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On consistent discretization of a homogeneous 2-SM controller

Andrey Polyakov¹, Denis Efimov¹, Xubin Ping²

Abstract—The paper presents a discretization scheme for a homogeneous second order sliding mode (2-SM) controller, which preserves the finite-time stability of the closed-loop system even in the case the sampled-time implementation of the control. Numerical simulations confirm the efficiency of the proposed scheme for chattering reduction.

I. INTRODUCTION

An asymptotically stable homogeneous system is finite-time stable in the case of negative homogeneity degree and nearly fixed-time stable in the case of the positive homogeneity degree (see, e.g. [1], [2], [3]). However, the finite/fixed-time stability is a fragile property in the sense that an improper discretization of a finite-time or a fixed-time stable ordinary differential equation (ODE) results in chattering [4], [5] or even in a finite-time blow up [6]. That is why the discretization issues are very important for practical implementation of finite/fixed-time sliding mode (SM) control/estimation algorithms [4], [7], [8], [9], [10], [11], [12], [13].

The concept of consistent discretization introduced in [14] postulates that stability properties of a continuous-time system are preserved in its discrete-time counterpart (approximation). Consistent discretizations for stable generalized homogeneous ODEs were developed in [14], [15] based on Lyapunov function theory. Some schemes with state dependent discretization step were given in [16]. Being efficient for numerical simulations, the mentioned schemes do not allow a consistent discretization (sampled-time implementation) of *finite-time controllers* in the general case. To the best of authors' knowledge, such implementations are developed only for the conventional (first order) SM algorithms [4], [17] as well as for the twisting algorithm [9] and super-twisting algorithm [18] based on the implicit method. This paper presents a consistent discretization for a homogeneous 2-SM controller studied in [19]. The latter controller is a version of the so-called quasi-continuous SM algorithm [20], which has a discontinuity only in the origin. It is shown that the sampled-time implementation of the controller according to the developed scheme preserves the finite-time stability property of the original closed-loop continuous-time system in the disturbance-free case. Numerical simulations show the

efficiency of this scheme for a chattering reduction even in the case of matched perturbations.

Notation: \mathbb{N} is the set of natural numbers; \mathbb{R} is the field of real numbers; \mathbb{C} is the field of complex numbers; $\mathbf{0}$ is the zero of a vector space (e.g., the zero vector in \mathbb{R}^n or the zero matrix in $\mathbb{R}^{n \times m}$); $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix; $P \succ 0$ denotes positive definiteness of a matrix $P = P^\top \in \mathbb{R}^{n \times n}$; $\lambda_{\max}(P)$ is a maximum eigenvalue of a symmetric matrix P ; $\|x\| = \sqrt{x^\top P x}$ with $P \succ 0$ denotes the weighted Euclidean norm in \mathbb{R}^n ; \mathcal{K} denotes a class of strictly increasing positive definite functions $[0, +\infty) \mapsto [0, +\infty)$; a function $\sigma \in \mathcal{K}$ of the class \mathcal{K}^∞ if $\sigma(s) \rightarrow +\infty$ as $s \rightarrow +\infty$; the set of continuous maps $X \mapsto Y$ between open connected sets X and Y of finite-dimensional vectors spaces is denoted by $C(X, Y)$.

II. PROBLEM STATEMENT

Let us consider a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is control input and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are known matrices.

Recall [21] that a system $\dot{x} = f(t, x)$, $t > 0$, $x(0) = x_0$ is globally uniformly *finite-time stable* if it is Lyapunov stable and there exists a locally bounded function $T : \mathbb{R}^n \mapsto [0, +\infty)$ such that any trajectory of the system vanishes to zero in a finite time: $x(t, x_0) = \mathbf{0}$, $\forall t \geq T(x_0)$, $\forall x_0 \in \mathbb{R}^n$.

Definition 1: Let the system (1) with a feedback $u \in C(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}^m)$ be globally uniformly finite-time stable. A family of functions $\tilde{u}_h : \mathbb{R}^n \mapsto \mathbb{R}^m$ parameterized by a scalar $h > 0$ is said to be a *consistent discretization* of u if

- **Finite-time stability:** the closed-loop system (1)

$$u(t) = \tilde{u}_h(x(t_i)), \quad \forall t \in [t_i, t_{i+1}), \quad t_i = ih, i = 0, 1, \dots \quad (2)$$

is globally uniformly finite-time stable;

- **Approximation:** $\forall r_1 > 0, \forall r_2 > r_1, \exists \omega_{r_1, r_2} \in \mathcal{K} :$

$$\sup_{r_1 \leq \|x\| \leq r_2} \|\tilde{u}_h(x) - u(x)\| \leq \omega_{r_1, r_2}(h). \quad (3)$$

The aim of this paper is to develop a consistent (in the sense of the above definition) discretization for a homogeneous 2-SM controller [19]. In this case, A and B satisfy the following assumption.

Assumption 1: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

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III. PRELIMINARIES: HOMOGENEOUS SYSTEMS

A. Linear dilation and homogeneous norm

In this paper we deal only with the so-called linear geometric dilation [22, Chapter 6] in \mathbb{R}^n given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{\infty} \frac{(sG_{\mathbf{d}})^i}{i!}, \quad s \in \mathbb{R}, \quad (4)$$

where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is an anti-Hurwitz matrix¹ known as the *generator of linear dilation*. The latter guarantees that \mathbf{d} satisfies the limit property ($\|\mathbf{d}(s)x\| \rightarrow 0$ as $s \rightarrow -\infty$ and $\|\mathbf{d}(s)x\| \rightarrow +\infty$ as $s \rightarrow +\infty$) required for a group to be a dilation in \mathbb{R}^n (see, e.g., [23]). The dilation introduces an alternative norm topology in \mathbb{R}^n by means the so-called canonical homogeneous norm.

Definition 2: [22] The functional $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$ given by $\|x\|_{\mathbf{d}} = 0$ for $x = \mathbf{0}$ and

$$\|x\|_{\mathbf{d}} = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (5)$$

is called the *canonical homogeneous norm* in \mathbb{R}^n , where \mathbf{d} is a monotone dilation².

Notice that $\|x\| = 1$ (resp. $\|x\| \leq 1$) is equivalent to $\|x\|_{\mathbf{d}} = 1$ (resp. $\|x\|_{\mathbf{d}} \leq 1$). For the uniform dilation $\mathbf{d}(s) = e^s I_n$, $s \in \mathbb{R}$ we have $\|\cdot\| = \|\cdot\|_{\mathbf{d}}$.

