



HAL
open science

Efficient a posteriori estimates for the wave equation

Théophile Chaumont-Frelet

► **To cite this version:**

Théophile Chaumont-Frelet. Efficient a posteriori estimates for the wave equation. European Finite Element Fair, Jun 2022, Helsinki, Finland. 2022. hal-03940628

HAL Id: hal-03940628

<https://inria.hal.science/hal-03940628>

Submitted on 16 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Efficient a posteriori estimates for the wave equation

T. Chaumont-Frelet*

European Finite Element Fair, 3-4 June 2022

*Inria Université Côte d'Azur, Atlantis project-team

1 The problem and available solutions

- The problem
- The goal
- The available solutions

2 The proposed error estimates

- The estimator
- The norm
- The estimates
- A numerical example

3 A glance at the proofs

- A quick detour: the Poisson problem
- The Helmholtz problem
- The wave equation

The problem and available solutions

The problem and available solutions

The problem

The wave equation

The wave equation

Given $f : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, find $u : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} \ddot{u}(t) - \Delta u(t) = f(t) & \text{in } \Omega, \\ u(t) = 0 & \text{on } \Gamma, \end{cases}$$

for $t > 0$, together with initial conditions

$$u(0) = \dot{u}(0) = 0 \text{ in } \Omega.$$

The wave equation

Given $f : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, find $u : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} \ddot{u}(t) - \Delta u(t) = f(t) & \text{in } \Omega, \\ u(t) = 0 & \text{on } \Gamma, \end{cases}$$

for $t > 0$, together with initial conditions

$$u(0) = \dot{u}(0) = 0 \text{ in } \Omega.$$

For simplicity, f is piecewise polynomial in space and Ω is convex.

The wave equation

Given $f : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, find $u : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} \ddot{u}(t) - \Delta u(t) = f(t) & \text{in } \Omega, \\ u(t) = 0 & \text{on } \Gamma, \end{cases}$$

for $t > 0$, together with initial conditions

$$u(0) = \dot{u}(0) = 0 \text{ in } \Omega.$$

For simplicity, f is piecewise polynomial in space and Ω is convex.

Importantly, we assume that f is smooth in time.

We (semi) discretize the problem with finite element in space.
This yields discrete solution $u_h : \mathbb{R}_+ \rightarrow V_h$ smooth in time.

We (semi) discretize the problem with finite element in space.
This yields discrete solution $u_h : \mathbb{R}_+ \rightarrow V_h$ smooth in time.

We want to find a suitable norm $\|\cdot\|$ such that estimates like

$$\int_0^{+\infty} \|(u - u_h)(t)\|_{\Omega}^2 \leq \int_0^{+\infty} \eta(t)^2 dt$$

and

$$\eta_{\kappa}(t) \lesssim \|(u - u_h)(t)\|_{\tilde{\kappa}}$$

hold true for a fully-computable estimator η .

The problem and available solutions

The available solutions



S. Adjerid, 2002

hierarchical estimator

✓ asymptotically exact

✗ requires Cartesian grids and smooth solutions



C. Bernardi and E. Süli, 2005

residual-based estimator

✓ upper bound and lower bounds, fully discrete problem

✗ weaker norm in the upper bound than in the lower bound



M. Picasso, 2010

elliptic reconstructions

✓ upper bound in $L^2(H^1)$

✗ no lower bound



E.H. Georgoulis, O. Lakkis and C. Makridakis, 2013

elliptic reconstructions

✓ upper bound in $L^\infty(L^2)$, fully discrete problem

✗ no lower bound



S. Adjerid, 2002

hierarchical estimator

✓ asymptotically exact

✗ requires Cartesian grids and smooth solutions



C. Bernardi and E. Süli, 2005

residual-based estimator

✓ upper bound and lower bounds, fully discrete problem

✗ weaker norm in the upper bound than in the lower bound



M. Picasso, 2010

elliptic reconstructions

✓ upper bound in $L^2(H^1)$

✗ no lower bound



E.H. Georgoulis, O. Lakkis and C. Makridakis, 2013

elliptic reconstructions

✓ upper bound in $L^\infty(L^2)$, fully discrete problem

✗ no lower bound

To the best of my knowledge, no rigorous efficiency proofs.

