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Scattering by finely-layered obstacles: frequency-explicit stability and homogenization

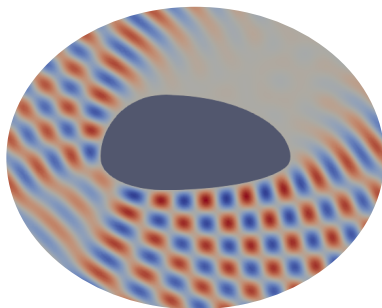
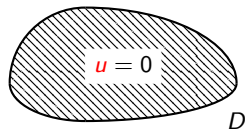
T. Chaumont-Frelet^{*} and E.A. Spence[†]

Wave Propagation in Periodic Structures: Analysis and Numerics
Conference on Mathematics of Wave Phenomena

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It is about “sending” a wave on a obstacle D , and modeling the reflections.



Time-harmonic means that we assume that wave has the form $u(\mathbf{x})e^{ikt}$, where $k > 0$ is a fixed frequency.

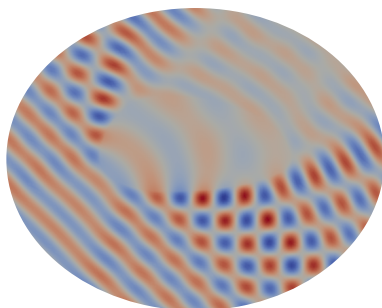
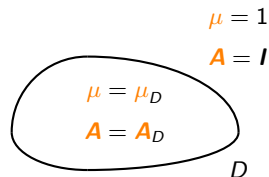
Mathematically, this is modeled by a PDE problem that looks like

$$\left\{ \begin{array}{ll} -k^2 u - \Delta u = f & \text{outside } D, \\ u = 0 & \text{on } \partial D, \\ u \text{ "radiates"} & \text{at infinity.} \end{array} \right.$$

“Fredholm + UCP” shows that there exists a unique u , and

$$k^2 \|u\| \leq C_s(k) \|f\|.$$

Frequency-explicit stability is about understanding how $C_s(k)$ depends on k .

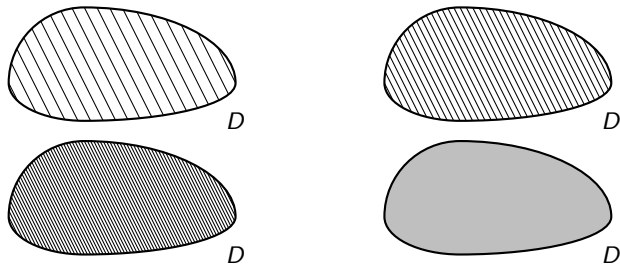


Mathematically, the problem now reads:

$$\begin{cases} -k^2 \mu u - \nabla \cdot (\mathbf{A} \nabla u) = f, \\ u \text{ "radiates" at infinity,} \end{cases}$$

and still, we are interested in $C_s(k)$.

(Highly) Heterogeneous composite obstacles



We now write this as

$$\begin{cases} -k^2 \mu_\varepsilon u_\varepsilon - \nabla \cdot (\mathbf{A}_\varepsilon \nabla u_\varepsilon) = f, \\ u_\varepsilon \text{ "radiates" at infinity,} \end{cases}$$

where ε denote the size of the layer. We want to analyze

$$k^2 \|u_\varepsilon\| \leq C_s(k, \varepsilon) \|f\|$$

“uniformly” as $k \rightarrow \infty$ and/or $\varepsilon \rightarrow 0$, and understand “ $\lim_{\varepsilon \rightarrow 0} u_\varepsilon$ ”.

- 1 Homogenization in the coercive case: why does stability matter?
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
- 3 Frequency-explicit homogenization of Helmholtz problems
- 4 Concluding remarks

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 - Mathematical setting
 - Two-scale expansion
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 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

- 1 Homogenization in the coercive case: why does stability matter?
 - **Mathematical setting**
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

We will start with the “easy” problem: find $u_\varepsilon : D \rightarrow \mathbb{R}$ such that

$$\begin{cases} -\nabla \cdot (\mathbf{A}^\varepsilon \nabla u_\varepsilon) &= f & \text{in } D, \\ u_\varepsilon &= 0 & \text{on } \partial D. \end{cases}$$

Formally, we assume that $f \in L^2(D)$, $\mathbf{A}^\varepsilon \in C^\infty(D, \mathbb{R}^{d \times d})$, and we search $u_\varepsilon \in H_0^1(D)$ such that

$$a^\varepsilon(u_\varepsilon, v) = (f, v)_D \quad \forall v \in H_0^1(D)$$

with

$$a^\varepsilon(u_\varepsilon, v) := (\mathbf{A}^\varepsilon \nabla u_\varepsilon, \nabla v)_D.$$

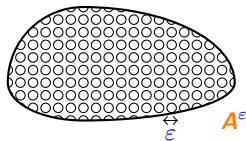
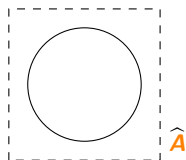
We will assume that A^ε is periodic, meaning that there exists \widehat{A} s.t.

$$A^\varepsilon(x) := \widehat{A}\left(\left\{\frac{x}{\varepsilon}\right\}\right).$$

We will also assume uniform continuity and ellipticity:

$$0 < a_* \preceq \widehat{A}(y) \preceq a^* < +\infty$$

for a.e. y in $(0, 1)^d$.



Poincaré inequality ensures that

$$\|u_\varepsilon\|_{H^1(D)}^2 \lesssim \|\nabla u_\varepsilon\|_{L^2(D)}^2 \leq \frac{1}{a_\star} \int_\Omega A^\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon = (f, u_\varepsilon)_D.$$

Hence, we have

$$\|u_\varepsilon\|_{H^1(D)} \lesssim \|f\|_{L^2(D)}$$

uniformly in ε , where the hidden constant depend only on D , a_\star .

In particular, the compact injection $H^1(D) \subset L^2(D)$ immediately ensures

$$u_{\varepsilon'} \rightarrow u_0 \text{ in } L^2(D).$$

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 - Mathematical setting
 - **Two-scale expansion**
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

We assume the following ansatz

$$u_\varepsilon(\mathbf{x}) = u_0\left(\mathbf{x}, \left\{\frac{\mathbf{x}}{\varepsilon}\right\}\right) + \varepsilon u_1\left(\mathbf{x}, \left\{\frac{\mathbf{x}}{\varepsilon}\right\}\right) + \varepsilon^2 u_2\left(\mathbf{x}, \left\{\frac{\mathbf{x}}{\varepsilon}\right\}\right) + \dots,$$

where $u_0(\mathbf{x}, \mathbf{y})$, $u_1(\mathbf{x}, \mathbf{y})$ and $u_2(\mathbf{x}, \mathbf{y})$ are \mathbf{y} -periodic.