Theorem 1: [22] If \mathbf{d} is a monotone dilation, the canonical homogeneous norm is continuous on \mathbb{R}^n , locally Lipschitz continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and there exist $\underline{\sigma}, \bar{\sigma} \in \mathcal{K}^\infty$ such that

$$\underline{\sigma}(\|x\|) \leq \|x\|_{\mathbf{d}} \leq \bar{\sigma}(\|x\|), \quad \forall x \in \mathbb{R}^n.$$

Moreover, $\|\cdot\|_{\mathbf{d}}$ is differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ provided that $\|\cdot\|$ is differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. For $\|x\| = \sqrt{x^\top P x}$ with the symmetric matrix $P \in \mathbb{R}^{n \times n}$ satisfying

$$P \succ 0, \quad P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0$$

we have

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}, \quad \forall x \neq \mathbf{0}. \quad (6)$$

Below the canonical homogeneous norm is used as a Lyapunov function for analysis and design of a homogeneous SMC.

B. Homogeneous functions and vector fields

Definition 3: [23] A vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ (resp. a function $h : \mathbb{R}^n \mapsto \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$ if $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s)f(x)$ (resp. $h(\mathbf{d}(s)x) = e^{\mu s} h(x)$), for all $x \in \mathbb{R}^n$, $s \in \mathbb{R}$.

Homogeneity of a function (operator) is inherited by any other object induced by this function. For example, the Euler homogeneous function theorem implies that the derivative of the homogeneous function is homogeneous as well. If a vector field f is \mathbf{d} -homogeneous of degree μ then solutions of the ODE

$$\dot{x} = f(x) \quad (7)$$

¹A matrix $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is called anti-Hurwitz if $-G_{\mathbf{d}}$ is Hurwitz.

²A dilation in \mathbb{R}^n is monotone if and only if for any $x \in \mathbb{R}^n$ the function $s \mapsto \|\mathbf{d}(s)x\|$, $s \in \mathbb{R}$ is monotone.

are symmetric [23]:

$$x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\mu s}t, x_0),$$

where $x(t, z)$ denotes a solution of (7) with the initial condition $x(0) = z$.

Example 1: [24] The linear vector field $x \mapsto Ax$, $A \in \mathbb{R}^{n \times n}$ is \mathbf{d} -homogeneous of the degree $\mu \neq 0 \Leftrightarrow A$ is nilpotent $\Leftrightarrow AG_{\mathbf{d}} = (\mu I_n + G_{\mathbf{d}})A$.

The homogeneity degree specifies the convergence rate.

Theorem 2: [2] Let the vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and \mathbf{d} -homogeneous of a degree $\mu \in \mathbb{R}$. If the system (7) is asymptotically stable then it is globally finite-time stable for $\mu < 0$.

The homogeneous control systems are robust (input-to-state stable) with respect to a rather large class of perturbations [25], [3].

C. Homogeneous stabilization of linear plant

The following theorem combines the results of [19], [22].

Theorem 3: Let the pair $\{A, B\}$ be controllable. Then

- 1) the linear algebraic equation

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = \mathbf{0} \quad (8)$$

has a solution with respect to the pair $Y_0 \in \mathbb{R}^{m \times n}$, $G_0 \in \mathbb{R}^{n \times n}$, and $G_0 - I_n$ is invertible, $G_{\mathbf{d}} = I_n + \mu G_0$ is anti-Hurwitz for any $\mu \in [-1, 1/k]$, where k is minimal natural number such that $\text{rank}[B, AB, \dots, A^{k-1}B] = n$, the matrix $A_0 = A + BY_0(G_0 - I_n)^{-1}$ satisfies

$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0, \quad G_{\mathbf{d}}B = B; \quad (9)$$

- 2) the system of linear matrix inequalities (LMIs) with the linear matrix equation

$$A_0X + XA_0^\top + BY + Y^\top B^\top + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top) = \mathbf{0}, \quad (10)$$

$$G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succ 0, \quad X = X^\top \succ 0$$

has a solution $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{m \times n}$ for any fixed $\rho > 0$;

- 3) the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ induced by the weighted Euclidean norm $\|x\| = \sqrt{x^\top P x}$ with $P = X^{-1}$ is a Lyapunov function of the system (1) with

$$u(x) = K_0x + \|x\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \quad (11)$$

$$K_0 = Y_0(G_0 - I_n)^{-1}, \quad K = YX^{-1}, \quad (12)$$

where \mathbf{d} is a dilation generated by $G_{\mathbf{d}}$; moreover,

$$\frac{d}{dt} \|x\|_{\mathbf{d}} = -\rho \|x\|_{\mathbf{d}}^{1+\mu}, \quad x \neq \mathbf{0}; \quad (13)$$

- 4) the feedback law u given by (11) is continuously differentiable on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, u is continuous at zero if $\mu > -1$ and u is discontinuous at zero but locally bounded if $\mu = -1$;
- 5) the closed-loop system (1), (11) is \mathbf{d} -homogeneous of the degree μ .

Obviously, the closed-loop system (1),(11) is finite-time stable if $\mu < 0$ and it is nearly fixed-time stable if $\mu > 0$. For $\mu = 0$ the control (11) becomes the linear exponentially stabilizing feedback $u = K_0x + Kx$.

Remark 1: For $\mu = -1$ the control (11) is discontinuous at $x = 0$. If, additionally, $K_0 = \mathbf{0}$ then u is uniformly bounded

$$\begin{aligned} u^\top u &= x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) K^\top K \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x \\ &\leq u_{\max}^2 := \lambda_{\max}(P^{-1/2} K^\top K P^{1/2}). \end{aligned}$$

The control u is a high order sliding mode control, which rejects matched bounded uncertainties (see, [19] for more details). Notice also that for $\mu = -1$ and $\rho = 1$ the canonical homogeneous norm is the settling-time function of the closed-loop system.

Such a control law (under some variation and/or simplifications) is well-known in the literature as a solution to a finite-time stabilization problem for linear plants [26], [27], [19]. Using the topological equivalence of any stable \mathbf{d} -homogeneous system to a standard homogeneous one we prove the following corollary which allows the explicit formula for solutions of the closed-loop system (1), (11) to be presented.

Corollary 1: Under conditions of Theorem 3 with $\mu \neq 0$, the solution of the closed-loop system (1), (11) is unique and satisfies the identity

$$x(t + \tau) = Q_\tau(\|x(t)\|_{\mathbf{d}}) x(t), \quad (14)$$

where $\tau, t \geq 0$ and $Q_\tau(0) = \mathbf{0}$,

$$\begin{aligned} Q_\tau(r) &= \begin{cases} e^{G_{\mathbf{d}} \ln r} \hat{Q}_\tau(r) e^{-G_{\mathbf{d}} \ln r} & \text{if } \frac{1}{r^\mu} > -\mu\rho\tau, \\ 0 & \text{if } \frac{1}{r^\mu} \leq -\mu\rho\tau, \end{cases} \quad r > 0, \quad (15) \\ \hat{Q}_\tau(r) &= e^{\frac{G_{\mathbf{d}}}{\mu} \ln \frac{1}{1+\mu\rho\tau r^\mu}} e^{\frac{(A+B(K_0+K)+\rho G_{\mathbf{d}}) \ln(1+\mu\rho\tau r^\mu)}{\rho\mu}}. \end{aligned}$$

