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Some Remarks on Counting Propositional Logic*

M. Antonelli

(part of a joint work with U. Dal Lago and P. Pistone)

Abstract

Counting propositional logic was recently introduced in relation to randomized computation and shown able to logically characterize the full counting hierarchy [1]. In this paper we aim to clarify the intuitive meaning and expressive power of its univariate fragment. On the one hand, we make the connection between this logic and stochastic experiments explicit, proving that the counting language can simulate *any* (and only) event associated with dyadic distributions. On the other, we provide an effective procedure to measure the probability of counting formulas.

1 Introduction

The need for reasoning about uncertain knowledge and probability has come out in several areas of research, from AI to economics, from linguistics to theoretical computer science (TCS, for short). For example, probabilistic models are crucial when considering randomized programs and algorithms or dealing with partial information, e.g. in expert systems. It was this concrete demand that led to the first attempts to analyze probabilistic reasoning *formally*, and to the development of a few logical systems, starting in 1986 with Nilsson’s pioneering (modal) proposal:

Because many artificial intelligence applications require the ability to reason with uncertain knowledge, it is important to seek appropriate generalizations of logic from this case. [17, p. 71]

For probabilistic algorithms behavioral properties, like termination or equivalence, have *quantitative* nature, that is computation terminates *with a certain probability*, or programs simulate the desired function *up to some probability* of error, for instance with learning algorithm. How can such properties be studied within a logical system? In a series of recent works [2, 1, 3], we introduce logics with counting and measure quantifiers, providing a new formal framework to study probability, and show them strongly related to several aspects of randomized computation. Specifically, counting formulas can be seen as expressing that a program behaves in a certain way with a given probability. In this short paper, we aim to clarify what is the expressive power (and limit) of the simple, univariate fragment of counting propositional logic [1], to better understand its connection with both randomized computation and other probability systems.

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1.1 On Logic and Randomized Computation

The development of counting logics is part of an overall study aiming at analyzing the interaction between (quantitative) logic and probabilistic computation, so to deepen our knowledge of both. This project was motivated by two main considerations. On the one hand, since their appearance in the 1970s, probabilistic computational models have become more and more pervasive in several fast-growing areas of computer science and technology, such as statistical learning or approximate computing. On the other, the development of (deterministic) models has considerably benefitted from interchanges between logic and TCS. Nevertheless, there is at least one crucial aspect of the theory of computation which was only marginally touched by such fruitful interactions, namely *randomized* computation. The global purpose of our study is to lay the foundation for a new approach to bridge this gap, and its key ingredient consists in considering new *inherently quantitative* logics, extended with non-standard quantifiers able to “measure” the probability of their argument formula.

So far, we have mostly focussed on a few specific aspects of the interaction between quantitative logics and randomized computation:

- * Complexity theory: it is well-known that classical propositional logic (**PL**, for short) provides the first example of an **NP**-complete problem [6], while its quantified version characterizes the full polynomial hierarchy [15, 16, 5]. Yet, no analogous logical counterpart was found for the probabilistic and counting classes [11, 20] and hierarchy [21]. In [1], we introduce a counting logic, called **CPL**, which is shown to be the probabilistic counterpart of quantified propositional logic (**QPL**, for short). Indeed, its formulas in a special prenex normal form, characterizes the corresponding level of Wagner’s hierarchy.
- * Programming language theory: type systems for randomized λ -calculi, also guaranteeing various forms of termination properties, were introduced in the last decades, e.g. in [18, 14, 9, 8], but these systems are not “logically oriented” and no Curry-Howard correspondence [7, 13, 19] is known for them. In [3], we define an intuitionistic counting logic, which captures quantitative behavioral properties and typed λ -calculi in which types reveal the actual probability of termination. In this way, we also provides a probabilistic version of the correspondence.
- * Computation theory: arithmetics and deterministic computation are linked by deep theorems from logic and recursion theory. In [2], we present a quantitative extension for the language of Peano Arithmetic, able to formalize basic results from probability theory which are not expressible in standard arithmetic. We also generalize classical theorems from recursion theorem to the quantitative realm, for instance due to our randomized version of Gödel’s arithmetization [12].

1.2 The Structure of the Paper

We try to clarify the intuitive meaning associated with our counting logic and the nature of its non-standard quantifiers, focussing on a few specific topics. In particular, the presentation is structured as follows. First, in Section 2 we briefly recap some crucial aspects of measure theory and the semantics of **CPL**₀. Then, in Section 3 we consider the relation between this logic and stochastic experiments explicit. Specifically, we prove that **CPL**₀ can simulate *any* event associated with dyadic distributions. Finally, in Section 4 we provide an effective procedure to measure the probability of counting formulas.

2 On (Univariate) Counting Propositional Logic

In order to avoid clash in terminology, we start by briefly recapping a few notions from basic probability theory. Then, we summarise the crucial aspects of our non-standard, univariate counting propositional logic as first introduced in [1].

2.1 Preliminaries

Probability Space. In probability theory, an *outcome* or *point* is the result of a single execution of an experiment, the *sample space* Ω is the set of all possible outcomes, and an event is then a subset of Ω . Two events, say E_1 and E_2 , are *disjoint* or *mutually exclusive*, when they cannot happen at the same time, that is $E_1 \cap E_2 = \emptyset$. Two events are (*stochastically*) *independent* when the occurrence of one does not affect the probability for the other to occur. A class \mathcal{F} of subsets of Ω is a (σ -)field if it contains Ω itself and is closed under the formation of complements and (in-)finite unions. The largest σ -field in Ω is the power class 2^Ω consisting of *all* the subsets of Ω . Given a σ -field \mathcal{F} , we call the σ -field generated by \mathcal{F} , denoted $\sigma(\mathcal{F})$, the smallest σ -algebra containing \mathcal{F} . A probability measure $\text{PROB}(\cdot)$, is a real-valued function defined on a field \mathcal{F} and satisfying Kolmogorov's axioms, i.e. associating each event E in the field with a number $\text{PROB}(E)$ so that: (i) for each $E \in \mathcal{F}$, $0 \leq \text{PROB}(E) \leq 1$, (ii) $\text{PROB}(\emptyset) = 0$ and $\text{PROB}(\mathbb{B}^\mathbb{N}) = 1$, (iii) if $E_1, E_2, \dots \in \mathcal{F}$ is a sequence of disjoint events, then $\text{PROB}(\bigcup_{k=1}^\infty E_k) = \sum_{k=1}^\infty \text{PROB}(E_k)$. Given two disjoint events, E_1 and E_2 , $\text{PROB}(E_1 \cup E_2) = \text{PROB}(E_1) + \text{PROB}(E_2)$, while for two independent events, E'_1 and E'_2 , $\text{PROB}(E'_1 \cap E'_2) = \text{PROB}(E'_1) \cdot \text{PROB}(E'_2)$.

Cylinder Measure. In the following, we will deal with a specific probability space, as defined in [4], where Billingsley considers a model to simultaneously fit random drawing of points from a segment and infinite sequence of coin tosses (so to be interesting for both geometry and probability). In particular, when tossing a coin, the set of the possible outcomes of the experiment is $S = \{\text{TAIL}, \text{HEAD}\}$. More in general, when dealing with a Bernoulli experiment, we can consider the set of its possible outcomes as simply $\mathbb{B} = \{0, 1\}$ (or even $2 = \{0, 1\}$).¹ The corresponding sample space is $\Omega = \mathbb{B}^\mathbb{N}$, i.e. the set of all infinite sequences of random bits (that is coin tosses) denoted as $\omega = \omega(1)\omega(2)\dots$, where for any $\omega \in \Omega$ and $i \geq \mathbb{N}$, $\omega(i) \in \mathbb{B}$.² Each sequence ω can be interpreted as the result of infinitely flipping a coin.

Definition 1 (Cylinder of Rank n). A *cylinder of rank n* is a set of the form $C_H = \{\omega \mid \omega(1), \dots, \omega(n) \in H\}$, with $H \subset \mathbb{B}^n$.

When H is a singleton, an event $E = \{\omega \mid \omega(1), \dots, \omega(n) = (u_1, \dots, u_n)\}$, such that the first n repetitions of the experiment give the outcomes u_1, \dots, u_n in sequence is called a *thin cylinder*.

