ .

Our model problem is to find $u \in H^1(\mathbb{R}^d)$ such that

$$-k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f$$

where

$f \in L^2(\mathbb{R}^d)$ is compactly supported,

μ and \mathbf{A} are $C^\infty(\mathbb{R}^d, \mathbb{C})$ and bounded along with all their derivatives.

In the second part of the talk, we will also specifically use the assumption that

$$f = \phi e^{ikd \cdot x}$$

for a smooth compactly supported function ϕ .

For the sake of simplicity, we assume that the coefficients are non-trapping:

$$\|\nabla u\|_{L^2(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)} \quad \forall k \geq 1.$$

Convergence for general right-hand sides

Convergence for general right-hand sides

The Gabor frame

Start with the Gaussian distribution

$$\pi^{-d/4} e^{-\frac{1}{2}|\boldsymbol{y}|^2}.$$

Start with the Gaussian distribution

$$\pi^{-d/4} e^{-\frac{1}{2}|\mathbf{y}|^2}.$$

Shift in position by $\mathbf{m}\sqrt{\pi}$, with $\mathbf{m} \in \mathbb{Z}^d$

$$\pi^{-d/4} e^{-\frac{1}{2}|\mathbf{y}-\mathbf{m}\sqrt{\pi}|^2}.$$

Start with the Gaussian distribution

$$\pi^{-d/4} e^{-\frac{1}{2}|\mathbf{y}|^2}.$$

Shift in position by $\mathbf{m}\sqrt{\pi}$, with $\mathbf{m} \in \mathbb{Z}^d$

$$\pi^{-d/4} e^{-\frac{1}{2}|\mathbf{y}-\mathbf{m}\sqrt{\pi}|^2}.$$

Shift in momentum by $\mathbf{n}\sqrt{\pi}$, with $\mathbf{n} \in \mathbb{Z}^d$

$$\Psi_{\mathbf{m},\mathbf{n}} := \pi^{-d/4} e^{-\frac{1}{2}|\mathbf{y}-\mathbf{m}\sqrt{\pi}|^2} e^{i\mathbf{n}\sqrt{\pi}\cdot\mathbf{y}}.$$

Start with the Gaussian distribution

$$\pi^{-d/4} e^{-\frac{1}{2}|\mathbf{y}|^2}.$$

Shift in position by $\mathbf{m}\sqrt{\pi}$, with $\mathbf{m} \in \mathbb{Z}^d$

$$\pi^{-d/4} e^{-\frac{1}{2}|\mathbf{y}-\mathbf{m}\sqrt{\pi}|^2}.$$

Shift in momentum by $\mathbf{n}\sqrt{\pi}$, with $\mathbf{n} \in \mathbb{Z}^d$

$$\Psi_{\mathbf{m},\mathbf{n}} := \pi^{-d/4} e^{-\frac{1}{2}|\mathbf{y}-\mathbf{m}\sqrt{\pi}|^2} e^{i\mathbf{n}\sqrt{\pi}\cdot\mathbf{y}}.$$

The resulting family $\{\Psi_{\mathbf{m},\mathbf{n}}\}_{[\mathbf{m},\mathbf{n}] \in \mathbb{Z}^{2d}}$ is the Gabor frame. It is dense in $L^2(\mathbb{R}^d)$.



Gabor, 1946.

We can scale the family with the change of variable $\mathbf{x} = k^{1/2}\mathbf{y}$, leading to

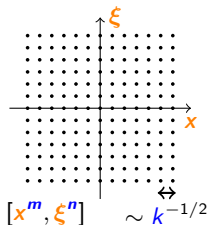
$$\Psi_{k,m,n}(\mathbf{x}) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|\mathbf{x} - \sqrt{\pi/km}|^2} e^{ik(\sqrt{\pi/kn})\cdot\mathbf{x}}.$$

We can scale the family with the change of variable $x = k^{1/2}y$, leading to

$$\Psi_{k,m,n}(x) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|x - \sqrt{\pi/km}|^2} e^{ik(\sqrt{\pi/kn}) \cdot x}.$$

Introducing the lattice $x^m := \sqrt{\pi/km}$ and $\xi^n := \sqrt{\pi/kn}$, we have

$$\Psi_{k,m,n}(x) = \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|x - x^m|^2} e^{ik\xi^n \cdot x}.$$

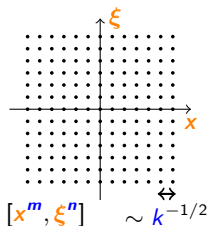


We can scale the family with the change of variable $x = k^{1/2}y$, leading to

$$\Psi_{k,m,n}(x) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|x - \sqrt{\pi/km}|^2} e^{ik(\sqrt{\pi/kn}) \cdot x}.$$

Introducing the lattice $x^m := \sqrt{\pi/km}$ and $\xi^n := \sqrt{\pi/kn}$, we have

$$\Psi_{k,m,n}(x) = \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|x - x^m|^2} e^{ik\xi^n \cdot x}.$$



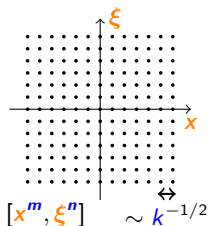
This is still a dense set in $L^2(\mathbb{R}^d)$.

We can scale the family with the change of variable $\mathbf{x} = k^{1/2}\mathbf{y}$, leading to

$$\Psi_{k,m,n}(\mathbf{x}) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|\mathbf{x} - \sqrt{\pi/km}|^2} e^{ik(\sqrt{\pi/kn}) \cdot \mathbf{x}}.$$

Introducing the lattice $\mathbf{x}^m := \sqrt{\pi/km}$ and $\boldsymbol{\xi}^n := \sqrt{\pi/kn}$, we have

$$\Psi_{k,m,n}(\mathbf{x}) = \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|\mathbf{x} - \mathbf{x}^m|^2} e^{ik\boldsymbol{\xi}^n \cdot \mathbf{x}}.$$



This is still a dense set in $L^2(\mathbb{R}^d)$.