Moreover, for $x \neq \mathbf{0}$ one holds

$$\frac{d}{d\tau} Q_\tau(\|x\|_{\mathbf{d}}) x = A Q_\tau(\|x\|_{\mathbf{d}}) x + B u(Q_\tau(\|x\|_{\mathbf{d}}) x). \quad (16)$$

Proof. Denoting $y = \|x\|_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x$, we derive $\|y\| = \|x\|_{\mathbf{d}} \|\mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x\| = \|x\|_{\mathbf{d}}$ and

$$\begin{aligned} \dot{y} &= \frac{d\|x\|_{\mathbf{d}}}{dt} (I_n - G_{\mathbf{d}}) \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x + \|x\|_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) \dot{x} \\ &= \rho \|y\|^\mu (G_{\mathbf{d}} - I_n) y \\ &\quad + \|x\|_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) (A_0 x + \|x\|_{\mathbf{d}}^{1+\mu} B K \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x). \end{aligned}$$

Since $\mathbf{d}(s) A_0 = e^{-\mu s} A_0 \mathbf{d}(s)$ and $\mathbf{d}(s) B = e^s B$ for all $s \in \mathbb{R}$ then $\dot{y} = \|y\|^\mu (A_0 + B K + \rho(G_{\mathbf{d}} - I_n)) y$.

In this case, using (10) we conclude

$$\begin{aligned} \frac{d}{dt} \|y(t)\| &= \frac{y^\top P \dot{y}(t)}{\|y(t)\|} = \|y\|^{\mu-1} y^\top P (A_0 + B K + \rho(G_{\mathbf{d}} - I_n)) y \\ &= \|y\|^{\mu-1} \left(\frac{y^\top \{P(A_0 + B K + \rho G_{\mathbf{d}}) + (A_0 + B K + \rho G_{\mathbf{d}})^\top P\} y}{2} - \rho y^\top P y \right) \\ &= -\rho \|y(t)\|^{\mu+1} \end{aligned}$$

and

$$\|y(t + \tau)\|^{-\mu} = \|y(t)\|^{-\mu} + \mu\rho\tau,$$

for $\|y(t)\|^{-\mu} + \mu\rho\tau \geq 0$. Obviously, $\|y(t + \tau)\| = 0$ if $\|y(t)\|^{-\mu} + \mu\rho\tau \leq 0$. The latter corresponds to the negative homogeneity degree $\mu < 0$ and the finite-time stability of

the closed-loop system. Hence, denoting $K_{lin} = K_0 + K$ we obtain

$$\begin{aligned} y(t + \tau) &= e^{(A+BK_{lin}+\rho(G_{\mathbf{d}}-I_n)) \int_0^\tau \|y(t+\sigma)\|^\mu d\sigma} y(t) \\ &= e^{(A+BK_{lin}+\rho(G_{\mathbf{d}}-I_n)) \int_0^\tau \frac{1}{\|y(t)\|^{-\mu+\mu\rho\sigma}} d\sigma} y(t) \\ &= e^{(A+BK_{lin}+\rho(G_{\mathbf{d}}-I_n)) \frac{1}{\mu\rho} \ln \frac{\|y(t)\|^{-\mu+\mu\rho\tau}}{\|y(t)\|^{-\mu}}} y(t) \\ &= e^{(A+BK_{lin}+\rho(G_{\mathbf{d}}-I_n)) \frac{1}{\mu\rho} \ln(1+\mu\rho\tau\|y(t)\|^\mu)} y(t). \end{aligned}$$

Returning to the original coordinates we derive

$$\begin{aligned} x(t + \tau) &= \mathbf{d}(\ln \|y(t + \tau)\|) \frac{y(t + \tau)}{\|y(t + \tau)\|} \\ &= \mathbf{d}(\ln(\|y(t)\|^{-\mu} + \mu\rho\tau)^{\frac{1}{-\mu}}) \\ &\quad \cdot e^{(A+BK_{lin}+\rho(G_{\mathbf{d}}-I_n)) \frac{1}{\rho\mu} \ln(1+\mu\rho\tau\|y(t)\|^\mu)} \\ &\quad \cdot \frac{y(t)}{(\|y(t)\|^{-\mu} + \mu\rho\tau)^{\frac{1}{-\mu}}} \\ &= \mathbf{d}(\ln(\|x(t)\|_{\mathbf{d}}^{-\mu} + \mu\rho\tau)^{\frac{1}{-\mu}}) \\ &\quad \cdot e^{(A+BK_{lin}+\rho G_{\mathbf{d}}) \frac{1}{\rho\mu} \ln(1+\mu\rho\tau\|x(t)\|_{\mathbf{d}}^\mu)} \\ &\quad \cdot \mathbf{d}(-\ln \|x(t)\|_{\mathbf{d}}) x(t) \\ &= Q_\tau(\|x(t)\|_{\mathbf{d}}) x(t) \end{aligned}$$

for all $t \geq 0$ and all $\tau \geq 0$. The identity (16) follows immediately from the fact that $\tau \mapsto x(\tau)$ is a solution of (1), (11). ■

The latter corollary presents an important result for a class of nonlinear homogeneous system under consideration, since this is a very rare case that a solution of an essentially nonlinear system in \mathbb{R}^n can be found explicitly (see the formula (14)). Such a property may be very useful for various analysis. In this paper, we design a consistent discretization for the homogeneous control (11) for the double integrator. In such a particular case, the control (11) may be defined analytically (with an explicit representation of the canonical homogeneous norm by means of Ferrari formulas [28]).

Remark 2: For $\mu = -1$ under Assumption 1 we have

$$K_0 = \mathbf{0}, \quad A_0 = A, \quad G_{\mathbf{d}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, for $\rho = 1$ the condition (10) with $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} \in \mathbb{R}^{2 \times 2}$ and $Y = [y_1, y_2] \in \mathbb{R}^{1 \times 2}$ becomes

$$x_{11} > 0, x_{12} = -2x_{11}, x_{22} > \frac{9}{2}x_{11}, y_1 = -3x_{12} - x_{22}, y_2 = -x_{22}. \quad (17)$$

and the control (11) has the form

$$u(x) = k_1 \|x\|_{\mathbf{d}}^{-2} x_1 + k_2 \|x\|_{\mathbf{d}}^{-1} x_2, \quad K = [k_1 \ k_2] = Y P, \quad (18)$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$, $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = X^{-1}$ and the canonical homogeneous norm $\|x\|_{\mathbf{d}}$ is induced by the norm $\|x\| = \sqrt{x^\top P x}$. From (5) we derive

$$x^\top \begin{bmatrix} \|x\|_{\mathbf{d}}^{-1+\mu} & 0 \\ 0 & \|x\|_{\mathbf{d}}^{-1} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} \|x\|_{\mathbf{d}}^{-1+\mu} & 0 \\ 0 & \|x\|_{\mathbf{d}}^{-1} \end{bmatrix} x = 1.$$

Therefore, $\|x\|_{\mathbf{d}} = r$ for $\mu = -1$, where $r \in \mathbb{R}$ is a unique positive root of the following quartic equation

$$r^4 = ar^2 + br + c, \quad a = p_{22}x_2^2, b = 2p_{12}x_1x_2, c = p_{11}x_1^2, \quad (19)$$

which can be solved using Ferrari formulas.