The proposed error estimates

The proposed error estimates
The estimator

An instantaneous estimator

The estimator is designed to control the “instantaneous residual”

$$|(\ddot{u}_h, v)_\Omega + (\nabla u_h(t), \nabla v)_\Omega - (f(t), v)_\Omega| \leq \eta(t) \|\nabla v\|_\Omega \quad \forall v \in H_0^1(\Omega),$$

at each $t \in \mathbb{R}_+$.

The estimator is designed to control the “instantaneous residual”

$$|(\ddot{u}_h, v)_\Omega + (\nabla u_h(t), \nabla v)_\Omega - (f(t), v)_\Omega| \leq \eta(t) \|\nabla v\|_\Omega \quad \forall v \in H_0^1(\Omega),$$

at each $t \in \mathbb{R}_+$.

The associated residual-based estimator would be

$$\eta_K(t) := h_K \|f(t) - \ddot{u}_h(t) + \Delta u_h(t)\|_K + h_K^{1/2} \| [\![\nabla u_h(t)]\!] \cdot \mathbf{n} \|_{\partial K \setminus \Gamma}.$$

The estimator is designed to control the “instantaneous residual”

$$|(\ddot{u}_h, v)_\Omega + (\nabla u_h(t), \nabla v)_\Omega - (f(t), v)_\Omega| \leq \eta(t) \|\nabla v\|_\Omega \quad \forall v \in H_0^1(\Omega),$$

at each $t \in \mathbb{R}_+$.

The associated residual-based estimator would be

$$\eta_K(t) := h_K \|f(t) - \ddot{u}_h(t) + \Delta u_h(t)\|_K + h_K^{1/2} \| [\![\nabla u_h(t)]\!] \cdot \mathbf{n} \|_{\partial K \setminus \Gamma}.$$

Here, we use an equilibrated flux $\sigma(t)$ with $\nabla \cdot \sigma(t) = f(t) - \ddot{u}_h(t)$, and

$$\eta_K(t) := \|\sigma(t) + \nabla u_h(t)\|_K.$$

The proposed error estimates
The norm

Fix a time-scale $T > 0$.

Fix a time-scale $T > 0$.

Let $\xi_h := u - u_h$. The “instantaneous” error measure is

$$\|\xi_h(t)\|_{\Omega}^2 := \underbrace{T^2 \|\ddot{\xi}_h(t)\|_{\Omega}^2}_{\text{optional}} + \underbrace{\|\dot{\xi}_h(t)\|_{\Omega}^2 + \|\nabla \xi_h(t)\|_{\Omega}^2}_{\text{physical energy}}.$$

Fix a time-scale $T > 0$.

Let $\xi_h := u - u_h$. The “instantaneous” error measure is

$$\|\xi_h(t)\|_{\Omega}^2 := \underbrace{T^2 \|\ddot{\xi}_h(t)\|_{\Omega}^2}_{\text{optional}} + \underbrace{\|\dot{\xi}_h(t)\|_{\Omega}^2 + \|\nabla \xi_h(t)\|_{\Omega}^2}_{\text{physical energy}}.$$

The “accumulated” measure is

$$\int_0^{+\infty} \|\xi_h(t)\|_{\Omega}^2 e^{-2t/T} dt.$$

Fix a time-scale $T > 0$.

Let $\xi_h := u - u_h$. The “instantaneous” error measure is

$$\|\xi_h(t)\|_{\Omega}^2 := \underbrace{T^2 \|\ddot{\xi}_h(t)\|_{\Omega}^2}_{\text{optional}} + \underbrace{\|\dot{\xi}_h(t)\|_{\Omega}^2 + \|\nabla \xi_h(t)\|_{\Omega}^2}_{\text{physical energy}}.$$

The “accumulated” measure is

$$\int_0^{+\infty} \|\xi_h(t)\|_{\Omega}^2 e^{-2t/T} dt.$$

The exponential weight means that we essentially control up to $t \sim T$.

