HAL open science

A multiscale hybrid-mixed method for Helmholtz problems in periodic structures

Théophile Chaumont-Frelet, zakaria kassali, Stéphane Lanteri

To cite this version:

Théophile Chaumont-Frelet, zakaria kassali, Stéphane Lanteri. A multiscale hybrid-mixed method for Helmholtz problems in periodic structures. ECCOMAS 2022, Jun 2022, Oslo, Norway. hal-03940660

HAL Id: hal-03940660

https://inria.hal.science/hal-03940660

Submitted on 16 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A multiscale hybrid-mixed method for Helmholtz problems in periodic structures

T. Chaumont-Frelet*, Z. Kassali* and S. Lanteri*

ECCOMAS, Structure preserving and adaptive polytopal methods, 6 June 2022

*Inria Université Côte d'Azur, Atlantis project-team

Is that good enough?

I did my best to find a method that:
handles polytopal meshes and preserves important invariants.

Is that good enough?

I did my best to find a method that:
handles polytopal meshes and preserves important invariants.

I failed! I am goind to talk about a method that:
handles on general meshes, but only preserves invariants on Cartesian grids!

Is that good enough?

I did my best to find a method that:
handles polytopal meshes and preserves important invariants.

I failed! I am goind to talk about a method that:
handles on general meshes, but only preserves invariants on Cartesian grids!

I hope this is good enough! :-)

More seriously...

We will be concerned with Helmholtz problems in periodic structures.

More seriously...

We will be concerned with Helmholtz problems in periodic structures.

A challenge with periodicity is the presence of specific "quasi-modes" which impact the problems stability at the continuous and discrete levels.

More seriously...

We will be concerned with Helmholtz problems in periodic structures.

A challenge with periodicity is the presence of specific "quasi-modes" which impact the problems stability at the continuous and discrete levels.

Multiscale hybrid-mixed (MHM) methods can:
deal with highly heterogeneous media and exactly reproduce some solutions.

More seriously...

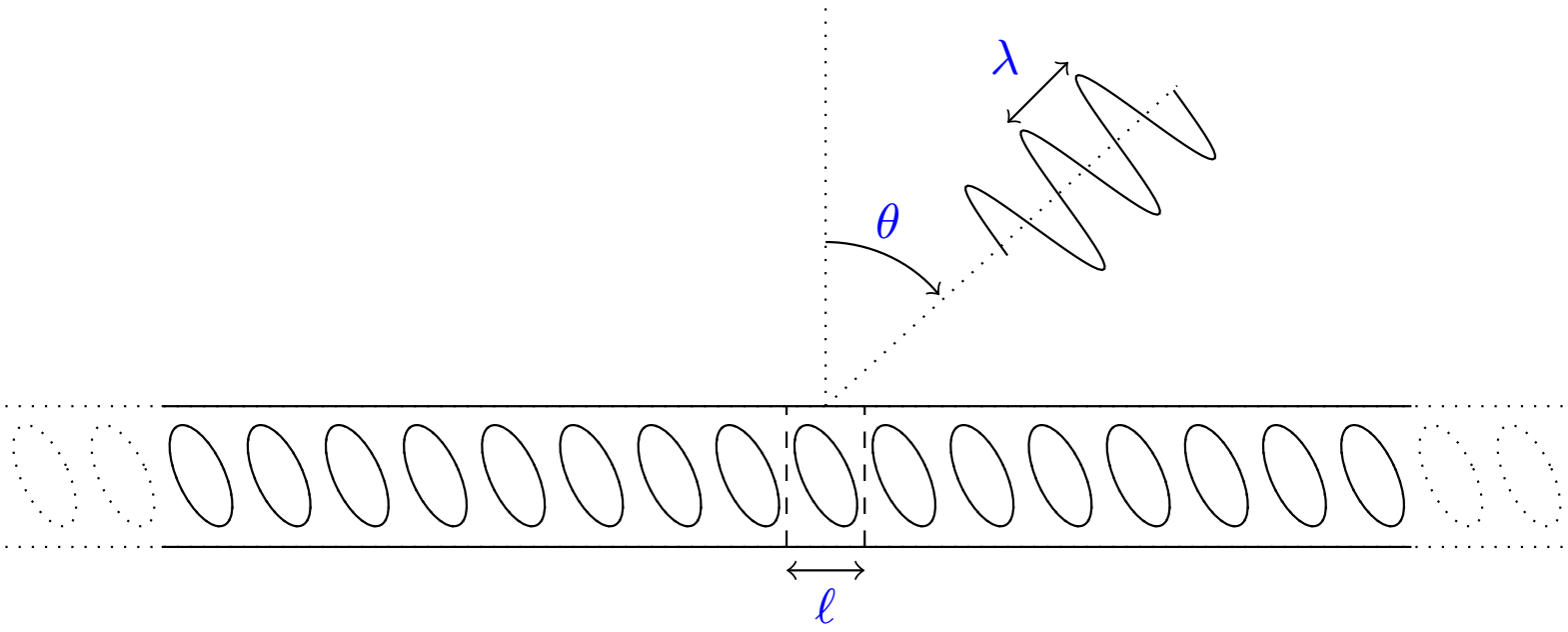
We will be concerned with Helmholtz problems in periodic structures.

A challenge with periodicity is the presence of specific "quasi-modes" which impact the problems stability at the continuous and discrete levels.

Multiscale hybrid-mixed (MHM) methods can:
deal with highly heterogeneous media and exactly reproduce some solutions.