The class of cylinders of all ranks is denoted by \mathcal{C}_0 , while \mathcal{C} is a field closed under complementation and union. It is thus possible to define a measure on it. In particular, a canonical one, consists in assigning the following probability measure $\mu_{\mathcal{C}}$, to any cylinder of rank n .

¹In what follows, we use all these three notations basing on the pertinence with the context. In particular, we use TAIL and HEAD when dealing with concrete examples concerning tossing. Of course, all these sets are equivalent for our goal.

²Notice that we've slightly modified Billingsley's notation, where e.g. the infinite-dimensional Cartesian product Ω as S^∞ and ω as $z_1(\omega)$.

Definition 2 (Cylinder Measure). Given $u \in \mathbb{B}$, p_u denote the (non-negative and summing to 1) probabilities of getting u . For any cylinder C_H

$$\mu_{\mathcal{C}}(C_H) = \sum_H p_{u_1} \cdots p_{u_n}.$$

In the special case of C_H being a thin cylinder $\mu_{\mathcal{C}}(\{\omega \mid (\omega(1), \dots, \omega(n)) = (u_1, \dots, u_n)\}) = p_{u_1} \cdots p_{u_n}$, providing a model for an infinite sequence of random bits or independent tosses, each with probability $p_{\mathbf{0}_i}$ of success and $p_{\mathbf{1}_i}$ of failure. Observe also that when the coin is fair, for each tossing $p_{\mathbf{0}} = p_{\mathbf{1}} = \frac{1}{2}$. In this case, since cylinders of rank n are finite sets, the following result is a straightforward consequence of Definition 1.

Corollary 1. *For any cylinder of rank n , call it C_H , and $p_{\mathbf{0}} = p_{\mathbf{1}} = \frac{1}{2}$, there are some $l, m \in \mathbb{N}$ such that $\mu_{\mathcal{C}}(C_H) = \frac{1}{2^m}$.*

Finally, going back to $\sigma(\mathcal{C})$, a well-defined probability measure can be assigned to it by simply generalizing Definition 2 in the natural way. Then, the probability space $(\mathbb{B}^{\mathbb{N}}, \sigma(\mathcal{C}), \mu_{\mathcal{C}})$, where $\mu_{\mathcal{C}}$ is such that, $p_{\mathbf{0}} = p_{\mathbf{1}} = \frac{1}{2}$, defines a standard model for infinite and independent tosses of a *fair* coin.

2.2 (Univariate) Counting Propositional Logic in a Nutshell

Grammar and Semantics. In standard **PL** formulas are interpreted as single truth-values. The core idea of our counting semantics consists in modifying this intuition in a quantitative sense, associating formulas with *measurable* sets of (satisfying) valuations. Given a counting formula F , its interpretation is the set $\llbracket F \rrbracket \subseteq \mathbb{B}^{\mathbb{N}}$, made of all maps $f \in \mathbb{B}^{\mathbb{N}}$ “making F true”. Such sets belong to the standard Borel algebra over $\mathbb{B}^{\mathbb{N}}$, $\mathcal{B}(\mathbb{B}^{\mathbb{N}})$, yielding a genuinely quantitative semantics. Specifically, atomic propositions correspond to *cylinder sets* [4] of the form:

$$Cyl(i) = \{f \in \mathbb{B}^{\mathbb{N}} \mid f(i) = \mathbf{1}\},$$

with $i \in \mathbb{N}$. Molecular expressions are interpreted in a natural way, via standard operations of complementation, finite intersection and union over the σ -algebra. So, formulas are all measurable and, in particular, associated with the unique cylinder measure $\mu_{\mathcal{C}}$, where $\mu_{\mathcal{C}}(Cyl(i)) = \frac{1}{2}$ for any $i \in \mathbb{N}$ [4].

We enrich this language with new formulas expressing the measure of such sets. By adapting the notion of counting operator by Wagner [21], we introduce two non-standard quantifiers \mathbf{C}^q and \mathbf{D}^q , with $q \in \mathbb{Q}_{[0,1]}$. Then, quantified formulas $\mathbf{C}^q F$ and $\mathbf{D}^q F$ express that F is satisfied in a certain portion of all its possible interpretations to be (resp.) greater or strictly smaller than the given one. For example, the formula $\mathbf{C}^{1/2} F$ says that F is satisfied by *at least* half of its valuations. Semantically, this amounts at checking that $\mu_{\mathcal{C}}(\llbracket F \rrbracket) \geq \frac{1}{2}$.

Definition 3. *Formulas of \mathbf{CPL}_0* are defined by the grammar below:

$$F := \mathbf{i} \mid \neg F \mid F \wedge F \mid F \vee F \mid \mathbf{C}^q F \mid \mathbf{D}^q F,$$

where $i \in \mathbb{N}$ and $q \in \mathbb{Q}_{[0,1]}$. Given the standard cylinder space $\mathcal{P} = (\mathbb{B}^{\mathbb{N}}, \sigma(\mathcal{C}), \mu_{\mathcal{C}})$, for each formula of \mathbf{CPL}_0 , F , its *interpretation* is the measurable set $\llbracket F \rrbracket \in \mathcal{B}(\mathbb{B}^{\mathbb{N}})$ defined as follows:

$$\begin{aligned}
\llbracket \mathbf{i} \rrbracket &= \text{Cyl}(i) & \llbracket \mathbf{C}^q G \rrbracket &= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{E}}(\llbracket G \rrbracket) \geq q \\ \emptyset & \text{otherwise} \end{cases} \\
\llbracket \neg G \rrbracket &= \mathbb{B}^{\mathbb{N}} - \llbracket G \rrbracket & \llbracket \mathbf{D}^q G \rrbracket &= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{E}}(\llbracket G \rrbracket) < q \\ \emptyset & \text{otherwise.} \end{cases} \\
\llbracket G_1 \wedge G_2 \rrbracket &= \llbracket G_1 \rrbracket \cap \llbracket G_2 \rrbracket \\
\llbracket G_1 \vee G_2 \rrbracket &= \llbracket G_1 \rrbracket \cup \llbracket G_2 \rrbracket
\end{aligned}$$

A formula of \mathbf{CPL}_0 F , is said *valid* when $\llbracket F \rrbracket = \mathbb{B}^{\mathbb{N}}$ while is said *invalid* when $\llbracket F \rrbracket = \emptyset$.

Proof Theory. In [1], we even define a sequent calculus, called $\mathbf{LK}_{\mathbf{CPL}_0}$, which is proved sound and complete for the semantics above. Its language is labelled, that is its formulas contains both a counting and a Boolean part.

Definition 4 (Boolean Formula). *Boolean formulas* are defined by the grammar below:

$$b := x_i \mid \top \mid \perp \mid \neg b \mid b \wedge b \mid b \vee b,$$

where $i \in \mathbb{N}$. The *interpretation of a Boolean formula* b , $\llbracket b \rrbracket \in \mathcal{B}(\mathbb{B}^{\mathbb{N}})$ is inductively defined as follows:

$$\begin{aligned}
\llbracket x_i \rrbracket &= \text{Cyl}(i) & \llbracket \neg b \rrbracket &= \mathbb{B}^{\mathbb{N}} - \llbracket b \rrbracket \\
\llbracket \top \rrbracket &= \mathbb{B}^{\mathbb{N}} & \llbracket b \wedge c \rrbracket &= \llbracket b \rrbracket \cap \llbracket c \rrbracket \\
\llbracket \perp \rrbracket &= \emptyset & \llbracket b \vee c \rrbracket &= \llbracket b \rrbracket \cup \llbracket c \rrbracket.
\end{aligned}$$

So, sequents of $\mathbf{LK}_{\mathbf{CPL}_0}$ are made of *labelled expressions* of the form $b \multimap F$ and $b \multimapleft F$, where b and F are (resp.) a Boolean and a counting formula. Intuitively, a labelled formula $b \multimap F$ (resp. $b \multimapleft F$) is true when the set of valuations satisfying b is included in (resp. includes) the interpretation of F . We use $b \models c$ for $\llbracket b \rrbracket \subseteq \llbracket c \rrbracket$. Then, a *labelled sequent* is a sequent of the form $\vdash L$, where L is a labelled formula. The proof system for \mathbf{CPL}_0 is defined by the rules illustrated in Figure 1. Observe that some of the rules include so-called *external hypotheses*, i.e. formulas such as $b \models c$ or $\mu(\llbracket b \rrbracket) \triangleright q$, where $\triangleright \in \{\geq, >, \leq, <, =\}$, b and c are Boolean formulas and $q \in \mathbb{Q}_{[0,1]}$. These hypotheses express semantic properties of Boolean formulas or conditions to be checked inside $\mathcal{B}(\mathbb{B}^{\mathbb{N}})$.

