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Silent sources in L^p and Helmholtz-type decompositions*

Laurent Baratchart[†], Juliette Leblond[‡], and Masimba Nemaire[§]

Abstract. We study the Poisson-Laplace equation of the form $\Delta P_M = \operatorname{div} M$ in \mathbb{R}^n with $M \in [L^p(\mathbb{R}^n)]^n$, $1 < p < \infty$. This equation appears in inverse potential problems of electroencephalography (EEG), magnetoencephalography (MEG) or problems of geomagnetism under the quasi-static assumption of Maxwell's equations. We aim to characterise those non-zero vector-fields M with bounded support, say Ω , that generate a constant P_M in each connected component of $\mathbb{R}^n \setminus \Omega$, such M are said to be Ω -silent. When Ω is open and its boundary has measure zero, we managed to show M is Ω -silent if and only if $M = \nabla\psi + D$ with D divergence-free is such the extension of $\nabla\psi$ (D) by zero to \mathbb{R}^n is a gradient (divergence-free) on \mathbb{R}^n , that is, all Ω -silent M have Helmholtz decompositions. For a given $M \in [L^p(\Omega)]^n$, with Ω say a Sobolev extension domain, we managed to show that $M = \nabla\psi + D + |\nabla h|^{q-2}\nabla h$, $1/p + 1/q = 1$, for h a harmonic function such that ∇h is orthogonal to both Ω -silent $\nabla\psi$ and D . This decomposition is a non-linear generalisation of the Helmholtz decomposition.

Key words. Inverse potential problems, silent sources, Helmholtz decomposition

MSC codes. 31B15, 31B20, 31C45, 35Q61

1. Introduction. The present paper elaborates on the remark that the magnetic field generated by a magnet in \mathbb{R}^3 is, in the absence of electric sources, given by the same formula (up to a sign) as the gradient component in the Helmholtz (Helmholtz-Hodge) decomposition of the vector field representing the magnetization. This yields mutual interaction between inverse magnetization problems and the decomposition of vector fields as the sum of a gradient and a divergence-free part. In fact, Helmholtz decompositions turn out to be a major tool to describe “silent” magnetizations (those generating the zero field), that hinder most approaches to inverse magnetization problems. In the opposite direction, we shall see that the natural search for a magnetization of minimum L^p -norm equivalent to a given one gives rise to a (nonlinear) generalization of the Hilbertian Helmholtz decomposition over fairly general bounded domains for $1 < p < \infty$, which extends [21, Theorem 4.5] and broadens the scope of boundary value problems of regularity type for the p -Laplacian. In this connection, it is worth recalling that the classical Helmholtz decomposition in L^p is only valid in the range $3/2 \leq p \leq 3$ over all Lipschitz domains. Though the results receive physical interpretation in \mathbb{R}^3 , we work in \mathbb{R}^n for $n \geq 3$ at no extra cost.

Having sketched the contents of the paper, let us describe in more details the underlying inverse problems.

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1.1. Silent sources. Inverse source problems of elliptic type with a source term in divergence form arise naturally in static electromagnetics, and enjoy several applications: for instance to Earth and planetary sciences, where the remanent magnetisation of rocks from the Earth or other celestial bodies must be recovered from measurements of their magnetic field, in order to study their past history and structure [18, 48]. This raises a so-called inverse magnetisation problem, where one measures the magnetic field generated by a magnetisation M (a \mathbb{R}^3 -valued density of magnetic moment) away from its support, and one seeks to reconstruct M . Under quasi-static assumptions on Maxwell's equations [36, Section 5.9.C], the scalar magnetic potential P_M of M satisfies the following Poisson-Laplace equation in \mathbb{R}^3 :

$$(1.1) \quad \Delta P_M = \operatorname{div} M ,$$

where $\Delta = \sum_{i=1}^3 \partial_{x_i}^2$ is the Euclidean Laplacian. The limiting condition is that P_M vanishes at infinity, and the magnetic field (which is the measured quantity) is minus the gradient ∇P_M of the potential.