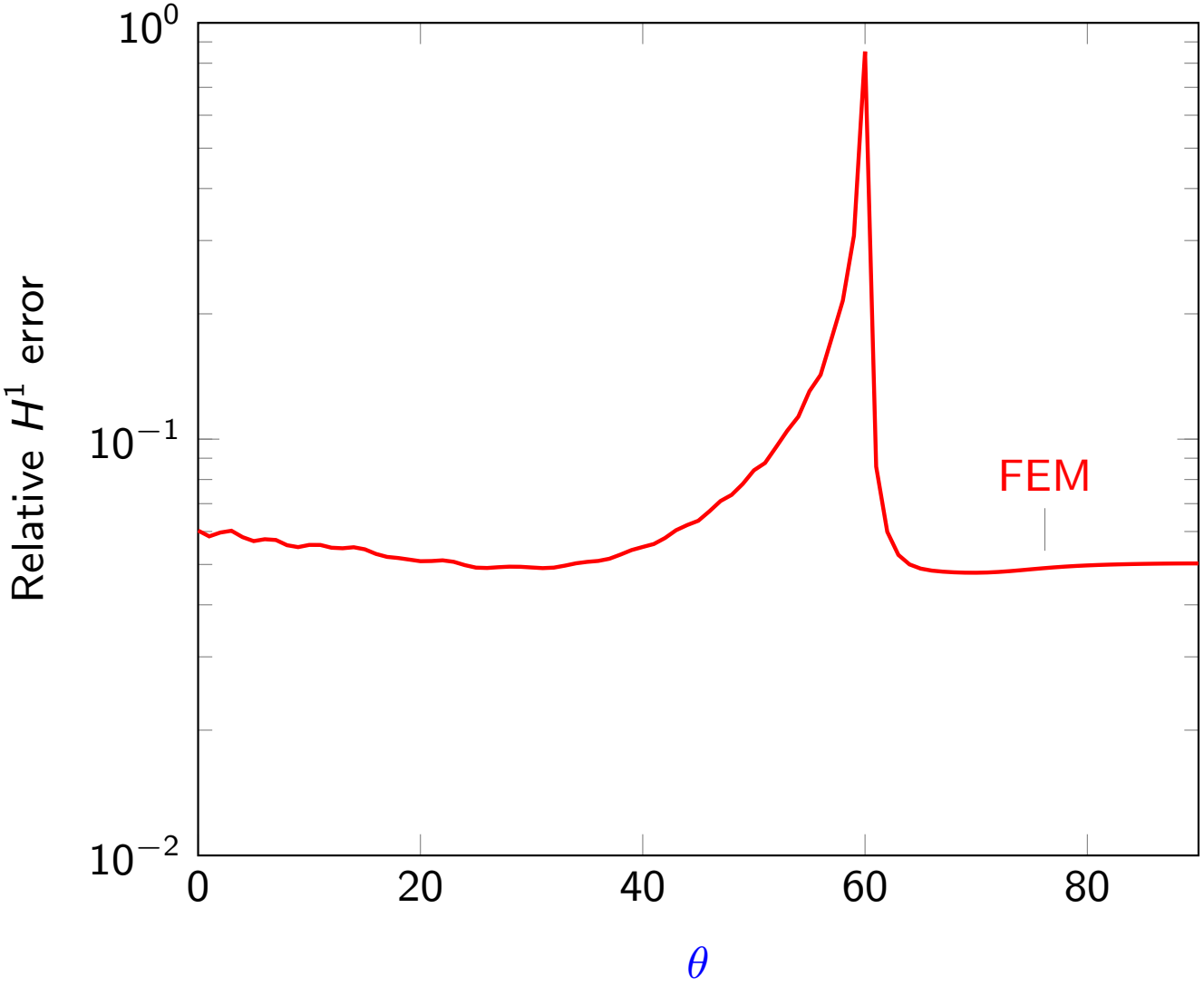
We will focus on the latter aspect, and see how it improves stability.

A small picture is better than a long explanation

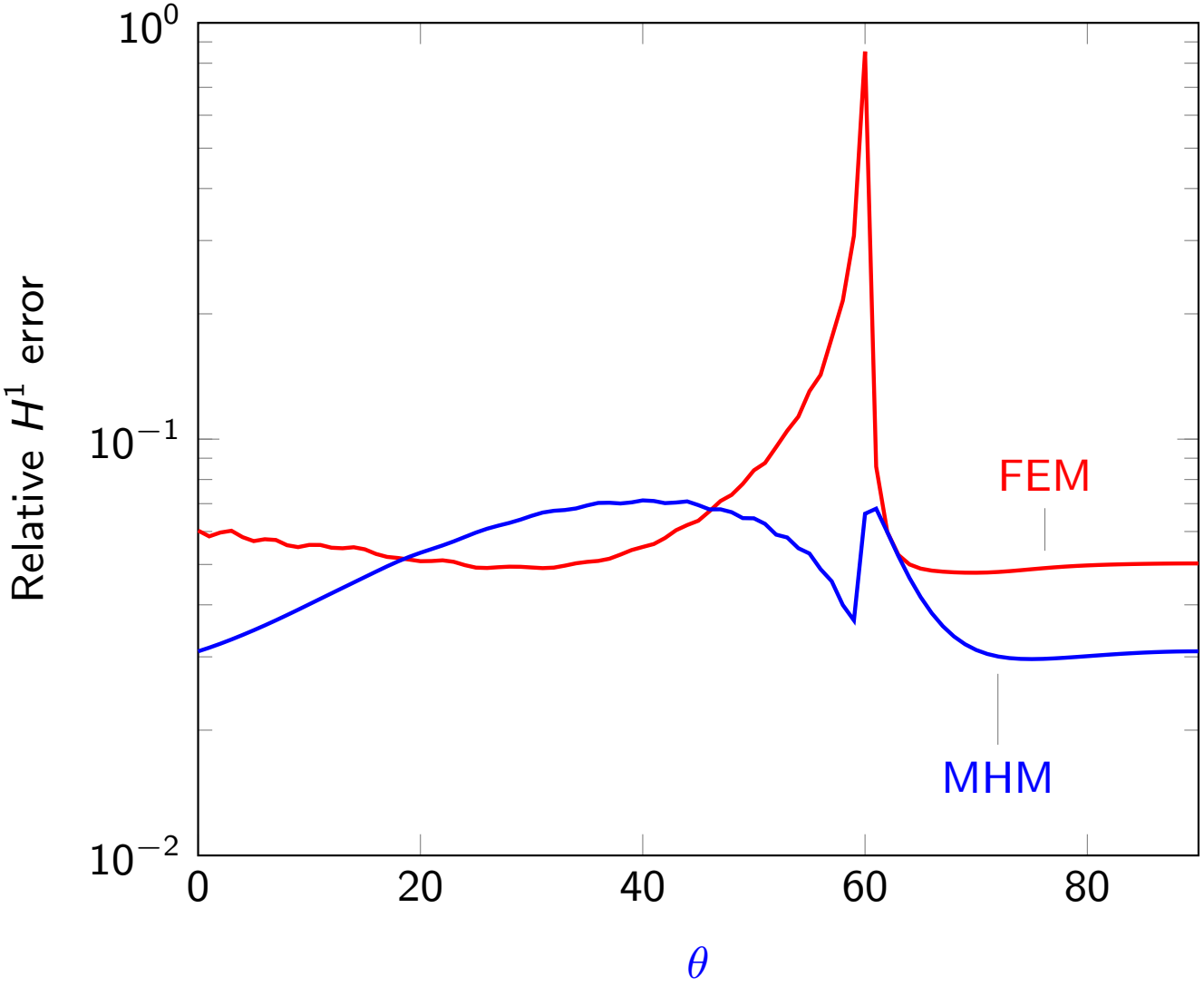


Particular combinations of ℓ, λ and θ generate "quasi-resonances".

An example



An example



1. The Helmholtz equation in periodic structures
   - Where does the boundary conditions come from?
   - Perfectly matched layer
   - A discussion on stability

2. The multiscale hybrid-mixed method
   - General presentation of the method
   - Well-posedness and convergence
   - Robustness on Cartesian grids

The Helmholtz equation in periodic structures

The Helmholtz equation in periodic structures

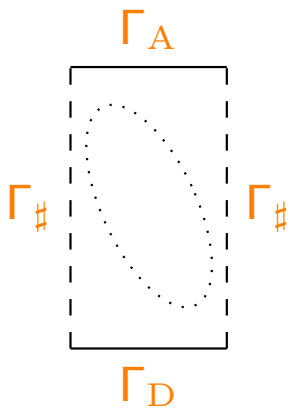
We set the problem on a single "periodic cell" Ω.

The Helmholtz equation in periodic structures

We set the problem on a single "periodic cell" Ω.

The problem is to find u such that

$$\left\{\begin{array}{rcll} -k^2 u - \Delta u & = & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \nabla u \cdot \boldsymbol{n} - \mathscr{R}_k u & = & g & \text{on } \Gamma_{\mathrm{A}}, \\ u^+ - e^{i\alpha} u^- & = & 0 & \text{on } \Gamma_\sharp, \end{array}\right.$$



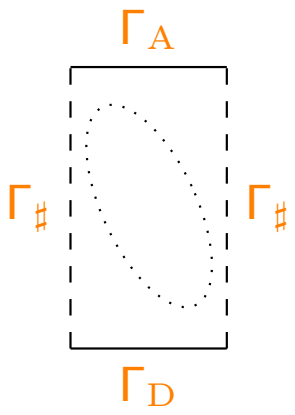
where $\alpha := k\ell \sin\theta$.