We have a grid $[\mathbf{x}^m, \boldsymbol{\xi}^n]$ on \mathbb{R}^{2d} instead of \mathbb{R}^d , spaced by $k^{-1/2}$ instead of k^{-1} .

The family $\{\Psi_{k,m,n}\}_{[m,n] \in \mathbb{Z}^{2d}}$ is dense in $L^2(\mathbb{R}^d)$.

The family $\{\Psi_{k,m,n}\}_{[m,n] \in \mathbb{Z}^{2d}}$ is dense in $L^2(\mathbb{R}^d)$.

It also forms a *frame* on $L^2(\mathbb{R}^d)$. It means that

$$\alpha \|v\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{[m,n] \in \mathbb{Z}^{2d}} |(v, \Psi_{k,m,n})_{L^2(\mathbb{R}^d)}|^2 \leq \beta \|v\|_{L^2(\mathbb{R}^d)}^2 \quad \forall v \in L^2(\mathbb{R}^d),$$

with $0 < \alpha \leq \beta < +\infty$, two constants independent of k .

The family $\{\psi_{k,m,n}\}_{[m,n] \in \mathbb{Z}^{2d}}$ is dense in $L^2(\mathbb{R}^d)$.

It also forms a *frame* on $L^2(\mathbb{R}^d)$. It means that

$$\alpha \|\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2 \leq \sum_{[m,n] \in \mathbb{Z}^{2d}} |(\mathbf{v}, \psi_{k,m,n})_{L^2(\mathbb{R}^d)}|^2 \leq \beta \|\mathbf{v}\|_{L^2(\mathbb{R}^d)}^2 \quad \forall \mathbf{v} \in L^2(\mathbb{R}^d),$$

with $0 < \alpha \leq \beta < +\infty$, two constants independent of k .

This property is a generalization of the concept of Hilbertian basis.



Daubechies, Grossman and Meyer, 1986.

Convergence for general right-hand sides
Approximating smooth functions in the Gabor frame

Think about $L^2(-\pi, \pi)$. Then the family $\{e^{inx}\}_{n \in \mathbb{Z}}$ is a Hilbertian basis.

Think about $L^2(-\pi, \pi)$. Then the family $\{e^{inx}\}_{n \in \mathbb{Z}}$ is a Hilbertian basis.

In particular, we have

$$v = \sum_{n \in \mathbb{Z}} (v, e^{inx})_{L^2(-\pi, \pi)} e^{inx} \quad \forall v \in L^2(-\pi, \pi).$$

Think about $L^2(-\pi, \pi)$. Then the family $\{e^{inx}\}_{n \in \mathbb{Z}}$ is a Hilbertian basis.

In particular, we have

$$v = \sum_{n \in \mathbb{Z}} (v, e^{inx})_{L^2(-\pi, \pi)} e^{inx} \quad \forall v \in L^2(-\pi, \pi).$$

If $v \in H^m(-\pi, \pi)$, we can actually show that

$$|(v, e^{inx})_{L^2(-\pi, \pi)}| \leq \frac{C}{|n|^m} |(v^{(m)}, e^{inx})_{L^2(-\pi, \pi)}|$$

Think about $L^2(-\pi, \pi)$. Then the family $\{e^{inx}\}_{n \in \mathbb{Z}}$ is a Hilbertian basis.

In particular, we have

$$v = \sum_{n \in \mathbb{Z}} (v, e^{inx})_{L^2(-\pi, \pi)} e^{inx} \quad \forall v \in L^2(-\pi, \pi).$$

If $v \in H^m(-\pi, \pi)$, we can actually show that

$$|(v, e^{inx})_{L^2(-\pi, \pi)}| \leq \frac{C}{|n|^m} |(v^{(m)}, e^{inx})_{L^2(-\pi, \pi)}|$$

An easy consequence is that

$$\|v - v_N\|_{L^2(-\pi, \pi)} \leq CN^{-m} \|v\|_{H^m(-\pi, \pi)}, \quad v_N := \sum_{\substack{n \in \mathbb{Z} \\ |n| < N}} (v, e^{inx})_{L^2(-\pi, \pi)} e^{inx}.$$

A general result on frames is the existence of a dual frame $\{\Psi_{k,m,n}^*\}_{[m,n] \in \mathbb{Z}^{2d}}$ s.t.

$$v = \sum_{[m,n] \in \mathbb{Z}^{2d}} (v, \Psi_{k,m,n}^*)_{L^2(\mathbb{R}^d)} \Psi_{k,m,n}.$$



Christensen, 2003.

A general result on frames is the existence of a dual frame $\{\Psi_{k,m,n}^*\}_{[m,n] \in \mathbb{Z}^{2d}}$ s.t.

$$v = \sum_{[m,n] \in \mathbb{Z}^{2d}} (v, \Psi_{k,m,n}^*)_{L^2(\mathbb{R}^d)} \Psi_{k,m,n}.$$



Christensen, 2003.

Our first contribution is to show that

$$\sum_{\substack{[m,n] \in \mathbb{Z}^{2d} \\ |x^m|^2 + |\xi^n|^2 \geq D^2}} |(v, \Psi_{k,m,n}^*)_{L^2(\mathbb{R}^d)}|^2 \leq CD^{-2m} \|v\|_{\check{H}_k^m(\mathbb{R}^d)}^2 \quad \forall v \in \check{H}^m(\mathbb{R}^d)$$

where $\check{H}^m(\mathbb{R}^d)$ is a weighted Sobolev space:

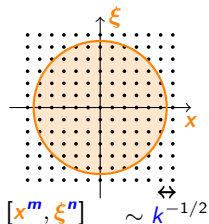
$$\|v\|_{\check{H}_k^m(\mathbb{R}^d)}^2 := \sum_{[a] \leq m} \sum_{q=0}^{p-[a]} k^{-2[a]} \| |x|^q \partial^a v \|_{L^2(\mathbb{R}^d)}^2.$$



Chaumont-Frelet and Ingremeau, 2022.