Another noteworthy application is to medical imaging, in particular electroencephalography (EEG), where the primary cerebral current is to be estimated from measurements of the electric potential on the scalp [31],[34, Chapter 3]. The situation, in this electrostatic setting, is slightly more involved. This time the electric potential P_J generated by a primary cerebral current J (to be recovered) satisfies the non-Euclidean Poisson Laplace equation:

$$(1.2) \quad \operatorname{div}(\sigma \nabla P_J) = \operatorname{div} J$$

where the conductivity σ is assumed piecewise constant in a sequence of consecutive layers: the encephalon (that carries J), the cerebrospinal fluid, the skull, the scalp, and finally the air in which the conductivity is identically zero. The potential P_J may be normalized so as to vanish at infinity, and measurements thereof are taken on the scalp.

A common feature to both inverse problems is that the source term on the right hand side of (1.1) and (1.2) is the divergence of a vector field: the magnetisation in the former case, and the primary cerebral current in the latter.

Such questions are severely ill-posed, due to the existence of nonzero *silent* sources; *i.e.*, magnetisations or primary currents that produce the zero field (or locally constant potential), wherever they can be measured. In the language of inverse problems, one would say that the forward operator has non-trivial kernel; hence, a fundamental uncertainty attaches to the solution. As a consequence, further assumptions are required to set up consistent regularisation schemes, and in order to derive such schemes one needs both working and general characterizations of silent sources. For definiteness, we discuss the magnetostatic framework only, but everything applies to the electrostatic setting of EEG. Indeed, under weak assumptions, a primary current J is silent if and only if the Newton potential of its divergence (that is: the solution of (1.1) where M is set to J) is constant outside the encephalon.

As is easy to see from (1.1), magnetisations that are divergence-free (as distributions on \mathbb{R}^3) are silent. Now, obvious questions arise: are there any others and how does the geometry of the sample impinge on their characterization?

Known characterizations of silent sources so far mainly deal with magnetizations supported on a surface. Namely, silent magnetisations supported on a plane in \mathbb{R}^3 were described in [9]

under very weak regularity assumptions; since the plane separates \mathbb{R}^3 , one must distinguish between silence from one side or from both sides, and a silent \mathbb{R}^3 -valued distribution from one side can be characterized in terms of its Hardy-Hodge decomposition (the latter decomposes a \mathbb{R}^3 -valued field on a surface as the sum of two harmonic gradients, one from each side, plus a tangent divergence-free field). Similar results can be obtained on the 2-D sphere, only replacing Riesz transforms by their spherical analogs; let us mention also that silent magnetisations of L^2 -class have long been described on the sphere *via* their expansion into spherical harmonics, see [28]. Such characterizations were adapted in [7, Theorems 3.3 & 3.16] to closed Lipschitz surface for L^2 -magnetisations, and formulated there in potential-theoretic terms; using results from [11], this carries over to magnetisations of L^p -class for $p \in (2 - \varepsilon, \frac{2-\varepsilon}{1-\varepsilon})$, where $\varepsilon \leq 1$ depends on the surface ($\varepsilon = 1$ when the latter is C^1 -smooth). Combining things, a magnetisation is seen to be silent from both sides if and only if it is tangent and divergence-free, as a vector-field on the surface; this amounts to say that it is divergence-free when viewed as a \mathbb{R}^3 -valued distribution on \mathbb{R}^3 , supported on that surface. In fact, this also characterizes silent \mathbb{R}^3 -valued *measures* supported on a slender set (a closed set of Lebesgue measure zero whose complement has all its connected components of infinite Lebesgue measure) [12]; in particular, tangent divergence-free vector fields correspond to silent magnetisations on non-closed connected surfaces (for which there is no notion of silence from one side).