The Helmholtz equation in periodic structures

We set the problem on a single "periodic cell" Ω.

The problem is to find u such that

$$\begin{cases} -k^2 u - \Delta u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \nabla u \cdot \boldsymbol{n} - \mathscr{R}_k u &= g & \text{on } \Gamma_{\mathrm{A}}, \\ u^+ - e^{i\alpha} u^- &= 0 & \text{on } \Gamma_\sharp, \end{cases}$$



where $\alpha := k\ell \sin\theta$.

The DtN operator is given by

$$\mathscr{R}_k u := i \sum_{m \in \mathbb{Z}} k_m \widehat{u}_m e^{i(\alpha - 2\pi m) x_1 / \ell}.$$

The Helmholtz equation in periodic structures

Where does the boundary conditions come from?

Where does the boundary conditions come from?

We assume the structure is on top of a metal,
this is modeled by the Dirichlet condition at the bottom.

Where does the boundary conditions come from?

We assume the structure is on top of a metal,
this is modeled by the Dirichlet condition at the bottom.

The incident field is the plane wave

$$u_{\mathrm{inc}} := e^{ik\boldsymbol{d}\cdot\boldsymbol{x}}, \qquad \boldsymbol{d} := (\cos\theta, \sin\theta).$$

We can easily see that

$$u_{\mathrm{inc}}(\boldsymbol{x}_1 + \ell, \cdot) = e^{ik\ell\sin\theta} u_{\mathrm{inc}}(\boldsymbol{x}_1, \cdot) = e^{i\alpha} u_{\mathrm{inc}}(\boldsymbol{x}_1, \cdot)$$

Where does the boundary conditions come from?

We assume the structure is on top of a metal,
this is modeled by the Dirichlet condition at the bottom.

The incident field is the plane wave

$$u_{\mathrm{inc}} := e^{ik\boldsymbol{d}\cdot\boldsymbol{x}}, \qquad \boldsymbol{d} := (\cos\theta, \sin\theta).$$

We can easily see that

$$u_{\mathrm{inc}}(\boldsymbol{x}_1 + \ell, \cdot) = e^{ik\ell\sin\theta} u_{\mathrm{inc}}(\boldsymbol{x}_1, \cdot) = e^{i\alpha} u_{\mathrm{inc}}(\boldsymbol{x}_1, \cdot)$$

The DtN ensures that

$$\widehat{u}_m{}' = i\beta_m u_m \qquad \forall m \in \mathbb{Z}$$

with $\beta_m < 0$ or $i\beta_m < 0$. Each mode is thus outgoing or evanescent.

The Helmholtz equation in periodic structures

Perfectly matched layer

Perfectly matched layer

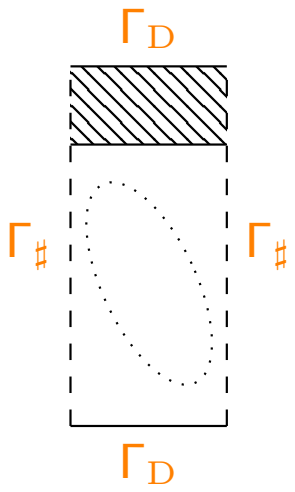
The DtN is not convenient for practical computations.
We transform the problem a perfetcly matched layers (PML).

Perfectly matched layer

The DtN is not convenient for practical computations.
We transform the problem a perfetcly matched layers (PML).

Find u such that

$$\begin{cases} -k^2 \mu u - \nabla \cdot (\boldsymbol{A} \nabla u) &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ u^+ - e^{i\alpha} u^- &= 0 & \text{on } \Gamma_\sharp. \end{cases}$$

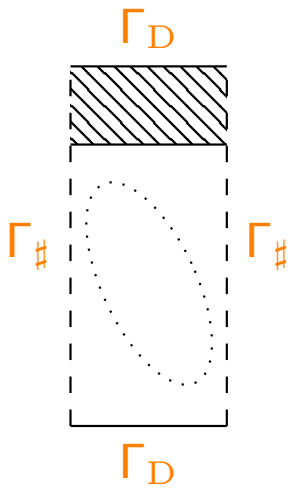


Perfectly matched layer

The DtN is not convenient for practical computations.
We transform the problem a perfetcly matched layers (PML).

Find u such that

$$\begin{cases} -k^2\mu u - \nabla\cdot(\boldsymbol{A}\nabla u) &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma_{\mathrm{D}}, \\ u^+ - e^{i\alpha}u^- &= 0 & \text{on } \Gamma_\sharp. \end{cases}$$



We do not touch the original domain where $\mu = 1$ and $\boldsymbol{A} = \boldsymbol{I}$.
In the extended layer, we set

$$\mu = \nu \qquad \boldsymbol{A} = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \qquad \nu = 1 + i.$$

Weak formulation

Dirichlet and quasi-periodic boundary conditions are imposed in the space

$$V := H^1_\sharp(\Omega) := \left\{ w \in H^1(\Omega) \,\middle|\, \begin{array}{rcll} w & = & 0 & \text{on } \Gamma_{\mathrm{D}} \\ w^+ - e^{i\alpha} w^- & = & 0 & \text{on } \Gamma_\sharp \end{array} \right\}.$$

Weak formulation

Dirichlet and quasi-periodic boundary conditions are imposed in the space

$$V := H^1_\sharp(\Omega) := \left\{ w \in H^1(\Omega) \,\middle|\, \begin{array}{rcll} w & = & 0 & \text{on } \Gamma_{\mathrm{D}} \\ w^+ - e^{i\alpha} w^- & = & 0 & \text{on } \Gamma_\sharp \end{array} \right\}.$$

There exists a unique $u \in V$ such that

$$b(u, v) = (f, v) \qquad \forall v \in V$$

where

$$b(u, v) := -k^2(\mu u, v) + (\boldsymbol{A}\boldsymbol{\nabla} u, \boldsymbol{\nabla} v).$$

The Helmholtz equation in periodic structures

A discussion on stability

Stability

Here, I want to discuss stability properties of this Helmholtz problem.

Stability

Here, I want to discuss stability properties of this Helmholtz problem.

Specifically, I am interested in the inf-sup constant

$$\inf_{\substack{w \in V \\ \|w\|_V = 1}} \sup_{\substack{v \in V \\ \|v\|_V = 1}} \operatorname{Re} b(w, v)$$

where

$$\|w\|_V^2 := k^2 \|w\|_\Omega^2 + \|\nabla w\|_\Omega^2.$$

Stability

Here, I want to discuss stability properties of this Helmholtz problem.

Specifically, I am interested in the inf-sup constant

$$\inf_{\substack{w \in V \\ \|w\|_V = 1}} \sup_{\substack{v \in V \\ \|v\|_V = 1}} \operatorname{Re} b(w, v)$$

where

$$\|w\|_V^2 := k^2 \|w\|_\Omega^2 + \|\nabla w\|_\Omega^2.$$

It is immediately related to the best constant $C_{\mathrm{st}}(k\ell)$ such that

$$\|w\|_V \leq C_{\mathrm{st}}(k\ell) \|\phi\|_\Omega,$$

where w solves the problem with rhs $\phi \in L^2(\Omega)$.