The \mathbf{y} -periodicity assumption is "natural", since A^ε is \mathbf{y} -periodic.



Cioranescu and Donato, *Oxford university press*, 1999.

Plugging the two-scale expansion in the PDE, and using the above identity, we obtain a cascade of equations for the u_j 's by identifying powers of ε .

Specifically, we formally obtain that

$$u_1(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^d \widehat{\chi}_j(\mathbf{y}) \frac{\partial u_0}{\partial \mathbf{x}_j}(\mathbf{x}),$$

where the $\widehat{\chi}_j$ only depend on $\widehat{\mathbf{A}}$, and

$$-\nabla \cdot (\mathbf{A}^H \nabla u_0) = f \text{ in } \Omega,$$

with

$$\mathbf{A}_{j\ell}^H := \int_{(0,1)^d} \widehat{\mathbf{A}}_{j\ell} + \frac{\partial \widehat{\chi}_j}{\partial \mathbf{y}_\ell} d\mathbf{y}.$$

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 - Mathematical setting
 - Two-scale expansion
 - **Justification of the ansatz and error estimate**
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

What have we achieved so far?

By itself, the two-scale ansatz does not do much.

It gives us “feel” of the correct quantities to look at.
In particular, $\hat{\chi}$, \mathbf{A}^H , u_0 and u_1 are “correctly” defined.

But we still need to rigorously show that $u_\epsilon \rightarrow u_0$!

The first step is to observe that

$$a^\varepsilon(u_\varepsilon - u_0, v) = (f, v) - a^\varepsilon(u_0, v) = ((A^H - A^\varepsilon)\nabla u_0, \nabla v)_D.$$

It means that we can identify the error as a solution of our starting PDE.

The stability of the oscillatory problem (involving a^ε) will be important!

We have that

$$a^\varepsilon(u_\varepsilon - u_0, v) = ((A^H - A^\varepsilon)\nabla u_0, \nabla v)_D, \quad \forall v \in H_0^1(D).$$

The problem is that $A^\varepsilon \rightarrow A^H$ only in a very weak sense (H- or G-convergence).

Employing u_1 and a “boundary” corrector θ_ε , we arrive at

$$|a^\varepsilon(u_\varepsilon - (u_0 + \varepsilon u_1^\varepsilon + \theta_\varepsilon), v)| \lesssim \varepsilon \|u_0\|_{H^2(D)} \|v\|_{H^1(D)}$$

showing that

$$\|u_\varepsilon - (u_0 + \varepsilon u_1^\varepsilon + \theta_\varepsilon)\|_{H^1(D)} \lesssim \varepsilon \|u_0\|_{H^2(D)} \lesssim \varepsilon \|f\|_{L^2(D)},$$

and in particular

$$\|u_\varepsilon - u_0\|_{L^2(D)} + \|u_\varepsilon - (u_0 + \varepsilon u_1^\varepsilon)\|_{H^1(D)} \lesssim \varepsilon^{1/2} \|f\|_{L^2(D)}.$$

The fact that the oscillating PDE problem is stable uniformly in ε is paramount.

For elliptic problems, it is trivial: only the min and max of A are important.

The situation is, unfortunately, more complicated for wave propagation: each “new layer” can create a reflection, which may trap the waves.

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 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - **Mathematical setting**
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

Let

$$\mu(\mathbf{x}) = \begin{cases} \mu_D(\mathbf{x}) & \text{in } D, \\ 1 & \text{outside,} \end{cases} \quad \mathbf{A}(\mathbf{x}) = \begin{cases} \mathbf{A}_D(\mathbf{x}) & \text{in } D, \\ \mathbf{I} & \text{outside.} \end{cases}$$

Let Ω be a ball of radius ℓ containing D and $\text{supp } f$. Then, we consider:

$$\begin{cases} -k^2 \mu \mathbf{u} - \nabla \cdot (\mathbf{A} \nabla \mathbf{u}) & = f & \text{in } \Omega, \\ \nabla \mathbf{u} \cdot \mathbf{n} - i k \mathbf{u} & = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, I will use a first-order absorbing BC on $\partial\Omega$ for the sake of simplicity. The “real” DtN operator can be handled as well, but it is slightly more technical.

We will need a technical assumption on the geometry of D .
To simplify for the moment, let us just assume that D is convex.

We also assume that $\mu_D \leq 1$ and $I \preceq A_D$ inside D .
It means that the wavespeed is higher inside D :

$$c_D = \sqrt{\frac{\min A_D}{\max \mu_D}} \geq 1 = c_{\text{out}}.$$

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - **An intuitive justification of our assumptions**
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

Our assumptions concern the geometry of D and the wavespeed inside.

These two assumptions are actually very natural.
They can be explained by considering geometrical optics.

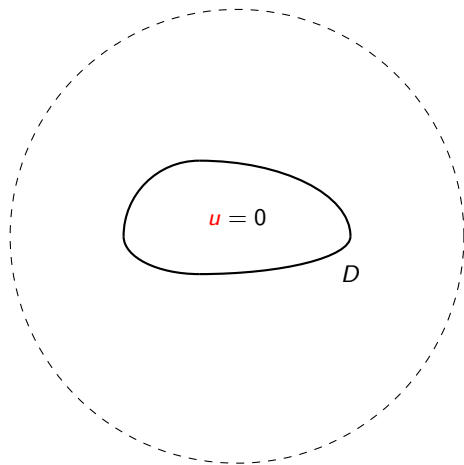
Here, I just want to give an “intuitive” feel of what’s going on.

However, a rigorous link between rays and stability of high-frequency problems can be established via semi-classical analysis.

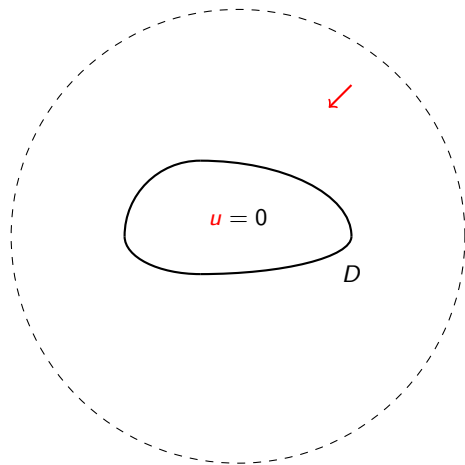


Zworski, *AMS*, 2012.

