Extension of the Observability Rank Condition to Time-Varying Nonlinear Systems

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***Abstract*—This note provides an extension of the observability rank condition to time-varying nonlinear systems. Previous conditions to check the state observability only hold for nonlinear systems that do not explicitly depend on time, or, for time-varying systems, they only account for the linear case. In this note, a general analytic condition is provided. The new condition reduces to the well known observability rank condition in the two simpler cases of time-varying linear systems and time-invariant nonlinear systems. The proposed condition is applied to aerospace robotics to study the observability of a lunar module type system equipped with monocular vision and inertial sensors.**

***Index Terms*—Nonlinear observability; Time-Varying Systems; Observability rank condition**

I. INTRODUCTION

Observability is a fundamental structural property of a control system. This property describes the possibility of inferring the state that characterizes the system from observing its inputs and outputs. As for controllability, the concept of observability was first introduced for linear systems [1], [2] and the analytic condition to check if a linear system satisfies this property has also been obtained. The nonlinear case is much more complex. In this note we refer to the *weak local observability*, as defined in [3], [4] (definitions 8, 9, 10, 11, in [4]). The analytic condition to check if a continuous Time-Invariant (TI) nonlinear system satisfies this property (the weak local observability) has also been introduced [3]–[5]. It is known as the *observability rank condition* and it is based on the computation of the *observability codistribution*. It is summarized in section III-B.

Based on this fundamental result, a new scheme to design Extended Kalman Filter estimators has been proposed [6], [7]. In addition, new methodologies for analysing the structural identifiability of dynamic models have also been proposed [8].

The observability rank condition has extensively been used in many application domains, ranging from computer vision (e.g., [9]) up to robotics (e.g., [10]–[12]) and calibration (e.g., [13]). Very recently, new analytic conditions, proposed in [14]–[17], extend the observability rank condition to the case when the dynamics are also driven by unknown inputs.

Unfortunately, all the conditions above cannot be used in the case when the system is time-varying (nonautonomous).

A Time-Varying (TV) system is a system whose behaviour changes with time. In particular, the system will respond differently to the same input at different times. In a general mathematical characterization of a nonlinear TV system, all the key scalar and vector fields that define its dynamics and/or its output functions, explicitly depend on time (see equation (2)).

In order to apply the observability rank condition to a nonautonomous system, one would have to transform the original system into an autonomous system. In many cases, this can be achieved by including new entries into the state. When possible, this strategy is viable if there are very few factors that make the system nonautonomous. However, when multiple and independent factors make the system nonautonomous, this strategy becomes laborious. Furthermore, its implementation depends on the specific case and cannot be set up as a general systematic procedure.

For TV systems, a general analytic condition to check observability has only been obtained in the linear case. This condition is summarized in section III-A. Very recently, a condition to check the weak local observability of nonlinear TV systems was provided in [16]. However, its derivation is based on very restrictive conditions (analytic functions that characterize the system and analytic controls). So far, no automatic procedure exists to check the weak local observability for general nonlinear TV systems[1]. This is precisely the goal of this note, i.e., the extension of the observability rank condition to nonlinear TV systems. Note that, in the case of controllability, a study about the TV case has been carried out in [18], where the author provides an extension of accessibility and strong accessibility conditions to the TV control-affine case. The analog of this extension for observability is still missing and is given in this note. As in the case of nonlinear TI systems, the new condition is based on the computation of the observability codistribution. The computation of this codistribution is the core of the new condition (Algorithm 2 and Theorem 2 in section III-C).

This note is organized as follows. Section II provides the basic equations that characterize the systems investigated here. It also provides the extension of the definitions of *Indistinguishability* and *Weak Local Observability* to the TV case. Section III provides the new analytic condition, which is illustrated in section IV by a simple application. Finally, our conclusion is given in section V.

II. DEFINITION OF OBSERVABILITY FOR TV SYSTEMS

We will refer to a nonlinear control system with m inputs $(u_1, \dots, u_m)$. The state is:

$$x \triangleq [x^1, \dots,\ x^n]^T \in \mathcal{M}, \tag{1}$$

with $\mathcal{M}$ a $\mathcal{C}^\infty-$ manifold of dimension n. We assume that the dynamics are nonlinear with respect to the state and affine with respect to the inputs. We account for an explicit time dependence, namely, all the functions that characterize the dynamics and/or the outputs, can explicitly depend on time. Finally, the system has $p(\geq 1)$ outputs. Our system is characterized by the following equations:

$$\begin{cases} \dot{x} &= f^0(x,\ t) + \sum_{i=1}^m f^i(x,\ t)u_i \\ y &= [h_1(x,\ t), \dots, h_p(x,\ t)]^T \end{cases} \tag{2}$$

where $f^i(x,\ t)$, $i = 0, 1, \dots, m$, are vector fields in $\mathcal{M}$ and the functions $h_1(x,\ t),\ \dots,\ h_p(x,\ t)$ are scalar fields defined on the manifold $\mathcal{M}$. All these vector and scalar fields explicitly depend on time. We assume that they are $\mathcal{C}^\infty$ functions of their arguments, $\forall (t,\ x) \in \underline{\mathcal{M}}$, where $\underline{\mathcal{M}} = (\mathcal{I} \times \mathcal{M})$ and $\mathcal{I} \subseteq \mathbb{R}$ is an open time interval. Finally, the results provided by this note (the five statements of Theorem 2), which are the extension of several results that hold

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[1]With *automatic procedure* we mean a procedure that can be executed by performing systematic calculations such as derivatives and the rank of matrices.

in the TI case (the five statements of Theorem 1), are obtained by using the same class of controls (as functions of time) used in the TI case (i.e., piecewise continuous controls).

Given $t, t_0 \in \mathcal{I}$ with $t \geq t_0$ and given the input u defined in $[t_0,\ t]$, we denote by $x(t;\ x_0,\ u)$ the value of the state at time t, when the initial state is $x(t_0) = x_0$ and the control input is u. We introduce the definition of observability for TV systems by following the same steps for TI systems [3], [4]. We start by providing the definition of indistinguishable states.