IV. CONSISTENT DISCRETIZATION OF HOMOGENEOUS CONTROL FOR DOUBLE INTEGRATOR

Let us consider the discrete-time version of the system (1) with the sample-time control implementation. Obviously, if $u(t) = u_k := u(t_k)$ for $t \in [t_k, t_{k+1})$ then

$$x_{k+1} = A_h x_k + B_h u(t_k), \quad k = 0, 1, \dots, \quad (20)$$

where $x_k = x(t_k)$, $t_k = kh$,

$$A_h = e^{hA} \quad \text{and} \quad B_h = \left(\int_0^h e^{sA} ds \right) B.$$

The consistent discretizations of twisting and super-twisting algorithms presented in [9] and [18] are based on the so-called implicit scheme. The application of the same approach to the system (1), (11) gives the generalized equation

$$x_{k+1} = Ax_k + Bu_k,$$

$$u_k \in R(x_{k+1}) := \begin{cases} u(x_{k+1}) & \text{if } x_{k+1} \neq \mathbf{0}, \\ [-u_{\max}, u_{\max}] & \text{if } x_{k+1} = \mathbf{0}, \end{cases}$$

which is difficult to solve since R is not a maximal monotone operator. This equation may not have a solution without a relaxation (e.g., a Yosida approximation [29]). In this case the finite-time stability of the discrete-time system may not be guaranteed. To overcome this difficulty, we introduce an alternative technique inspired by Corollary 1.

Using (20) we derive

$$x_{k+2} = B_h u(t_{k+1}) + A_h B_h u(t_k) + A_h^2 x_k. \quad (21)$$

For the double integrator (see Assumption 1) we have

$$A_h = I_2 + hA, \quad B_h = \left(hI_2 + \frac{h^2}{2}A \right) B$$

and, obviously, the matrix

$$W_h = [B_h, A_h B_h] \quad (22)$$

is invertible. Since by Corollary 1 for any solution of the closed-loop continuous-time system (1) with the homogeneous control (11) we have

$$x(t+2h) = Q_{2h}(\|x(t)\|_{\mathbf{d}})x(t)$$

then to guarantee $x_{k+2} = Q_{2h}(\|x_k\|_{\mathbf{d}})x_k$ we may select

$$\begin{bmatrix} u(t_{k+1}) \\ u(t_k) \end{bmatrix} = W_h^{-1} (Q_{2h}(\|x_k\|_{\mathbf{d}}) - A_h^2) x_k, \quad (23)$$

where Q_{τ} is given by (15). By construction, the discrete-time system (20), (23) tracks the states of the closed-loop continuous-time system (1), (11) at discrete time instances t_{2k} . This means that the discretization (23) preserves the convergence rate of the continuous time system. However, the control (23) is not a conventional feedback law, since $u(t_{k+1})$ depends on $x_k = x(t_k)$ but not on x_{k+1} . Inspired by (23), we consider the control law:

$$\tilde{u}_h(x_k) = \tilde{K}_h(\|x_k\|_{\mathbf{d}})x_k, \quad \tilde{K}_h(\cdot) = [0, 1]W_h^{-1}(Q_{2h}(\cdot) - A_h^2), \quad (24)$$

which selects only $u(t_k)$ by the formula (23) leaving $u(t_{k+1})$ to be defined at the next sampling instant. To show that the static nonlinear feedback (24) is a consistent discretization of the finite-time stabilizing feedback (11), the discrete-time closed-loop system (20), (24) has to be globally finite-time stable for $\mu < 0$ and the feedback (24) must tend to the control (11) as $h \rightarrow 0^+$.

Theorem 4: For $n = 2, \mu = -1, \rho = 1$ the control u defined by (11) under conditions of Theorem 3 has a consistent discretization \tilde{u}_h given by (24) provided that

$$F^\top(\tilde{h})X^{-1}F(\tilde{h}) < X^{-1}, \quad \forall \tilde{h} \in (0, 1/2] \quad (25)$$

where

$$F(\tilde{h}) = I_2 + \tilde{h}A + (I_2 + \frac{\tilde{h}}{2}A)B \begin{bmatrix} 1 - \frac{\tilde{h}}{2} \\ \frac{e^{G_{\mathbf{d}} \ln(1-2\tilde{h})} e^{-(A+BK+G_{\mathbf{d}}) \ln(1-2\tilde{h})} - I - 2\tilde{h}A}{\tilde{h}} \end{bmatrix}. \quad (26)$$

Proof. 1) First, let us show that $\tilde{u}_h \rightarrow u$ as $h \rightarrow 0^+$. Since

$$A_h = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}, B_h = \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix}, W_h = \begin{bmatrix} \frac{h^2}{2} & \frac{3h^2}{2} \\ h & h \end{bmatrix}, W_h^{-1} = \begin{bmatrix} -\frac{1}{h^2} & \frac{3}{2h} \\ \frac{1}{h^2} & -\frac{1}{2h} \end{bmatrix}$$

then

$$\tilde{K}_h(r) = \begin{bmatrix} \frac{1}{h^2} & -\frac{1}{2h} \end{bmatrix} (Q_{2h}(r) - A_h^2).$$

Denoting $s_h = \ln(1 - 2h/r)^{-1}$ and $s = \ln r$ with $r = \|x\|_{\mathbf{d}} > 0$ we derive

$$Q_{2h}(r) = \mathbf{d}(s)\hat{Q}_{2h}(r)\mathbf{d}(-s),$$

where

$$\begin{aligned} \hat{Q}_{2h}(r) &= \left(I_2 - s_h G_{\mathbf{d}} + \frac{s_h^2}{2} G_{\mathbf{d}}^2 + O(s_h^3) \right) \cdot \\ &\quad \cdot \left(I_2 + (A+BK+G_{\mathbf{d}})s_h + \frac{(A+BK+G_{\mathbf{d}})^2 s_h^2}{2} + O(s_h^3) \right) \\ &= I_2 + (A+BK)s_h - s_h^2 G_{\mathbf{d}}(A+BK) - \frac{s_h^2}{2} G_{\mathbf{d}}^2 \\ &\quad + \frac{s_h^2}{2} (A+BK+G_{\mathbf{d}})^2 + O(s_h^3). \end{aligned}$$