3 On the Expressive Power of \mathbf{CPL}_0

Our counting logics are strongly related to probabilistic reasoning and, indeed, \mathbf{CPL}_0 offers a natural model for events corresponding to Bernoulli distributions. In particular, we show how counting formulas can simulate experiments associated to dyadic distribution. To do so, we start by introducing auxiliary quantifiers to express exact probability in a compact way (Section 3.1). Then, we show that our formulas provide a natural formalism to express (quantitative properties of) events associated to *dyadic* distributions (Section 3.2). In particular, it is proved that counting formulas can simulate events associated with any such distribution, but those related to non-dyadic ones only in an approximate way. Finally, it is sketched a natural generalization of \mathbf{CPL}_0 able to simulate events associated to any discrete distribution (Section 3.3).

Initial Sequents	
$\frac{b \vDash x_n}{\vdash b \multimap \mathbf{n}} Ax1$	$\frac{x_n \vDash b}{\vdash b \leftarrow \mathbf{n}} Ax2$
Set Rules	
$\frac{\vdash d \multimap F \quad \vdash e \multimap F \quad b \vDash c \vee d}{\vdash b \multimap F} R_{\cup}^{\multimap}$	
$\frac{\vdash c \leftarrow F \quad \vdash d \leftarrow F \quad c \wedge d \vDash b}{\vdash b \leftarrow F} R_{\cap}^{\leftarrow}$	
Logical Rules	
$\frac{\vdash c \leftarrow F \quad b \vDash \neg c}{\vdash b \multimap \neg F} R_{\neg}^{\multimap}$	$\frac{\vdash c \multimap F \quad \neg c \vDash b}{\vdash b \leftarrow \neg F} R_{\neg}^{\leftarrow}$
$\frac{\vdash b \multimap F}{\vdash b \multimap F \vee G} R1_{\vee}^{\multimap}$	$\frac{\vdash b \multimap G}{\vdash b \multimap F \vee G} R2_{\vee}^{\multimap}$
$\frac{\vdash b \leftarrow F \quad \vdash b \leftarrow G}{\vdash b \leftarrow F \vee G} R_{\vee}^{\leftarrow}$	$\frac{\vdash b \multimap F \quad \vdash b \multimap G}{\vdash b \multimap F \wedge G} R_{\wedge}^{\multimap}$
$\frac{\vdash b \leftarrow F}{\vdash b \leftarrow F \wedge G} R1_{\wedge}^{\leftarrow}$	$\frac{\vdash b \leftarrow G}{\vdash b \leftarrow F \wedge G} R2_{\wedge}^{\leftarrow}$
Counting Rules	
$\frac{\mu(\llbracket b \rrbracket) = 0}{\vdash b \multimap F} R_{\mu}^{\multimap}$	$\frac{\mu(\llbracket b \rrbracket) = 1}{\vdash b \leftarrow F} R_{\mu}^{\leftarrow}$
$\frac{\vdash c \multimap F \quad \mu(\llbracket c \rrbracket) \geq q}{\vdash b \multimap \mathbf{C}^q F} R_{\mathbf{C}}^{\multimap}$	$\frac{\vdash c \leftarrow F \quad \mu(\llbracket c \rrbracket) < q}{\vdash b \leftarrow \mathbf{C}^q F} R_{\mathbf{C}}^{\leftarrow}$
$\frac{\vdash c \leftarrow F \quad \mu(\llbracket c \rrbracket) < q}{\vdash b \multimap \mathbf{D}^q F} R_{\mathbf{D}}^{\multimap}$	$\frac{\vdash c \multimap F \quad \mu(\llbracket c \rrbracket) \geq q}{\vdash b \leftarrow \mathbf{D}^q F} R_{\mathbf{D}}^{\leftarrow}$

Figure 1: Sequent Calculus $\mathbf{LK}_{\mathbf{CPL}_0}$

3.1 Expressing Exact Probability

In \mathbf{CPL}_0 , we can easily express that the probability for a formula to be true is *precisely* the given one. In fact, for the sake of readability, we introduce auxiliary quantifiers, \mathbf{C}^q and \mathbf{D}^q , intuitively meaning that their argument formula is true with probability (resp.) strictly greater or smaller than q .

Notation 1. So-called *white counting quantifiers* are interpreted as follows:

$$\llbracket \mathbf{C}^q F \rrbracket := \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{E}}(\llbracket F \rrbracket) > q \\ \emptyset & \text{otherwise} \end{cases} \quad \llbracket \mathbf{D}^q F \rrbracket := \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{E}}(\llbracket F \rrbracket) \leq q \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, these quantifiers do not extend the expressive power of \mathbf{CPL}_0 , as they are easily definable

in terms of the primitive \mathbf{C}^q and \mathbf{D}^q , see Proposition 1.

Lemma 1. For every formula of \mathbf{CPL}_0 F , and $q \in \mathbb{Q}_{[0,1]}$,

$$\mu_{\mathcal{E}}(\llbracket F \rrbracket) \triangleright q \quad \Leftrightarrow \quad \mu_{\mathcal{E}}(\llbracket \neg F \rrbracket) \triangleleft 1 - q,$$

with $(\triangleright, \triangleleft) \in \{(\geq, \leq), (\leq, \geq), (>, <), (<, >)\}$.

Proof Sketch. Let us consider the case \leq, \geq only. Since $\mu_{\mathcal{E}}(\llbracket \neg F \rrbracket) = \mu_{\mathcal{E}}(\mathbb{B}^{\mathbb{N}} - \llbracket F \rrbracket) = 1 - \mu_{\mathcal{E}}(\llbracket F \rrbracket)$, trivially $\mu_{\mathcal{E}}(\llbracket F \rrbracket) \geq q \Leftrightarrow 1 - \mu_{\mathcal{E}}(\llbracket F \rrbracket) \leq 1 - q \Leftrightarrow \mu_{\mathcal{E}}(\llbracket \neg F \rrbracket) \leq 1 - q$. \square

Proposition 1. For every formula of \mathbf{CPL}_0 F , and $q \in \mathbb{Q}_{[0,1]}$:

$$\begin{aligned} \mathbf{C}^q \neg F &\equiv \mathbb{D}^{1-q} F & \mathbf{D}^q \neg F &\equiv \mathbf{C}^{1-q} F \\ \mathbf{C}^q \neg F &\equiv \neg \mathbf{C}^{1-q} F & \mathbf{D}^q \neg F &\equiv \neg \mathbb{D}^{1-q} F. \end{aligned}$$

Proof. The proof is based on semantic definition and Lemma 1 above:

$$\begin{aligned} \llbracket \mathbf{C}^q \neg F \rrbracket &= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{E}}(\llbracket \neg F \rrbracket) \geq q \\ \emptyset & \text{otherwise} \end{cases} & \llbracket \mathbf{D}^q \neg F \rrbracket &= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{E}}(\llbracket \neg F \rrbracket) < q \\ \emptyset & \text{otherwise} \end{cases} \\ \stackrel{L1}{=} \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{E}}(\llbracket F \rrbracket) \leq 1 - q \\ \emptyset & \text{otherwise} \end{cases} & \stackrel{L1}{=} \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{E}}(\llbracket F \rrbracket) > 1 - q \\ \emptyset & \text{otherwise} \end{cases} \\ = \llbracket \mathbb{D}^{1-q} F \rrbracket & & = \llbracket \mathbf{C}^{1-q} F \rrbracket. \\ \\ \llbracket \mathbf{C}^q \neg F \rrbracket &= \llbracket \neg \mathbf{D}^q \neg F \rrbracket & \llbracket \mathbf{C}^q \neg F \rrbracket &= \llbracket \neg \mathbf{C}^q \neg F \rrbracket \\ = \llbracket \neg \mathbf{C}^{1-q} F \rrbracket & & = \llbracket \neg \mathbb{D}^{1-q} F \rrbracket. \end{aligned}$$

\square

Nevertheless, via \mathbf{C}^q and \mathbf{D}^q , we can express exact probability *in a compact way*. For example,

Example 1. We formalize that the formula $F = \mathbf{1} \wedge \mathbf{2}$ is true with probability $\frac{1}{4}$ as:

$$F_{ex} = \mathbf{C}^{1/4}(\mathbf{1} \wedge \mathbf{2}) \wedge \mathbb{D}^{1/4}(\mathbf{1} \wedge \mathbf{2}).$$

We can even extend $\mathbf{LK}_{\mathbf{CPL}_0}$ with the derivable rules for \mathbf{C} and \mathbb{D} illustrated in Figure 2.