Definition 1 (Indistinguishable States) *Given the system in (2), two states, $x_1,\ x_2\ \in \mathcal{M}$, are indistinguishable at $t_0 \in \mathcal{I}$ if, $\forall u$ and $\forall t \in \mathcal{I} \bigcap [t_0, \infty)$, the system produces the same output $y(t)$ for the initial states $x(t_0) = x_1$ and $x(t_0) = x_2$.*

In the above definition we made the assumption that the solutions $x(t;\ x_1,\ u)$ and $x(t;\ x_2,\ u)$ exist on the same open interval $\mathcal{I} \bigcap [t_0, \infty)$. The only difference with respect to the definition of indistinguishable states in the TI case resides in the fact that here we must specify the time t_0 and the time interval $\mathcal{I}$ (e.g., see Definition 8 in [4]). To this regard, we emphasize the fact that the functions in (2) are $\mathcal{C}^\infty$ functions of their arguments, which is less restrictive than considering analytic functions. The following trivial example shows that the indistinguishability property depends on the considered time t_0. Let us suppose that the state has a single component, $x = x^1 \in \mathbb{R}$, $m = 0$ (no input), and that Equation (2) is:

$$\left\{ \begin{array}{ll} \dot{x} & = 1 \\ y & = x q(t) \end{array} \right. \quad \text{with } q(t) = \left\{ \begin{array}{ll} e^{-\frac{1}{(t-t^*)^2}} & \text{if } t < t^* \\ 0 & \text{if } t \geq t^* \end{array} \right. \tag{3}$$

For this example, $\underline{\mathcal{M}} = \mathbb{R} \times \mathbb{R}$. It is immediate to realize that any $x_1 \neq x_2 \in \mathbb{R}$ are indistinguishable at any $t_0 \geq t^*$ and they are not indistinguishable at any $t_0 < t^*$. Note that both $f^0(x,t)(=1)$ and $h_1(x,t)(= xq(t))$ are $\mathcal{C}^\infty$ functions of their arguments in the entire definition set (i.e., $\forall x \in \mathbb{R}$ and $\forall t \in \mathbb{R}$). On the other hand, $h_1(x,t)$ is not an analytic function of t at $t = t^*$.

Starting from Definition 1, we introduce the definition of observability, local observability, weak observability, and weak local observability, by proceeding as in [3], [4].

Definition 2 (Observability) *The system in (2) is observable at a given $x_0 \in \mathcal{M}$ and at a given $t_0 \in \mathcal{I}$ if it does not admit any state indistinguishable from x_0 at t_0. It is observable on a subset $\underline{A} \subseteq \underline{\mathcal{M}}$, if it is observable at any $(t,\ x) \in \underline{A}$.*

Again, compared with the TI case, we must specify the time t_0. The system defined in (3), is observable at any $t_0 < t^*$ and at any x. Conversely, independently of x, it is not observable at any $t_0 \geq t^*$. Therefore, it is observable on the subset $\underline{A} = (-\infty,\ t^*) \times \mathbb{R}$. Note that, the structure of $\underline{A}$ is in general more complex than the Cartesian product of a subset of $\mathbb{R}$ and a subset of $\mathcal{M}$ (e.g., this would be the case of the system in (3) when the parameter t^* depends on x).

The concept of local observability is obtained by restricting the trajectories necessary to distinguish two states, in particular, by requiring that we remain close to the initial state. Let us consider an open neighbourhood of $x_0 \in \mathcal{M}$ and let us denote it by B. In addition, given $T > 0$, we denote by $\mathcal{I}_T$ the time interval $[t_0,\ t_0 + T] \bigcap \mathcal{I}$. We introduce the concept of $(B,\ T)-$indistinguishability as follows. We say that two states, $x_1,\ x_2\ \in \mathcal{M}$, are $(B,\ T)$-indistinguishable at $t_0 \in \mathcal{I}$ if, $\forall u$ defined on the interval $\mathcal{I}_T$, and such that both $x(t;\ x_1,\ u)$ and $x(t;\ x_2,\ u)$ exist and lie in B, the system produces the same output $y(t)$, $\forall t \in \mathcal{I}_T$, for the initial states x_1 and x_2.

Definition 3 (Local Observability) *The system in (2) is locally observable at a given $x_0 \in \mathcal{M}$ and at a given $t_0 \in \mathcal{I}$, if for any open neighbourhood B of x_0 and for any $T > 0$, there is no $(B,\ T)$-indistinguishable state from x_0 at t_0. It is locally observable on a subset $\underline{A} \subseteq \underline{\mathcal{M}}$, if it is locally observable at any $(t,\ x) \in \underline{A}$.*

It is immediate to realize that, the system defined in (3), is locally observable where it is observable (i.e., on $\underline{A} = (-\infty,\ t^*) \times \mathbb{R}$).

In many practical situations, one might need only to distinguish x_0 from their neighbours. For this reason, the notion of weak observability was introduced. In the TV case, this definition becomes:

Definition 4 (Weak Observability) *The system in (2) is weakly observable at a given $x_0 \in \mathcal{M}$ and at a given $t_0 \in \mathcal{I}$, if there exists an open neighbourhood B of x_0 such that there is no indistinguishable state from x_0 in B at t_0. It is weakly observable on a subset $\underline{A} \subseteq \underline{\mathcal{M}}$, if it is weakly observable at any $(t,\ x) \in \underline{A}$.*

On the other hand, distinguishing two states in B can require to go far from B. For this reason, the notion of weak local observability was introduced. In the TV case, this definition becomes:

Definition 5 (Weak Local Observability) *The system in (2) is weakly locally observable at a given $x_0 \in \mathcal{M}$ and at a given $t_0 \in \mathcal{I}$, if there exist an open neighbourhood B_1 of x_0 and a given $T_1 > 0$ such that, for every open neighbourhood $B_2 \subseteq B_1$ of x_0 and for any $0 < T_2 \leq T_1$, there is no $(B_2,\ T_2)$-indistinguishable state from x_0 in B_2 at t_0, It is weakly locally observable on a subset $\underline{A} \subseteq \underline{\mathcal{M}}$, if it is weakly locally observable at any $(t,\ x) \in \underline{A}$.*

We conclude this section by mentioning a definition of observability for TV systems in the state of the art. This definition was introduced in [20] and is based on the concept of indistinguishable states on a given time interval, $[t_0,\ t_0 + T)$ $(T > 0)$. This definition considers two states indistinguishable if the outputs are identical on the considered time interval, i.e., for any $t \in [t_0,\ t_0 + T)$. This definition is equivalent to our definition (i.e., Definition 1), once we set the interval $\mathcal{I}$ right-bounded by $t_0 + T$. Note that, as the functions that define our system are not necessarily analytic functions (they are $\mathcal{C}^\infty$) the observability depends on the chosen interval $\mathcal{I}$. Let us consider again the example given in (3) and let us set $\mathcal{I} = (T_0,\ \infty)$. If $T_0 \geq t^*$ the system is neither observable nor weakly observable on the entire $\underline{\mathcal{M}}$. If $T_0 < t^*$, then it is observable and also locally observable for any $t < t^*$ and is neither observable nor weakly observable for any $t \geq t^*$ (independently of x_0).