Using $\text{Ad}(s) = e^{-s}\mathbf{d}(s)A$ we derive

$$\begin{aligned} Q_{2h}(r) - A_h^2 &= \mathbf{d}(s)\hat{Q}_{2h}(r)\mathbf{d}(-s) - I_2 - 2hA = \\ &= \mathbf{d}(s) \left(\hat{Q}_{2h}(r) - I_2 - 2h/rA \right) \mathbf{d}(-s) = O(s_h^3) + \end{aligned}$$

$$\mathbf{d}(s) \left(\frac{A(rs_h - 2h)}{r} + \frac{2BKs_h + ((A+BK+G_{\mathbf{d}})^2 - 2G_{\mathbf{d}}(A+BK) - G_{\mathbf{d}}^2)s_h^2}{2} \right) \mathbf{d}(-s)$$

Taking into account

$$s_h = \ln(1 - \frac{2h}{r})^{-1} = -\ln(1 - \frac{2h}{r}) = \frac{2h}{r} + \frac{(2h/r)^2}{2} + O(h^3)$$

we derive

$$\begin{aligned} \tilde{K}_h(r) &= O(h) + \begin{bmatrix} \frac{1}{h^2} & -\frac{1}{2h} \end{bmatrix} \frac{2hBK}{r} + \left(\frac{2h}{r} \right)^2 \begin{bmatrix} \frac{1}{h^2} & -\frac{1}{2h} \end{bmatrix} \cdot \\ &\quad \cdot \mathbf{d}(s) \left(\frac{A}{2} + \frac{((A+BK+G_{\mathbf{d}})^2 - 2G_{\mathbf{d}}(A+BK) - G_{\mathbf{d}}^2)}{2} \right) \mathbf{d}(-s) = O(h) + \end{aligned}$$

$$\frac{2 \left[1 - \frac{h}{2} \right] \mathbf{d}(s) (A - 2G_{\mathbf{d}}(A+BK) - G_{\mathbf{d}}^2 + (A+BK+G_{\mathbf{d}})^2) \mathbf{d}(-s) - r^2 K \mathbf{d}(-s)}{r^2}.$$

Since $[1\ 0]B=0$, $[1\ 0]G_d=[2\ 0]$, $[1\ 0]\mathbf{d}(s)=r^2[1\ 0]$ then

$$\begin{aligned}\tilde{K}_h(r) &= 2[1\ 0](A-2G_dA-G_d^2+(A+G_d)^2+ABK)\mathbf{d}(-s)+ \\ &\quad +O(h) - K\mathbf{d}(-s) = \\ 2[1\ 0](A-2G_dA+AG_d+G_dA)\mathbf{d}(-s)+K\mathbf{d}(-s) &+ O(h) = \\ K\mathbf{d}(-\ln r) + O(h),\end{aligned}$$

where the identity (9) is utilized on the last step. Hence, $\tilde{u} \rightarrow u$ as $h \rightarrow 0^+$ uniformly on compacts from $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

2) **Finite-time stability.** a) Let us denote

$$\begin{aligned}z(h,x) &= x+hAx+(hB+\frac{h^2}{2}AB)\left[\frac{1}{h^2}-\frac{1}{2h}\right] \cdot \\ &\left(\mathbf{d}(\ln\|x\|_d)e^{G_d\ln\frac{\|x\|_d-2h}{\|x\|_d}}e^{-(A+BK+G_d)\ln\frac{\|x\|_d-2h}{\|x\|_d}}\mathbf{d}(-\ln\|x\|_d)-I_2-2hA\right)x.\end{aligned}$$

The function $z: \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is well defined on the set

$$D = \{h > 0, x \in \mathbb{R}^n : \|x\|_d > 2h\}.$$

Obviously, the closed-loop system (20), (24) can be rewritten as $x_{k+1} = z(h, x_k)$ as long as $x_k \notin \Omega_h$,

$$\Omega_h = \{x : \|x\|_d \leq 2h\}.$$

Notice that $A\mathbf{d}(s) = e^{-s}\mathbf{d}(s)A$, $\mathbf{d}(s)B = e^sB$, $\|\mathbf{d}(s)x\|_d = e^s\|x\|_d$ and $\mathbf{d}(-\ln\|\mathbf{d}(s)x\|_d) = \mathbf{d}(-\ln\|x\|_d)\mathbf{d}(-s)$. Hence, we have

$$z(e^s h, \mathbf{d}(s)x) = \mathbf{d}(s)z(h, x), \forall s \in \mathbb{R}, \forall (h, x) \in D. \quad (27)$$

Indeed,

$$\begin{aligned}z(e^s h, \mathbf{d}(s)x) &= \mathbf{d}(s)x + e^s h A \mathbf{d}(s)x + (e^s h B + \frac{e^{2s} h^2}{2} AB) \left[\frac{1}{e^{2s} h^2} - \frac{1}{2e^s h} \right] \cdot \\ &\left(\mathbf{d}(\ln\|\mathbf{d}(s)x\|_d) e^{G_d \ln\left(1 - \frac{2e^s h}{\|\mathbf{d}(s)x\|_d}\right)} \right) \cdot \\ &\cdot e^{-(A+BK+G_d)\ln\left(1 - \frac{2e^s h}{\|\mathbf{d}(s)x\|_d}\right)} \mathbf{d}(-\ln\|\mathbf{d}(s)x\|_d) - I_2 - 2e^s h A \Big) \mathbf{d}(s)x = \\ &\mathbf{d}(s)x + h \mathbf{d}(s)A x + \mathbf{d}(s)(hI_2 + \frac{h^2}{2}A)B \left[\frac{1}{(e^s h)^2} - \frac{1}{2e^s h} \right] \cdot \\ &\cdot \left(\mathbf{d}(s)\mathbf{d}(\ln\|x\|_d) e^{G_d \ln(1-2h/\|x\|_d)} \right) \cdot \\ &\cdot e^{-(A+BK+G_d)\ln(1-2h/\|x\|_d)} \mathbf{d}(-\ln\|x\|_d) - \mathbf{d}(s) - 2h\mathbf{d}(s)A \Big) x = \\ &\mathbf{d}(s)z(h, x), \quad \forall s \in \mathbb{R}, \forall x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}.\end{aligned}$$

Recall that $\|\cdot\|_d$ is a Lyapunov function of the closed-loop continuous time system (1), (11) such that

$$\|x(t+h)\|_d = \|x(t)\|_d - h.$$

Since $\tilde{u} \rightarrow u$ as $h \rightarrow 0$ uniformly on compacts from $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ then away from the origin a solution of the sampled-time (discrete-time) system uniformly (on compact intervals of time) converge to a solution of the continuous-time system, so there exist a natural number $n_0 \geq 2$ and $\varepsilon_0 > 0$ such that

$$\|z(h, x)\|_d \leq -\varepsilon_0 h + \|x\|_d, \quad \forall x : \|x\|_d = 1, \forall h \in (0, 1/2^{n_0}).$$