Stability in the "open" case

Stability in the "open" case

Consider the function $w(\boldsymbol{x}) := \chi(\boldsymbol{x}) e^{ik\boldsymbol{x}_1}$, where χ is a smooth cutoff function.

Stability in the "open" case

Consider the function $w(\boldsymbol{x}) := \chi(\boldsymbol{x})e^{ik\boldsymbol{x}_1}$, where χ is a smooth cutoff function.

Then,

$$-k^2 w - \Delta w = \left(2ik\frac{\partial \chi}{\partial \boldsymbol{x}_1} + \Delta\chi\right) e^{ik\boldsymbol{x}_1} =: \phi.$$

and since χ does not depend on k, we have

$$\|w\|_\Omega \simeq \frac{1}{k\ell}\left(\ell^2\|\phi\|_\Omega\right).$$

Stability in the "open" case

Consider the function $w(\boldsymbol{x}) := \chi(\boldsymbol{x})e^{ik\boldsymbol{x}_1}$, where χ is a smooth cutoff function.

Then,

$$-k^2 w - \Delta w = \left(2ik\frac{\partial \chi}{\partial \boldsymbol{x}_1} + \Delta\chi\right) e^{ik\boldsymbol{x}_1} =: \phi.$$

and since χ does not depend on k, we have

$$\|w\|_\Omega \simeq \frac{1}{k\ell}\left(\ell^2\|\phi\|_\Omega\right).$$

This tells us that

$$\inf_{\substack{w\in V\\ \|w\|_V=1}} \sup_{\substack{v\in V\\ \|v\|_V=1}} \operatorname{Re} a(w,v) \lesssim \frac{1}{k\ell}.$$

and actually, we can show that equility holds.

Stability with QPBC

Consider now $w(\boldsymbol{x}) := \chi(\boldsymbol{x}_2)e^{ik\boldsymbol{x}_1}$. Then,

$$-k^2 w - \Delta w = \Delta\chi e^{ik\boldsymbol{x}_1} =: \phi.$$

Stability with QPBC

Consider now $w(\boldsymbol{x}) := \chi(\boldsymbol{x}_2)e^{ik\boldsymbol{x}_1}$. Then,

$$-k^2 w - \Delta w = \Delta\chi e^{ik\boldsymbol{x}_1} =: \phi.$$

We only have

$$\|w\|_V \simeq \ell^2 \|\phi\|_\Omega$$

and

$$\inf_{\substack{w\in V \\ \|w\|_V=1}} \sup_{\substack{v\in V \\ \|v\|_V=1}} \operatorname{Re} a(w,v) \sim (k\ell)^{-2}$$

whenever $k\ell = \alpha + 2m\pi$.

Stability with QPBC

Consider now $w(\boldsymbol{x}) := \chi(\boldsymbol{x}_2)e^{ik\boldsymbol{x}_1}$. Then,

$$-k^2 w - \Delta w = \Delta\chi e^{ik\boldsymbol{x}_1} =: \phi.$$

We only have

$$\|w\|_V \simeq \ell^2 \|\phi\|_\Omega$$

and

$$\inf_{\substack{w\in V \\ \|w\|_V=1}} \sup_{\substack{v\in V \\ \|v\|_V=1}} \operatorname{Re} a(w,v) \sim (k\ell)^{-2}$$

whenever $k\ell = \alpha + 2m\pi$.

We loose one power of frequency in the stability constant at quasi-resonant frequencies. The stability issue of FEM are a consequence of this phenomenon.

The multiscale hybrid-mixed method

The multiscale hybrid-mixed method

General presentation of the method

Mesh and broken space

We consider a mesh $\mathcal{T}_h$ of Ω made of polygons K.
We assume that the faces of K match on both sides of $\Gamma_\sharp$.

Mesh and broken space

We consider a mesh $\mathcal{T}_h$ of Ω made of polygons K.
We assume that the faces of K match on both sides of $\Gamma_\sharp$.

We would like to seek a solution in the broken space

$$W := H^1(\mathcal{T}_h)$$

that misses compatibility and boundary conditions.

Mesh and broken space

We consider a mesh $\mathcal{T}_h$ of Ω made of polygons K.
We assume that the faces of K match on both sides of $\Gamma_\sharp$.

We would like to seek a solution in the broken space

$$W := H^1(\mathcal{T}_h)$$

that misses compatibility and boundary conditions.

We will enfore that $[\![u]\!]_F = 0$ by introducing a suitable Lagrange multiplier.
This is a subtle issue that I want to correctly explain.

Continuity via duality

Consider a function $v \in W := H^1(\mathcal{T}_h)$, when do we have $v \in V := H^1_\sharp(\Omega)$?

Continuity via duality

Consider a function $v \in W := H^1(\mathcal{T}_h)$, when do we have $v \in V := H^1_\sharp(\Omega)$?

When $v \in V$, we have

$$(\boldsymbol{\nabla} v, \boldsymbol{\sigma})_\Omega + (v, \boldsymbol{\nabla} \cdot \boldsymbol{\sigma})_\Omega = 0$$

for all $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\boldsymbol{\sigma}^- \cdot \boldsymbol{n} - e^{i\alpha} \boldsymbol{\sigma}^+ \cdot \boldsymbol{n} = 0$ on $\Gamma_\sharp$.

Continuity via duality

Consider a function $v \in W := H^1(\mathcal{T}_h)$, when do we have $v \in V := H^1_\sharp(\Omega)$?

When $v \in V$, we have

$$(\boldsymbol{\nabla} v, \boldsymbol{\sigma})_\Omega + (v, \boldsymbol{\nabla} \cdot \boldsymbol{\sigma})_\Omega = 0$$

for all $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\boldsymbol{\sigma}^- \cdot \boldsymbol{n} - e^{i\alpha} \boldsymbol{\sigma}^+ \cdot \boldsymbol{n} = 0$ on $\Gamma_\sharp$.

Actually, the converse holds true, and $v \in W$ belongs to V iff

$$\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\sigma} \cdot \boldsymbol{n}, v \rangle_{\partial K} = (\boldsymbol{\nabla} v, \boldsymbol{\sigma})_{\mathcal{T}_h} + (v, \boldsymbol{\nabla} \cdot \boldsymbol{\sigma})_{\mathcal{T}_h} = 0$$

for all $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\boldsymbol{\sigma}^- \cdot \boldsymbol{n} - e^{i\alpha} e^+ \cdot \boldsymbol{n} = 0$ on $\Gamma_\sharp$.

An easier explanation

If $\boldsymbol{\sigma}$ is smooth, we actually have

$$\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\sigma} \cdot \boldsymbol{n}, v \rangle_{\partial K} = \sum_{F \in \mathcal{F}_h} (\boldsymbol{\sigma} \cdot \boldsymbol{n}_F, [\![v]\!]_F)_F,$$

but we cannot break the duality pairing in general.

An easier explanation

If $\boldsymbol{\sigma}$ is smooth, we actually have

$$\sum_{K \in \mathcal{T}_h} \langle \boldsymbol{\sigma} \cdot \boldsymbol{n}, v \rangle_{\partial K} = \sum_{F \in \mathcal{F}_h} (\boldsymbol{\sigma} \cdot \boldsymbol{n}_F, [\![v]\!]_F)_F,$$

but we cannot break the duality pairing in general.