The proposed error estimates
The estimates

The estimator is space-time locally efficient:

$$\eta_{\kappa}(t) \lesssim \|\xi_h(t)\|_{\tilde{K}}.$$

The estimator is space-time locally efficient:

$$\eta_{\kappa}(t) \lesssim \|\xi_h(t)\|_{\tilde{K}}.$$

The estimator is also reliable:

$$\int_0^{+\infty} \|\xi_h(t)\|_{\Omega}^2 e^{-2t/T} dt \leq \underbrace{\left(1 + o(h^{1/2})\right)^2}_{\text{pre factor}} \int_0^{+\infty} \eta(t)^2 e^{-2t/T} dt + \underbrace{\text{osc}^2(f)}_{\text{data oscillation}}.$$

The estimator is space-time locally efficient:

$$\eta_{\kappa}(t) \lesssim \|\xi_h(t)\|_{\tilde{K}}.$$

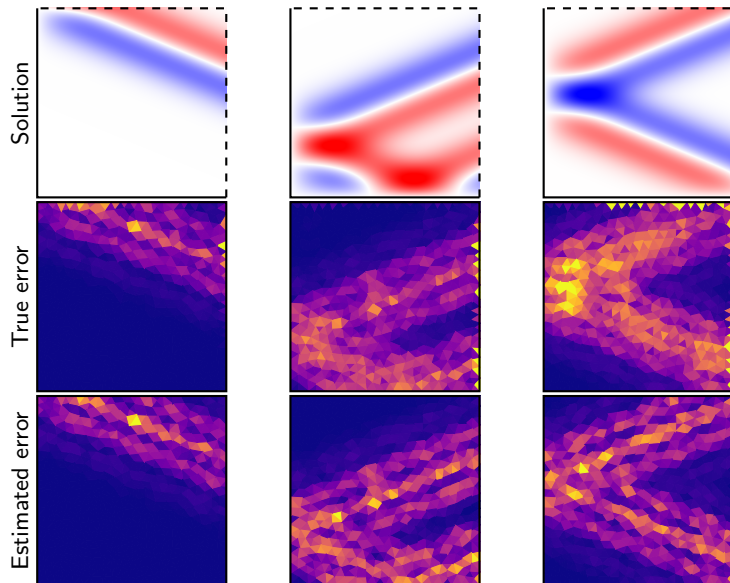
The estimator is also reliable:

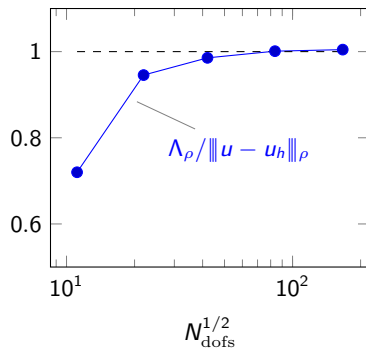
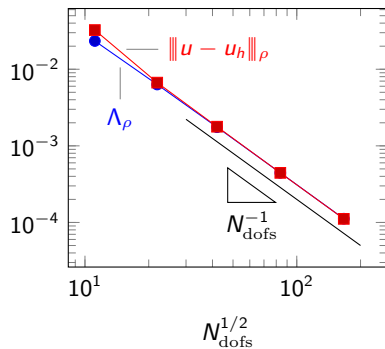
$$\int_0^{+\infty} \|\xi_h(t)\|_{\Omega}^2 e^{-2t/T} dt \leq \underbrace{\left(1 + o(h^{1/2})\right)^2}_{\text{pre factor}} \int_0^{+\infty} \eta(t)^2 e^{-2t/T} dt + \underbrace{\text{osc}^2(f)}_{\text{data oscillation}}.$$

$$\text{osc}(f) = h^{p+1} \times \text{“high power of } T\text{”} \times \text{“high order time derivatives of } f\text{”}.$$

The proposed error estimates
A numerical example

Reflection by a wall with $\rho = 2$





A glance at the proofs

A glance at the proofs

A quick detour: the Poisson problem

Consider the Poisson problem

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega)$$

and its conforming FEM discretization u_h .

With constructions for η similar to what I presented, we have

$$|(\nabla \xi_h, \nabla v)_\Omega| \leq \eta \|\nabla v\|_\Omega \quad \forall v \in H_0^1(\Omega).$$

Because $(\nabla \cdot, \nabla \cdot)_\Omega$ is actually an energy norm, we have

$$\|\nabla \xi_h\|_\Omega \leq \eta.$$

What makes the wave equation tough is the lack of inf-sup stability.