III. Analytic condition for observability

This section introduces the analytic condition to check the state observability for systems that satisfy equation (2).

Before introducing this new condition we remind the reader the existing results for less general systems. Specifically, in section III-A, we provide the analytic condition that holds in the case of linear TV systems and, in section III-B, we provide the analytic condition that holds in the case of nonlinear TI systems. In section III-C, we provide the new condition that holds in general, i.e., for nonlinear TV systems.

A. Time-varying linear systems

This special case is obtained by setting in (2):

- $f^0(x,\ t) = A(t)x$, where A is a matrix of dimension $n \times n$.
- $f^i(x,\ t) = b^i(t)$, where $b^1(t), \ldots, b^m(t)$ are m column vectors of dimension n.

- $h_j(x,\ t) = c_j(t)x$, where $c_1(t), \ldots, c_p(t)$ are p row-vectors of dimension n.

We can write (2) as follows:

$$\left\{ \begin{array}{ll} \dot{x} & = A(t)x + B(t)u \\ y & = C(t)x \end{array} \right. \tag{4}$$

where the columns of B are the vectors $b^1, \ldots, b^m$ above and the rows of C are the vectors $c_1, \ldots, c_p$ above. In this case, by a direct integration of the first equation in (4) and by substituting in the second equation in (4), it is possible to express the output at any t as the sum of two terms. The former depends on the initial state and is independent of all the inputs while the latter depends on the inputs and is independent of the initial state. This separation between the initial state and the system inputs makes observability independent of the inputs. An interesting case is when the above $A(t)$ and $C(t)$ have analytic dependence on t. In this case, the above first term of the output (the one that is independent of all the inputs) has an analytic dependence on the time. Hence, observability becomes independent of the time interval $\mathcal{I}$.

In [20], it is proved that, observability on a given time interval $[t_0,\ t_0+T)$ is equivalent to the following algebraic condition. There exists $\bar{t}$ in this interval and a positive integer k such that:

$$rank \left[\begin{array}{c} N_0(\bar{t}) \\ N_1(\bar{t}) \\ \ldots \\ N_k(\bar{t}) \end{array} \right] = n \tag{5}$$

where $N_0(t) \triangleq C(t)$ and $N_i(t)$ is defined recursively as:

$$N_i(t) = N_{i-1}(t)A(t) + \frac{dN_{i-1}(t)}{dt}, \quad i = 1, \ldots, k \tag{6}$$

This result was obtained long time ago in [19]. The reader is addressed to [20] for further details and for analytic derivations to prove the validity of the above equivalence.

In the linear case, all observability notions hold globally.

B. Time-invariant nonlinear systems

This special case is obtained when all the vector and scalar fields that appear in (2) do not explicitly depend on time. The analytic condition to check the weak local observability at a given $x_0 \in \mathcal{M}$ of the state x that satisfies (2) is obtained by computing the *observability codistribution*. When all the vector and scalar fields do not explicitly depend on time, the observability codistribution is generated by the recursive algorithm 1 (see [3]–[5]).

We use the following notation:

- A covector field ω will be denoted by $\omega = \omega_1 dx_1 + \omega_2 dx_2 + \ldots + \omega_n dx_n$ or, more simply, by the row vector $[\omega_1,\ \omega_2,\ \ldots,\ \omega_n]$.
- Given a scalar field h, dh is its differential.
- Given a vector field f (defined on the manifold $\mathcal{M}$), $\mathcal{L}_f$ denotes the Lie derivative along f. We remind the reader that the Lie derivative along f of a tensor field of type $(p,\ q)$ is a tensor field of the same type. In particular, for the scalar field $h(x)$ (tensor of type $(0,\ 0)$) we have:

$$\mathcal{L}_f h = \frac{\partial h}{\partial x} \cdot f. \tag{7}$$

The Lie derivative along f of a given covector field ω (tensor of type $(0,\ 1)$) is [5]:

$$\mathcal{L}_f \omega = f^T \left(\frac{\partial \omega^T}{\partial x} \right)^T + \omega \left(\frac{\partial f}{\partial x} \right) \tag{8}$$

In the special case when $\omega = dh$ (i.e., it is an exact 1-form), (7) and (8) simplify as follows:

$$\mathcal{L}_f dh = d\mathcal{L}_f h \tag{9}$$

- Given a codistribution[2] Ω and a vector field f (both defined on the manifold $\mathcal{M}$), we define $\mathcal{L}_f \Omega \triangleq \text{span}\{\mathcal{L}_f \omega \mid \omega \in \Omega\}$, where the span is over the ring of the $\mathcal{C}^\infty$ functions on $\mathcal{M}$. In addition, we say that Ω is invariant under $\mathcal{L}_f$ if $\mathcal{L}_f \Omega \subseteq \Omega$.
- Given two vector spaces V_1 and V_2, $V_1 + V_2$ is their sum, i.e., the span of all the generators of V_1 and V_2, (again, the span is over the ring of the $\mathcal{C}^\infty$ functions on $\mathcal{M}$).

We summarize the main results for nonlinear TI systems. Let us consider the system in (2) when all the functions do not explicitly depend on time. We consider the codistributions constructed by Algorithm 1, for any positive integer k.