In contrast, silent magnetisations supported in a volume have been less investigated. The only reference so far is apparently [8, Theorem 3.3], where certain silent magnetisations of L^2 -class carried by a bounded Lipschitz domain Ω are described in terms of their Helmholtz decomposition. The role of the latter can be surmised from (1.1), since ∇P_M is the gradient component in the Helmholtz decomposition on \mathbb{R}^3 of the extension by zero of M (see Lemma 3.1 further below). However, even in the Lipschitz setting, the results from [8] differ from the present ones when $\overline{\Omega}$ has non-connected complement, because there the definition of silence is different from ours; namely, we call silent a magnetisation generating the zero field almost everywhere in the complement of Ω (so that the potential is *locally constant* in the complement of $\overline{\Omega}$), whereas [8] terms silent a magnetisation having *constant* potential in the complement of $\overline{\Omega}$. And if we drop the Lipschitz assumption, it may happen that the two definitions disagree even when Ω has connected complement. We feel that the present definition is justified in connection with inverse problems, for the magnetic field (more accurately: the electric current it generates) is all one can measure in practice.

Let us now discuss in a more comprehensive, but still somewhat informal manner the results obtained in the paper.

1.2. Outline of the results. For convenience, we say that a gradient (resp. divergence-free) vector field on an open set is *extendable by zero* if its extension by zero is still a gradient (resp. divergence-free) field on the whole of \mathbb{R}^n .

We will show that silent magnetisations of L^p -class carried by a bounded open set Ω are, for $1 < p < \infty$, the sum of a gradient and a divergence-free field, each of which is extendable by zero (Theorem 5.2). For instance when Ω is Lipschitz, silent magnetisations are the sum in $L^p(\Omega, \mathbb{R}^n)$ of a gradient vector field with zero tangential boundary component and of a divergence free field with zero normal boundary component. Note that this characterisation

is valid for every $p \in (1, \infty)$, even though a Helmholtz decomposition may not always exist in that range.

When Ω is too rough to define boundary behaviour easily, Helmholtz decompositions are typically normalized so that their divergence-free component (or else their gradient component) lies in the closure of smooth divergence-free (or else gradient) fields compactly supported in Ω [26, 45]. So, in view of what precedes, the question arises as to when Ω is such that every divergence-free (or gradient) field in $L^p(\Omega, \mathbb{R}^n)$ which is extendable by zero necessarily lies in that closure. When the answer is positive for divergence-free (resp. gradient) fields, we say that Ω is *div-regular* (resp. *grad-regular*) for the zero extension in L^p ; if Ω is both div-regular and grad-regular, we say that it is *regular* for the zero extension (in L^p). To avoid pathologies, we restrict our discussion to cases where the boundary $\partial\Omega$ has Lebesgue measure zero.

The issue of grad-regularity for the zero extension can be viewed as a variation on the spectral synthesis problem for homogeneous Sobolev functions, asking whether a L^p -gradient which is zero outside a bounded open set with boundary of zero measure can be approximated by a smooth one, compactly supported therein. We prove in Theorem 2.2 that the answer is positive when $n < p < \infty$, and also when $1 < p \leq n$ modulo weak potential-theoretic assumptions on $\mathbb{R}^n \setminus \bar{\Omega}$ (it should be *unstretched*, see Definition 2.1).

As to div-regularity for the zero extension (the property that a divergence-free field in $L^p(\Omega, \mathbb{R}^n)$ which is extendable by zero can be approximated by a smooth compactly supported divergence-free field in Ω), all we could say is that it holds when traces of functions from $W^{1,q}(\mathbb{R}^n)$ are dense in the Sobolev space $W^{1,q}(\Omega)$ for $1/p + 1/q = 1$. One can then appeal to known sufficient conditions for this to hold, like the segment condition or more refined criteria based on hyperbolic geometry [39] (see Lemma 2.5). Even though such conditions may seem far from being sharp, they imply altogether that regularity for the zero extension in L^p is a mild assumption: for instance it is met for every $p \in (1, \infty)$ by bounded open sets whose boundary is locally isometric to a continuous graph with no thin inward-pointing cusps quasi everywhere, or by (finite unions of) quasiconformal images of bounded uniform domains whose closure has unstretched complement; see definitions in Sections 2.1 and 2.3. In particular, open sets regular for the zero extension are considerably more general than, say Lipschitz domains.