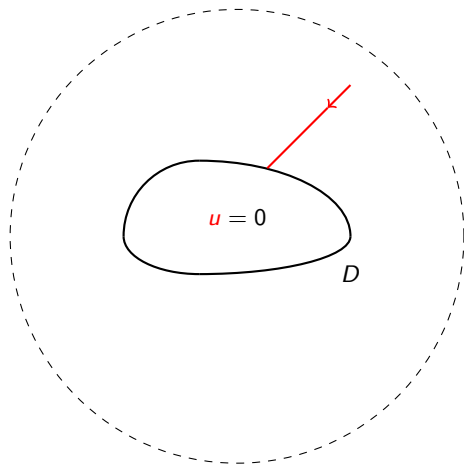
A non-trapping obstacle



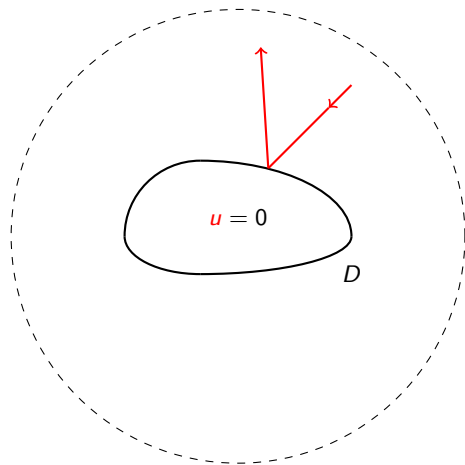
A non-trapping obstacle



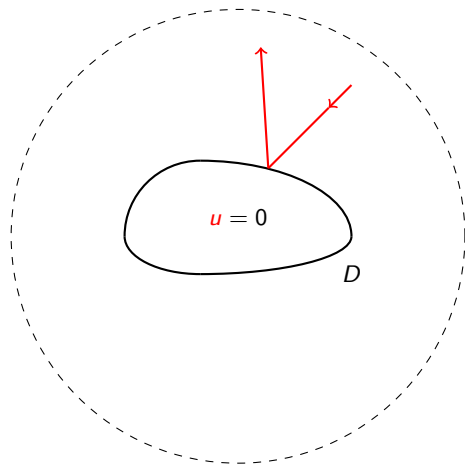
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A non-trapping obstacle

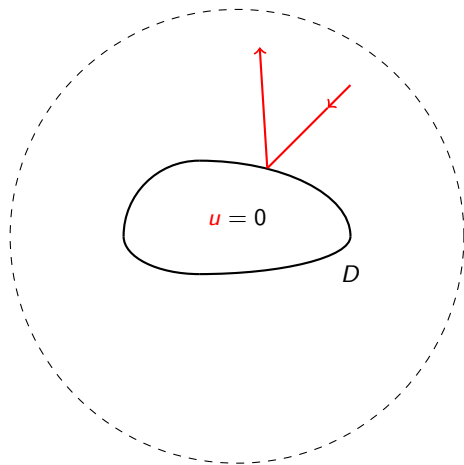


A non-trapping obstacle



All rays escape Ω .

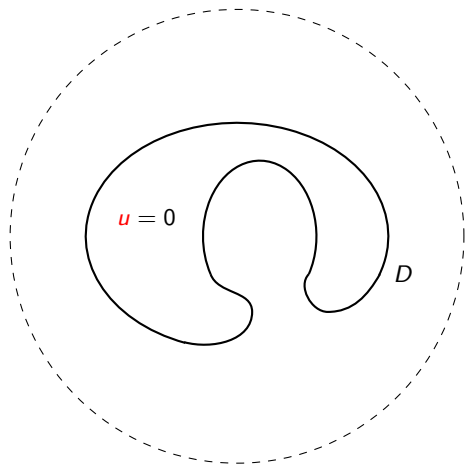
A non-trapping obstacle



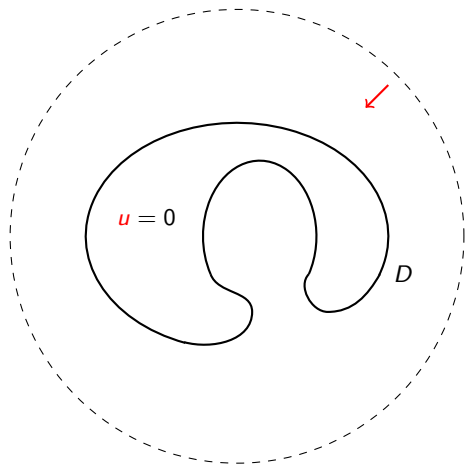
All rays escape Ω .