The issue is subtle, and related to the difference between $H^{1/2}(F)$ and $H^{1/2}_{00}(F)$.

Quotient space of traces

We actually do not need the "volume" information of $\boldsymbol{\sigma}$.
The only thing that really matters is $\mu = \boldsymbol{\sigma} \cdot \boldsymbol{n}$.

Quotient space of traces

We actually do not need the "volume" information of $\boldsymbol{\sigma}$.
The only thing that really matters is $\mu = \boldsymbol{\sigma} \cdot \boldsymbol{n}$.

We thus set

$$\Lambda := \left\{ \mu \in H^{-1/2}(\partial \mathcal{T}_h);\ \mu|_{\partial K} = \boldsymbol{\sigma} \cdot \boldsymbol{n}_K \quad \forall K \in \mathcal{T}_h. \right\},$$

and the pairing

$$\langle \mu, v \rangle = \sum_{K \in \mathcal{T}_h} \langle \mu, v \rangle_{\partial K} \quad \forall (\mu, v) \in \Lambda \times W.$$

Quotient space of traces

We actually do not need the "volume" information of $\boldsymbol{\sigma}$.
The only thing that really matters is $\mu = \boldsymbol{\sigma} \cdot \boldsymbol{n}$.

We thus set

$$\Lambda := \left\{ \mu \in H^{-1/2}(\partial \mathcal{T}_h);\ \mu|_{\partial K} = \boldsymbol{\sigma} \cdot \boldsymbol{n}_K \quad \forall K \in \mathcal{T}_h. \right\},$$

and the pairing

$$\langle \mu, v \rangle = \sum_{K \in \mathcal{T}_h} \langle \mu, v \rangle_{\partial K} \quad \forall (\mu, v) \in \Lambda \times W.$$

For a smooth $\mu \in \Lambda \cap L^2(\partial \mathcal{T}_h)$, we do have

$$\langle \mu, v \rangle = \sum_{F \in \mathcal{F}_h} (\mu, [\![v]\!])_F.$$

As a mental model, you can think of Λ as a single valued face-based space.

The primal-hybrid formulation

Imagine for a moment we were solving a coercive problem. We would have

$$u := \arg\min_{w \in V} \frac{1}{2} b(w, w) - (f, w)_\Omega,$$

and weak formulation would be the Euler-Lagrange equations

$$b(u, v) = (f, v) \qquad \forall v \in V.$$

The primal-hybrid formulation

Imagine for a moment we were solving a coercive problem. We would have

$$u := \arg\min_{w \in V} \frac{1}{2} b(w, w) - (f, w)_\Omega,$$

and weak formulation would be the Euler-Lagrange equations

$$b(u, v) = (f, v) \qquad \forall v \in V.$$

Equivalently, we can consider the constrained problem

$$u := \arg \min_{\substack{w \in W \\ \langle \mu, w \rangle = 0 \ \forall \mu \in \Lambda}} \frac{1}{2} a(w, w) - (f, w)_\Omega.$$

The primal-hybrid formulation

Imagine for a moment we were solving a coercive problem. We would have

$$u := \arg\min_{w \in V} \frac{1}{2} b(w, w) - (f, w)_\Omega,$$

and weak formulation would be the Euler-Lagrange equations

$$b(u, v) = (f, v) \qquad \forall v \in V.$$

Equivalently, we can consider the constrained problem

$$u := \arg \min_{\substack{w \in W \\ \langle \mu, w \rangle = 0 \ \forall \mu \in \Lambda}} \frac{1}{2} a(w, w) - (f, w)_\Omega.$$

Then, the Euler-Lagrange equations are to find $(u, \lambda) \in W \times \Lambda$

$$\begin{cases} b(u, v) + \langle \lambda, v \rangle &= (f, v)_{\mathcal{T}_h} \\ \langle \mu, u \rangle &= 0 \end{cases}$$

for all $(v, \mu) \in W \times \Lambda$.

The primal hybrid formulation

The primal hybrid formulation is to find $(u, \lambda) \in W \times \Lambda$

$$\left\{ \begin{array}{rcl} b(u, v) + \langle \lambda, v \rangle & = & (f, v)_{\mathcal{T}_h} \\ \langle \mu, u \rangle & = & 0 \end{array} \right.$$

for all $(v, \mu) \in W \times \Lambda$.

The primal hybrid formulation

The primal hybrid formulation is to find $(u, \lambda) \in W \times \Lambda$

$$\left\{ \begin{array}{rcl} b(u, v) + \langle \lambda, v \rangle & = & (f, v)_{\mathcal{T}_h} \\ \langle \mu, u \rangle & = & 0 \end{array} \right.$$

for all $(v, \mu) \in W \times \Lambda$.

Here, it can not be interpreted as a saddle-point originated from an optimization problem, but it is still equivalent to the "conforming" formulation.

The primal hybrid formulation

The primal hybrid formulation is to find $(u, \lambda) \in W \times \Lambda$

$$\left\{ \begin{array}{rcl} b(u, v) + \langle \lambda, v \rangle & = & (f, v)_{\mathcal{T}_h} \\ \langle \mu, u \rangle & = & 0 \end{array} \right.$$

for all $(v, \mu) \in W \times \Lambda$.

Here, it can not be interpreted as a saddle-point originated from an optimization problem, but it is still equivalent to the "conforming" formulation.

We actually have $\lambda|_{\partial K} = \boldsymbol{\nabla} u \cdot \boldsymbol{n}_K$.

The primal hybrid formulation

The primal hybrid formulation is to find $(u, \lambda) \in W \times \Lambda$

$$\left\{ \begin{array}{rcl} b(u, v) + \langle \lambda, v \rangle & = & (f, v)_{\mathcal{T}_h} \\ \langle \mu, u \rangle & = & 0 \end{array} \right.$$

for all $(v, \mu) \in W \times \Lambda$.

Here, it can not be interpreted as a saddle-point originated from an optimization problem, but it is still equivalent to the "conforming" formulation.

We actually have $\lambda|_{\partial K} = \boldsymbol{\nabla} u \cdot \boldsymbol{n}_K$.

To obtain the MHM method, we want to eliminate u in terms of λ.
This operation amounts to solving element-wise local problems.

The primal hybrid formulation

The primal hybrid formulation is to find $(u, \lambda) \in W \times \Lambda$

$$\left\{ \begin{array}{rcl} b(u, v) + \langle \lambda, v \rangle & = & (f, v)_{\mathcal{T}_h} \\ \langle \mu, u \rangle & = & 0 \end{array} \right.$$

for all $(v, \mu) \in W \times \Lambda$.

Here, it can not be interpreted as a saddle-point originated from an optimization problem, but it is still equivalent to the "conforming" formulation.

We actually have $\lambda|_{\partial K} = \boldsymbol{\nabla} u \cdot \boldsymbol{n}_K$.

To obtain the MHM method, we want to eliminate u in terms of λ.
This operation amounts to solving element-wise local problems.

It can be interpreted as construction multiscale basis functions.

The MHM formulation

Consider the operators $T : \Lambda \to W$ and $\widehat{T} : L^2(\Omega) \to W$

$$b(T\mu, v) = -\langle \lambda\mu, v \rangle \qquad b(\widehat{T}\phi, v) = (\phi, v)_{\mathcal{T}_h} \quad \forall v \in W$$

for $\mu \in \Lambda$ and $\phi \in L^2(\Omega)$. This is a well-posed local definition.