For $v \in L^2(\mathbb{R}^d)$ let us set

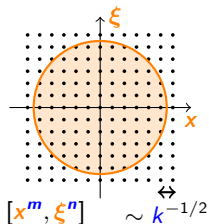
$$\Pi_D v := \sum_{\substack{[m,n] \in \mathbb{Z}^{2d} \\ |x^m|^2 + |\xi^n|^2 < D^2}} (v, \Psi_{k,m,n}^*) \Psi_{k,m,n}.$$



Approximation of smooth functions by Gaussian coherent states

For $v \in L^2(\mathbb{R}^d)$ let us set

$$\Pi_D v := \sum_{\substack{[m,n] \in \mathbb{Z}^{2d} \\ |x^m|^2 + |\xi^n|^2 < D^2}} (v, \Psi_{k,m,n}^*) \Psi_{k,m,n}.$$



Then, we have

$$\|v - \Pi_D v\|_{\dot{H}^p(\mathbb{R}^d)} \leq C D^{-r} \|v\|_{\dot{H}^{p+r}(\mathbb{R}^d)}$$

for all $v \in \dot{H}^{p+r}(\mathbb{R}^d)$ and $0 \leq p \leq r$.



Chaumont-Frelet and Ingremau, 2022.

Convergence for general right-hand sides
Application to the Helmholtz equation

By combining standard arguments and decay in the PML, we can show that

$$\|u\|_{\check{H}_k^2(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}.$$

By combining standard arguments and decay in the PML, we can show that

$$\|u\|_{\check{H}_k^2(\mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}.$$

An easy consequence is

$$\|u - \Pi_D u\|_{\check{H}_k^1(\mathbb{R}^d)} \leq C D^{-1} \|f\|_{L^2(\mathbb{R}^d)} \quad \forall k \geq 1.$$

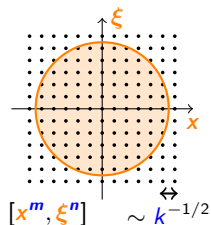
Convergence for general right-hand sides

Let

$$W_\Lambda := \text{Vect} \{ \Psi_{k,m,n}; \quad [m, n] \in \Lambda \}$$

where

$$\Lambda := \left\{ [m, n] \in \mathbb{Z}^{2d}; \quad |x^m|^2 + |\xi^n|^2 \leq D^2 \right\}.$$



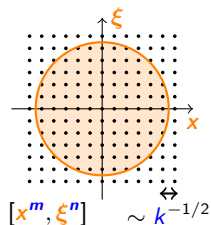
Convergence for general right-hand sides

Let

$$W_\Lambda := \text{Vect} \{ \Psi_{k,m,n}; \quad [m, n] \in \Lambda \}$$

where

$$\Lambda := \left\{ [m, n] \in \mathbb{Z}^{2d}; \quad |x^m|^2 + |\xi^n|^2 \leq D^2 \right\}.$$



Then, we have

$$\dim W_\Lambda \sim D^{2d} k^d, \quad \text{and} \quad \min_{w \in W_\Lambda} \|u - w\|_{\dot{H}_k^1(\mathbb{R}^d)} \leq CD^{-1} \|f\|_{L^2(\mathbb{R}^d)}.$$

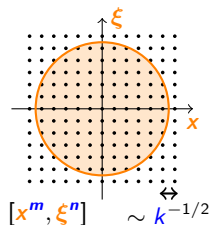
Convergence for general right-hand sides

Let

$$W_\Lambda := \text{Vect} \{ \Psi_{k,m,n}; \quad [m, n] \in \Lambda \}$$

where

$$\Lambda := \left\{ [m, n] \in \mathbb{Z}^{2d}; \quad |x^m|^2 + |\xi^n|^2 \leq D^2 \right\}.$$



Then, we have

$$\dim W_\Lambda \sim D^{2d} k^d, \quad \text{and} \quad \min_{w \in W_\Lambda} \|u - w\|_{\check{H}_k^1(\mathbb{R}^d)} \leq CD^{-1} \|f\|_{L^2(\mathbb{R}^d)}.$$

For general $L^2(\mathbb{R}^d)$ rhs, same number of dofs than standard FEM: k^d .

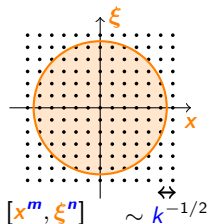
Convergence for general right-hand sides

Let

$$W_\Lambda := \text{Vect} \{ \Psi_{k,m,n}; \quad [m, n] \in \Lambda \}$$

where

$$\Lambda := \left\{ [m, n] \in \mathbb{Z}^{2d}; \quad |x^m|^2 + |\xi^n|^2 \leq D^2 \right\}.$$



Then, we have

$$\dim W_\Lambda \sim D^{2d} k^d, \quad \text{and} \quad \min_{w \in W_\Lambda} \|u - w\|_{\check{H}_k^1(\mathbb{R}^d)} \leq CD^{-1} \|f\|_{L^2(\mathbb{R}^d)}.$$

For general $L^2(\mathbb{R}^d)$ rhs, same number of dofs than standard FEM: k^d .

This is expected: for a general $f \in L^2(\mathbb{R}^d)$, u is a general $H^2(\mathbb{R}^d)$ function.

Convergence for scattering problems

Convergence for scattering problems

Semi-classical concepts: symbols and micro-localization

Assuming for simplicity that \mathbf{A} is constant, we have:

$$\left\{ -k^2 \mu u + \nabla \cdot (\mathbf{A} \nabla u) \right\}(\mathbf{x}) = C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ik(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\xi}} p(\mathbf{x}, \boldsymbol{\xi}) u(\mathbf{y}) d\mathbf{y} d\boldsymbol{\xi},$$

where $p : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ with

$$p(\mathbf{x}, \boldsymbol{\xi}) := \mathbf{A}(\mathbf{x}) \boldsymbol{\xi} \cdot \boldsymbol{\xi} - \mu(\mathbf{x})$$

is the *symbol* associated with the Helmholtz differential operator.