$\frac{\vdash c \rightarrow F \quad \mu(\llbracket c \rrbracket) > q}{\vdash b \rightarrow \mathbf{C}^q F} R_{\mathbf{C}}^{\rightarrow}$	$\frac{\vdash c \leftarrow F \quad \mu(\llbracket c \rrbracket) \leq q}{\vdash b \leftarrow \mathbf{C}^q F} R_{\mathbf{C}}^{\leftarrow}$
$\frac{\vdash c \rightarrow F \quad \mu(\llbracket c \rrbracket) \leq q}{\vdash b \rightarrow \mathbb{D}^q F} R_{\mathbb{D}}^{\rightarrow}$	$\frac{\vdash c \leftarrow F \quad \mu(\llbracket c \rrbracket) > q}{\vdash b \leftarrow \mathbb{D}^q F} R_{\mathbb{D}}^{\leftarrow}$

Figure 2: Rules for \mathbf{C} and \mathbb{D}

3.2 Simulating Dyadic Distributions

It is natural to see atomic formulas of \mathbf{CPL}_0 as corresponding to infinite sequences of *fair* coin tosses and, more in general, counting formulas as formalizing experiments associated with *specific* probability distributions. For instance, when tossing an unbiased coin twice, the probability that it returns HEAD both times is $\frac{1}{4}$. This fact is expressed in \mathbf{CPL}_0 by F_{ex} above, (which is easily proved valid in our semantics and derivable in the corresponding proof system, see Appendix A). Generally speaking, counting formulas can simulate events associated with *any dyadic* distribution, but those related to non-dyadic ones only in an approximate way.³

Dyadic Distributions. In particular, we formalize atomic sampling from a Bernoulli distribution of *non-reducible* parameter $p = \frac{m}{2^n}$ by molecular formulas of \mathbf{CPL}_0 , while events are expressed combining such formulas in the usual way. Let us consider a simple example of an experiment associated with a dyadic distribution.

Example 2. *Let a biased coin return HEAD only 25% of the time. Clearly, in this case, a single toss is not simulated by an atomic formula of \mathbf{CPL}_0 , but by a complex one, namely $(\mathbf{i} \wedge \mathbf{j})$, with $i, j \in \mathbb{N}$ “fresh”. Then, we can even express properties concerning events. For instance, that the probability for at least one of two subsequent biased tosses to return HEAD is greater than $\frac{1}{3}$ is formalized by the (valid) formula:*

$$F_{bias} = \mathbf{C}^{1/3}((\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{3} \wedge \mathbf{4})).$$

Specifically, we prove that formulas of \mathbf{CPL}_0 are interpreted as events associated with *dyadic distribution* relying on the notion of cylinder of rank n (and on Corollary 1). We start by showing that any counting formula (as finite) is interpreted as a cylinder of a proper rank.

Lemma 2. *For any formula of \mathbf{CPL}_0 F , there is a cylinder of rank k C_K such that $\llbracket F \rrbracket = C_K$.*

Proof. The proof is by induction on the structure of F :

- * $F = \mathbf{i}$ for some $i \in \mathbb{N}$. Then, by Definition 3, $\llbracket \mathbf{i} \rrbracket = \text{Cyl}(i)$, which is a thin cylinder.
- * $F = \neg G$. By IH, there is a $k \in \mathbb{N}$ and a cylinder of rank k C_K , such that $\llbracket G \rrbracket = C_K$. Let $K' = \mathbb{B}^k \setminus K$. Then, $\llbracket \neg G \rrbracket = \mathbb{B}^{\mathbb{N}} - \llbracket G \rrbracket = \mathbb{B}^{\mathbb{N}} - C_K = C_{K'}$ is clearly a cylinder of rank k as well.
- * $F = G \wedge H$. By IH, there are exist $k_1, k_2 \in \mathbb{N}$ and cylinders of rank k_1, k_2 , such that (resp.) $\llbracket G \rrbracket = C_{K_1}$ and $\llbracket H \rrbracket = C_{K_2}$. Then, if $k_1 = k_2$, $\llbracket F \rrbracket = \llbracket G \rrbracket \cap \llbracket H \rrbracket = C_{K_1} \cap C_{K_2} = C_{K_1 \cap K_2}$ which is a cylinder of rank k_1 as well. Otherwise, assume $k_1 > k_2$ (the case $k_2 > k_1$ is equivalent). Let K'_2 consists of the sequences (u_1, \dots, u_{k_1}) in \mathbb{B}^{k_1} such that the truncated sequence (u_1, \dots, u_{k_2}) is in K_2 . Then, we obtain an alternative, but equivalent representation for C_{K_2} : $C'_{K_2} = \{\omega : (\omega(1), \dots, \omega(k)) \in K'_2\}$. So, $\llbracket F \rrbracket = \llbracket G \rrbracket \cap \llbracket H \rrbracket = C_{K_1} \cap C'_{K_2} = C_{K_1 \cap K'_2}$, which is a cylinder of rank k_1 .
- * $F = G \vee H$ is similar to the case above.
- * $F = \mathbf{C}^q G$. Then, by Definition 3, either $\llbracket F \rrbracket = \mathbb{B}^{\mathbb{N}}$ or $\llbracket F \rrbracket = \emptyset$ which are both cylinder of rank (resp.) 0 and k .

³As we shall see, a generalization of \mathbf{CPL}_0 associated with a probability space $(2^{\mathbb{N}}, \sigma(\mathcal{C}), \mu_{\mathcal{C}})$, where $\mu_{\mathcal{C}}$ is not necessarily the measure of i.i.d. sequences are cursorily presented in Section 3.3. Clearly, these logics can also express events related to non-dyadic distributions.

* $F = \mathbf{D}^q G$. Equivalent to the case above.

□

Then, since by Corollary 1, for any n the measure of a cylinder of rank n is dyadic, we conclude that for any formula of \mathbf{CPL}_0 its measure is dyadic as well.⁴

Lemma 3. *For any formula of \mathbf{CPL}_0 F , there are $n, m \in \mathbb{N}$, such that $\mu_{\mathcal{E}}(\llbracket F \rrbracket) = \frac{m}{2^n}$.*

Proof. By putting Corollary 1 and Lemma 3 together. □

Discrete Distributions with $\sharp X = 2^n$ In the same way, we can (quantitatively) simulate any discrete distribution with $\sharp X = 2^n$, (which are possibly non-Bernoulli ones). Of course, any information concerning the nature of variables involved in the experiment is lost, but the quantitative aspects, that is the probability of events, are all preserved through the formalization. Let us consider an example of such simulation, which, as we shall see, passes through (somehow arbitrary) “conventions”.