In this case, if $\|x\|_d > 1$ then $h/\|x\|_d < h$ and for $s = -\ln\|x\|_d$ we have $\|\mathbf{d}(s)x\|_d = 1$,

$$\|z(h, x)\|_d = \|\mathbf{d}(-s)z(e^s h, \mathbf{d}(s)x)\|_d =$$

$$\begin{aligned}e^{-s}\|z(h/\|x\|_d, \mathbf{d}(s)x)\|_d &\leq e^{-s}(\|\mathbf{d}(s)x\|_d - \varepsilon_0 h/\|x\|_d) = \\ &\|x\|_d - \varepsilon_0 h.\end{aligned}$$

Therefore, we have

$$\|z(h, x)\|_d \leq \|x\|_d - \varepsilon_0 h, \forall x : \|x\|_d \geq 1, \forall h \in (0, 1/2^{n_0}). \quad (28)$$

b) On the one hand, since the first identity from (10) guarantees that

$$X^{-\frac{1}{2}}(A+BK+G_d)X^{\frac{1}{2}} + X^{\frac{1}{2}}(A+BK+G_d)^\top X^{-\frac{1}{2}} = \mathbf{0}$$

then the matrix $X^{-1/2}(A+BK+G_d)X^{1/2}$ is skew-symmetric and

$$e^{-X^{-1/2}(A+BK+G_d)X^{1/2}\phi} = R(\phi), \forall \phi \in \mathbb{R}$$

is a rotation matrix $R^\top R = I_2$, so $\limsup_{\phi \rightarrow \pm\infty} \|R(\phi)\| < +\infty$. Since

$$e^{G_d \ln(1-2\tilde{h})} = \begin{bmatrix} (1-2\tilde{h})^2 & 0 \\ 0 & 1-2\tilde{h} \end{bmatrix}, \quad \tilde{h} \in [0, 1/2]$$

then

$$e^{G_d \ln(1-2\tilde{h})} e^{-(A+BK+G_d)\ln(1-2\tilde{h})} \rightarrow \mathbf{0} \text{ as } \tilde{h} \rightarrow 1/2.$$

Hence, the function z can be continuously prolonged to the set $\{h > 0, x \in \mathbb{R}^n : \|x\|_d \leq 2h\}$ as follows

$$z(h, x) := \left(I_2 + hA + (B + \frac{h}{2}AB) \left[\frac{1}{h} - \frac{1}{2} \right] (-I_2 - 2hA) \right) x.$$

Since any continuous function on a compact reaches the maximal value on this compact then there exists $x^* : \|x^*\|_d = 1$ and $h^* \in [1/2^{n_0}, 1/2]$ such that

$$\max_{h \in [1/2^{n_0}, 1/2], \|x\|_d = 1} \|z(h, x)\|_d = \|z(h^*, x^*)\|_d = c \geq 0.$$

On the other hand, the matrix inequality (25) implies that $z^\top(h, x)X^{-1}z(h, x) < x^\top X^{-1}x = 1, \forall x : \|x\|_d = 1, \forall h \in [\frac{1}{2^{n_0}}, \frac{1}{2}]$.

or, equivalently,

$$\|z(h, x)\|_d < \|x\|_d = 1, \quad \forall x : \|x\|_d = 1, \forall h \in [\frac{1}{2^{n_0}}, \frac{1}{2}].$$

Therefore, we derive

$$\|z(h, x)\|_d \leq c = \|z(h^*, x^*)\|_d < 1$$

or equivalently,

$$\|z(h, x)\|_d \leq 1 - \varepsilon, \quad \forall x : \|x\|_d = 1, \forall h \in [\frac{1}{2^{n_0}}, \frac{1}{2}]$$

where $\varepsilon = 1 - c > 0$. If $\|x\|_d \in [1/2^{n_0-1}, 1]$ then for $s = -\ln\|x\|_d$ we have $e^s/2^{n_0} \in [1/2^{n_0}, 1/2]$ and

$$\begin{aligned}\|z(1/2^{n_0}, x)\|_d &= \|\mathbf{d}(-s)z(e^s/2^{n_0}, \mathbf{d}(s)x)\|_d = \\ e^{-s}\|z(e^s/2^{n_0}, \mathbf{d}(s)x)\|_d &\leq \|x\|_d(1 - \varepsilon) \leq \|x\|_d - \varepsilon/2^{n_0-1}\end{aligned}$$

for all $\forall x : \|x\|_d \in [1/2^{n_0-1}, 1]$. Taking into account the inequality (28) we derive

$$\|z(h_0, x)\|_d \leq \|x\|_d - \tilde{\varepsilon}h_0, \quad \forall x : \|x\|_d \geq 2h_0,$$

where

$$h_0 = 1/2^{n_0}, \quad \tilde{\varepsilon} = \min\{\varepsilon, \varepsilon_0\}.$$

This property implies a global finite-time convergence to the set Ω_{h_0} .

c) Let us show that the set Ω_{h_0} is invariant. Notice that for $x \in \Omega_{h_0}$ we have $Q_{2h_0}(\|x\|_{\mathbf{d}}) = 0$ and

$$z(h_0, x) = (I_2 + h_0 A + (B + \frac{h_0}{2} AB) [\frac{1}{h_0} \quad -\frac{1}{2}]) (-I_2 - 2h_0 A)x.$$

Since $\|x\|_{\mathbf{d}} \leq 2h_0$ is equivalent to

$$x^\top \mathbf{d}(-\ln(2h_0)) X^{-1} \mathbf{d}(-\ln(2h_0)) x \leq 1,$$

then to prove that $\|x\|_{\mathbf{d}} \leq 2h_0 \Rightarrow \|z\|_{\mathbf{d}} \leq 2h_0$ it is sufficient to fulfill the inequality

$$\begin{aligned} z^\top(h_0, x) \mathbf{d}(-\ln(2h_0)) X^{-1} \mathbf{d}(-\ln(2h_0)) z(h_0, x) \leq \\ x^\top \mathbf{d}(-\ln(2h_0)) X^{-1} \mathbf{d}(-\ln(2h_0)) x, \end{aligned}$$

The latter follows from (25) since

$$\begin{aligned} \mathbf{d}(-\ln(2h_0)) z(h_0, x) = \\ \mathbf{d}(-\ln(2h_0)) (I_2 + h_0 A + (B + \frac{h_0}{2} AB) [\frac{1}{h_0} \quad -\frac{1}{2}]) (-I_2 - 2h_0 A)x \\ = F(0.5) \mathbf{d}(-\ln(2h)) x. \end{aligned}$$

Therefore, Ω_{h_0} is a finite-time stable invariant set of the system (20), (24).