Assuming for simplicity that \mathbf{A} is constant, we have:

$$\left\{ -k^2 \mu u + \nabla \cdot (\mathbf{A} \nabla u) \right\}(\mathbf{x}) = C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ik(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\xi}} p(\mathbf{x}, \boldsymbol{\xi}) u(\mathbf{y}) d\mathbf{y} d\boldsymbol{\xi},$$

where $p : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ with

$$p(\mathbf{x}, \boldsymbol{\xi}) := \mathbf{A}(\mathbf{x}) \boldsymbol{\xi} \cdot \boldsymbol{\xi} - \mu(\mathbf{x})$$

is the *symbol* associated with the Helmholtz differential operator.

We can see the action of operators through a function defined over \mathbb{R}^{2d} .

Assuming for simplicity that \mathbf{A} is constant, we have:

$$\left\{ -k^2 \mu u + \nabla \cdot (\mathbf{A} \nabla u) \right\}(\mathbf{x}) = C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ik(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\xi}} p(\mathbf{x}, \boldsymbol{\xi}) u(\mathbf{y}) d\mathbf{y} d\boldsymbol{\xi},$$

where $p : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ with

$$p(\mathbf{x}, \boldsymbol{\xi}) := \mathbf{A}(\mathbf{x}) \boldsymbol{\xi} \cdot \boldsymbol{\xi} - \mu(\mathbf{x})$$

is the *symbol* associated with the Helmholtz differential operator.

We can see the action of operators through a function defined over \mathbb{R}^{2d} .

We can define the micro-local support of a function $v \in L^2(\mathbb{R}^d)$ by duality

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ik(\mathbf{x}-\mathbf{y}) \cdot \boldsymbol{\xi}} a(\mathbf{x}, \boldsymbol{\xi}) v(\mathbf{y}) d\mathbf{y} d\boldsymbol{\xi}$$

as a subset of \mathbb{R}^{2d} .

Convergence for scattering problems
Micro-localization of high-frequency solutions

We can show that

$$\begin{aligned}
 (a^W u, u)_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ik(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y)\bar{u}(x) dy d\xi dx \\
 &\xrightarrow[k \rightarrow \infty]{} \\
 &\int_{\mathbb{R}^{2d}} a(x, \xi) d\lambda(x, \xi)
 \end{aligned}$$

for all “smooth” symbols a , where λ is a measure over \mathbb{R}^{2d} .

We can show that

$$\begin{aligned}
 (a^W u, u)_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ik(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y)\bar{u}(x) dy d\xi dx \\
 &\xrightarrow[k \rightarrow \infty]{} \\
 &\int_{\mathbb{R}^{2d}} a(x, \xi) d\lambda(x, \xi)
 \end{aligned}$$

for all “smooth” symbols a , where λ is a measure over \mathbb{R}^{2d} .

It is called the “defect measure”, and we have

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\} \cup \text{supp } f.$$

We can show that

$$(a^W u, u)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ik(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y)\bar{u}(x) dy d\xi dx$$

$$\xrightarrow[k \rightarrow \infty]$$

$$\int_{\mathbb{R}^{2d}} a(x, \xi) d\lambda(x, \xi)$$

for all “smooth” symbols a , where λ is a measure over \mathbb{R}^{2d} .

It is called the “defect measure”, and we have

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\} \cup \text{supp } f.$$

More generally, u is micro-localized close to $\text{supp } \lambda$ for large frequencies.

We can show that

$$(a^W u, u)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ik(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right) u(y)\bar{u}(x) dy d\xi dx$$

$$\xrightarrow[k \rightarrow \infty]{} \int_{\mathbb{R}^{2d}} a(x, \xi) d\lambda(x, \xi)$$

for all “smooth” symbols a , where λ is a measure over \mathbb{R}^{2d} .

It is called the “defect measure”, and we have

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\} \cup \text{supp } f.$$

More generally, u is micro-localized close to $\text{supp } \lambda$ for large frequencies.



Zworski, 2012

In an appropriate sense, we have

$$u \xrightarrow[k \rightarrow \infty]{} \lambda,$$

where λ is a measure on \mathbb{R}^{2d} such that

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\} \cup \text{supp } f.$$

In an appropriate sense, we have

$$u \xrightarrow[k \rightarrow \infty]{} \lambda,$$

where λ is a measure on \mathbb{R}^{2d} such that

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\} \cup \text{supp } f.$$

In general, this information is not that useful, because $\text{supp } f$ can be anything.

In an appropriate sense, we have

$$u \xrightarrow[k \rightarrow \infty]{} \lambda,$$

where λ is a measure on \mathbb{R}^{2d} such that

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\} \cup \text{supp } f.$$

In general, this information is not that useful, because $\text{supp } f$ can be anything.

For scattering problems, however, $\text{supp } f \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\}$, and

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\}.$$

In an appropriate sense, we have

$$u \xrightarrow[k \rightarrow \infty]{} \lambda,$$

where λ is a measure on \mathbb{R}^{2d} such that

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\} \cup \text{supp } f.$$

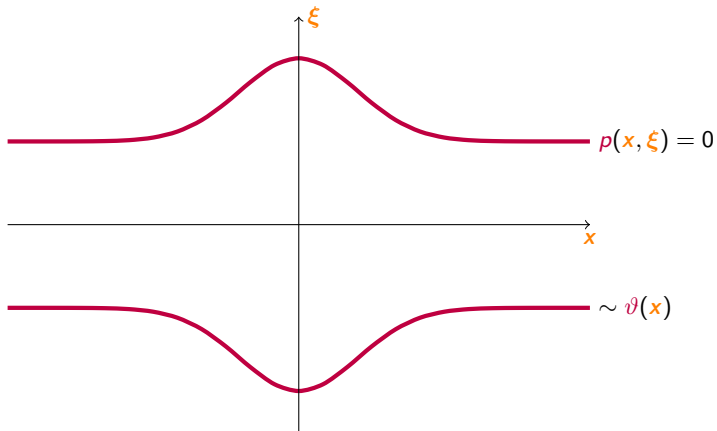
In general, this information is not that useful, because $\text{supp } f$ can be anything.