$$k^2 \|u\|_{\Omega} \lesssim k\ell \|f\|_{\Omega}$$

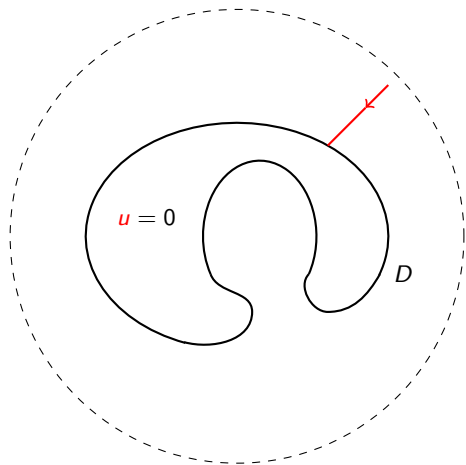
A trapping obstacle



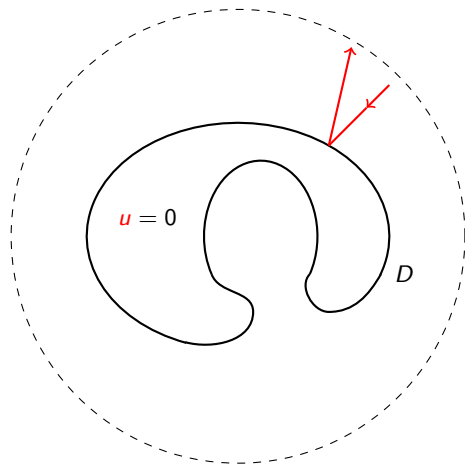
A trapping obstacle



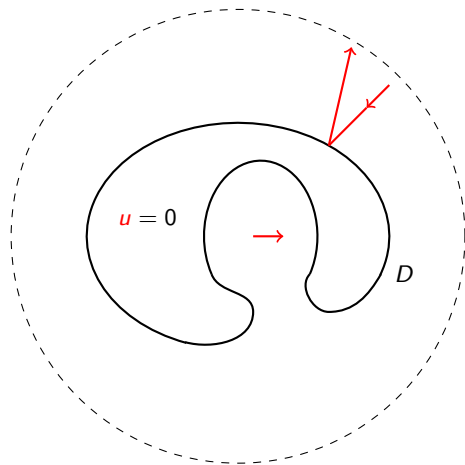
A trapping obstacle



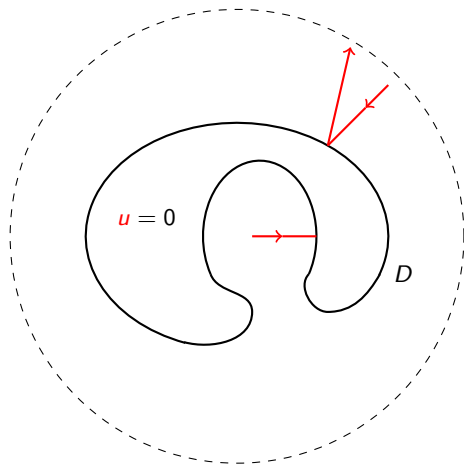
A trapping obstacle



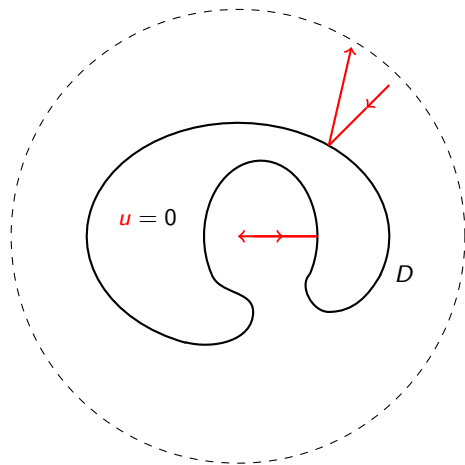
A trapping obstacle



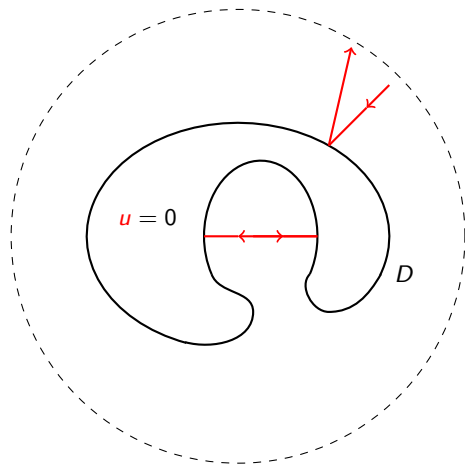
A trapping obstacle



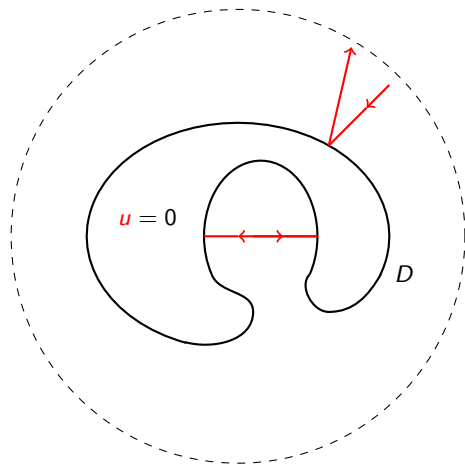
A trapping obstacle



A trapping obstacle

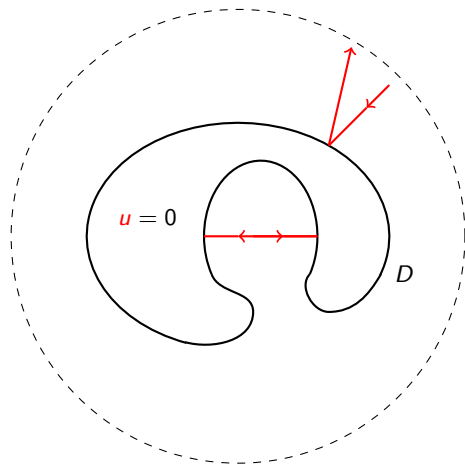


A trapping obstacle



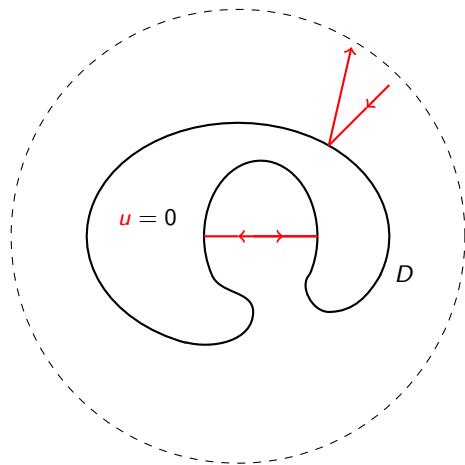
Some rays are trapped.

A trapping obstacle



Some rays are trapped.
Their trajectory is stable.

A trapping obstacle



Some rays are trapped.
Their trajectory is stable.

$$k^2 \|u\|_{\Omega} \lesssim \exp(\alpha k \ell) \|f\|_{\Omega}$$

Our assumption on the geometry of D prevents strongly trapped rays.

It avoids the “exponential only” stability.

We won't need ε “exponentially small” as k increases.

How do the rays behave when leaving the obstacle?
The answer is provided by Snell's law.

Recalling that $c = 1$ outside D , Snell's law reads

$$\sin(\theta_{\text{in}}) = c_D \sin(\theta_{\text{out}}).$$

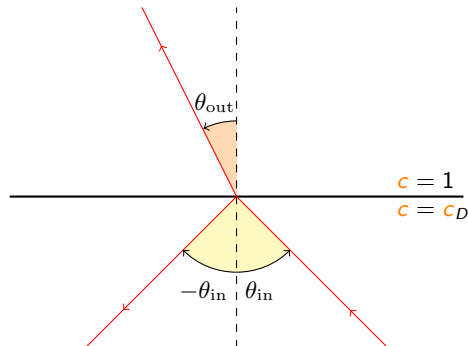
It will be important to distinguish between the cases where $c_D > 1$ and $c_D < 1$.

Leaving the obstacle when $c_D > 1$

Given θ_{in} , the equation

$$\sin(\theta_{\text{in}}) = c_D \sin(\theta_{\text{out}})$$

is always solvable for θ_{out} .