Next, elaborating on what precedes, we will characterize the magnetisation of minimum L^p -norm equivalent to a given one, when $1 < p < \infty$ and Ω is regular for the zero extension. This provides one with an Ansatz on the form of the unknown vector field that can be used to restore uniqueness, when addressing the inverse magnetization problem. It also gives rise to a new (non-linear) decomposition asserting that every L^p -vector field on Ω is the sum of a gradient and a divergence-free field, both extendable by zero, plus the duality mapping of a harmonic gradient; see Theorem 3.3 further below. When $p = 2$ the duality mapping reduces to the identity and the above decomposition subsumes the Helmholtz one, but it differs from it when $p \neq 2$ and unlike Helmholtz it exists for all $p \in (1, \infty)$. This new decomposition generalizes [21, Theorem 4.5] to arbitrary vector fields in $L^2(\Omega, \mathbb{R}^n)$ (not just gradients), and also to $L^p(\Omega, \mathbb{R}^n)$ for $p \neq 2$. Just like the standard Helmholtz decomposition connects to the well-posedness of a certain Neumann problem [26, Chapter III], our decomposition makes contact with solutions to certain boundary value problems with rough boundary for the p -Laplacian, in the style of [21, Remark 4.4].

The paper is organized as follows. In Section 2, we set up notation and recall some properties of Sobolev functions, Lipschitz domains and divergence-free vector fields. Section 3 reviews Helmholtz decompositions, while Section 4 compiles known facts on layer potentials. In Section 5 we characterize silent vector fields carried by bounded open sets, and use it in Section 6 to describe norm-minimal equivalent sources. Appendix 7 gives sufficient conditions on an open set for grad-regularity to hold, derives criteria ensuring non-thinness of a cusp in an epigraph, and gives an example of a silent magnetisation carried by a bounded open set with connected complement, whose potential is non constant in that complement. Finally, we show that silent sources for EEG are the same as for inverse magnetisation problems.

2. Notation and preliminaries. Let \mathbb{R}^n denote Euclidean space of dimension n . Hereafter, we assume that $n \geq 3$. We write $x = (x_1, \dots, x_n)^t$ to display the coordinates of $x \in \mathbb{R}^n$, with superscript “ t ” to mean “transpose”, and $x \cdot y$ for the scalar product of $x, y \in \mathbb{R}^n$; $|x| = (x \cdot x)^{1/2}$ is the Euclidean norm of x . We let $B(x, r)$ be the open ball centered at x of radius r , and $S(x, r)$ the boundary sphere. A domain is a connected open set. We put χ_E for the characteristic function of E , and $d(E_1, E_2)$ for the distance between $E_1, E_2 \subset \mathbb{R}^n$.

2.1. Function spaces. For E a Lebesgue-measurable subset of \mathbb{R}^n and $1 \leq p \leq \infty$, we denote by $L^p(E)$ the familiar Lebesgue space of (equivalence classes of a.e. coinciding) \mathbb{R} -valued measurable functions on E whose absolute value to the p -th power is integrable, with norm $\|g\|_{L^p(E)} = (\int_E |g(y)|^p dy)^{1/p}$ (ess. sup $_E |g|$ if $p = \infty$). For $1 \leq p < \infty$ and $1/p + 1/q = 1$, the dual of $L^p(E)$ is $L^q(E)$, isometrically under the pairing $\langle f, g \rangle = \int_E fg$. We set $L^p_{loc}(E)$ to consist of functions f whose restriction $f|_K$ lies in $L^p(K)$ for every compact $K \subset E$.