Since $x_{k+1} = z(h, x_k)$ then from the identity (27) we derive the symmetry of solutions of the system with respect to simultaneous the scaling of the sampling time $h \mapsto e^s h$ and the initial $x_0 \mapsto \mathbf{d}(s)x_0$ (see, e.g., [30] for more details). Therefore, for any $h > 0$ the set $\Omega_h = \{x : \|x\|_{\mathbf{d}} \leq 2h\}$ is a finite-time stable invariant set of the system (20), (24) and for any $x_0 \in \mathbb{R}^2$ there exists k^* such that $x_k \in \Omega_h, \forall k \geq k^*$.

d) Since for $x_k \in \Omega_h$ we have

$$x_{k+1} = \tilde{F}(h)x_k$$

where

$$\begin{aligned} \tilde{F}(h) = I_2 + hA + (I_2 + \frac{h}{2}A)B \begin{bmatrix} \frac{1}{h} & -\frac{1}{2} \\ -\frac{1}{h} & -\frac{1}{2} \end{bmatrix} (-I_2 - 2hA) = \\ \begin{bmatrix} \frac{1}{2} & \frac{h}{4} \\ -\frac{1}{h} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

then to complete the proof we just need to show that the matrix $\tilde{F}(h)$ is Schur stable and nilpotent (to have finite-time convergence). The required conclusion follows from the fact that the matrix $\tilde{F}(h)$ has only zero eigenvalues for all $h > 0$. Therefore, we have shown that the system (20), (24) is finite-time stable for any $h > 0$. ■

To design a consistent discretization for (11), the matrix inequality (10) has to be fulfilled together with (25).

Remark 3 (On feasibility of matrix inequalities): If $n = 2, \rho = 1, \mu = -1$ then the matrix

$$X = x_{22} \begin{bmatrix} \frac{1}{32} & -\frac{1}{16} \\ -\frac{1}{16} & 1 \end{bmatrix}, \quad x_{22} > 0$$

satisfy both conditions (10) and (25). Indeed, in this case, we have

$$X^{-1} = \frac{256}{7x_{22}} \begin{bmatrix} 1 & \frac{1}{16} \\ \frac{1}{16} & 32 \end{bmatrix}$$

and the gain matrix K is independent of X :

$$K = \begin{bmatrix} 6x_{22}/32 - x_{22} & -x_{22} \end{bmatrix} X^{-1} =$$

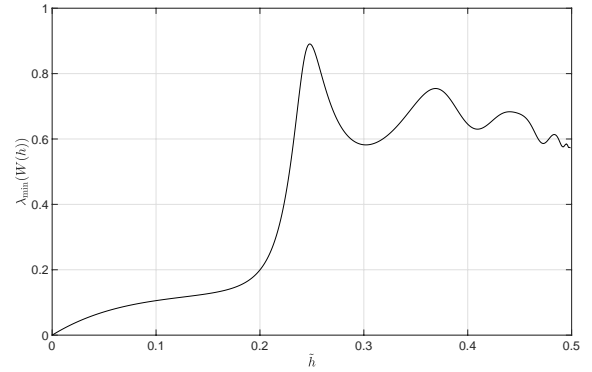


Fig. 1: The minimal eigenvalue of the matrix $W(\tilde{h})$

$$\frac{256}{7} \begin{bmatrix} -13/16 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & \frac{1}{32} \\ \frac{1}{16} & 32 \end{bmatrix} = \begin{bmatrix} -32 & -3 \end{bmatrix}.$$

Let us consider the matrix-valued function $W : (0, 1/2] \rightarrow \mathbb{R}^2$ given by

$$W(\tilde{h}) = \begin{bmatrix} \frac{1}{16} & \frac{1}{32} \\ \frac{1}{16} & \frac{1}{32} \end{bmatrix} - F(\tilde{h})^\top \begin{bmatrix} \frac{1}{16} & \frac{1}{32} \\ \frac{1}{16} & \frac{1}{32} \end{bmatrix} F(\tilde{h}).$$

The minimal eigenvalue of $W(\tilde{h})$ is always positive for all $\tilde{h} \in (0, 1/2]$ (see, Fig. 1).

The latter means

$$X^{-1} - F(\tilde{h})^\top X^{-1} F(\tilde{h}) \succ 0, \forall \tilde{h} \in (0, 1/2].$$

Notice, despite that the gain K is fixed the settling time of the closed-loop system can be tuned by selection of x_{22} . Indeed, the settling time of the closed-loop system (1), (11) with $n = 2, \mu = -1, \rho = 1$ is given by the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ which depends on X . It is easy to see that larger x_{22} faster the convergence of the system.

Remark 4 (Practical implementation of sampled control): The sampled control (24) is explicit for $\mu = -1$. Indeed, the canonical homogeneous norm $\|x\|_{\mathbf{d}}$ is defined explicitly (see Remark 2). The matrix $Q_{2h}(\|x\|_{\mathbf{d}})$ has the explicit representation as well since for $\mu = -1, \rho = 1$ one holds

$$e^{-\frac{G_{\mathbf{d}}}{\mu} \ln \frac{1+\mu\rho 2h\|x(t)\|_{\mathbf{d}}^\mu}{\|x(t)\|_{\mathbf{d}}^\mu}} = \begin{bmatrix} \phi^2 & 0 \\ 0 & \phi \end{bmatrix},$$

$$\phi = \|x\|_{\mathbf{d}} - 2h, \quad e^{-G_{\mathbf{d}} \ln \|x(t)\|_{\mathbf{d}}} = \begin{bmatrix} 1/\|x\|_{\mathbf{d}}^2 & 0 \\ 0 & 1/\|x\|_{\mathbf{d}} \end{bmatrix}$$

and there exists $J \in \mathbb{R}^{2 \times 2}$ (see, the proof of Theorem 4) such that

$$e^{\frac{(A+B(K_0+K)+\rho G_{\mathbf{d}}) \ln(1+\mu\rho 2h\|x(t)\|_{\mathbf{d}}^\mu)}{\rho\mu}} = J^{-1} \begin{bmatrix} \cos(\ln \phi) & \sin(\ln \phi) \\ -\sin(\ln \phi) & \cos(\ln \phi) \end{bmatrix} J.$$

Hence, for $\|x\|_{\mathbf{d}} > 2h$ we, obviously, have

$$Q_{2h}(\|x\|_{\mathbf{d}}) = \begin{bmatrix} \phi & 0 \\ 0 & 1 \end{bmatrix} J^{-1} \begin{bmatrix} \phi \cos(\ln \phi) & \phi \sin(\ln \phi) \\ -\phi \sin(\ln \phi) & \phi \cos(\ln \phi) \end{bmatrix} J \begin{bmatrix} \frac{1}{\|x\|_{\mathbf{d}}^2} & 0 \\ 0 & \frac{1}{\|x\|_{\mathbf{d}}} \end{bmatrix},$$

with $\phi = \|x\|_{\mathbf{d}} - 2h$, so $Q_{2h} \rightarrow \mathbf{0}$ as $\phi \rightarrow 0^+$.

V. NUMERICAL EXAMPLE

Let us consider the control law u given by the formula (11) and designed according to Remarks 2 and 3 for $n = 2, \mu = -1, \rho = 1$ and $x_{22} = 1$. The sampling period is $h = 0.1$. The initial condition is $x_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The results of the consistent discretization u_h of the control u are depicted in Fig. 3. For comparison purposes, we present the simulation results (see Fig. 2) of the explicit discretization

$$u(t) = u(x(t_i)), t \in [t_i, t_{i+1}]$$

of the homogeneous 2SM control (11).