For scattering problems, however, $\text{supp } f \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\}$, and

$$\text{supp } \lambda \subset \{[x, \xi] \in \mathbb{R}^{2d}; p(x, \xi) = 0\}.$$

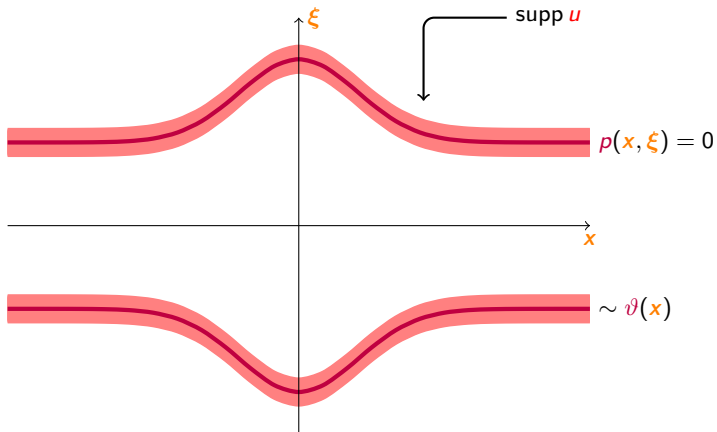
In the high-frequency regime, we know “where” to look for u !

A small picture is often better than a long explanation



$$p(x, \xi) = 0 \iff \mu(x) - \alpha(x)\xi^2 = 0 \iff \xi^2 = v(x)^2$$

A small picture is often better than a long explanation



$$p(x, \xi) = 0 \iff \mu(x) - \alpha(x)\xi^2 = 0 \iff \xi^2 = \vartheta(x)^2$$

Convergence for scattering problems

Micro-localization properties of Gaussian states

Consider the Gaussian distribution

$$\Psi(\mathbf{x}) := \pi^{-d/4} e^{-\frac{1}{2}|\mathbf{x}|^2}.$$

It is localized around $\mathbf{0}$ in position.

Consider the Gaussian distribution

$$\Psi(\mathbf{x}) := \pi^{-d/4} e^{-\frac{1}{2}|\mathbf{x}|^2}.$$

It is localized around $\mathbf{0}$ in position. But we also have

$$\widehat{\Psi}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} \Psi(\mathbf{x}) e^{i\mathbf{x} \cdot \boldsymbol{\xi}} d\mathbf{x} = (2\pi)^{d/2} \pi^{-d/4} e^{-\frac{1}{2}|\boldsymbol{\xi}|^2},$$

so that it is also localized around $\mathbf{0}$ in momentum.

Consider the Gaussian distribution

$$\Psi(\mathbf{x}) := \pi^{-d/4} e^{-\frac{1}{2}|\mathbf{x}|^2}.$$

It is localized around $\mathbf{0}$ in position. But we also have

$$\widehat{\Psi}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} \Psi(\mathbf{x}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} = (2\pi)^{d/2} \pi^{-d/4} e^{-\frac{1}{2}|\boldsymbol{\xi}|^2},$$

so that it is also localized around $\mathbf{0}$ in momentum.

Actually, it has the “best” micro-localization property, as it saturates Heisenberg’s uncertainty principle:

$$\|\mathbf{x}\Psi\|_{L^2(\mathbb{R}^d)} \|\boldsymbol{\xi}\widehat{\Psi}\|_{L^2(\mathbb{R}^d)} \underset{(\geq)}{=} \frac{1}{2} \|\Psi\|_{L^2(\mathbb{R}^d)}^2 \|\widehat{\Psi}\|_{L^2(\mathbb{R}^d)}^2.$$

Recall that our discretization space is based on the k -scaled version

$$\psi_{k,m,n}(x) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|x-x^m|^2} e^{ikx \cdot \xi^n},$$

where $[x^m, \xi^n] := \sqrt{\pi/k}[m, n]$.

Recall that our discretization space is based on the k -scaled version

$$\psi_{k,m,n}(x) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|x-x^m|^2} e^{ikx \cdot \xi^n},$$

where $[x^m, \xi^n] := \sqrt{\pi/k}[m, n]$.

The scaling is adapted to the scaled Fourier transform employed in symbols.

Recall that our discretization space is based on the k -scaled version

$$\psi_{k,m,n}(x) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|x-x^m|^2} e^{ikx \cdot \xi^n},$$

where $[x^m, \xi^n] := \sqrt{\pi/k}[m, n]$.

The scaling is adapted to the scaled Fourier transform employed in symbols.

In fact, we have

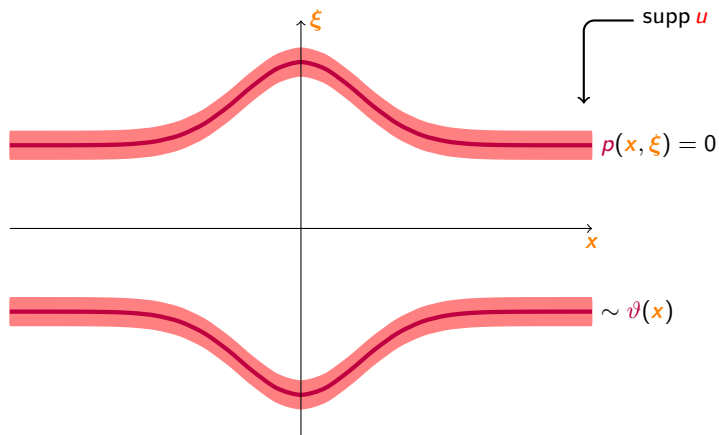
$$\psi_{k,x_0,\xi_0}(x) := \left(\frac{k}{\pi}\right)^{d/4} e^{-\frac{k}{2}|x-x_0|^2} e^{ikx \cdot \xi_0} \xrightarrow{k \rightarrow \infty} \delta_{[x_0, \xi_0]},$$