Algorithm 1

$\Omega_0 = \text{span}\{dh_1, \ldots, dh_p\}$
$\Omega_k = \Omega_{k-1} + \mathcal{L}_{f^0}\Omega_{k-1} + \mathcal{L}_{f^1}\Omega_{k-1} + \ldots + \mathcal{L}_{f^m}\Omega_{k-1}$

We also denote by Ω^* the smallest codistribution that contains Ω_0 and that is invariant under $\mathcal{L}_{f^0}, \mathcal{L}_{f^1}, \ldots, \mathcal{L}_{f^m}$. This minimal codistribution exists and is smooth (see the first statement of Theorem 1). We remind the reader the definition of *Observability Rank Condition* for TI systems.

Definition 6 (Observability Rank Condition) *The system in (2), when all functions do not explicitly depend on time, satisfies the observability rank condition at $x_0 \in \mathcal{M}$ if the rank of Ω^* at x_0 equals n. If the rank of Ω^* equals n $\forall x \in A \subseteq \mathcal{M}$, then the system satisfies the observability rank condition on A.*

We have:

Theorem 1 *When the system in (2) has not an explicit time-dependence (i.e., is TI), the following statements hold:*

1) *The family of codistributions that are invariant under $\mathcal{L}_{f^0}, \mathcal{L}_{f^1}, \ldots, \mathcal{L}_{f^m}$ and contain Ω_0 has a minimal element (Ω^*), which is a smooth codistribution.*
2) *The codistributions generated by Algorithm 1 are such that $\Omega_k \subseteq \Omega_{k+1} \subseteq \Omega^*$, $\forall k \geq 0$. If there exists an integer k^* such that $\Omega_{k^*} = \Omega_{k^*-1}$, then $\Omega_{k^*-1} = \Omega^*$. In addition, if the codistributions constructed by Algorithm 1 are non singular, the above condition is attained at an integer $k^* \leq n$.*
3) *There exists an open and dense set $\mathcal{M}^* \subseteq \mathcal{M}$ such that $\Omega^*(x) = \Omega_{n-1}(x)$, $\forall x \in \mathcal{M}^*$ (this also holds when the codistributions constructed by Algorithm 1 are singular).*
4) *If the system satisfies the observability rank condition at x_0, it is weakly locally observable at x_0 (sufficient condition for the weak local observability at x_0).*
5) *If the system is weakly locally observable on a given open set $\mathcal{M}_0 \subseteq \mathcal{M}$, then it satisfies the observability rank condition on a dense subset of $\mathcal{M}_0$ (necessary condition for the weak local observability on a given open set).*

Proof: All these statements are very well known results. The first three statements are proved in [5] (Lemma 1.9.1, Lemma 1.9.2, and Lemma 1.9.3, respectively). The proof of the sufficient condition for the weak local observability at x_0 is available in [3], Theorem 3.1 (or

[2] We address the reader to [5] for the definitions of distribution and codistribution.

[4], Theorem 18). The proof of the necessary condition for the weak local observability on a given open set, is available in [3], Theorem 3.11 (or [4], Theorem 19). ◄

C. Time-varying nonlinear systems

We start by introducing the following new operator (which will replace the Lie derivative along the drift f^0):

$$\dot{\mathcal{L}}_{f^0} \triangleq \frac{\partial}{\partial t} + \mathcal{L}_{f^0} \tag{10}$$

Note that for $\omega = \sum_{i=1}^n \omega_i dx_i$, we have: $\frac{\partial \omega}{\partial t} = \sum_{i=1}^n \frac{\partial \omega_i}{\partial t} dx_i$.

Let us consider the codistributions constructed by Algorithm 2, for any non negative integer k. Now, the span is over the ring of the $\mathcal{C}^\infty$ functions on $\underline{\mathcal{M}}$ (e.g., at the initialization, any element of span$\{dh_1, \ldots, dh_p\}$ is a covector field, ω, that can be expressed as follows: $\omega = \sum_{l=1}^p c_l(t,\ x)\ dh_l$, with $c_1(t,\ x), \ldots, c_p(t,\ x)$ scalar and $\mathcal{C}^\infty$ functions in $\underline{\mathcal{M}}$, and $dh_l = \sum_{i=1}^n \frac{\partial h_l}{\partial x_i} dx_i$.

Algorithm 2

$\Omega_0 =$span$\{dh_1, \ldots, dh_p\}$
$\Omega_k = \Omega_{k-1} + \dot{\mathcal{L}}_{f^0}\Omega_{k-1} + \mathcal{L}_{f^1}\Omega_{k-1} + \ldots + \mathcal{L}_{f^m}\Omega_{k-1}$

Compared with Algorithm 1, in the recursive step we replaced $\mathcal{L}_{f^0}$, with $\dot{\mathcal{L}}_{f^0}$. Note that, for TI systems, the recursive step of Algorithm 2 coincides with the recursive step of Algorithm 1, as Ω_k is independent of time, $\forall k$. We also denote by Ω^* the smallest codistribution that contains Ω_0 and that is invariant under $\dot{\mathcal{L}}_{f^0}, \mathcal{L}_{f^1}, \ldots, \mathcal{L}_{f^m}$. This minimal element exists and is smooth in t and x (see the first statement of Theorem 2).

We extend Definition 6 to TV systems, as follows:

Definition 7 (Extended Observability Rank Condition) *The system in (2) satisfies the observability rank condition at $(t_0,\ x_0) \in \underline{\mathcal{M}}$ if the rank of Ω^* at $(t_0,\ x_0)$ equals n. If the rank of Ω^* equals n $\forall (t,\ x) \in \underline{A} \subseteq \underline{\mathcal{M}}$, then the system satisfies the observability rank condition on $\underline{A}$.*

Theorem 2 extends the statements of Theorem 1 to TV systems.