The numerical simulations confirm our theoretical conclusion then the consistent discretization of the homogeneous SM control guarantees the finite time stability of the continuous-time system even in the case of the sampled-time implementation of the control law. The settling-time obtained by the simulation (3.6 sec) is close to the theoretical settling-time of the continuous-time system ($T(x_0) = \|x_0\|_d \approx 3.7442$ sec). Computations for various sampling periods show that the error of the settling-time estimate is $O(h)$. The system with the explicitly discretized control has a large chattering (see Fig. 2) even in the disturbance-free case. The consistent discretization scheme allows a chattering reduction even in the case of perturbations. To show this the system with the matched additive perturbation $d(t) = B \sin(t)$ has been simulated as well. Fig. 4 presents the simulation results for the perturbed system with the consistently discretized controller. The zero set-point is well tracked even in the case of matched perturbation. It is worth stressing that the control does not use any knowledge about the perturbation.

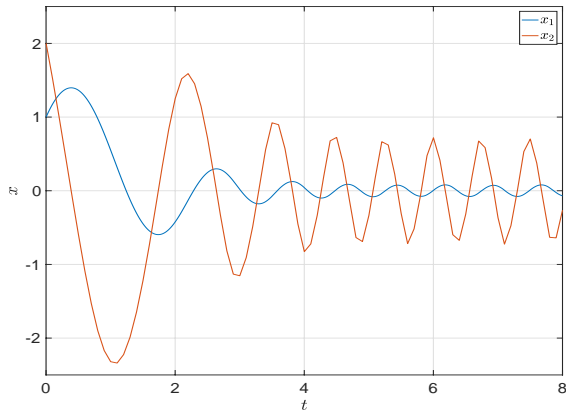
VI. CONCLUSIONS

The paper introduces a new concept of a digital (sampled-time) implementation of the control law, which is called a "consistent discretization of a control". It consists in a digital (sampled) implementation of the control law in such a way that the closed-loop system with a sampled control implementation preserves convergence rate of the system with the original (continuous-time) control law. In this paper, we present such a discretization for a homogeneous finite-time 2-SM controller. Numerical simulations confirm essential improvement of the control precision (comparing with conventional explicit scheme). Development of consistent discretization schemes for other popular finite-time and fixed-time controllers seems an interesting direction for the further research. A rigorous robustness analysis of the consistently discretized control is an important problem as well.

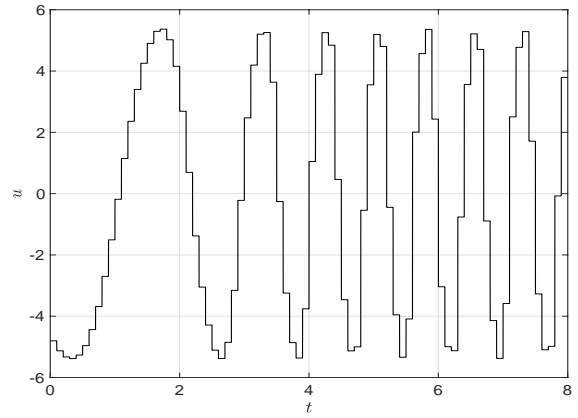
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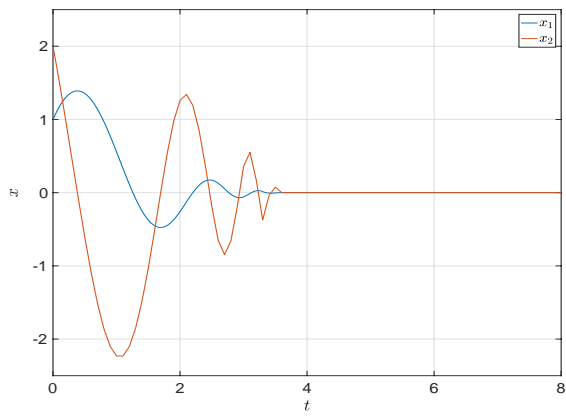


(a) System states

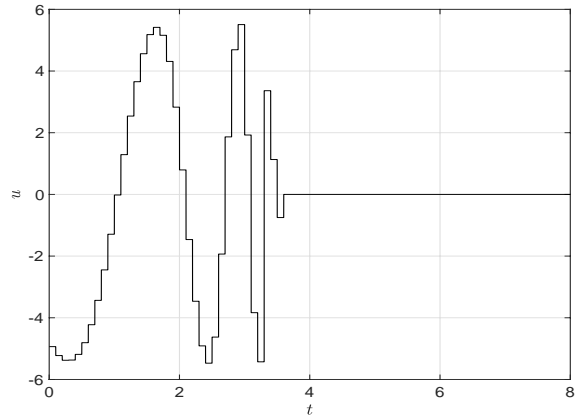


(b) Control signal

Fig. 2: Evolution of the system with the *explicitly* discretized control (the disturbance-free case).

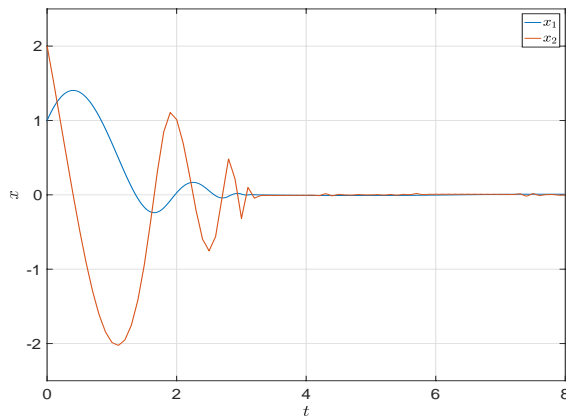


(a) System states

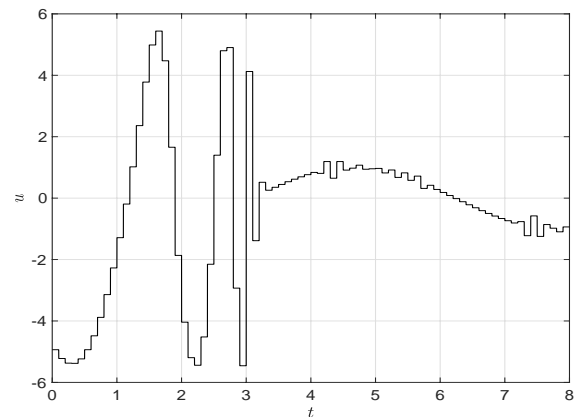


(b) Control signal

Fig. 3: Evolution of the system with the *consistently* discretized control (the disturbance-free case).



(a) System states



(b) Control signal

Fig. 4: Evolution of the system with the *consistently* discretized control (the perturbed case).

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