Example 3. *We have four cards, each representing a lynx from an existing species, i.e. lynx canadensis, lynx lynx, lynx pardinus and lynx rufus. We can express that, by randomly choosing a card, the probability not to catch the lynx lynx is $\frac{3}{4}$, for example, by the following (valid) formula:*

$$F_{nll} = \mathbf{C}^{3/4}(\neg(\mathbf{1} \wedge \mathbf{2})) \wedge \mathbb{D}^{3/4}(\neg(\mathbf{1} \wedge \mathbf{2})),$$

whereas the probability that the card represents a European lynx, that is either the lynx lynx or the lynx pardinus, is $\frac{1}{2}$ is formalized by the (valid) formula:

$$\begin{aligned} F_{el} &= \mathbf{C}^{1/2}((\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{1} \wedge \neg\mathbf{2})) \wedge \mathbb{D}^{1/2}((\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{1} \wedge \neg\mathbf{2})) \\ &= \mathbf{C}^{1/2}\mathbf{1} \wedge \mathbb{D}^{1/2}\mathbf{1}. \end{aligned}$$

If one repeats twice the (random) picking of a card from the four available ones, then the probability that a lynx rufus is picked both times is $\frac{1}{16}$ as simulated by the formula below:

$$F_{lr2} = \mathbf{C}^{1/16}((\neg\mathbf{1} \wedge \neg\mathbf{2}) \wedge (\neg\mathbf{3} \wedge \neg\mathbf{4})) \wedge \mathbb{D}^{1/16}((\neg\mathbf{1} \wedge \neg\mathbf{2}) \wedge (\neg\mathbf{3} \wedge \neg\mathbf{4})).$$

Non-Dyadic Bernoulli Distributions. Something different happens when considering experiments related to non-dyadic distributions. Indeed, by Lemma 3, formulas of \mathbf{CPL}_0 cannot express these events as they are always interpreted as sets of dyadic measure. Nevertheless, we can somehow simulate events associated to non-dyadic distributions *in an approximate way*.

Example 4. *Let us consider a biased coin returning HEAD only $\frac{1}{3}$ of the time. We cannot simulate this event in \mathbf{CPL}_0 . We can however “approximate” it with $n = 2m$ variables of \mathbf{CPL}_0 in the following sense. For instance, if $m = 2$ we can down-approximate a single toss of the biased coin as follows:*

$$F_{ndy} = (\mathbf{1} \wedge \mathbf{2}) \vee ((\neg\mathbf{1} \wedge \mathbf{2}) \wedge (\mathbf{3} \wedge \mathbf{4})).$$

⁴A “syntactic proof” of this result is obtained as a corollary of Lemmas presented in Section 4.

Notably, disjuncts are mutually contradictory so $\mu_{\mathcal{C}}(\llbracket F_{ndy} \rrbracket) = \frac{1}{2^2} + \frac{1}{2^4} = \frac{5}{16}$.
 If $m = 3$, (down-)approximation is obtained as follows,

$$F'_{ndy} = (\mathbf{1} \wedge \mathbf{2}) \vee ((-\mathbf{1} \wedge \mathbf{2}) \wedge (\mathbf{3} \wedge \mathbf{4})) \vee ((-\mathbf{1} \wedge \mathbf{2}) \wedge (-\mathbf{3} \wedge \mathbf{4}) \wedge (\mathbf{5} \wedge \mathbf{6})).$$

In this case, $\mu_{\mathcal{C}}(\llbracket F_{ndy} \rrbracket') = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} = \frac{21}{64}$.

In general, increasing n , our formula of \mathbf{CPL}_0 approximates the desired atomic event in a more and more precise way.

Although events associated with non-dyadic distributions cannot be expressed in \mathbf{CPL}_0 in a precise way, when switching to the measure quantified language \mathbf{MQPA} , as presented in [2], this formalization becomes possible.⁵ In the following Section 3.3, we provide an alternative, more natural way to generalize \mathbf{CPL}_0 so to express such events.

3.3 Generalizing \mathbf{CPL}_0

As seen, the semantics for \mathbf{CPL}_0 is associated with a canonical cylinder space $\mathcal{P} = (\mathbb{B}^{\mathbb{N}}, \sigma(\mathcal{C}), \mu_{\mathcal{C}})$, where $\mu_{\mathcal{C}}$ is the standard measure $\mu_{\mathcal{C}}(\text{Cyl}(i)) = \frac{1}{2}$ for any $i \in \mathbb{N}$, intuitively corresponding to tossing *fair* coins [4]. We can generalize this framework in a natural way, so to allow the cylinder measure to possibly express experiments associated to distributions other than dyadic ones. So, we define extended \mathbf{CPL}_0^* , this time associated with $\mathcal{P}^* = (2^{\mathbb{N}}, \sigma(\mathcal{C}), \mu^*)$, where μ^* can be *any* (well-defined) probability measure over $\sigma(\mathcal{C})$. In particular, the grammar and semantics for \mathbf{CPL}_0^* is as in Definition 3, except for quantifiers, which are now defined using the probability measure, μ^* .

Definition 5 (Grammar and Semantics of \mathbf{CPL}_0^*). Extended formulas are defined by substituting standard counting quantifiers with $\mathbf{C}_{\mu^*}^q$ and $\mathbf{D}_{\mu^*}^q$, the interpretation of which is generalized as basing on \mathcal{P}^* :

$$\llbracket \mathbf{C}_{\mu^*}^q F \rrbracket = \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu^*(\llbracket F \rrbracket) \geq q \\ \emptyset & \text{otherwise} \end{cases} \quad \llbracket \mathbf{D}_{\mu^*}^q F \rrbracket = \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu^*(\llbracket F \rrbracket) < q \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, \mathbf{CPL}_0 -formulas $\mathbf{C}^q F$ and $\mathbf{D}^q F$ become special cases of the extended ones, (resp.) $\mathbf{C}_{\mu^*}^q F$ and $\mathbf{D}_{\mu^*}^q F$ where $\mu^* = \mu_{\mathcal{C}}$. On the other hand, in \mathbf{CPL}_0^* we can simulate experiments corresponding to tossing (arbitrarily) *biased* coins in a very simple way. Let us consider the experiment of Example 4 once again.

Example 5. Let a biased coin again return HEAD only $\frac{1}{3}$ of the time. We consider a specific μ^* such that $\mu^*(\text{Cyl}(i)) = \frac{1}{3}$ for any $i \in \mathbb{N}$. Then, we can express that the probability for subsequent tosses to be successful is greater than $\frac{1}{9}$ by the formula of \mathbf{CPL}_0^* : $F_* = \mathbf{C}_{\mu^*}^{1/9}(\mathbf{1} \wedge \mathbf{2})$. Furthermore, since $\mu^*(\llbracket \mathbf{1} \wedge \mathbf{2} \rrbracket) = \frac{1}{3} \cdot \frac{1}{3}$, the formula is valid, i.e. $\llbracket \mathbf{C}_{\mu^*}^{1/9}(\mathbf{1} \wedge \mathbf{2}) \rrbracket = \mathbb{B}^{\mathbb{N}}$.

Observe that it is easy to define a sound and complete proof system for \mathbf{CPL}_0^* . Indeed, no substantial change is required with respect to the calculus $\mathbf{LK}_{\mathbf{CPL}_0}$ of [1, Sec. 2.2]. Indeed, only semantic conditions are related to probability measure, so one only needs to generalize μ -rules substituting $\mu_{\mathcal{C}}$ with μ^* , in the natural way illustrated by Figure 3.

⁵This fact is coherent with our randomized arithmetization result [2, Theorem 3].

$$\boxed{\frac{\mu^*(\llbracket b \rrbracket) = 0}{\vdash b \rightsquigarrow F} R_{\mu^*}^{\rightarrow} \qquad \frac{\vdash b \rightsquigarrow F \quad \mu^*(\llbracket c \rrbracket) \geq q}{\vdash b \rightsquigarrow \mathbf{C}_{\mu^*}^q F} R_{\mathbf{C}}^{\rightarrow}}$$

Figure 3: Examples of Rules in $\mathbf{LK}_{\mathbf{CPL}_0^c}$

4 Measuring Formulas of \mathbf{CPL}_0

In [1], the validity of counting formulas is decided accessing an oracle for $\#\text{SAT}$, i.e. counting the satisfying models of Boolean formulas. Here, we provide an effective procedure to measure formulas of \mathbf{CPL}_0 , *without appealing for an external source*, so somehow making explicit the task which, in the proof system $\mathbf{LK}_{\mathbf{CPL}_0}$ is done by the oracle. In this way, there is no need for so-called external hypotheses and a purely syntactical (though labelled) calculus can be defined. In our opinion, this result also makes the comparison with other probability logics, such as [10], more clear and help clarifying the nature of counting quantifiers. Furthermore, we hope this is also the first step to shed new lights on the study of the complexity of deciding formulas of \mathbf{CPL}_0 .