Theorem 2 *The following statements hold:*

1) *The family of codistributions that are invariant under $\dot{\mathcal{L}}_{f^0}, \mathcal{L}_{f^1}, \ldots, \mathcal{L}_{f^m}$ and contain Ω_0 has a minimal element Ω^*, which is a smooth codistribution.*
2) *The codistributions generated by Algorithm 2 are such that $\Omega_k \subseteq \Omega_{k+1} \subseteq \Omega^*$, $\forall k \geq 0$. If there exists an integer k^* such that $\Omega_{k^*} = \Omega_{k^*-1}$, then $\Omega_{k^*-1} = \Omega^*$. In addition, if the codistributions constructed by Algorithm 2 are non singular[3], the above condition is attained at an integer $k^* \leq n$.*
3) *There exists an open and dense set $\underline{\mathcal{M}}^* \subseteq \underline{\mathcal{M}}$ such that $\Omega^*(t,\ x) = \Omega_{n-1}(t,\ x)$, $\forall (t,\ x) \in \underline{\mathcal{M}}^*$ (this also holds when the codistributions constructed by Algorithm 2 are singular).*
4) *If the system satisfies the observability rank condition at $(t_0,\ x_0)$, it is weakly locally observable at $(t_0,\ x_0)$ (sufficient condition for the weak local observability at $(t_0,\ x_0)$).*
5) *If the system is weakly locally observable on a given open set $\underline{\mathcal{M}}_0 \subseteq \underline{\mathcal{M}}$, then it satisfies the observability rank condition on a dense subset of $\underline{\mathcal{M}}_0$ (necessary condition for the weak local observability).*

[3]In the TV case, a codistribution is defined *singular* when its rank is not constant on $\underline{\mathcal{M}}$ (i.e., a non singular codistribution has the rank independent of t and x).

Proof: We include the variable time in the state and we use the underline to denote quantities in the extended space. We denote the new extended state by $\underline{x}$. We have:

$$\underline{x} = \left[\tau,\ x^1,\ \ldots,\ x^n\right]^T \tag{11}$$

The system in (2) can be characterized by using the extended state. From equation (2), we obtain:

$$\begin{cases} \dot{\underline{x}} & = \underline{f}^0(\underline{x}) + \sum_{i=1}^m \underline{f}^i(\underline{x}) u_i \\ y & = [h_0(\underline{x}) = \tau,\ h_1(\underline{x}), \ldots, h_p(\underline{x})]^T \end{cases} \tag{12}$$

where:

$$\underline{f}^0(\underline{x}) \equiv \begin{bmatrix} 1 \\ f^0 \end{bmatrix}, \quad \underline{f}^i(\underline{x}) \equiv \begin{bmatrix} 0 \\ f^i \end{bmatrix}, \tag{13}$$

and we set $d\tau = dt$. Note that we also included a new output. Indeed, it is a common (and implicit) assumption that all the system inputs and the outputs are synchronized. Hence, the system includes a new output function that is precisely $h_0(\underline{x}) = t = \tau$ (we set $\tau(t_0) = t_0$).

A covector field in the extended space is $\underline{\omega} = [\omega_0,\ \omega] = [\omega_0,\ \omega_1,\ \ldots,\ \omega_n]$ (or $\underline{\omega} = \omega_0 dt + \omega_1 dx_1 + \omega_2 dx_2 + \ldots + \omega_n dx_n$). We have $\mathcal{L}_{\underline{f}^0}\underline{\omega} = \left[\frac{\partial \omega_0}{\partial t} + \frac{\partial \omega_0}{\partial x} f^0 + \omega \frac{\partial f^0}{\partial t},\ \left(\frac{\partial \omega^T}{\partial x} f^0\right)^T + \omega \frac{\partial f^0}{\partial x} + \frac{\partial \omega}{\partial t}\right]$. Hence:

$$\mathcal{L}_{\underline{f}^0}\underline{\omega} = \left[\frac{\partial \omega_0}{\partial t} + \frac{\partial \omega_0}{\partial x} f^0 + \omega \frac{\partial f^0}{\partial t},\ \dot{\mathcal{L}}_{f^0}\omega\right] \tag{14}$$

Similarly, for any $k = 1, \ldots, m$, we have:

$$\mathcal{L}_{\underline{f}^k}\underline{\omega} = \left[\frac{\partial \omega_0}{\partial x} f^k + \omega \frac{\partial f^k}{\partial t},\ \mathcal{L}_{f^k}\omega\right], \tag{15}$$

By introducing the extended state we transformed our original nonautonomous system in (2) into the autonomous system in (12). We are allowed to use the results stated by Theorem 1.

Algorithm 1 becomes Algorithm 3. We denote by $\underline{d}$ the differential in the extended state. In our notation, $\underline{d}h_l = \frac{\partial h_l}{\partial t} dt + dh_l = \frac{\partial h_l}{\partial t} dt + \sum_{i=1}^n \frac{\partial h_l}{\partial x_i} dx_i$, $(l = 0, 1, \ldots, p)$.

Algorithm 3

$\underline{\Omega}_0 =$span$\{\underline{d}h_0, \underline{d}h_1, \ldots, \underline{d}h_p\}$
$\underline{\Omega}_k = \underline{\Omega}_{k-1} + \mathcal{L}_{\underline{f}^0}\underline{\Omega}_{k-1} + \ldots + \mathcal{L}_{\underline{f}^m}\underline{\Omega}_{k-1}$

By construction, $dt \in \underline{\Omega}_0$. Hence, we have:

$$\underline{\Omega}_0 = \text{span}\{dt\} \oplus \Omega_0, \tag{16}$$

where the symbol $\oplus$ denotes the direct sum (i.e., it is used for the sum of two vector spaces whose intersection is empty) and Ω_0 is the codistribution at the initialization step of Algorithm 2.

We also denote by $\underline{\Omega}^*$ the smallest codistribution that contains $\underline{\Omega}_0$ and that is invariant under $\mathcal{L}_{\underline{f}^0},\ \mathcal{L}_{\underline{f}^1}, \ldots, \mathcal{L}_{\underline{f}^m}$. From the first statement of Theorem 1, we know that this codistribution exists and is smooth in t and x. Starting from $\underline{\Omega}^*$, we define Ω^*, for any $(t,\ x) \in \underline{\mathcal{M}}$, as follows:

$$\Omega^*(t,\ x) \triangleq \{\omega \in T_x^* \mid \exists\ \omega_o \in \mathbb{R} \text{ and } \omega_0 dt + \omega \in \underline{\Omega}^*(t,\ x)\}$$

with T_x^* the cotangent space of $\mathcal{M}$ at x. On the other hand, $dt \in \underline{\Omega}_0 \subseteq \underline{\Omega}^*$. As a result:

$$\underline{\Omega}^* = \text{span}\{dt\} \oplus \Omega^* \tag{17}$$

From (14), (15), (17), and the invariance of $\underline{\Omega}^*$ under $\mathcal{L}_{\underline{f}^0}, \mathcal{L}_{\underline{f}^1}, \ldots, \mathcal{L}_{\underline{f}^m}$, it follows that Ω^* is invariant under

