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A multiscale hybrid-mixed method for Helmholtz problems in periodic structures

T. Chaumont-Frelet*, Z. Kassali* and S. Lanteri*

ECCOMAS, Structure preserving and adaptive polytopal methods, 6 June 2022

*Inria Université Côte d'Azur, Atlantis project-team

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I hope this is good enough! :-)

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Multiscale hybrid-mixed (MHM) methods can:
deal with highly heterogeneous media and exactly reproduce some solutions.

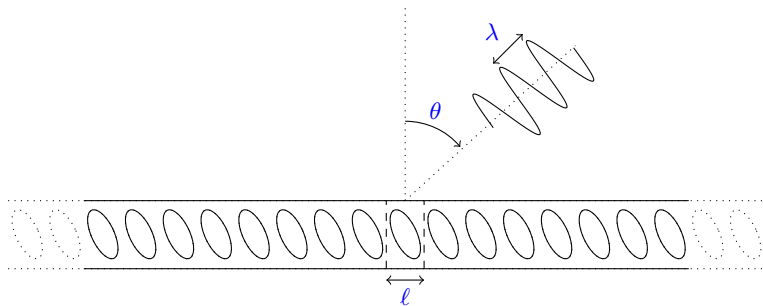
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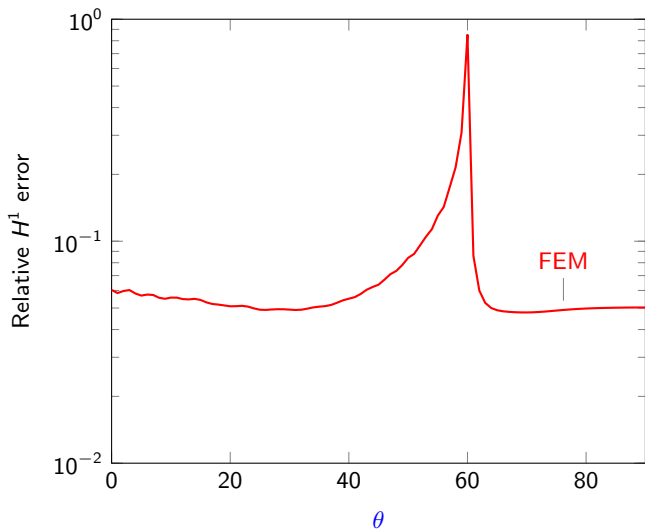
We will focus on the latter aspect, and see how it improves stability.

A small picture is better than a long explanation

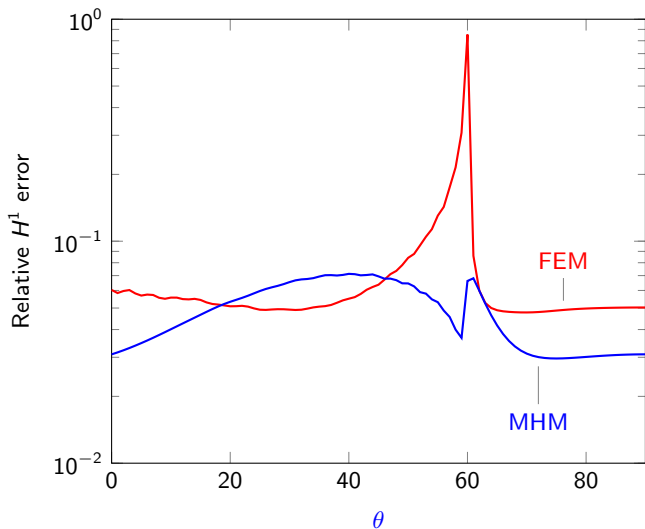


Particular combinations of ℓ , λ and θ generate “quasi-resonances”.

An example



An example



- 1 The Helmholtz equation in periodic structures
 - Where do the boundary conditions come from?
 - Perfectly matched layer
 - A discussion on stability

- 2 The multiscale hybrid-mixed method
 - General presentation of the method
 - Well-posedness and convergence
 - Robustness on Cartesian grids

The Helmholtz equation in periodic structures

We set the problem on a single “periodic cell” Ω .