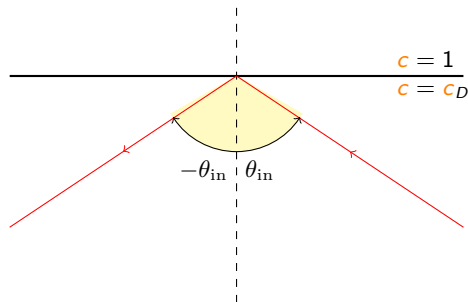


Leaving the obstacle when $c_D < 1$

Now, the equation

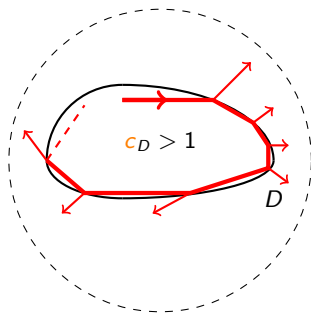
$$\sin(\theta_{\text{in}}) = c_D \sin(\theta_{\text{out}})$$

does not have a solution if $\sin(\theta_{\text{in}}) > c_D$.

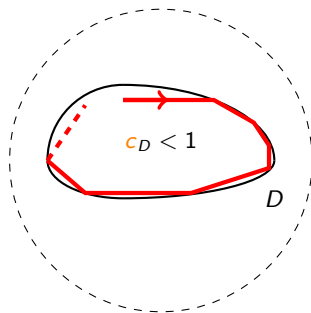


In such case, total reflection occurs, similar to a Dirichlet boundary.

When $c_D < 1$ rays can be trapped inside can be trapped inside the obstacle.



$$k^2 \|u\|_{\Omega} \lesssim kl \|f\|_{\Omega}$$



$$k^2 \|u\|_{\Omega} \lesssim \exp(\alpha kl) \|f\|_{\Omega}$$

We assume that $\mu_D \leq 1$ and $l \preceq A_D$, which actually implies that $c_D \geq 1$.

This second assumption rules out whispering gallery modes, and again prevents “exponential only” stability.

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - **Basic theory of Morawerz multipliers**
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

Semi-classical (or micro-local) techniques. (not good as $\varepsilon \rightarrow 0$)



Zworski, *AMS*, 2012.

Semi-classical (or micro-local) techniques. (not good as $\varepsilon \rightarrow 0$)



Zworski, *AMS*, 2012.

Morawetz multiplier: multiply the PDE by $x \cdot \nabla u$. (does not quite work)



Morawetz, *Comm. Pure Appl. Math.*, 1975.



Melenk, *Ph.D. thesis*, 1995.



Barucq, Chaumont-Frelet and Gout, *Math. Comp.*, 2017.



Spence and Moiola, *Models Meth. Appl. Math.*, 2019.

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Spence and Moiola, *Models Meth. Appl. Math.*, 2019.

Directional multiplier: multiply the PDE by $\mathbf{x}_d \partial_d u$. (the correct idea)



Chandler-Wilde, Monk, *SIAM J. Math. Anal.*, 2005.



Chandler-Wilde, Spence, Gibbs and Smyshlyaev, *SIAM J. Math. Anal.*, 2020.



Chaumont-Frelet and Spence, *arxiv*, 2021.

For all $u \in H^1(\Omega)$, it holds that

$$b(u, u) = -k^2 \|u\|_{\Omega}^2 - ik \|u\|_{\partial\Omega}^2 + \|\nabla u\|_{\Omega}^2.$$

The imaginary part gives us control over the boundary

$$-\operatorname{Im} b(u, u) = k \|u\|_{\partial\Omega}^2.$$

The real part does not give immediate information, as it is indefinite

$$\operatorname{Re} b(u, u) = \|\nabla u\|_{\Omega}^2 - k^2 \|u\|_{\Omega}^2.$$

Other test functions are needed to obtain more information.

The Morawetz multiplier in homogeneous media

The Morawetz multiplier is the test function $\mathbf{x} \cdot \nabla u$.

Let us first consider a homogeneous media.

It integrates by parts as follows in the (u, v) term

$$2 \operatorname{Re} \int_{\Omega} \overline{u \mathbf{x} \cdot \nabla u} = \int_{\Omega} \mathbf{x} \cdot \nabla |u|^2 = \int_{\partial\Omega} \mathbf{x} \cdot \mathbf{n} |u|^2 - 2 \int_{\Omega} |u|^2,$$

and in the $(\nabla u, \nabla v)$ term

$$2 \operatorname{Re} \int_{\Omega} \nabla u \cdot \nabla (\overline{\mathbf{x} \cdot \nabla u}) = \int_{\partial\Omega} \mathbf{x} \cdot \mathbf{n} |\nabla u|^2.$$

Using the variational formulation

$$b(\mathbf{u}, \mathbf{x} \cdot \nabla \mathbf{u}) = (f, \mathbf{x} \cdot \nabla \mathbf{u})_{\Omega},$$

we arrive at the “Morawetz identity”

$$2k^2 \|\mathbf{u}\|_{\Omega}^2 + \int_{\partial\Omega} |\nabla \mathbf{u}|^2 \mathbf{x} \cdot \mathbf{n} = \\ 2 \operatorname{Re}(f, \mathbf{x} \cdot \nabla \mathbf{u})_{\Omega} + 2 \operatorname{Re}\{ik(\mathbf{u}, \mathbf{x} \cdot \nabla \mathbf{u})_{\partial\Omega}\} + k^2 \int_{\partial\Omega} |\mathbf{u}|^2 \mathbf{x} \cdot \mathbf{n},$$

and the rhs can be controlled using $b(\mathbf{u}, \mathbf{u}) = (f, \mathbf{u})$, leading to

$$k^2 \|\mathbf{u}\|_{\Omega} \lesssim k\ell \|f\|_{\Omega}.$$

What happens in heterogeneous media?

Let us consider, for simplicity, that $\mathbf{A} = \mathbf{I}$, but that μ varies.