$\dot{\mathcal{L}}_{f^0}, \mathcal{L}_{f^1}, \ldots, \mathcal{L}_{f^m}$. In addition, $\Omega_0 \subseteq \Omega^*$. As $\underline{\Omega}^*$ is the smallest codistribution that contains $\underline{\Omega}_0$ and that is invariant under $\mathcal{L}_{\underline{f}^0}$, $\mathcal{L}_{\underline{f}^1}$ and $\mathcal{L}_{\underline{f}^m}$, from (17) we obtain that Ω^* is the smallest codistribution that contains Ω_0 and that is invariant under $\dot{\mathcal{L}}_{f^0}, \mathcal{L}_{f^1}, \ldots, \mathcal{L}_{f^m}$. This proves the first statement.

For any nonnegative integer k, we have:

$$\underline{\Omega}_k = \text{span}\{dt\} \oplus \Omega_k \tag{18}$$

This can be proved by induction. It is true for $k = 0$, as it is stated in (16). Let us assume that it holds for a given k. By using (14) and (15) it follows that it holds for $k + 1$. By using (18), the condition $\underline{\Omega}_{k^*} = \underline{\Omega}_{k^*-1}$ is equivalent to the condition $\Omega_{k^*} = \Omega_{k^*-1}$. In addition, by also using (17), the condition $\underline{\Omega}_{k^*-1} = \underline{\Omega}^*$ is equivalent to $\Omega_{k^*-1} = \Omega^*$. This proves the second statement with $k^* \leq n+1$ (which is the dimension of $\underline{x}$). On the other hand, as the rank of $\underline{\Omega}_0$ is the sum of the rank of span$\{dt\}$ plus the rank of Ω_0, it is ≥ 2 and, consequently, $k^* \leq n$. From (18) it also follows the convergence property of Algorithm 2 on a dense set of $\underline{\mathcal{M}}$.

From the separation stated by (17), it also follows that the condition rank $(\underline{\Omega}^*) = n + 1$ is equivalent to rank $(\Omega^*) = n$. In addition, the weak local observability of the extended system is equivalent to the weak local observability of the original system (see Lemma 1). This proves the last two statements. ◄

Lemma 1 *The system defined by (12) is weakly locally observable at a given* $\underline{x}_0 = (t_0,\ x_0) \in \underline{\mathcal{M}}$ *if and only if the system defined by (2) is weakly locally observable at* $(t_0,\ x_0)$.

Proof: ($\Rightarrow$) We know that there exists an open neighbourhood $\underline{B}_1 \subseteq \underline{\mathcal{M}}$ of $\underline{x}_0$ and $T_0 > 0$ such that, for any open neighbourhood $\underline{B}_2 \subseteq \underline{B}_1$ of $\underline{x}_0$ and for any $T \leq T_0$ there are no $(\underline{B}_2,\ T)$-indistinguishable states from $\underline{x}_0$. As $\underline{B}_1$ is an open set, there exist an open neighbourhood B_1 of x_0 and $0 < T_1 \leq T_0$, such that $(\mathcal{I}_1 \times B_1) \subseteq \underline{B}_1$, where $\mathcal{I}_1$ is the open interval $(t_0 - T_1,\ t_0 + T_1)$. For any positive $T_2 \leq T_1$ and any open neighbourhood $B_2 \subseteq B_1$ of x_0 we have $(\mathcal{I}_2 \times B_2) \subseteq (\mathcal{I}_1 \times B_1) \subseteq \underline{B}_1$, where $\mathcal{I}_2 = (t_0 - T_2,\ t_0 + T_2)$. As a result, there are no $(B_2,\ T_2)$-indistinguishable states from x_0 at t_0 and this proves that the system defined by (2) is weakly locally observable at $(t_0,\ x_0)$.

($\Leftarrow$) We know that there exist an open neighbourhood B_1 of x_0 and $T_1 > 0$ such that for any open neighbourhood $B \subseteq B_1$ of x_0 and for any positive $T \leq T_1$ there are no $(B,\ T)$-indistinguishable states from x_0 at t_0. We set $\underline{B}_1$ the Cartesian product $(t_0 - T_1,\ t_0 + T_1) \times B_1$, which is an open neighbourhood of $\underline{x}_0$. Let us consider any open neighbourhood $\underline{B}_2 \subseteq \underline{B}_1$ of $\underline{x}_0$. We must prove that, for any initial state $(t_2,\ x_2) \in \underline{B}_2$, the outputs of the system in (12) do not coincide with the outputs with initial state $(t_0,\ x_0)$, when considering input controls u such that both $(t,\ x(t;\ x_2,\ u))$ and $(t,\ x(t;\ x_0,\ u))$ lie in $\underline{B}_2$. As one of the outputs of the system in (12) is $h_0 = \tau = t$, the only candidates of $\underline{B}_2$ that can produce the same outputs are: $\underline{x}_2 \triangleq (t_0,\ x_2)$ with $x_2 \in B_2$ and $B_2 \triangleq \{x \in \mathcal{M}, |(t_0,\ x) \in \underline{B}_2\}$. By construction, $B_2 \subseteq B_1$. Now, let us consider any point $x \in B_2$. We must show that is possible to distinguish $(t_0,\ x_0)$ from $(t_0,\ x)$, by remaining in $\underline{B}_2$. On the other hand, we know that, for any $T_2 < T_1$, we can distinguish x_0 from x. Hence, it suffices to choose T_2 sufficiently small such that for $t < T_2$, both $(t,\ x(t;\ x_2,\ u))$ and $(t,\ x(t;\ x_0,\ u))$ lie in $\underline{B}_2$. This is certainly possible because $\underline{B}_2$ is an open set and both $(t,\ x(t;\ x_0,\ u))$ and $(t,\ x(t;\ x_2,\ u))$ are continuous functions of t. ◄

Note that the necessary condition for the weak local observability provided by the last statement of Theorem 2 is that the system satisfies the observability rank condition on a dense set of $\underline{\mathcal{M}}_0$. Hence, if the rank of Ω^* does not equal the state dimension at a given time t_0, this does not necessarily mean that the system is not weakly locally observable at t_0. Let us consider the same example given in (3), where we replace the function $q(t)$ with $\tilde{q}(t) = q(2t^* - t)$. The resulting system is:

$$\begin{cases} \dot{x} &= 1 \\ y &= x\tilde{q}(t) \end{cases} \quad \text{with } \tilde{q}(t) = \begin{cases} 0 & \text{if } t \leq t^* \\ e^{-\frac{1}{(t-t^*)^2}} & \text{if } t > t^* \end{cases} \tag{19}$$

Let us set $\mathcal{I} = (-\infty,\ T_0)$ and $T_0 > t^*$. The system is weakly locally observable at t^* (it is even locally observable). On the other hand, the rank of Ω^* at t^* is $0 < 1 = n$.