in the sense that

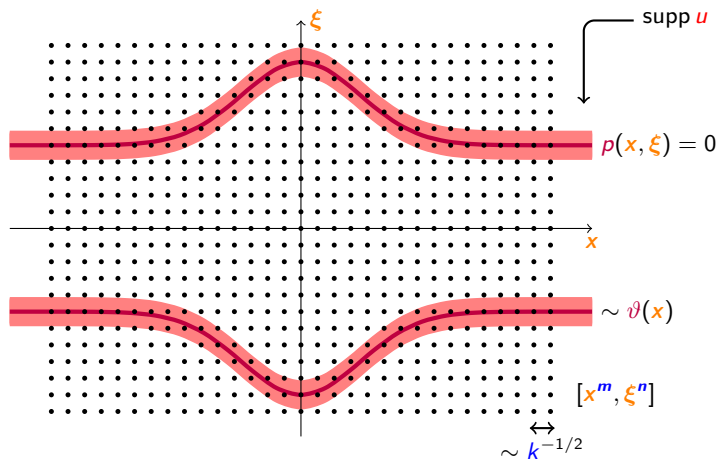
$$(a^W \psi_{k,x_0,\xi_0}, \psi_{k,x_0,\xi_0})_{L^2(\mathbb{R}^d)} \xrightarrow{k \rightarrow \infty} a(x_0, \xi_0)$$

for all “smooth” symbols a .

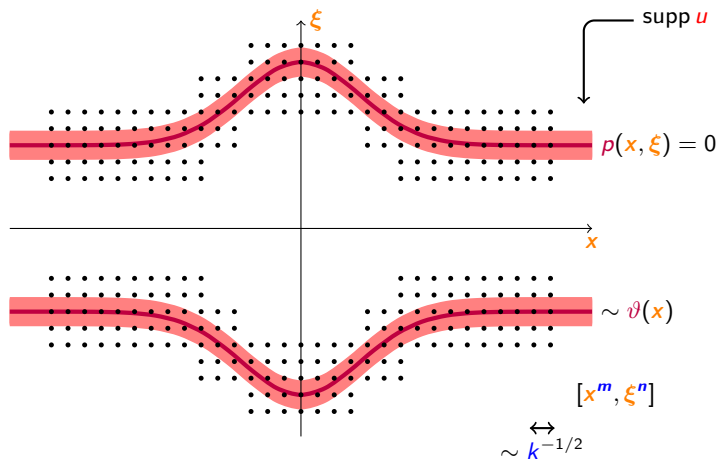
The same thing, graphically...



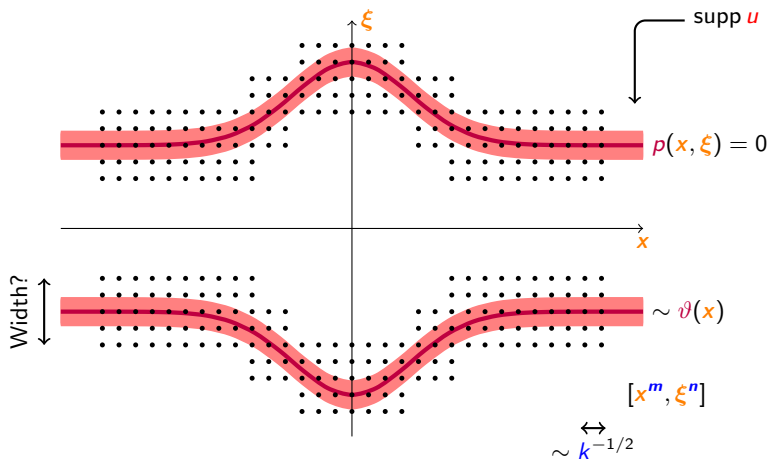
The same thing, graphically...



The same thing, graphically...



The same thing, graphically...



Convergence for scattering problems
Approximation of high-frequency solutions

Fix $\varepsilon > 0$, and consider the set of indices

$$\Lambda := \left\{ [m, n] \in \mathbb{Z}^{2d} \mid |p(x^m, \xi^n)| \leq k^{-1/2+\varepsilon} \right\},$$

and the associated space

$$W_\Lambda := \text{Vect} \{ \Psi_{k,m,n}; \quad [m, n] \in \Lambda \}.$$

The approximability result

Fix $\varepsilon > 0$, and consider the set of indices

$$\Lambda := \left\{ [m, n] \in \mathbb{Z}^{2d} \mid |p(x^m, \xi^n)| \leq k^{-1/2+\varepsilon} \right\},$$

and the associated space

$$W_\Lambda := \text{Vect} \{ \Psi_{k,m,n}; \quad [m, n] \in \Lambda \}.$$

Then, we have

$$\dim W_\Lambda \sim k^{d-1/2+\varepsilon},$$

and

$$\min_{w \in W_\Lambda} \|u - w\|_{H_k^1(\Omega)} \leq C_{\varepsilon,m} k^{-m} \quad \forall m \in \mathbb{N}.$$



T. Chaumont-Frelet, V. Dolean and M. Ingremeau, 2022

Convergence for scattering problems

Takeaways

We consider an hypersurface in \mathbb{R}^{2d} . Specifically, the $2d - 1$ manifold

$$\{[\mathbf{x}, \boldsymbol{\xi}] \in \mathbb{R}^{2d}; p(\mathbf{x}, \boldsymbol{\xi}) = 0\}.$$

We consider an hypersurface in \mathbb{R}^{2d} . Specifically, the $2d - 1$ manifold

$$\{[\mathbf{x}, \boldsymbol{\xi}] \in \mathbb{R}^{2d}; p(\mathbf{x}, \boldsymbol{\xi}) = 0\}.$$

Since \mathbb{R}^{2d} is paved with a lattice spaced in $k^{-1/2}$, we have about

$$(k^{1/2})^{2d-1} = k^{d-1/2}$$

dofs in our discretization space.