Our proof is based on some crucial steps. Given a counting formula, we start by considering its (inner) quantified formulas. To calculate their measure, we pass through a special form, the measure of which can be computed in a straightforward way (Lemma 5), and prove that formulas of \mathbf{CPL}_0 *without quantifiers* can be converted into such measurable form (Lemma 6). Notably, this procedure is effective, but not necessarily “feasible” as requiring argument formulas to be in disjunctive normal form (DNF, for short). Finally, nested quantifications can be taken into account. It is easy to see that by measuring argument formulas, we can simply substitute the corresponding (non-nested) counting-quantified expressions is either valid or invalid (see [1, p. 7]).

4.1 Preliminaries

On the logical side, we enrich the language of \mathbf{CPL}_0 by the two standard symbols \top and \perp , to be interpreted as predictable, i.e. $\llbracket \top \rrbracket = \mathbb{B}^{\mathbb{N}}$ and $\llbracket \perp \rrbracket = \emptyset$.

Notation 2. We use L_1, L_2, \dots to denote literals, that is either a counting variable or its negation. Given a literal L_i ,

$$\overline{L}_i = \begin{cases} \neg \mathbf{k} & \text{if } L_i = \mathbf{k} \\ \mathbf{k} & \text{otherwise.} \end{cases}$$

4.2 Polite Normal Form

For simplicity, before defining *measurable normal form*, we introduce the auxiliary, polite form. Intuitively, a conjunction of literals is in polite form if (is in $\{\perp, \top\}$ or) each variable occurs in it at most once, i.e. repetitions are removed and it is not possible for any literal to appear in the conjunction both in atomic L_i , and in negated form, \overline{L}_i . Observe that the measure of this conjunction can be easily computed. Trivially, when the formula is \top , its measure is 1 and when is \perp , its measure is 0. Otherwise, since literals correspond to mutually *independent* events (each of measure $\frac{1}{2}$), the measure of the polite conjunction of n formulas is simply $\frac{1}{2^n}$. On the other hand,

a formula in DNF is in *disjunctive polite form* when it is either \top or \perp or each of its disjunct is in polite form, without being in $\{\perp, \top\}$. Formally,

Definition 6 (Polite Form). A formula of \mathbf{CPL}_0 , which is a conjunction of literals $C = \bigwedge_{i \in \{1, \dots, n\}} L_i$, is in *conjunctive polite form* (CPF, for short) if either $C \in \{\perp, \top\}$ or for any $k \neq k' \in \{1, \dots, n\}$, $L_k \neq L_{k'}$ and $L_k \neq \overline{L_{k'}}$. A formula of \mathbf{CPL}_0 in DNF $D = \bigvee_{j \in \{1, \dots, m\}} C_j$, is in *disjunctive polite form* (DPF, for short) if either $D \in \{\perp, \top\}$ or for each $k \in \{1, \dots, m\}$, C_k is in CPF and $C_k \notin \{\perp, \top\}$.

Then, we prove that every formula of \mathbf{CPL}_0 (without quantifiers) can be converted in DPF. In particular, as for standard \mathbf{PL} , we know that for any counting formula *without quantifiers*, there is an equivalent formula of \mathbf{CPL}_0 in DNF. Then, by Lemma 4 below, it is established that for every DNF-formula, there is an equivalent formula of \mathbf{CPL}_0 in DPF.⁶

Lemma 4. *Given a formula of \mathbf{CPL}_0 in DNF D , there is a D^* such that D^* is in DPF and $D \equiv D^*$.*

Proof. Let $D = \bigvee_{i \in \{1, \dots, n\}} C_i$ be in DNF. For any $C_i = \bigwedge_{j \in \{1, \dots, m\}} L_j$, with $i \in \{1, \dots, n\}$, we define C_i^* applying the transformations below:

- i.* if $C_i = \top$, then $C_i^* = \top$.
- ii.* otherwise, consider each $j \in \{1, \dots, m\}$, starting from $j = 1$:
 - a.* if $L_j = \perp$, then $C_i^* = \perp$.
 - b.* if $L_j = \top$, then L_j is removed and $j + 1$ is considered.
 - c.* if $L_j \notin \{\perp, \top\}$, we consider each pedex $k \neq j \in \{1, \dots, m\}$, starting from the first possible one:
 - * if $L_j = L_k$, then L_k is removed and the subsequent pedex (different from j and k) is considered.
 - * if $L_j = \overline{L_k}$, then $C_i^* = \perp$.
 - * otherwise, L_j is left unchanged and $k + 1$ is considered.

It is clear that $C_i \equiv C_i^*$. Now we consider $D' = \bigvee_{i \in \{1, \dots, n'\}} C_i^*$ and define D^* applying the following transformations:

- i.* if $C_i^* = \perp$ for any $i \in \{1, \dots, n'\}$, then $D^* = \perp$.
- ii.* otherwise, we consider each $i \in \{1, \dots, n'\}$, starting from $i = 1$:
 - a.* if $C_i = \top$, then $D^* = \top$.
 - b.* if $C_i = \perp$, then C_i is removed and $i + 1$ is considered.
 - c.* if $C_i \notin \{\top, \perp\}$, we consider each pedex starting from the first $k' \neq i \in \{1, \dots, n'\}$:
 - * if C_i and $C_{k'}$ contain exactly the same literals, then $C_{k'}$ is removed and the subsequent pedex ($\neq k', i$) is considered.
 - * Otherwise, C_i is (maybe temporarily) left unchanged and the subsequent pedex ($\neq k', i$) is considered.

Again, it is clear that $D \equiv D^*$. □

⁶Actually, the proof of Lemma 4, is so defined that the DPF-formula D^* does not contain repetitions of disjuncts.

Observe that for any formula F in DPF, either $F \in \{\perp, \top\}$ or no instance of \perp, \top occurs in it. Furthermore, as anticipated, it is easy to measure the probability of a formula in CPF.

Proposition 2. *Given a formula C in CPF:*

- i. if $C = \top$, then $\mu_{\mathcal{E}}(\llbracket C \rrbracket) = 1$.*
- ii. if $C = \perp$, then $\mu_{\mathcal{E}}(\llbracket C \rrbracket) = 0$.*
- iii. Otherwise, $C = \bigwedge_{i \in \{1, \dots, n\}} L_i$, and $\mu_{\mathcal{E}}(\llbracket C \rrbracket) = \frac{1}{2^n}$.*

Proof. Case *i.* and *ii.* are trivial consequences of Definition 3 and basic measure theory: $\mu_{\mathcal{E}}(\llbracket \top \rrbracket) = \mu_{\mathcal{E}}(\mathbb{B}^N) = 1$ and $\mu_{\mathcal{E}}(\llbracket \perp \rrbracket) = \mu_{\mathcal{E}}(\emptyset) = 0$. Case *iii.* relies on Definition 6, that is C does not contain \perp or \top (or contradictions) or repetitions. Thus, by semantic definition, its literals have to be interpreted as *independent* events the measure of which is known, i.e. $\llbracket L_i \rrbracket$'s for $i \in \{1, \dots, n\}$ are *independent* events. So, as seen, for basic measure theory, $\mu_{\mathcal{E}}(\llbracket \bigwedge_{i \in \{1, \dots, n\}} L_i \rrbracket) = \mu_{\mathcal{E}}(\bigcap_{i \in \{1, \dots, n\}} \llbracket L_i \rrbracket) = \frac{1}{2^n}$. \square

4.3 Measurable Normal Form

Now, we introduce a special form, such that formulas in this form can be “measured” in a straightforward way. We start by considering the (logical) notion of *contradictory pair*. Two formulas in CPF are mutually contradictory if their conjunction is an invalid formula.