We conclude this section with the following remarks:

1) Algorithm 2 differs from algorithm 1 only for the recursive step. In particular, the operator given in (10) replaces the Lie derivative along f^0. In other words, the new algorithm is obtained with the substitution:

$$\mathcal{L}_{f^0} \rightarrow \dot{\mathcal{L}}_{f^0} \tag{20}$$

If f^0 is null (driftless system), in the recursive step we need to add the term $\frac{\partial}{\partial t}\Omega_{k-1}$.

2) The extended observability rank condition reduces to the observability rank condition provided in section III-B when all the vector and scalar fields that appear in (2) do not explicitly depend on time.
3) The extended observability rank condition reduces to the condition provided in section III-A in the linear case.

IV. Application

We consider a vehicle, like a lunar module, that moves in the presence of gravity and in the absence of an atmosphere. It is equipped with a monocular camera able to detect a point feature on the ground. In this section, we adopt a $2D$ simplified version of the problem, where we do not account for the orientation, which is assumed to be constant, and we do not account for the moment of inertia tensor. The $3D$ study is much more complex and is available in [21]. Fig 1 provides an illustration of our simplified system.

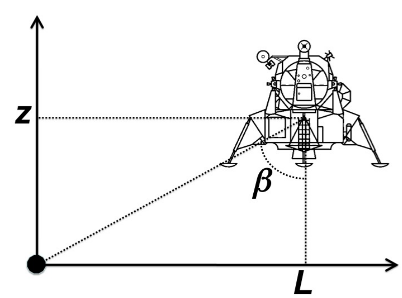

Fig. 1. The rocket moves along the vertical axis and observes a point feature at the origin by its on-board monocular camera.

Note that, through this simplified model we can achieve the same main results obtained in [21]. The complexity of the true $3D$ scenario is due to the state dimension ($n = 14$ instead of $n = 4$) and to the fact that, to achieve a compact expression of the dynamics (which allows us an easy implementation of Algorithm 2) we adopt quaternions (see equations (20) of [21], where, the subscript q denotes the quaternion associated to a $3D$ vector).

We characterize the system by the following state:

$$X = [x,\ z,\ v,\ g]^T \tag{21}$$

where $(x,\ z)$ is the rocket position with respect to our reference frame, v is the magnitude of the speed and g the magnitude of the gravity, which is unknown. We assume that the rocket accomplishes movements which are small enough to consider g constant (in general, for larger motions, g may vary and, to use the same set up in these cases, we include it in the state). Finally, the camera provides the angle $\beta = \tan^{-1}\frac{x}{z}$. For the observability analysis, we can set the output $y = z/x$. Under these assumptions, the system is characterized by the following equations:

$$\left[\begin{array}{ll} \dot{x} = & 0 \\ \dot{z} = & v \\ \dot{v} = & F/\mu - g \\ \dot{g} = & 0 \\ y = & h(X) = z/x \end{array}\right. \tag{22}$$

where F is the force provided by the main rocket engine and μ the mass of the rocket. The system in (22) is defined on $\mathcal{M} = \{[x,\ z,\ v,\ g]^T \in \mathbb{R}^4,\ x \neq 0\}$ (actually, the angle β could be defined also when $x = 0$ provided that, when $x = 0$, $z \neq 0$). We investigate the case when $F = F_0$, namely F is independent of time (the main rocket engine delivers constant power). By comparing (21-22) with (2) we obtain: $f^0(X) = [0, v,\ a,\ 0]^T$, where $a \triangleq F_0/\mu - g$. In addition, the system has no input and it has a single output. We consider three distinct scenarios. The first scenario (Section IV-A) is characterized by a constant value of the mass (μ). In this case, a is independent of time and we set $a = a_0$. In the second and in the third scenario (Sections IV-B, and IV-C, respectively), the mass decreases during the maneuver, due to the fuel consumption. As a result, $\mu = \mu(t)$ and we assume that $\mu(t)$ is known. In the third scenario, we assume that the point feature observed by the monocular camera is not on the ground and it moves. Hence, in this last case, the output depends on time and we have two independent factors that make the system nonautonomous.

A. *Constant mass*

In this case, neither the dynamics nor the output depend explicitly on time. Therefore, we obtain the observability codistribution by running algorithm 1. We obtain, at the initialization, $\Omega_0 = \text{span}\{dh\} = \text{span}\left\{\frac{1}{x}dz - \frac{z}{x^2}dx\right\}$. Additionally, $\mathcal{L}_{f^0}dh = \frac{1}{x}dv - \frac{v}{x^2}dx$, $\mathcal{L}^2_{f^0}dh = -\frac{1}{x}dg - \frac{a_0}{x^2}dx$ and $\mathcal{L}^3_{f^0}dh = 0$ (note that $\frac{\partial a_0}{\partial g} = -1$). This means that Algorithm 1 converges at its second step and the observability codistribution is: $\Omega_2 = \Omega^* = \text{span}\left\{dh,\ \mathcal{L}_{f^0}dh,\ \mathcal{L}^2_{f^0}dh\right\}$. As a result, the rank of Ω^* is smaller than the state dimension for any $x \in \mathcal{M}$. In accordance with the fifth statement of Theorem 1, there is no open set of $\mathcal{M}$ where the system is weakly locally observable. By computing the orthogonal distribution we obtain: $\Omega^\perp = \text{span}\{[x,\ z,\ v,\ \ -a_0]^T\}$. Its rank is 1 and it characterizes the system invariance with respect to the scale (see Section 4.6.2 in [16]). In other words, to make the state weakly locally observable during this maneuver (characterized by a constant F) we need to equip the system with a further sensor that provides the scale (e.g., a laser range finder).