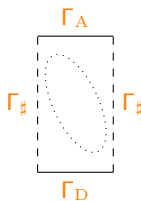
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The problem is to find u such that

$$\left\{ \begin{array}{ll} -k^2 u - \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \nabla u \cdot \mathbf{n} - \mathcal{R}_k u = g & \text{on } \Gamma_A, \\ u^+ - e^{i\alpha} u^- = 0 & \text{on } \Gamma_\# \end{array} \right.$$

where $\alpha := k\ell \sin \theta$.



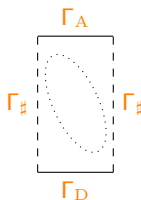
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where $\alpha := k\ell \sin \theta$.



The DtN operator is given by

$$\mathcal{R}_k u := i \sum_{m \in \mathbb{Z}} k_m \hat{u}_m e^{i(\alpha - 2\pi m)x_1/\ell}.$$

The Helmholtz equation in periodic structures

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The incident field is the plane wave

$$u_{\text{inc}} := e^{ik\mathbf{d}\cdot\mathbf{x}}, \quad \mathbf{d} := (\cos \theta, \sin \theta).$$

We can easily see that

$$u_{\text{inc}}(\mathbf{x}_1 + \ell, \cdot) = e^{ik\ell \sin \theta} u_{\text{inc}}(\mathbf{x}_1, \cdot) = e^{i\alpha} u_{\text{inc}}(\mathbf{x}_1, \cdot)$$

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The DtN ensures that

$$\widehat{u}_m' = i\beta_m u_m \quad \forall m \in \mathbb{Z}$$

with $\beta_m < 0$ or $i\beta_m < 0$. Each mode is thus outgoing or evanescent.

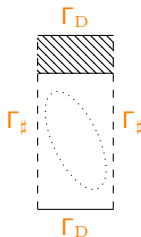
The Helmholtz equation in periodic structures
Perfectly matched layer

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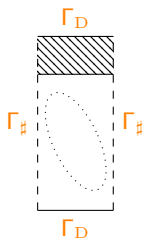
$$\begin{cases} -k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ u^+ - e^{i\alpha} u^- = 0 & \text{on } \Gamma_\# . \end{cases}$$



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We do not touch the original domain where $\mu = 1$ and $\mathbf{A} = \mathbf{I}$.
In the extended layer, we set

$$\mu = \nu \quad \mathbf{A} = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \quad \nu = 1 + i.$$

Dirichlet and quasi-periodic boundary conditions are imposed in the space

$$V := H_{\#}^1(\Omega) := \left\{ w \in H^1(\Omega) \mid \begin{array}{l} w^+ = 0 \quad \text{on } \Gamma_D \\ w^+ - e^{i\alpha} w^- = 0 \quad \text{on } \Gamma_{\#} \end{array} \right\}.$$

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There exists a unique $u \in V$ such that

$$b(u, v) = (f, v) \quad \forall v \in V$$

where

$$b(u, v) := -k^2(\mu u, v) + (A \nabla u, \nabla v).$$

The Helmholtz equation in periodic structures
A discussion on stability

Here, I want to discuss stability properties of this Helmholtz problem.

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Specifically, I am interested in the inf-sup constant

$$\inf_{\substack{w \in V \\ \|w\|_V=1}} \sup_{\substack{v \in V \\ \|v\|_V=1}} \operatorname{Re} b(w, v)$$

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It is immediately related to the best constant $C_{\text{st}}(kl)$ such that

$$\|w\|_V \leq C_{\text{st}}(kl) \|\phi\|_\Omega,$$

where w solves the problem with rhs $\phi \in L^2(\Omega)$.

Stability in the “open” case

Consider the function $w(\mathbf{x}) := \chi(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}$, where χ is a smooth cutoff function.

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Then,

$$-k^2 w - \Delta w = \left(2ik \frac{\partial \chi}{\partial x_1} + \Delta \chi \right) e^{ikx_1} =: \phi.$$

and since χ does not depend on k , we have

$$\|w\|_{\Omega} \simeq \frac{1}{kl} \left(\ell^2 \|\phi\|_{\Omega} \right).$$

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This tells us that

$$\inf_{\substack{w \in V \\ \|w\|_V=1}} \sup_{\substack{v \in V \\ \|v\|_V=1}} \operatorname{Re} a(w, v) \lesssim \frac{1}{kl}.$$

and actually, we can show that equility holds.