We consider an hypersurface in \mathbb{R}^{2d} . Specifically, the $2d - 1$ manifold

$$\{[\mathbf{x}, \boldsymbol{\xi}] \in \mathbb{R}^{2d}; p(\mathbf{x}, \boldsymbol{\xi}) = 0\}.$$

Since \mathbb{R}^{2d} is paved with a lattice spaced in $k^{-1/2}$, we have about

$$(k^{1/2})^{2d-1} = k^{d-1/2}$$

dofs in our discretization space.

Micro-localization properties of high-frequency solutions and of Gaussian states are the key to allow this “sparse” selection of shape functions.

Numerical examples

Numerical examples

A least-squares formulation

A least-squares formulation

Define $P_k : H^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ as $P_k v := -k^2 \mu v - \nabla \cdot (\mathbf{A} \nabla v)$.

A least-squares formulation

Define $P_k : H^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ as $P_k v := -k^2 \mu v - \nabla \cdot (A \nabla v)$.

The least-squares formulation is to find $u_\Lambda \in W_\Lambda$ such that

$$(P_k u_\Lambda, P_k v)_{L^2(\mathbb{R}^d)} = (f, P_k v)_{L^2(\mathbb{R}^d)}$$

for all $v \in W_\Lambda$.

A least-squares formulation

Define $P_k : H^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ as $P_k v := -k^2 \mu v - \nabla \cdot (A \nabla v)$.

The least-squares formulation is to find $u_\Lambda \in W_\Lambda$ such that

$$(P_k u_\Lambda, P_k v)_{L^2(\mathbb{R}^d)} = (f, P_k v)_{L^2(\mathbb{R}^d)}$$

for all $v \in W_\Lambda$.

We can show that there exists a unique discrete solution, and that

$$\|u - u_\Lambda\|_{\check{H}_k^1(\mathbb{R}^d)} \leq C_{\varepsilon, m} k^{-m} \quad \forall m \in \mathbb{N}.$$

A least-squares formulation

Define $P_k : H^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ as $P_k v := -k^2 \mu v - \nabla \cdot (A \nabla v)$.

The least-squares formulation is to find $u_\Lambda \in W_\Lambda$ such that

$$(P_k u_\Lambda, P_k v)_{L^2(\mathbb{R}^d)} = (f, P_k v)_{L^2(\mathbb{R}^d)}$$

for all $v \in W_\Lambda$.

We can show that there exists a unique discrete solution, and that

$$\|u - u_\Lambda\|_{\dot{H}_k^1(\mathbb{R}^d)} \leq C_{\varepsilon, m} k^{-m} \quad \forall m \in \mathbb{N}.$$

In addition, we can also show that

$$|(P_k \Psi_{k, m, n}, P_k \Psi_{k, m', n'})_{L^2(\mathbb{R}^d)}| \leq C_m (|m - m'| + |n - n'|)^{-m} \quad \forall m \in \mathbb{N},$$

so that the matrix is essentially sparse.

We solve the equation

$$-k^2 \mu \nu u - (\nu^{-1} u')' = f,$$

where ν is the PML scaling function.

We solve the equation

$$-k^2 \mu \nu u - (\nu^{-1} u')' = f,$$

where ν is the PML scaling function.

We only consider scattering problems, hence

$$f(x) := \phi e^{ikx},$$

with a smooth ϕ supported where $\mu = 1$.

We solve the equation

$$-k^2 \mu \nu u - (\nu^{-1} u')' = f,$$

where ν is the PML scaling function.

We only consider scattering problems, hence

$$f(x) := \phi e^{ikx},$$

with a smooth ϕ supported where $\mu = 1$.

In the first example, $\mu \equiv 1$.

In the second $\mu = 2$ on $(-1/2, 1/2)$ and $\mu = 1$ outside $(-1, 1)$.

We solve the equation

$$-k^2 \mu \nu u - (\nu^{-1} u')' = f,$$

where ν is the PML scaling function.

We only consider scattering problems, hence

$$f(x) := \phi e^{ikx},$$

with a smooth ϕ supported where $\mu = 1$.

In the first example, $\mu \equiv 1$.

In the second $\mu = 2$ on $(-1/2, 1/2)$ and $\mu = 1$ outside $(-1, 1)$.

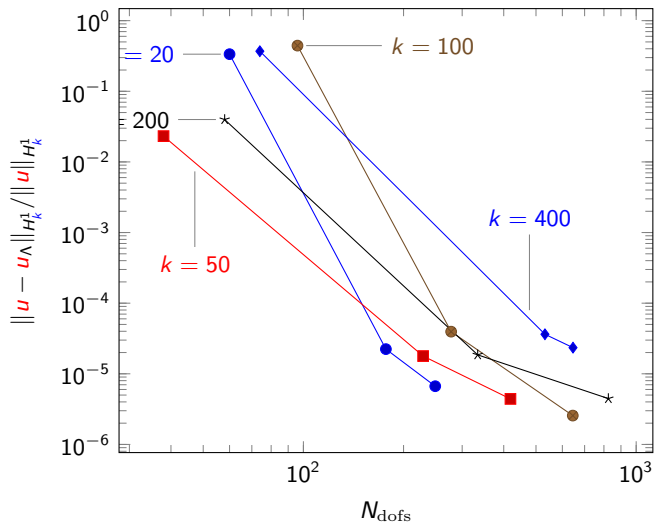
We solve the problem for different values of k and δ where

$$\Lambda := \left\{ [x^m, \xi^n] \in \mathbb{R}^2; |p(x^m, \xi^n)| \leq \delta \right\}.$$

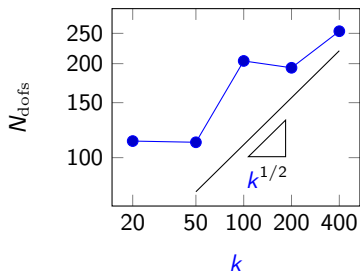
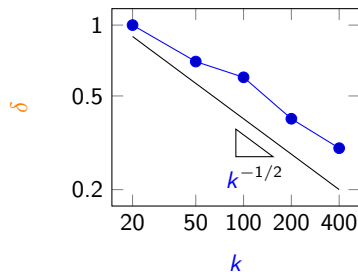
Numerical examples