Then, the $(\nabla \mathbf{u}, \nabla \mathbf{v})$ term does not change. For the $(\mu \mathbf{u}, \mathbf{v})$ term, we have

$$2 \operatorname{Re} \int_{\Omega} \mu \mathbf{u} \overline{\nabla \mathbf{u}} = \int_{\Omega} \mu \mathbf{x} \cdot \nabla |\mathbf{u}|^2 = \int_{\partial\Omega} \mu \mathbf{x} \cdot \mathbf{n} |\mathbf{u}|^2 - \int_{\Omega} \nabla \cdot (\mu \mathbf{x}) |\mathbf{u}|^2,$$

Then, $\nabla \cdot (\mu \mathbf{x}) = 2\mu + \mathbf{x} \cdot \nabla \mu$, leading to

$$2 \operatorname{Re} \int_{\Omega} \mu \mathbf{u} \overline{\nabla \mathbf{u}} = \int_{\partial\Omega} \mu \mathbf{x} \cdot \mathbf{n} |\mathbf{u}|^2 - 2 \int_{\Omega} \mu |\mathbf{u}|^2 - \underbrace{\int_{\Omega} (\mathbf{x} \cdot \nabla \mu) |\mathbf{u}|^2}_{\text{Additional term}}.$$

The new term is not a problem if $\mathbf{x} \cdot \nabla \mu \geq 0$, i.e.

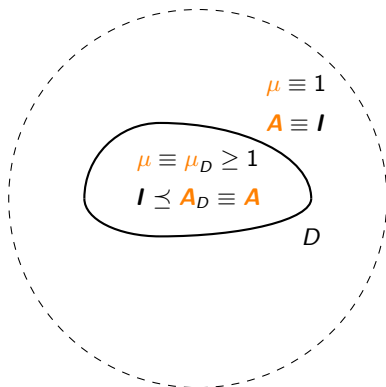
$$\frac{\partial \mu}{\partial r} \geq 0.$$

Under the assumption that

$$\frac{\partial \mu}{\partial r} \geq 0 \quad \frac{\partial \mathbf{A}}{\partial r} \preceq 0$$

the following estimate holds true

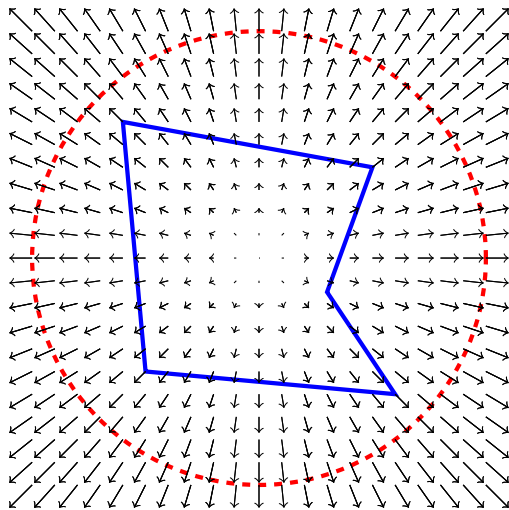
$$k^2 \|u\|_{\Omega} \lesssim k \ell \|f\|_{\Omega}.$$



This technique gives the correct estimate for an homogeneous obstacle.

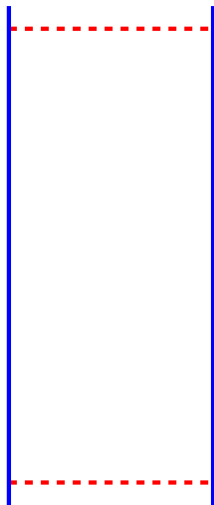
Still, we cannot handle oscillating coefficients with this technique.

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - **Directional multipliers**
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

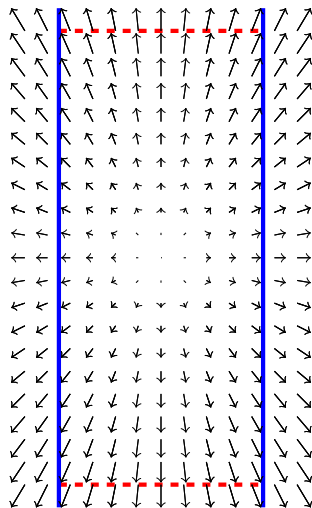


The vector field \mathbf{x} (recall the test function $\mathbf{x} \cdot \nabla u$)

Directional multiplier in a waveguide

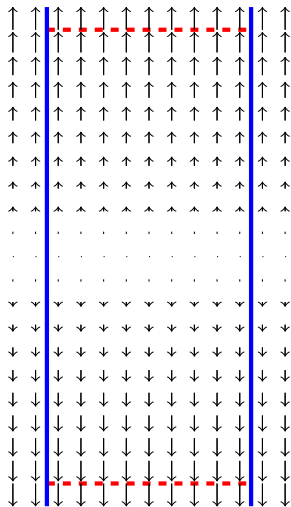


Directional multiplier in a waveguide



The vector field x

Directional multiplier in a waveguide



The vector field $x_d \mathbf{e}_d$

Observe that $(\mathbf{x}_d \mathbf{e}_d) \cdot \nabla \mathbf{u} = \mathbf{x}_d \partial_d \mathbf{u}$. Then

$$2 \operatorname{Re} \int_{\Omega} \mathbf{u} \overline{\mathbf{x}_d \partial_d \mathbf{u}} = \int_{\Omega} \mathbf{x}_d \partial_d |\mathbf{u}|^2 = \int_{\partial\Omega} \mathbf{x}_d \mathbf{n}_d |\mathbf{u}|^2 - \int_{\Omega} |\mathbf{u}|^2$$

and

$$2 \operatorname{Re} \int_{\Omega} \nabla \mathbf{u} \cdot \nabla (\overline{\mathbf{x}_d \partial_d \mathbf{u}}) = \int_{\partial\Omega} \mathbf{x}_d \mathbf{n}_d |\nabla \mathbf{u}|^2 - \int_{\Omega} |\partial_1 \mathbf{u}|^2$$

$\mathbf{x} \cdot \mathbf{n}$ is replaced by $\mathbf{x}_d \mathbf{n}_d$ in the boundary terms.

We have no boundary terms when $\mathbf{n}_d = 0$ (vertical boundaries on the slide).

Using that $b(\mathbf{u}, \mathbf{x}_d \partial_d \mathbf{u}) = (\mathbf{f}, \mathbf{x}_d \partial_d \mathbf{u})$, we have

$$k^2 \|\mathbf{u}\|_{\Omega}^2 + \int_{\partial\Omega} |\nabla \mathbf{u}|^2 \mathbf{x}_d \mathbf{n}_d = \underbrace{\|\partial_1 \mathbf{u}\|_{\Omega}^2}_{\text{new term}} + 2 \operatorname{Re}(\mathbf{f}, \mathbf{x}_d \partial_d \mathbf{u})_{\Omega} + 2 \operatorname{Re}\{ik(\mathbf{u}, \mathbf{x}_d \partial_d \mathbf{u})_{\partial\Omega}\} + k^2 \int_{\partial\Omega} |\mathbf{u}|^2 \mathbf{x}_d \mathbf{n}_d,$$

and we eventually arrive at

$$k^2 \|\mathbf{u}\|_{\Omega} \lesssim (kl)^3 \|\mathbf{f}\|_{\Omega}.$$

We lose two powers kl , but we can handle other geometries.