Consider now $w(\mathbf{x}) := \chi(\mathbf{x}_2)e^{ikx_1}$. Then,

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We lose one power of frequency in the stability constant at quasi-resonant frequencies. The stability issue of FEM are a consequence of this phenomenon.

The multiscale hybrid-mixed method

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General presentation of the method

We consider a mesh \mathcal{T}_h of Ω made of polygons K .

We assume that the faces of K match on both sides of $\Gamma_{\#}$.

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We would like to seek a solution in the broken space

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that misses compatibility and boundary conditions.

We will enforce that $[[u]]_F = 0$ by introducing a suitable Lagrange multiplier.

This is a subtle issue that I want to correctly explain.

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When $v \in V$, we have

$$(\nabla v, \sigma)_{\Omega} + (v, \nabla \cdot \sigma)_{\Omega} = 0$$

for all $\sigma \in \mathbf{H}(\text{div}, \Omega)$ with $\sigma^{-} \cdot \mathbf{n} - e^{i\alpha} \sigma^{+} \cdot \mathbf{n} = 0$ on Γ_{\sharp} .

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Actually, the converse holds true, and $v \in W$ belongs to V iff

$$\sum_{K \in \mathcal{T}_h} \langle \sigma \cdot \mathbf{n}, v \rangle_{\partial K} = (\nabla v, \sigma)_{\mathcal{T}_h} + (v, \nabla \cdot \sigma)_{\mathcal{T}_h} = 0$$

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If σ is smooth, we actually have

$$\sum_{K \in \mathcal{T}_h} \langle \sigma \cdot \mathbf{n}, \mathbf{v} \rangle_{\partial K} = \sum_{F \in \mathcal{F}_h} (\sigma \cdot \mathbf{n}_F, \llbracket \mathbf{v} \rrbracket_F)_F,$$

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The issue is subtle, and related to the difference between $H^{1/2}(F)$ and $H_{00}^{1/2}(F)$.

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The only thing that really matters is $\mu = \sigma \cdot \mathbf{n}$.

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We thus set

$$\Lambda := \left\{ \mu \in H^{-1/2}(\partial\mathcal{T}_h); \mu|_{\partial K} = \sigma \cdot \mathbf{n}_K \quad \forall K \in \mathcal{T}_h \right\},$$

and the pairing

$$\langle \mu, \nu \rangle = \sum_{K \in \mathcal{T}_h} \langle \mu, \nu \rangle_{\partial K} \quad \forall (\mu, \nu) \in \Lambda \times W.$$

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For a smooth $\mu \in \Lambda \cap L^2(\partial\mathcal{T}_h)$, we do have

$$\langle \mu, \mathbf{v} \rangle = \sum_{F \in \mathcal{F}_h} (\mu, \llbracket \mathbf{v} \rrbracket)_F.$$

As a mental model, you can think of Λ as a single valued face-based space.

Imagine for a moment we were solving a coercive problem. We would have

$$u := \arg \min_{w \in V} \frac{1}{2} b(w, w) - (f, w)_\Omega,$$

and weak formulation would be the Euler-Lagrange equations

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Equivalently, we can consider the constrained problem

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Then, the Euler-Lagrange equations are to find $(u, \lambda) \in W \times \Lambda$

$$\begin{cases} b(u, v) + \langle \lambda, v \rangle & = (f, v)_{\mathcal{T}_h} \\ \langle \mu, u \rangle & = 0 \end{cases}$$

for all $(v, \mu) \in W \times \Lambda$.

The primal hybrid formulation

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It can be interpreted as construction multiscale basis functions.

Consider the operators $T : \Lambda \rightarrow W$ and $\widehat{T} : L^2(\Omega) \rightarrow W$

$$b(T\mu, v) = -\langle \lambda\mu, v \rangle \quad b(\widehat{T}\phi, v) = (\phi, v)_{\mathcal{T}_h} \quad \forall v \in W$$

for $\mu \in \Lambda$ and $\phi \in L^2(\Omega)$. This is a well-posed local definition.

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The first equation of the primal-hybrid formulation is

$$b(u, v) = (f, v) - \langle \lambda, v \rangle \quad \forall v \in W,$$

and as a result

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Plugging this expression in the second equation ($\langle \mu, u \rangle = 0$), we have

$$\langle \mu, T\lambda \rangle = -\langle \mu, \widehat{T}f \rangle \quad \forall \mu \in \Lambda.$$

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Recalling that $\langle \mu, \nu \rangle = b(T\mu, \nu)$, we can recognize the original problem

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Consider $W_h := T(\Lambda_h)$, we see that we look for $w_h \in W_h$

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This is fairly similar to HDG, or a “V” EM with a computed the “virtual” space.