B. *Varying mass*

In this case, the dynamics explicitly depend on time and, in order to obtain the observability codistribution, we need to run Algorithm 2. We obtain the same generator of Ω_0, which is $dh = \frac{1}{x}dz - \frac{z}{x^2}dx$. In addition: $\dot{\mathcal{L}}_{f^0}dh = \mathcal{L}_{f^0}dh + \frac{\partial}{\partial t}dh = \mathcal{L}_{f^0}dh + 0 = \mathcal{L}_{f^0}dh$, $\dot{\mathcal{L}}^2_{f^0}dh = \dot{\mathcal{L}}_{f^0}\mathcal{L}_{f^0}dh = \mathcal{L}^2_{f^0}dh + \frac{\partial}{\partial t}\mathcal{L}_{f^0}dh = -\frac{1}{x}dg - \frac{a}{x^2}dx$. $\dot{\mathcal{L}}^2_{f^0}dh$ depends on time, since $a = a(t)$. We have: $\dot{\mathcal{L}}^3_{f^0}dh = \mathcal{L}_{f^0}\dot{\mathcal{L}}^2_{f^0}dh + \frac{\partial}{\partial t}\dot{\mathcal{L}}^2_{f^0}dh = -\frac{\dot{a}}{x^2}dx$. It is immediate to check that the above covectors, $dh,\ \dot{\mathcal{L}}_{f^0}dh,\ \dot{\mathcal{L}}^2_{f^0}dh$ and $\dot{\mathcal{L}}^3_{f^0}dh$, are independent where $\dot{a} \neq 0$, which is the case because of the fuel consumption during this maneuver . As a result, the rank of the observability codistribution is 4 and the state is weakly locally observable. This means that, during this maneuver, we do not need a further (range) sensor.

C. *Varying mass and moving point feature*

We now consider the more general case where the point feature observed by the monocular visual sensor is not static but moves (e.g., it is a satellite). We provide here a very simple study by still referring to a $2D$ model. The position of the point feature in our frame will be denoted by $[S_x(t),\ S_z(t)]^T$. As a result, the output of this system will be the bearing angle $\beta = \tan^{-1}\left(\frac{x-S_x}{z-S_z}\right)$. For the observability analysis, we can set the output $y = \frac{z-S_z(t)}{x-S_x}$. Our system is characterized by the first four equations in (22) and the above output. As a result, there are two independent factors that make it time-dependent: (i) the mass variation, which is described by the scalar function $\mu(t)$, and which is due to the fuel consumption, and (ii) the motion of the observed feature, which is described by the two scalar functions $S_x(t)$ and $S_z(t)$. The former acts on the state dynamics, the latter on the system output. For generic motions of the observed feature, the rank of Ω_3 is equal to 4, which is the state dimension. An interesting case is when the feature moves at constant speed, i.e., when $\ddot{S}_x = \ddot{S}_z = 0$. Under this assumption, the computation is easier, as we only need to compute the first order time derivatives of S_x and S_z. It is immediate to verify that the rank of Ω_3 is equal to 4, even when the fuel consumption is negligible and the only time dependence of the system is in the output. In this case (constant speed and negligible variation of the mass), the observability of the state can be seen as a generalization of the triangulation, where both the feature and the camera move. In addition, the baseline is obtained by the known motion of the feature, instead of the camera[4], and the unknown to be estimated is the position of the camera, instead of the position of the feature.

V. Conclusion

This note extended the observability rank condition to time-varying nonlinear systems. The note showed that when this condition is applied to time-invariant nonlinear systems, it coincides with the observability rank condition. In addition, when applied to time-varying linear systems, it coincides with a condition to check the observability in the state of the art. The note first extended the definition of indistinguishable states to time-varying systems and, based on that, it extended the definition of observability. The proposed extension of the observability rank condition is very simple and can be implemented automatically (i.e., by performing systematic calculations such as derivatives and the rank of matrices). This makes it preferable with respect to other methods, based on a state extension that transforms the original nonautonomous system into an autonomous system. In particular, these methods cannot be implemented automatically as the state extension depends on the specific case, and they can be laborious in the presence of multiple and independent factors that make the system nonautonomous. The new condition was applied to an aerospace system, which consists of a lunar module during the take-off in the absence of an atmosphere, when the main rocket engine delivers constant power.

In this note we did not discuss the problem of observer design. For linear time-invariant systems, observability is sufficient to guarantee

[4]Note that, the monocular camera only provides scale-invariant quantities. The system acquires metric information from the known motion of the feature.

the existence of observers with globally exponentially stable error dynamics. In the time-varying case, a stronger condition is required that is the uniform complete observability (UCO). A folk result (e.g., see Theorem 4 in [22]), provides a sufficient condition for a linear time-varying system to be UCO. In [23], this condition was relaxed (Theorem 3 in [23]). An interesting aspect is that the two above conditions (Theorem 4 in [22] and Theorem 3 in [23]) are based on the same quantities that, in the nonlinear time-varying case, become the generators of the observability codistribution (i.e., Ω^* in Theorem 2 of this note). In other words, it is possible to introduce these conditions for any nonlinear system. On the other hand, their meaning in the nonlinear case becomes unclear. For nonlinear observable systems, even in the TI case, it does not exist a complete theory which allows the design of an observer . The main reason is that, in contrast with the linear case, observability depends on the system inputs. It would be interesting to investigate about the meaning of the two above conditions in the nonlinear case (e.g., if they allow the design of a nonlinear observer, at least in some special cases). Note that, in the special case of a system with a single output and no input, if the extended observability rank condition introduced by Definition 7 is satisfied on an open set $\underline{A} \subseteq \underline{M}$, then, there exists a local diffeomorphism that transforms the system into the nonlinear TV observability canonical form introduced in [24], [25]. Specifically, the observability canonical form given by Equation (15) in [24] is obtained by setting $x^* \triangleq [h,\ \dot{\mathcal{L}}_{f^0} h, \dots, \dot{\mathcal{L}}_{f^0}^{n-1} h]^T$.

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