Definition 7 (Contradictory Pair). Two formulas of \mathbf{CPL}_0 in CPF, $C_i = \bigwedge_{j \in \{1, \dots, n\}} L_j$ and $C'_i = \bigwedge_{k \in \{1, \dots, m\}} L_k$ are said to be *mutually contradictory* when there exist $j \in \{1, \dots, n\}, k \in \{1, \dots, m\}$ such that $L_j = \overline{L_k}$ (or $L_k = \overline{L_j}$).

Clearly, contradictory formulas are interpreted as *disjoint* events (and, as seen, the measure of the union of two disjoint events is the sum of the measure of each event). So by Definition 3 plus basic probability theory, the measure of the disjunction of two contradictory formulas ($\notin \{\perp, \top\}$) in CPF is the sum of the measure of each disjunct (which, being themselves in CPF, are easily measurable as well by Proposition 2). By generalizing this intuition we obtain the definition below.

Definition 8 (Measurable Normal Form). A formula of \mathbf{CPL}_0 $F = \bigvee_{i \in \{1, \dots, n\}} C_i$ is in *measurable normal form* (MNF, for short) if either $F \in \{\perp, \top\}$ or F is in DPF and for each $j \neq k \in \{1, \dots, n\}$, C_j and C_k are mutually contradictory.

Observe that, when a formula is in MNF, as disjuncts are mutually contradictory, no disjunct is repeated in it and no disjunct can be a sub-formula of another.⁷

As seen, by Definition 8 a formula of MNF contains disjunct corresponding to mutually disjoint events, so Lemma 5 naturally follows.

Lemma 5. *Given a formula of \mathbf{CPL}_0 in MNF $F = \bigvee_{i \in \{1, \dots, n\}} C_i$:*

- i. if $F = \perp$, then $\mu_{\mathcal{E}}(\llbracket F \rrbracket) = 0$,*
- ii. if $F = \top$, then $\mu_{\mathcal{E}}(\llbracket F \rrbracket) = 1$,*
- iii. otherwise, $\mu_{\mathcal{E}}(\llbracket F \rrbracket) = \sum_{i \in \{1, \dots, n\}} \mu_{\mathcal{E}}(\llbracket C_i \rrbracket)$.*

⁷Further details on the notion of sub-formula are presented in Section 4.4.

Proof Sketch. Case *i.* and *ii.* hold by Definition 3 and basic measure theory: $\mu_{\mathcal{E}}(\llbracket \perp \rrbracket) = \mu_{\mathcal{E}}(\emptyset) = 0$ and $\mu_{\mathcal{E}}(\llbracket \top \rrbracket) = \mu_{\mathcal{E}}(\mathbb{B}^{\mathbb{N}}) = 1$. Case *iii.* is proved relying on the Definition 8: for any $j \neq k \in \{1, \dots, n\}$, (C_j, C_k) is a contradictory pair. Clearly, $\llbracket C_j \rrbracket \cap \llbracket C_k \rrbracket = \emptyset$ for any j and k . So, $\mu_{\mathcal{E}}(\llbracket \bigvee_{i \in \{1, \dots, n\}} C_i \rrbracket) = \mu_{\mathcal{E}}(\bigcup_{i \in \{1, \dots, n\}} \llbracket C_i \rrbracket) = \sum_{i \in \{1, \dots, n\}} \mu_{\mathcal{E}}(\llbracket C_i \rrbracket)$. \square

As said above, each disjunct is in CPF, so its measure is easily computable as well.

Corollary 2. *Given a formula of \mathbf{CPL}_0 in MNF $F = \underbrace{\bigwedge_{i \in \{1, \dots, m_1\}} L_i \vee \dots \vee \bigwedge_{j \in \{1, \dots, m_n\}} L_j}_{n \text{ times}}$:*

- if $F = \top$, then $\mu_{\mathcal{E}}(\llbracket F \rrbracket) = 1$,
- if $F = \perp$, then $\mu_{\mathcal{E}}(\llbracket F \rrbracket) = 0$,
- otherwise, $\mu_{\mathcal{E}}(\llbracket F \rrbracket) = \underbrace{1/2^{m_1} + \dots + 1/2^{m_n}}_{n \text{ times}}$.

Proof. By putting Proposition 2 and Lemma 5 together. \square

4.4 Conversion into MNF

To conclude our proof, we show that each formula in DPF can actually be “converted” into MNF. To do so, we notice that two disjuncts can be mutually related in three ways only: (1) one is a sub-formula of the other, in this case, the former is simply removed; (2) they are a contradictory pair, so are already in the desired form and another pair can be considered; (3) one of the two disjuncts, say C_i , contains a literal L_k , such that neither L_k or $\overline{L_k}$ occurs in the other disjunct, say C_j , in this case C_j is substituted by $C'_j = C_j \wedge L_k$ and $C''_j = C_j \wedge \overline{L_k}$. Notice that a formula in CPF, say $C_i = \bigwedge_{k \in \{1, \dots, n\}} L_k$ is said to be a *sub-formula* of another formula in CPF $C_j = \bigwedge_{k' \in \{1, \dots, m\}} L_{k'}$ when (they are not a contradictory pair and) for each $k' \in \{1, \dots, m\}$, there is a $k \in \{1, \dots, n\}$ such that $L_k = L_{k'}$. For example, the formula $\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{3}$ is a sub-formula of $\mathbf{1} \wedge \mathbf{2}$.

Lemma 6. *For each formula of \mathbf{CPL}_0 in DPF F , there is a formula in MNF F^{**} such that $F \equiv F^{**}$.*

Proof. Given a formula of \mathbf{CPL}_0 in DPF $F = \bigvee_{i \in \{1, \dots, n\}} C_i$, we define a formula F^{**} in MNF as follows.

- * if $F \in \{\perp, \top\}$, then $F^{**} = F$.
- * Otherwise, we consider each $i \in \{1, \dots, n\}$, starting from $i = 1$:
 - i.* if there is a $j \neq i \in \{1, \dots, n\}$, such that C_i is a sub-formula of C_j , then C_i is removed and $i + 1$ is considered.
 - ii.* otherwise, we consider each pedexes $j \neq i \in \{1, \dots, n\}$ starting from the first possible j :
 - a.* if C_i and C_j are mutually contradictory, then $j + 1$ is considered.
 - b.* otherwise, for $C_i = \bigwedge_{k \in \{1, \dots, l\}} L_k$ and $C_j = \bigwedge_{k' \in \{1, \dots, l'\}} L_{k'}$, we consider each $k \in \{1, \dots, l\}$ starting from $k = 1$.
 - if there is a $k' \in \{1, \dots, l'\}$ such that $L_k = L_{k'}$, then L_k is left unchanged and $k + 1$ is considered.

- if there is no $k' \in \{1, \dots, l'\}$ such that $L_k = L_{k'}$, then we replace C_i with two formulas $C'_i = C_i \wedge L_{k'}$ and $C''_i = C_i \wedge \overline{L_{k'}}$ and apply *a.-b.* to both.

We consider each $k' \in \{1, \dots, l'\}$ starting from 1:

- if there is a $k \in \{1, \dots, l\}$ such that $L_k = L_{k'}$, then $L_{k'}$ is left unchanged and $k + 1$ is considered.
- if there is no $k \in \{1, \dots, l\}$ such that $L_k = L_{k'}$, then we replace C_j with $C'_j = C_j \wedge L_{k'}$ and $C''_j = C_j \wedge \overline{L_{k'}}$ and apply *a.-b.* to both.

Then, we consider $j + 1$.

When $j + 1 = n$, $i + 1$ is considered (until also $i + 1 = n$).

□

So, for any formula of **CPL**₀ *without quantifier*, by Lemma 6, we can “convert” it into (DNF, then DPF, so) MNF and this concludes our proof. Indeed, by Lemma 5, given such MNF-formula there is a procedure to measure it. As anticipated, this result can be extended to any formulas of **CPL**₀, as quantified expressions are simply substituted by either \perp or \top (depending, of course, on the quantification involved and on the fact that the measure of the argument formula is greater or strictly smaller than the probability index). Then, the procedure is repeated as long as required.