The multiscale hybrid-mixed method

Well-posedness and convergence

Gårding inequality

$$\langle \mu_h, T\lambda_h \rangle = b(T\mu_h, T\lambda_h) = \|T\mu_h\|_W^2 - ck^2 \|T\mu_h\|_{L^2(\Omega)}^2.$$

The discrete inf-sup follows from a “Schatz” argument.

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Specifically, we have

$$\inf_{\substack{\lambda_h \in \Lambda_h \\ \|\lambda_h\|_\Lambda = 1}} \sup_{\substack{\mu_h \in \Lambda_h \\ \|\mu_h\|_\Lambda = 1}} \operatorname{Re} \langle \mu_h, T\lambda_h \rangle \geq (1 - c\gamma_h) \underbrace{(k\ell)^{-2}}_{\Lambda \text{ inf-sup}}$$

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with

$$\gamma_h := \sup_{\substack{\phi \in L^2(\Omega) \\ \|\phi\|=1}} \inf_{\mu_h \in \Lambda_h} \|\nabla u^*(\phi) \cdot \mathbf{n} - \mu_h\|_\Lambda$$

where $u^*(\phi) \in V$ solves

$$a(v, u^*(\phi)) = (v, \phi)_\Omega \quad \forall v \in V.$$

Pick $\phi \in L^2(\Omega)$ with $\|\phi\|_{\Omega} = 1$. We have

$$\begin{aligned}\|\nabla u^*(\phi) \cdot \mathbf{n} - \mu_h\|_{\Lambda} &= \sup_{\substack{v \in W \\ \|v\|_W = 1}} \operatorname{Re} \langle \nabla u^*(\phi) \cdot \mathbf{n} - \mu_h, v \rangle. \\ &\lesssim h^{1/2} \|\nabla u^*(\phi) \cdot \mathbf{n} - \mu_h\|_{\partial \mathcal{T}_h}\end{aligned}$$

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Pick $\mu_h|_{\partial K} := \pi_{\partial K}(\nabla u^*(\phi) \cdot \mathbf{n}_K)$, then

$$\begin{aligned} \|\nabla u^*(\phi) \cdot \mathbf{n} - \mu_h\|_{\Lambda} &\lesssim h \|\nabla u^*(\phi)\|_{H^{1/2}(\partial \mathcal{T}_h)} \\ &\lesssim h \|\nabla^2 u^*(\phi)\|_{\mathcal{T}_h}. \\ &\lesssim C_{\text{st}}(kl) kh \|\phi\|_{\Omega} = C_{\text{st}}(kl) kh. \end{aligned}$$

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Hence $\gamma_h \lesssim C_{\text{st}}(kl)kh$.

In the absence of trapped modes, we simply have $C_{\text{st}}(k\ell) \simeq k\ell$, so that

$$\gamma_h \simeq (k\ell)(kh) \sim k^2 h.$$

Stability holds if $k^2 h$ is small, which is the usual condition for FEM also.

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However, with trapped modes, we have $C_{\text{st}}(k\ell) \simeq (k\ell)^2$, thus stability is only ensured if $k^3 h$ is small.

For FEM we cannot do better, which explains the stability loss. This can, however, be improved for MHM on Cartesian grids.

The multiscale hybrid-mixed method
Robustness on Cartesian grids

Consider a Cartesian grid mesh.

The intuitive explanation

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Imagine the solution $u := e^{ikx_1}$, and its Neumann trace $\lambda = \nabla u \cdot \mathbf{n}$.

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Then,

$$\lambda = \frac{\partial u}{\partial \mathbf{x}_2}(\mathbf{x}_1) = 0 \quad \text{on horizontal faces}$$

and

$$\lambda = \frac{\partial u}{\partial \mathbf{x}_1}(\mathbf{x}_2) = C \quad \text{on vertical faces.}$$

so that $\lambda \in \mathcal{P}_0(\mathcal{T}_h)$.

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Horizontal (and vertical) plane waves are exactly reproduced.

The “bad modes” $\chi(\mathbf{x}_2)e^{ikx_1}$ are very well approximated.