The MHM formulation

Consider the operators $T : \Lambda \to W$ and $\widehat{T} : L^2(\Omega) \to W$

$$b(T\mu, v) = -\langle \lambda\mu, v\rangle \qquad b(\widehat{T}\phi, v) = (\phi, v)_{\mathcal{T}_h} \quad \forall v \in W$$

for $\mu \in \Lambda$ and $\phi \in L^2(\Omega)$. This is a well-posed local definition.

The first equation of the primal-hyrbid formulation is

$$b(u, v) = (f, v) - \langle \lambda, v\rangle \quad \forall v \in W,$$

and as a result

$$u = T\lambda + \widehat{T}f.$$

The MHM formulation

Consider the operators $T : \Lambda \to W$ and $\widehat{T} : L^2(\Omega) \to W$

$$b(T\mu, v) = -\langle \lambda\mu, v\rangle \qquad b(\widehat{T}\phi, v) = (\phi, v)_{\mathcal{T}_h} \quad \forall v \in W$$

for $\mu \in \Lambda$ and $\phi \in L^2(\Omega)$. This is a well-posed local definition.

The first equation of the primal-hyrbid formulation is

$$b(u, v) = (f, v) - \langle \lambda, v\rangle \quad \forall v \in W,$$

and as a result

$$u = T\lambda + \widehat{T}f.$$

Plugging this expression in the second equation ($\langle \mu, u\rangle = 0$), we have

$$\langle \mu, T\lambda\rangle = -\langle \mu, \widehat{T}f\rangle \quad \forall \mu \in \Lambda.$$

The MHM formulation

The MHM formulation is to find $\lambda \in \Lambda$ such that

$$\langle \mu, T\lambda \rangle = -\langle \mu, \widehat{T} f \rangle \quad \forall \mu \in \Lambda,$$

and retrieve the "primal" solution by

$$u = T\lambda + \widehat{T} f.$$

The MHM formulation

The MHM formulation is to find $\lambda \in \Lambda$ such that

$$\langle \mu, T\lambda \rangle = -\langle \mu, \widehat{T} f \rangle \quad \forall \mu \in \Lambda,$$

and retrieve the "primal" solution by

$$u = T\lambda + \widehat{T} f.$$

Recalling that $\langle \mu, v \rangle = b(T\mu, v)$, we can recognize the original problem

$$b(T\mu, T\lambda) = -\langle \mu, \widehat{T} f \rangle \quad \forall \mu \in \Lambda.$$

Discretization

Discretization

We will consider the simplest possible discretization.
We select $\Lambda_h := \Lambda \cap \mathcal{P}_0(\mathcal{T}_h) \subset \Lambda$.

Discretization

We will consider the simplest possible discretization.
We select $\Lambda_h := \Lambda \cap \mathcal{P}_0(\mathcal{T}_h) \subset \Lambda$.

The discrete problem is to find $\lambda_h \in \Lambda_h$ such that

$$\langle \mu_h, T\lambda_h \rangle = \langle \mu_h, \widehat{T} f \rangle \qquad \mu_h \in \Lambda_h,$$

and then recover the solution by setting $u_h := T\lambda_h + \widehat{T} f$.

Discretization

We will consider the simplest possible discretization.
We select $\Lambda_h := \Lambda \cap \mathcal{P}_0(\mathcal{T}_h) \subset \Lambda$.

The discrete problem is to find $\lambda_h \in \Lambda_h$ such that

$$\langle \mu_h, T\lambda_h \rangle = \langle \mu_h, \widehat{T} f \rangle \qquad \mu_h \in \Lambda_h,$$

and then recover the solution by setting $u_h := T\lambda_h + \widehat{T} f$.

Consider $W_h := T(\Lambda_h)$, we see that we look for $w_h \in W_h$

$$b(v_h, w_h) = \text{rhs}$$

Discretization

We will consider the simplest possible discretization.
We select $\Lambda_h := \Lambda \cap \mathcal{P}_0(\mathcal{T}_h) \subset \Lambda$.

The discrete problem is to find $\lambda_h \in \Lambda_h$ such that

$$\langle \mu_h, T\lambda_h \rangle = \langle \mu_h, \widehat{T} f \rangle \qquad \mu_h \in \Lambda_h,$$

and then recover the solution by setting $u_h := T\lambda_h + \widehat{T} f$.

Consider $W_h := T(\Lambda_h)$, we see that we look for $w_h \in W_h$

$$b(v_h, w_h) = \text{rhs}$$

This is fairly similar to HDG, or a "V"EM with a computed the "virtual" space.

The multiscale hybrid-mixed method

Well-posedness and convergence

Schatz argument

Gårding inequality

$$\langle \mu_h, T\lambda_h \rangle = b(T\mu_h, T\lambda_h) = \| T\mu_h \|_W^2 - ck^2 \| T\mu_h \|_{L^2(\Omega)}^2.$$

The discrete inf-sup follows from a "Schatz" argument.

Schatz argument

Gårding inequality

$$\langle \mu_h, T\lambda_h \rangle = b(T\mu_h, T\lambda_h) = \|T\mu_h\|_W^2 - ck^2 \|T\mu_h\|_{L^2(\Omega)}^2.$$

The discrete inf-sup follows from a "Schatz" argument.

Specifically, we have

$$\inf_{\substack{\lambda_h \in \Lambda_h \\ \|\lambda_h\|_\Lambda = 1}} \sup_{\substack{\mu_h \in \Lambda_h \\ \|\mu_h\|_\Lambda = 1}} \mathrm{Re}\langle \mu_h, T\lambda_h \rangle \geq (1 - c\gamma_h) \underbrace{(k\ell)^{-2}}_{\Lambda \text{ inf-sup}}$$

with

Schatz argument

Gårding inequality

$$\langle \mu_h, T\lambda_h \rangle = b(T\mu_h, T\lambda_h) = \|T\mu_h\|_W^2 - ck^2 \|T\mu_h\|_{L^2(\Omega)}^2.$$

The discrete inf-sup follows from a "Schatz" argument.

Specifically, we have

$$\inf_{\substack{\lambda_h \in \Lambda_h \\ \|\lambda_h\|_\Lambda = 1}} \sup_{\substack{\mu_h \in \Lambda_h \\ \|\mu_h\|_\Lambda = 1}} \mathrm{Re}\langle \mu_h, T\lambda_h \rangle \geq (1 - c\gamma_h) \underbrace{(k\ell)^{-2}}_{\Lambda \text{ inf-sup}}$$

with

$$\gamma_h := \sup_{\substack{\phi \in L^2(\Omega) \\ \|\phi\| = 1}} \inf_{\mu_h \in \Lambda_h} \|\boldsymbol{\nabla} u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_\Lambda$$

where $u^\star(\phi) \in V$ solves

$$a(v, u^\star(\phi)) = (v, \phi)_\Omega \quad \forall v \in V.$$

Naïve approach

Pick $\phi \in L^2(\Omega)$ with $\|\phi\|_\Omega = 1$. We have

$$\|\boldsymbol{\nabla} u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_\wedge = \sup_{\substack{v \in W \\ \|v\|_W = 1}} \mathrm{Re}\langle \boldsymbol{\nabla} u^\star(\phi) \cdot \boldsymbol{n} - \mu_h, v\rangle.$$

$$\lesssim h^{1/2} \|\boldsymbol{\nabla} u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_{\partial \mathcal{T}_h}$$