What about heterogeneous media?

With the Morawetz multiplier, we had requirements of the form

$$\mathbf{x} \cdot \nabla \mu \geq 0 \quad 0 \preceq \mathbf{x} \cdot \nabla \mathbf{A},$$

leading to radial growth conditions.

Here, we only integrate by part along \mathbf{x}_d , leading to the conditions

$$\mathbf{x}_d \frac{\partial \mu_d}{\partial \mathbf{x}_d} \geq 0 \quad 0 \preceq \mathbf{x}_d \frac{\partial \mathbf{A}_d}{\partial \mathbf{x}_d}.$$

The coefficients are free to oscillate in $d - 1$ directions:

- finely layered obstacles in 2D,
- finely fibred obstacles in 3D.

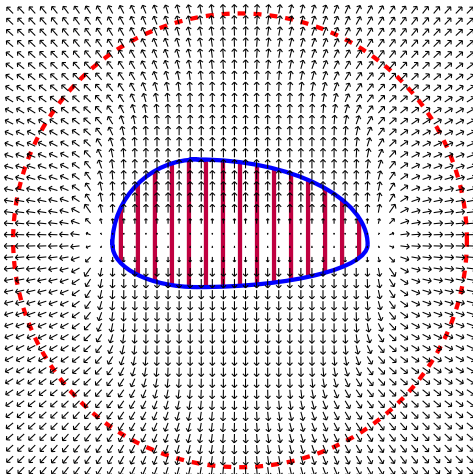
The last key idea consists in mixing the two multipliers.

Introducing a cutoff χ , we multiply the PDE with

$$\chi(|\mathbf{x}|)\mathbf{x} \cdot \nabla \mathbf{u} + (1 - \chi(|\mathbf{x}|))\mathbf{x}_d \partial_d \mathbf{u}.$$

That way, we have the “usual” multiplier close to the absorbing boundary, and we have the directional multiplier close to the obstacle.

Mixing the two kind of multipliers: finely layered obstacle

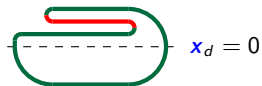
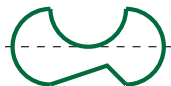
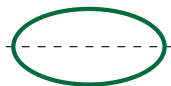


- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - **Precise statement of our stability result**
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

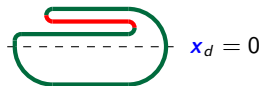
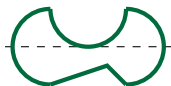
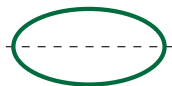
$D \subset \mathbb{R}^d$ is a Lipschitz domain such that $\mathbf{x}_d \mathbf{n}_d \geq 0$ on ∂D .

Precise assumptions

$D \subset \mathbb{R}^d$ is a Lipschitz domain such that $x_d \mathbf{n}_d \geq 0$ on ∂D .



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$\mu = 1$ and $\mathbf{A} = \mathbf{I}$ outside D . $0 < \mu_* \leq \mu \leq 1$ and $\mathbf{I} \preceq \mathbf{A}$ in D .

$D \subset \mathbb{R}^d$ is a Lipschitz domain such that $x_d \mathbf{n}_d \geq 0$ on ∂D .



$\mu = 1$ and $\mathbf{A} = \mathbf{I}$ outside D . $0 < \mu_* \leq \mu \leq 1$ and $\mathbf{I} \preceq \mathbf{A}$ in D .

$\mu|_D$ is a $L^\infty(D)$ function such that

$$x_d(\mu(\mathbf{x} + h\mathbf{e}_d) - \mu(\mathbf{x})) \geq 0 \quad h > 0, \mathbf{x} \in D.$$

$\mathbf{A}|_D$ is a $L^\infty(D, \mathbb{R}^{d \times d})$ symmetric matrix such that

$$x_d(\mathbf{A}(\mathbf{x} + h\mathbf{e}_d) - \mathbf{A}(\mathbf{x})) \preceq 0 \quad h > 0, \mathbf{x} \in D,$$

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and $\mathbf{A}_{\ell,d} = 0$ if $\ell \neq d$, i.e.

$$\mathbf{A}|_D = \begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix} \quad \text{or} \quad \mathbf{A}|_D = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{pmatrix}.$$

Main stability result

Let $f \in L^2(\mathbb{R}^d)$ with $\text{supp } f \subset \Omega := B(0, \ell)$.

Let T_k be the DtN operator with radiation condition outside Ω .

Assume that $k\ell \geq C_{\text{wave}}$.

Under the above assumptions, there is a unique $u \in H^1(\Omega)$ such that

$$-k^2(\mu u, v)_\Omega - \langle T_k u, v \rangle_{\partial\Omega} + (A \nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H^1(\Omega).$$

In addition, the following stability estimate holds true

$$k^2 \|u\|_\Omega \leq C_{\text{stab}} (k\ell)^3 \|f\|_\Omega,$$

where C_{stab} only depends on the minimum and maximum values of μ and A .

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - **Mathematical setting**
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

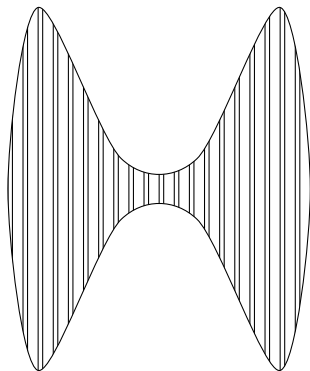
We assume that ∂D is $C^{1,1}$ with $\mathbf{x}_d \mathbf{n}_d \geq 0$ on ∂D .

We consider coefficients of the form

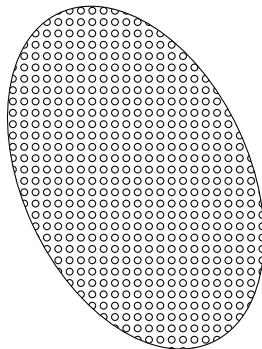
$$\mu_\varepsilon(\mathbf{x}) = \begin{cases} \widehat{\mu}(\{\mathbf{x}/\varepsilon\}) & \text{in } D \\ 1 & \text{outside} \end{cases} \quad \mathbf{A}_\varepsilon(\mathbf{x}) = \begin{cases} \widehat{\mathbf{A}}(\{\mathbf{x}/\varepsilon\}) & \text{in } D \\ \mathbf{I} & \text{outside} \end{cases}$$

where $\widehat{\mu}$ and $\widehat{\mathbf{A}}$ are independent of \mathbf{y}_d , piecewise smooth, and $\widehat{\mathbf{A}}_{\ell,d} = 0$ if $\ell \neq d$.