Consider the Fourier series expansion along x_1 :

$$\phi(\mathbf{x}) := \sum_{m \in \mathbb{Z}} \widehat{\phi}_m(\mathbf{x}_2) e^{i(\alpha + 2m\pi)x_1/\ell} \quad \mathbf{u}^*(\phi)(\mathbf{x}) := \sum_{m \in \mathbb{Z}} \widehat{u}_m(\mathbf{x}_2) e^{i(\alpha + 2m\pi)x_1/\ell}.$$

Then, we have

$$-k_m^2 \widehat{u}_m - \widehat{u}_m' = \widehat{\phi}_m \quad (+ \text{BC})$$

with $(k_m \ell)^2 := (k\ell)^2 - (\alpha + 2\pi m)^2$.

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The “problems” arise when k_m approaches zero.

A rigorous proof using a splitting

Consider the Fourier series expansion along \mathbf{x}_1 :

$$\phi(\mathbf{x}) := \sum_{m \in \mathbb{Z}} \widehat{\phi}_m(\mathbf{x}_2) e^{i(\alpha + 2m\pi)\mathbf{x}_1/\ell} \quad \mathbf{u}^*(\phi)(\mathbf{x}) := \sum_{m \in \mathbb{Z}} \widehat{\mathbf{u}}_m(\mathbf{x}_2) e^{i(\alpha + 2m\pi)\mathbf{x}_1/\ell}.$$

Then, we have

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with $(k_m \ell)^2 := (k\ell)^2 - (\alpha + 2\pi m)^2$.

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We thus split

$$\phi(\mathbf{x}) := \underbrace{\sum_{|k_m| < k} \widehat{\phi}_m(\mathbf{x}_2) e^{i(\alpha + 2m\pi)\mathbf{x}_1/\ell}}_{=:\check{\phi}} + \underbrace{\sum_{|k_m| > k} \widehat{\phi}_m(\mathbf{x}_2) e^{i(\alpha + 2m\pi)\mathbf{x}_1/\ell}}_{=:\tilde{\phi}}$$

and similarly $\mathbf{u} = \check{\mathbf{u}} + \tilde{\mathbf{u}}$

For \tilde{u} we actually have the “good” stability estimate

$$\|\nabla^2 \tilde{u}\|_{\tau_h} \lesssim (k\ell) \|\tilde{\phi}\|_{\Omega},$$

and we can follow the “naïve” approach.

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and we can follow the “naïve” approach.

For the other term, we only have

$$\|\nabla^2 \check{u}\|_{\tau_h} \lesssim (k\ell)^2 \|\check{\phi}\|_{\Omega}$$

so we need to be creative!

Because the grid is Cartesian, we actually have

$$\nabla w(\mathbf{x}) \cdot \mathbf{n} = \partial_2 w(\mathbf{x}_1, \cdot) \quad \text{or} \quad \partial_1 w(\cdot, \mathbf{x}_2).$$

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$$\nabla w(\mathbf{x}) \cdot \mathbf{n} = \partial_2 w(\mathbf{x}_1, \cdot) \quad \text{or} \quad \partial_1 w(\cdot, \mathbf{x}_2).$$

Looking at one particular mode \hat{u}_m , we have

$$\nabla \hat{u}_m(\mathbf{x}) \cdot \mathbf{n} = \hat{u}_m' e^{ik_m x_1} \quad \text{or} \quad ik_m u_m(\mathbf{x}_2).$$

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We then write

$$\|\nabla \widehat{u}_m \cdot \mathbf{n} - \mu_h\|_F \lesssim k_m h \|\widehat{u}_m\|_{L^\infty(F)} \quad \text{or} \quad h \|\widehat{u}_m'\|_F$$

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and we can actually show that

$$\|\nabla \hat{u}_m \cdot \mathbf{n} - \mu_h\| \lesssim k_m h \|\phi_m\|,$$

leading to

$$\|\nabla \check{u} \cdot \mathbf{n} - \mu_h\|_F \lesssim kh \|\check{\phi}\|.$$

Concluding remarks

Helmholtz problems in periodic structures suffer from quasi-resonances.

This resonances affect the stability of FEM and MHM on general meshes.

On Cartesian grids the MHM exactly preserves horizontal plane waves.

This preservation property ensures the robustness of the method.