Naïve approach

Pick $\phi \in L^2(\Omega)$ with $\|\phi\|_\Omega = 1$. We have

$$\|\nabla u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_\wedge = \sup_{\substack{v \in W \\ \|v\|_W = 1}} \mathrm{Re}\langle \nabla u^\star(\phi) \cdot \boldsymbol{n} - \mu_h, v\rangle.$$
$$\lesssim h^{1/2} \|\nabla u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_{\partial \mathcal{T}_h}$$

Pick $\mu_h|_{\partial K} := \pi_{\partial K}(\nabla u^\star(\phi) \cdot \boldsymbol{n}_K)$, then

$$\|\nabla u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_\wedge \lesssim h \|\nabla u^\star(\phi)\|_{H^{1/2}(\partial \mathcal{T}_h)}$$
$$\lesssim h \|\nabla^2 u^\star(\phi)\|_{\mathcal{T}_h}.$$
$$\lesssim C_{\mathrm{st}}(k\ell) k h \|\phi\|_\Omega = C_{\mathrm{st}}(k\ell) k h.$$

Naïve approach

Pick $\phi \in L^2(\Omega)$ with $\|\phi\|_\Omega = 1$. We have

$$\|\nabla u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_\wedge = \sup_{\substack{v \in W \\ \|v\|_W = 1}} \mathrm{Re}\langle \nabla u^\star(\phi) \cdot \boldsymbol{n} - \mu_h, v\rangle.$$
$$\lesssim h^{1/2} \|\nabla u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_{\partial \mathcal{T}_h}$$

Pick $\mu_h|_{\partial K} := \pi_{\partial K}(\nabla u^\star(\phi) \cdot \boldsymbol{n}_K)$, then

$$\|\nabla u^\star(\phi) \cdot \boldsymbol{n} - \mu_h\|_\wedge \lesssim h \|\nabla u^\star(\phi)\|_{H^{1/2}(\partial \mathcal{T}_h)}$$
$$\lesssim h \|\nabla^2 u^\star(\phi)\|_{\mathcal{T}_h}.$$
$$\lesssim C_{\mathrm{st}}(k\ell) k h \|\phi\|_\Omega = C_{\mathrm{st}}(k\ell) k h.$$

Hence $\gamma_h \lesssim C_{\mathrm{st}}(k\ell) k h$.

Stability conditions

In the absence of trapped modes, we simply have $C_{\mathrm{st}}(k\ell) \simeq k\ell$, so that

$$\gamma_h \simeq (k\ell)(kh) \sim k^2 h.$$

Stability holds if $k^2 h$ is small, which is the usual condition for FEM also.

Stability conditions

In the absence of trapped modes, we simply have $C_{\mathrm{st}}(k\ell) \simeq k\ell$, so that

$$\gamma_h \simeq (k\ell)(kh) \sim k^2 h.$$

Stability holds if $k^2 h$ is small, which is the usual condition for FEM also.

However, with trapped modes, we have $C_{\mathrm{st}}(k\ell) \simeq (k\ell)^2$,
thus stability is only ensured if $k^3 h$ is small.

Stability conditions

In the absence of trapped modes, we simply have $C_{\mathrm{st}}(k\ell) \simeq k\ell$, so that

$$\gamma_h \simeq (k\ell)(kh) \sim k^2 h.$$

Stability holds if $k^2 h$ is small, which is the usual condition for FEM also.

However, with trapped modes, we have $C_{\mathrm{st}}(k\ell) \simeq (k\ell)^2$,
thus stability is only ensured if $k^3 h$ is small.

For FEM we cannot do better, which explains the stability loss.
This can, however, be improved for MHM on Cartesian grids.

The multiscale hybrid-mixed method

Robustness on Cartesian grids

The intuitive explanation

Consider a Cartesian grid mesh.

The intuitive explanation

Consider a Cartesian grid mesh.

Imagine the solution $u := e^{ik\boldsymbol{x}_1}$, and its Neumann trace $\lambda = \boldsymbol{\nabla} u \cdot \boldsymbol{n}$.

The intuitive explanation

Consider a Cartesian grid mesh.

Imagine the solution $u := e^{ikx_1}$, and its Neumann trace $\lambda = \nabla u \cdot \boldsymbol{n}$.

Then,

$$\lambda = \frac{\partial u}{\partial x_2}(x_1) = 0 \qquad \text{on horizontal faces}$$

and

$$\lambda = \frac{\partial u}{\partial x_1}(x_2) = C \qquad \text{on vertical faces.}$$

so that $\lambda \in \mathcal{P}_0(\mathcal{T}_h)$.

The intuitive explanation

Consider a Cartesian grid mesh.

Imagine the solution $u := e^{ikx_1}$, and its Neumann trace $\lambda = \nabla u \cdot \boldsymbol{n}$.

Then,

$$\lambda = \frac{\partial u}{\partial x_2}(x_1) = 0 \qquad \text{on horizontal faces}$$

and

$$\lambda = \frac{\partial u}{\partial x_1}(x_2) = C \qquad \text{on vertical faces.}$$

so that $\lambda \in \mathcal{P}_0(\mathcal{T}_h)$.

Horizontal (and vertical) plane waves are exactly reproduced.

The intuitive explanation

Consider a Cartesian grid mesh.

Imagine the solution $u := e^{ik\boldsymbol{x}_1}$, and its Neumann trace $\lambda = \boldsymbol{\nabla} u \cdot \boldsymbol{n}$.

Then,

$$\lambda = \frac{\partial u}{\partial \boldsymbol{x}_2}(\boldsymbol{x}_1) = 0 \qquad \text{on horizontal faces}$$

and

$$\lambda = \frac{\partial u}{\partial \boldsymbol{x}_1}(\boldsymbol{x}_2) = C \qquad \text{on vertical faces.}$$

so that $\lambda \in \mathcal{P}_0(\mathcal{T}_h)$.

Horizontal (and vertical) plane waves are exactly reproduced.

The "bad modes" $\chi(\boldsymbol{x}_2)e^{ik\boldsymbol{x}_1}$ are very well approximated.

A rigorous proof using a splitting

Consider the Fourier series expansion along $\boldsymbol{x}_1$:

$$\phi(\boldsymbol{x}) := \sum_{m\in\mathbb{Z}} \widehat{\phi}_m(\boldsymbol{x}_2) e^{i(\alpha+2m\pi)\boldsymbol{x}_1/\ell} \quad u^\star(\phi)(\boldsymbol{x}) := \sum_{m\in\mathbb{Z}} \widehat{u}_m(\boldsymbol{x}_2) e^{i(\alpha+2m\pi)\boldsymbol{x}_1/\ell}.$$

Then, we have

$$-{k_m}^2 \widehat{u}_m - \widehat{u}_m{}' = \widehat{\phi}_m \qquad (+\text{ BC})$$

with $(k_m\ell)^2 := (k\ell)^2 - (\alpha + 2\pi m)^2$.