We further assume that $\widehat{\mu} \geq 1$ and $\mathbf{I} \preceq \widehat{\mathbf{A}}$.



2D case



$x_3 = 0$ cut of a 3D case

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - **Main results**
 - A discussion on the error estimate
- 4 Concluding remarks

Assume that $kl \geq C_{\text{wave}}$ and let $f \in L^2(\Omega)$.

For all $\varepsilon > 0$, there exists a unique $u_\varepsilon \in H^1(\Omega)$ such that

$$-k^2(\mu_\varepsilon u_\varepsilon, v)_\Omega - \langle T_k u_\varepsilon, v \rangle_{\partial\Omega} + (A_\varepsilon \nabla u_\varepsilon, \nabla v)_\Omega = (f, v)_\Omega \quad \forall v \in H^1(\Omega).$$

In addition, we have

$$k^2 \|u_\varepsilon\|_\Omega \leq C_{\text{stab}} (kl)^3 \|f\|_\Omega,$$

where C_{stab} does not depend on ε .

Consider the usual homogenized coefficients

$$\mu^H|_D := \int_{(0,1)^d} \hat{\mu}, \quad \mathbf{A}_{j\ell}^H|_D := \int_{(0,1)^d} \hat{\mathbf{A}}_{j\ell} + \frac{\partial \chi_j}{\partial \mathbf{y}_\ell},$$

with $\mu^H = 1$ and $\mathbf{A}^H = \mathbf{I}$ outside D .

The homogenized solution \mathbf{u}_0 is the only element of $H^1(\Omega)$ such that

$$-k^2(\mu^H \mathbf{u}_0, \mathbf{v})_\Omega - \langle T_k \mathbf{u}_0, \mathbf{v} \rangle_{\partial\Omega} + (\mathbf{A}^H \nabla \mathbf{u}_0, \nabla \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega \quad \forall \mathbf{v} \in H^1(\Omega).$$

The homogenized solution satisfies

$$k^2 \|\mathbf{u}_0\|_\Omega \leq C_{\text{stab}} (k\ell)^3 \|\mathbf{f}\|_\Omega.$$

We also obtain the usual error estimate using the first-order corrector.

Specifically, the following estimate holds true:

$$k^2 \|u_\varepsilon - u_0\|_\Omega + k \|\nabla(u_\varepsilon - (u_0 + \varepsilon u_1^\varepsilon))\|_\Omega \lesssim (kl)^6 (k\varepsilon)^{1/2} \|f\|_\Omega.$$

Importantly, we obtain frequency-explicit constants.

$$(kl)^6 = (kl)^3 \times (kl)^3 \Rightarrow \text{we used the stability estimate twice!}$$

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

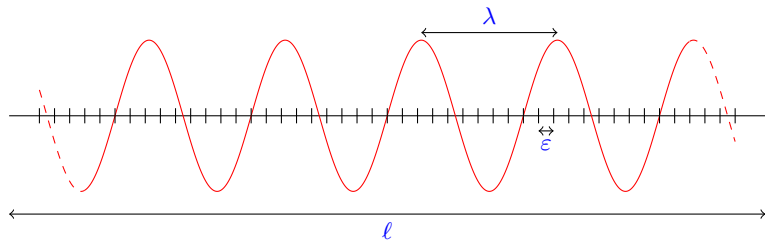
Recall the estimate

$$k^2 \|u_\varepsilon - u_0\|_\Omega \lesssim (kl)^6 (k\varepsilon)^{1/2} \|f\|_\Omega.$$

Since $k \simeq \lambda^{-1}$, $k\varepsilon \ll 1$ means that $\varepsilon \ll \lambda$:
the size of the layers (or fibers) is small w.r.t. the wavelength.

On the other hand, we have $kl \gg 1$, if $\lambda \ll \ell$:
the obstacle and/or the support of f are big w.r.t to the wavelength.

Key physical quantities



The presence of $k\epsilon \simeq \epsilon/\lambda$ is “intuitively” expected in the estimate.

$kl \simeq l/\lambda$ appears because the error accumulates over wavelengths.

Recall the error estimate

$$k^2 \|u_\varepsilon - u_0\|_\Omega \lesssim (kl)^6 (k\varepsilon)^{1/2} \|f\|_\Omega.$$

We can make the following interpretation

$$(kl)^{1/2} (k\varepsilon)^{1/2} \simeq \sqrt{\text{"\# wavelengths"} \times \text{"\varepsilon/\lambda"}.$$

The presence of both kl and $k\varepsilon$ seems to be natural, but it is not clear whether the exponent 6 in our estimate is sharp.

- 1 Homogenization in the coercive case: why does stability matter?
 - Mathematical setting
 - Two-scale expansion
 - Justification of the ansatz and error estimate
- 2 Stability of Helmholtz problems with finely layered/fibred obstacles
 - Mathematical setting
 - An intuitive justification of our assumptions
 - Basic theory of Morawerz multipliers
 - Directional multipliers
 - Precise statement of our stability result
- 3 Frequency-explicit homogenization of Helmholtz problems
 - Mathematical setting
 - Main results
 - A discussion on the error estimate
- 4 Concluding remarks

Consider a non-trapping homogeneous penetrable with $c_D > 1$. The estimate

$$k^2 \|u\| \lesssim k\ell \|f\|$$

is well-known.

Consider a non-trapping homogeneous penetrable with $c_D > 1$. The estimate

$$k^2 \|u\| \lesssim kl \|f\|$$

is well-known.

Our new estimate also applies if the coefficients oscillate in $d - 1$ directions

$$k^2 \|u\| \lesssim (kl)^3 \|f\|,$$

and is suited for finely layered/fibred obstacles.

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and is suited for finely layered/fibred obstacles.

In this case, we provide a new homogenization error estimate

$$k^2 \|u_\varepsilon - u_0\| + k \|\nabla u_\varepsilon - (u_0 + \varepsilon u_1^\varepsilon)\| \lesssim (kl)^6 (k\varepsilon)^{1/2} \|f\|,$$

and the sharpness of the exponent 6 is an open question.

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Chaumont-Frelet and Spence, *arxiv*, 2021.