Homogeneous test case

Homogeneous experiment: convergence



Homogeneous experiment: cost for a fixed accuracy

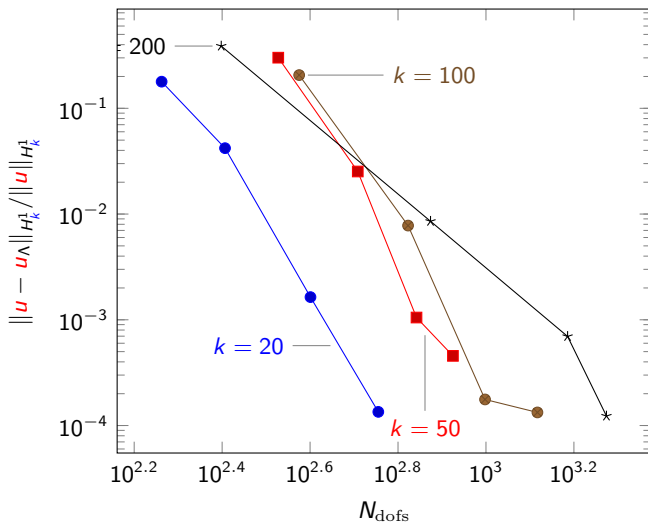


Computational cost for a 10^{-5} relative error.

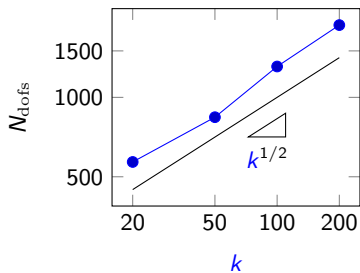
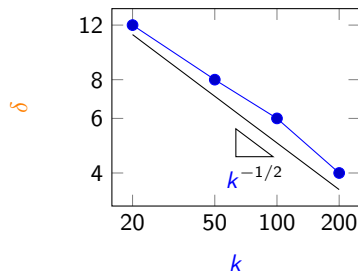
Numerical examples

Heterogeneous test case

Heterogeneous experiment: convergence



Heterogeneous experiment: cost for a fixed accuracy



Computational cost for a 10^{-5} relative error

Concluding remarks

Concluding remarks

Open questions and challenges

We currently only propose approximability estimates,
which we only know how to apply to least-squares formulations.

We currently only propose approximability estimates, which we only know how to apply to least-squares formulations. How can we cover the convergence of Galerkin discretizations?

We currently only propose approximability estimates, which we only know how to apply to least-squares formulations. How can we cover the convergence of Galerkin discretizations?

I lied: the matrices are *not* sparse!

But... The entries decay super-algebraically away from the diagonal.

We currently only propose approximability estimates, which we only know how to apply to least-squares formulations. How can we cover the convergence of Galerkin discretizations?

I lied: the matrices are *not* sparse!

But... The entries decay super-algebraically away from the diagonal. How should we properly sparsify and/or compress the matrices?

We currently only propose approximability estimates, which we only know how to apply to least-squares formulations. How can we cover the convergence of Galerkin discretizations?

I lied: the matrices are *not* sparse!

But... The entries decay super-algebraically away from the diagonal. How should we properly sparsify and/or compress the matrices?

Actually, even the computation of the entries is challenging.

We currently only propose approximability estimates, which we only know how to apply to least-squares formulations. How can we cover the convergence of Galerkin discretizations?

I lied: the matrices are *not* sparse!
But... The entries decay super-algebraically away from the diagonal. How should we properly sparsify and/or compress the matrices?

Actually, even the computation of the entries is challenging. How can we derive suited quadrature rules for Gabor wavelets?

We currently only propose approximability estimates, which we only know how to apply to least-squares formulations. How can we cover the convergence of Galerkin discretizations?

I lied: the matrices are *not* sparse!
But... The entries decay super-algebraically away from the diagonal. How should we properly sparsify and/or compress the matrices?

Actually, even the computation of the entries is challenging. How can we derive suited quadrature rules for Gabor wavelets?

The frame properties of Gabor wavelets are established in the whole \mathbb{R}^d . This enables us to treat (globally) smooth coefficients (including PML).

We currently only propose approximability estimates, which we only know how to apply to least-squares formulations. How can we cover the convergence of Galerkin discretizations?

I lied: the matrices are *not* sparse!
But... The entries decay super-algebraically away from the diagonal. How should we properly sparsify and/or compress the matrices?

Actually, even the computation of the entries is challenging. How can we derive suited quadrature rules for Gabor wavelets?

The frame properties of Gabor wavelets are established in the whole \mathbb{R}^d . This enables us to treat (globally) smooth coefficients (including PML). How can we incorporate boundary and/or interface conditions?

Concluding remarks

Main takeaways

Gabor wavelets form a frame over $L^2(\mathbb{R}^d)$.
They actually correspond to a $k^{-1/2}$ lattice of \mathbb{R}^{2d} .

Gabor wavelets form a frame over $L^2(\mathbb{R}^d)$.
They actually correspond to a $k^{-1/2}$ lattice of \mathbb{R}^{2d} .

For generic right-hand sides, they require k^d dofs.

Gabor wavelets form a frame over $L^2(\mathbb{R}^d)$.
They actually correspond to a $k^{-1/2}$ lattice of \mathbb{R}^{2d} .

For generic right-hand sides, they require k^d dofs.

A key aspect of Gaussian states is that they are localized in phase-space.
For scattering problems, we only need to “pave” the hypersurface $p(\mathbf{x}, \boldsymbol{\xi}) = 0$.

Gabor wavelets form a frame over $L^2(\mathbb{R}^d)$.
They actually correspond to a $k^{-1/2}$ lattice of \mathbb{R}^{2d} .

For generic right-hand sides, they require k^d dofs.

A key aspect of Gaussian states is that they are localized in phase-space.
For scattering problems, we only need to “pave” the hypersurface $p(\mathbf{x}, \boldsymbol{\xi}) = 0$.

For scattering problems, we only need $k^{d-1/2}$ dofs with sparse matrices.