A rigorous proof using a splitting

Consider the Fourier series expansion along $\boldsymbol{x}_1$:

$$\phi(\boldsymbol{x}) := \sum_{m\in\mathbb{Z}} \widehat{\phi}_m(\boldsymbol{x}_2) e^{i(\alpha+2m\pi)\boldsymbol{x}_1/\ell} \quad u^\star(\phi)(\boldsymbol{x}) := \sum_{m\in\mathbb{Z}} \widehat{u}_m(\boldsymbol{x}_2) e^{i(\alpha+2m\pi)\boldsymbol{x}_1/\ell}.$$

Then, we have

$$-{k_m}^2 \widehat{u}_m - \widehat{u}_m{}' = \widehat{\phi}_m \qquad (+\text{ BC})$$

with $(k_m\ell)^2 := (k\ell)^2 - (\alpha + 2\pi m)^2$.

The "problems" arise when k_m approaches zero.

A rigorous proof using a splitting

Consider the Fourier series expansion along $\boldsymbol{x}_1$:

$$\phi(\boldsymbol{x}) := \sum_{m\in\mathbb{Z}} \widehat{\phi}_m(\boldsymbol{x}_2) e^{i(\alpha+2m\pi)\boldsymbol{x}_1/\ell} \qquad u^\star(\phi)(\boldsymbol{x}) := \sum_{m\in\mathbb{Z}} \widehat{u}_m(\boldsymbol{x}_2) e^{i(\alpha+2m\pi)\boldsymbol{x}_1/\ell}.$$

Then, we have

$$-{k_m}^2 \widehat{u}_m - \widehat{u}_m{}' = \widehat{\phi}_m \qquad (+\text{ BC})$$

with $(k_m\ell)^2 := (k\ell)^2 - (\alpha + 2\pi m)^2$.

The "problems" arise when k_m approaches zero.

We thus split

$$\phi(\boldsymbol{x}) := \underbrace{\sum_{|k_m|<k} \widehat{\phi}_m(\boldsymbol{x}_2) e^{i(\alpha+2m\pi)\boldsymbol{x}_1/\ell}}_{=:\breve{\phi}} + \underbrace{\sum_{|k_m|>k} \widehat{\phi}_m(\boldsymbol{x}_2) e^{i(\alpha+2m\pi)\boldsymbol{x}_1/\ell}}_{=:\widetilde{\phi}}$$

and similarly $u = \breve{u} + \widetilde{u}$

Large frequencies

For $\widetilde{u}$ we actually have the "good" stability estimate

$$\|\nabla^2 \widetilde{u}\|_{\mathcal{T}_h} \lesssim (k\ell)\|\widetilde{\phi}\|_{\Omega},$$

and we can follow the "naïve" approach.

Large frequencies

For $\widetilde{u}$ we actually have the "good" stability estimate

$$\|\boldsymbol{\nabla}^2 \widetilde{u}\|_{\mathcal{T}_h} \lesssim (k\ell)\|\widetilde{\phi}\|_\Omega,$$

and we can follow the "naïve" approach.

For the other term, we only have

$$\|\boldsymbol{\nabla}^2 \breve{u}\|_{\mathcal{T}_h} \lesssim (k\ell)^2\|\breve{\phi}\|_\Omega$$

so we need to be creative!

Small frequencies

Because the grid is Cartesian, we actually have

$$\boldsymbol{\nabla} w(\boldsymbol{x}) \cdot \boldsymbol{n} = \partial_2 w(\boldsymbol{x}_1, \cdot) \quad \text{or} \quad \partial_1 w(\cdot, \boldsymbol{x}_2).$$

Small frequencies

Because the grid is Cartesian, we actually have

$$\nabla w(\boldsymbol{x}) \cdot \boldsymbol{n} = \partial_2 w(\boldsymbol{x}_1, \cdot) \quad \text{or} \quad \partial_1 w(\cdot, \boldsymbol{x}_2).$$

Looking at one particular mode $\widehat{u}_m$, we have

$$\nabla \widehat{u}_m(\boldsymbol{x}) \cdot \boldsymbol{n} = \widehat{u}_m{}' e^{i k_m \boldsymbol{x}_1} \text{ or } i k_m u_m(\boldsymbol{x}_2).$$

Small frequencies

Because the grid is Cartesian, we actually have

$$\nabla w(\boldsymbol{x}) \cdot \boldsymbol{n} = \partial_2 w(\boldsymbol{x}_1, \cdot) \quad \text{or} \quad \partial_1 w(\cdot, \boldsymbol{x}_2).$$

Looking at one particular mode $\widehat{u}_m$, we have

$$\nabla \widehat{u}_m(\boldsymbol{x}) \cdot \boldsymbol{n} = \widehat{u}_m{}' e^{i k_m \boldsymbol{x}_1} \text{ or } i k_m u_m(\boldsymbol{x}_2).$$

We then write

$$\|\nabla \widehat{u}_m \cdot \boldsymbol{n} - \mu_h\|_F \lesssim k_m h \|\widehat{u}_m\|_{L^\infty(F)} \text{ or } h \|\widehat{u}_m'\|_F$$

Small frequencies

Because the grid is Cartesian, we actually have

$$\nabla w(\boldsymbol{x}) \cdot \boldsymbol{n} = \partial_2 w(\boldsymbol{x}_1, \cdot) \quad \text{or} \quad \partial_1 w(\cdot, \boldsymbol{x}_2).$$

Looking at one particular mode $\widehat{u}_m$, we have

$$\nabla \widehat{u}_m(\boldsymbol{x}) \cdot \boldsymbol{n} = \widehat{u}_m{}' e^{i k_m \boldsymbol{x}_1} \text{ or } i k_m u_m(\boldsymbol{x}_2).$$

We then write

$$\|\nabla \widehat{u}_m \cdot \boldsymbol{n} - \mu_h\|_F \lesssim k_m h \|\widehat{u}_m\|_{L^\infty(F)} \text{ or } h \|\widehat{u}_m'\|_F$$

and we can actually show that

$$\|\nabla \widehat{u}_m \cdot \boldsymbol{n} - \mu_h\| \lesssim k_m h \|\phi_m\|,$$

Small frequencies

Because the grid is Cartesian, we actually have

$$\nabla w(\boldsymbol{x}) \cdot \boldsymbol{n} = \partial_2 w(\boldsymbol{x}_1, \cdot) \quad \text{or} \quad \partial_1 w(\cdot, \boldsymbol{x}_2).$$

Looking at one particular mode $\widehat{u}_m$, we have

$$\nabla \widehat{u}_m(\boldsymbol{x}) \cdot \boldsymbol{n} = \widehat{u}_m{}' e^{ik_m \boldsymbol{x}_1} \text{ or } ik_m u_m(\boldsymbol{x}_2).$$

We then write

$$\|\nabla \widehat{u}_m \cdot \boldsymbol{n} - \mu_h\|_F \lesssim k_m h \|\widehat{u}_m\|_{L^\infty(F)} \text{ or } h\|\widehat{u}_m'\|_F$$

and we can actually show that

$$\|\nabla \widehat{u}_m \cdot \boldsymbol{n} - \mu_h\| \lesssim k_m h \|\phi_m\|,$$

leading to

$$\|\nabla \breve{u} \cdot \boldsymbol{n} - \mu_h\|_F \lesssim kh \|\breve{\phi}\|.$$

Concluding remarks

Concluding remarks

Helmholtz problems in periodic structures suffer from quasi-resonances.

This resonances affect the stability of FEM and MHM on general meshes.

On Cartesian grids the MHM exactly preserves horizontal plane waves.

This preservation property ensures the robustness of the method.