A glance at the proofs
The Helmholtz problem

We next consider the Helmholtz problem with complex frequency $s := \nu + i\rho$

$$-s^2(u, v)_\Omega + (\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega).$$

The sesquilinear form is not an energy norm. We only have

$$|s|^2 \|\xi_h\|_\Omega^2 + \|\nabla \xi_h\|_\Omega^2 \leq \left(1 + \frac{|\nu|^2}{\rho^2}\right) \eta^2.$$

Using duality techniques, this may be improved to

$$|s|^2 \|\xi_h\|_\Omega^2 + \|\nabla \xi_h\|_\Omega^2 \leq \underbrace{\left(1 + C \frac{|\nu|}{\rho} \frac{|\nu|h}{c}\right)^2}_{\text{pre factor}} \eta^2.$$



W. Dörfler and S. Sauter, 2013



T. Chaumont-Frelet, A. Ern and M. Vohralík, 2021

A glance at the proofs

The wave equation

Key observation: letting $\rho := 1/T$, we have

$$\int_0^{+\infty} \|\xi(t)\|_{\Omega}^2 e^{-2t/T} dt = \int_{\rho-i\infty}^{\rho+i\infty} \left(\frac{|s|^4}{\rho^2} \|\widehat{\xi}(s)\|_{\Omega}^2 + |s|^2 \|\widehat{\xi}(s)\|_{\Omega}^2 + \|\nabla \widehat{\xi}(s)\|_{\Omega}^2 \right) ds.$$

The Laplace transform $\widehat{u}(s)$ of $u(t)$ solves a Helmholtz problem:

$$\frac{|s|^4}{\rho} \|\widehat{\xi}_h(s)\|_{\Omega}^2 + |s|^2 \|\widehat{\xi}_h(s)\|_{\Omega}^2 + \|\nabla \widehat{\xi}_h(s)\|_{\Omega}^2 \leq \left(1 + C \frac{|\nu|^2}{\rho^2} |\nu| h \right)^2 \widehat{\eta}(s)^2.$$

Problem: the pre factor is not bounded as $\nu \rightarrow +\infty$.

Solution: use a cutoff frequency ω and do something else for $|\nu| > \omega$.

For the high-frequency content, we simply use “plain” stability:

$$\|\widehat{\xi}(s)\|_s \leq \|\widehat{u}(s)\|_s + \|\widehat{u}_h(s)\|_s \leq 4 \frac{|s|^2}{\rho^2} \|\widehat{f}(s)\|_\Omega \leq 4 \frac{1}{\rho^2} \|\widehat{f}^{(2)}(s)\|_\Omega$$

Next, we use that

$$\int_{\rho-i\infty}^{\rho+i\infty} \|\widehat{\phi}\|_\Omega^2 \chi_{|s|>\omega} ds \leq \frac{1}{\omega^{2r}} \int_0^{+\infty} \|\phi^{(r)}(t)\|_\Omega^2 e^{-2\rho t} dt$$

It easily follows that term

$$\int_{\rho-i\infty}^{\rho+i\infty} \|\widehat{\xi}_h\|_\Omega^2 \chi_{|s|>\omega} ds \leq \frac{1}{\omega^{2r}} \int_0^{+\infty} \|f^{(r+2)}(t)\|_\Omega^2 e^{-2\rho t} dt$$

can be made small by approximately picking ω and r .

Reliability follows by regrouping the low- and high-frequency terms:

$$\begin{aligned} \int_0^{+\infty} \|\xi_h(t)\|^2 e^{-2\rho t} dt &= \int_{\rho-i\infty}^{\rho+i\infty} \|\widehat{\xi}_h\|_{\Omega}^2 \chi_{|s| < \omega} ds + \int_{\rho-i\infty}^{\rho+i\infty} \|\widehat{\xi}_h\|_{\Omega}^2 \chi_{|s| > \omega} ds \\ &\leq \\ \left(1 + C \frac{\omega^2}{\rho^2} \frac{\omega h}{c}\right) \int_0^{+\infty} \eta(t)^2 e^{-2\rho t} dt &+ \frac{4}{\rho^2} \frac{1}{\omega^{2r}} \int_0^{+\infty} \|f^{(r+1)}(t)\|_{\Omega}^2 e^{-2\rho t} dt. \end{aligned}$$

The estimates I presented earlier simply rely on a clever choice of ω and r .

Concluding remarks

This is a new framework to derive a posteriori estimates for the wave equation.

It can handle space-varying coefficients and dissipative terms (with \dot{u}).

With equilibration, it gives asymptotically constant-free and p -robust estimates.

Similar (slightly weaker) results are possible with residual-based estimators.

Future challenges include time-discretization and adaptivity.



T. Chaumont-Frelet, [arxiv:2205.13939](https://arxiv.org/abs/2205.13939), 2022.