Example 6. *Let us consider the formula $F = \mathbf{1} \vee (\mathbf{1} \wedge \mathbf{3})$. Its MNF is $F' = \mathbf{1}$. Then, $\mu_{\mathcal{C}}(F) = \frac{1}{2}$. As a second example, let $G = \mathbf{1} \vee (\neg \mathbf{1} \wedge \mathbf{2})$. This formula is already in MNF and $\mu_{\mathcal{C}}(\llbracket G \rrbracket) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Finally, let $H = \mathbf{1} \vee (\neg \mathbf{1} \wedge \mathbf{2}) \vee (\neg \mathbf{2} \wedge \mathbf{3})$. Its MNF is $H' = (\mathbf{1} \wedge \mathbf{2} \wedge \mathbf{3}) \vee (\mathbf{1} \wedge \mathbf{2} \wedge \neg \mathbf{3}) \vee (\mathbf{1} \wedge \neg \mathbf{2} \wedge \mathbf{3}) \vee (\mathbf{1} \wedge \neg \mathbf{2} \wedge \neg \mathbf{3}) \vee (\neg \mathbf{1} \wedge \mathbf{2} \wedge \mathbf{3}) \vee (\neg \mathbf{1} \wedge \mathbf{2} \wedge \neg \mathbf{3}) \vee (\neg \mathbf{1} \wedge \neg \mathbf{2} \wedge \mathbf{3})$. Thus, $\mu_{\mathcal{C}}(\llbracket H \rrbracket) = \frac{7}{8}$.*

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A Appendix

Proof of Example 1. The formula $\mathbf{C}^{1/4}(\mathbf{1} \wedge \mathbf{2}) \wedge \mathbb{D}^{1/4}(\mathbf{1} \wedge \mathbf{2})$ is shown valid as follows:

$$\begin{aligned}
\llbracket F_{ex} \rrbracket &= \llbracket \mathbf{C}^{1/4}(\mathbf{1} \wedge \mathbf{2}) \rrbracket \cap \llbracket \mathbb{D}^{1/4}(\mathbf{1} \wedge \mathbf{2}) \rrbracket \\
&= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}(\llbracket \mathbf{1} \wedge \mathbf{2} \rrbracket) \geq \frac{1}{4} \\ \emptyset & \text{otherwise} \end{cases} \cap \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}(\llbracket \mathbf{1} \wedge \mathbf{2} \rrbracket) \leq \frac{1}{4} \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}(\llbracket \mathbf{1} \rrbracket \cap \llbracket \mathbf{2} \rrbracket) \geq \frac{1}{4} \\ \emptyset & \text{otherwise} \end{cases} \cap \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}(\llbracket \mathbf{1} \rrbracket \cap \llbracket \mathbf{2} \rrbracket) \leq \frac{1}{4} \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}(Cyl(1) \cap Cyl(2)) \geq \frac{1}{4} \\ \emptyset & \text{otherwise} \end{cases} \cap \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}(Cyl(1) \cap Cyl(2)) \leq \frac{1}{4} \\ \emptyset & \text{otherwise} \end{cases} \\
&= \mathbb{B}^{\mathbb{N}} \cap \mathbb{B}^{\mathbb{N}} \\
&= \mathbb{B}^{\mathbb{N}}.
\end{aligned}$$

The formula is proved derivable in $\mathbf{LK}_{\mathbf{CPL}_0}$:

$$\frac{\frac{\mathcal{D} \quad \mu(\llbracket x_1 \wedge x_2 \rrbracket) \geq \frac{1}{4}}{\vdash \top \multimap \mathbf{C}^{1/4}(\mathbf{1} \wedge \mathbf{2})} R_{\mathbf{C}}^{\multimap} \quad \frac{\mathcal{D} \quad \mu(\llbracket x_1 \wedge x_2 \rrbracket) < \frac{1}{4}}{\vdash \top \multimap \mathbb{D}^{1/4}(\mathbf{1} \wedge \mathbf{2})} R_{\mathbb{D}}^{\multimap}}{\vdash x_1 \wedge x_2 \multimap \mathbf{1} \wedge \mathbf{2}} R_{\wedge}^{\multimap}$$

where \mathcal{D} is defined as,

$$\frac{\frac{\frac{x_1 \vDash x_1}{\vdash x_1 \multimap \mathbf{1}} Ax1 \quad \frac{x_2 \vDash x_2}{\vdash x_2 \multimap \mathbf{2}} Ax1}{\vdash x_1 \wedge x_2 \multimap \mathbf{1}} R_{\cup}^{\multimap} \quad \frac{\frac{x_2 \vDash x_2}{\vdash x_2 \multimap \mathbf{2}} Ax1}{\vdash x_1 \wedge x_2 \multimap \mathbf{2}} R_{\cup}^{\multimap}}{\vdash x_1 \wedge x_2 \multimap \mathbf{1} \wedge \mathbf{2}} R_{1\wedge}^{\multimap}$$

□

Proof of Example 2. The formula F_{bias} of Example 2 can be easily proved valid.

$$\begin{aligned}
\llbracket F_{bias} \rrbracket &= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}(\llbracket (\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{3} \wedge \mathbf{4}) \rrbracket) \geq \frac{1}{3} \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}(\llbracket \mathbf{1} \wedge \mathbf{2} \rrbracket \cup \llbracket \mathbf{3} \wedge \mathbf{4} \rrbracket) \geq \frac{1}{3} \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathcal{C}}((\llbracket \mathbf{1} \rrbracket \cap \llbracket \mathbf{2} \rrbracket) \cup (\llbracket \mathbf{3} \rrbracket \cap \llbracket \mathbf{4} \rrbracket)) \geq \frac{1}{3} \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \mu_{\mathbf{C}}((Cyl(1) \cap Cyl(2)) \cup (Cyl(3) \cap Cyl(4))) \geq \frac{1}{3} \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} \mathbb{B}^{\mathbb{N}} & \text{if } \frac{1}{2} \geq \frac{1}{3} \\ \emptyset & \text{otherwise} \end{cases} \\
&= \mathbb{B}^{\mathbb{N}}.
\end{aligned}$$

Let \mathcal{D} be defined as in Example 1. Then, the derivation of F_{bias} in $\mathbf{LK}_{\mathbf{CPL}_0}$ is as follows:

$$\begin{array}{c}
\frac{\frac{\frac{x_3 \vDash x_3}{\vdash x_3 \multimap \mathbf{3}} Ax1}{\vdash x_3 \wedge x_4 \multimap \mathbf{3}} R1_{\cup} \quad \frac{\frac{x_4 \vDash x_4}{\vdash x_4 \multimap \mathbf{4}} Ax1}{\vdash x_3 \wedge x_4 \multimap \mathbf{4}} R_{\cup}^{\rightarrow}}{\vdash x_3 \wedge x_4 \multimap \mathbf{3} \wedge \mathbf{4}} R_{\wedge}^{\rightarrow}} \\
\frac{\mathcal{D}}{\vdash x_1 \wedge x_2 \multimap (\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{3} \wedge \mathbf{4})} R_{\vee}^{\rightarrow} \quad \frac{\vdash x_3 \wedge x_4 \multimap \mathbf{3} \wedge \mathbf{4}}{\vdash x_3 \wedge x_4 \multimap (\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{3} \wedge \mathbf{4})} R_{\vee}^{\rightarrow}}{\vdash (x_1 \wedge x_2) \vee (x_3 \wedge x_4) \multimap (\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{3} \wedge \mathbf{4})} R_{\cup}^{\rightarrow}} \\
\frac{\vdash (x_1 \wedge x_2) \vee (x_3 \wedge x_4) \multimap (\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{3} \wedge \mathbf{4})}{\vdash \top \multimap \mathbf{C}^{1/3}(\mathbf{1} \wedge \mathbf{2}) \vee (\mathbf{3} \wedge \mathbf{4})} R_{\mathbf{C}}^{\rightarrow}
\end{array}$$

Indeed, $\mu(\llbracket (x_1 \wedge x_2) \vee (x_3 \wedge x_4) \rrbracket) = \frac{1}{2} \geq \frac{1}{3}$.

□