Given a functional space X , we write $[X]^k$ for the corresponding space of vector-fields with k components, each of which lies in X . For example, $[L^p(E)]^k$ is the space of \mathbb{R}^k -valued vector fields M on E whose components belong to $L^p(E)$, with norm

$$(2.1) \quad \|M\|_{[L^p(E)]^k} = \left(\int_E |M|^p dy \right)^{1/p} \quad (\text{ess. sup}_E |M| \quad \text{if } p = \infty),$$

and for $1 \leq p < \infty$ the dual of $[L^p(E)]^k$ is $[L^q(E)]^k$, $1/p + 1/q = 1$, isometrically under the pairing

$$(2.2) \quad \langle F, G \rangle_E = \int_E F(y) \cdot G(y) dy.$$

If $E \subset \mathbb{R}^n$ and $f : E \rightarrow \mathbb{R}^k$, we designate by $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}^k$ the extension of f by zero outside E . The notation stands irrespective of m, n and E .

When $\Omega \subset \mathbb{R}^n$ is open, $W^{1,p}(\Omega)$ indicates the Sobolev space of functions lying in $L^p(\Omega)$ together with their first distributional derivatives. It is a Banach space with norm

$$\|g\|_{W^{1,p}(\Omega)} = \left(\|g\|_{L^p(\Omega)}^p + \|\nabla g\|_{[L^p(\Omega)]^n}^p \right)^{1/p},$$

where $\nabla g = (\partial_1 g, \dots, \partial_n g)^t$ denotes the gradient of g and $\partial_j g$ the derivative with respect to the j -th variable. We let $W_0^{1,p}(\Omega)$ stand for the closure in $W^{1,p}(\Omega)$ of $C_c^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω . A function belongs to $W_0^{1,p}(\Omega)$

if and only if its extension by zero outside Ω belongs to $W^{1,p}(\mathbb{R}^n)$, see [3, Thm. 9.1.3]; this result pertains to the so-called *spectral synthesis* of Sobolev functions [3, Section 9.13]. When Ω is bounded, the Poincaré inequality implies that

$$(2.3) \quad \|\varphi\|_{L^p(\Omega)} \leq C \|\nabla \varphi\|_{[L^p(\Omega)]^n}, \quad \varphi \in W_0^{1,p}(\Omega),$$

where C depends only on the diameter of Ω [42, Thm. 15.4.1]. The space $W_{loc}^{1,p}(\Omega)$ is comprised of functions lying in $L_{loc}^p(\Omega)$ together with their first order derivatives; it is a Fréchet space for the seminorms defined by the Sobolev norms on relatively compact open subsets of Ω exhausting the latter.

If $f \in W^{1,p}(\Omega)$ then a.e. $x \in \Omega$ is a Lebesgue point of f , meaning that $f(x) = \lim_{r \rightarrow 0} m(B(x,r))^{-1} \int_{B(x,r)} f(y) dy$ where m indicates Lebesgue measure. More in fact is true: f is continuous on Ω when $n < p \leq \infty$, by the Sobolev embedding theorem, and when $1 < p \leq n$ it is continuous outside a set of arbitrary small $B_{1,p}$ -Bessel capacity, as follows easily from [3, Proposition 6.1.2]. The Bessel capacity $B_{1,p}(E)$ of a set $E \subset \mathbb{R}^n$ is

$$B_{1,p}(E) := \inf \{ \|f\|_{L^p(\mathbb{R}^n)}^p : g_1 * f \geq 1 \text{ on } E, f \geq 0 \}, \quad p \in (1, \infty),$$

where g_1 is the Bessel kernel and $*$ indicates convolution, see [50, Definition 2.6.2]. In particular, the set of non-Lebesgue points has $B_{1,p}$ -capacity zero. The following are standard estimates [50, Theorems 2.6.13 & 2.6.14]:

$$(2.4) \quad \frac{r^{n-p}}{C} \leq B_{1,p}(B(x,r)) \leq Cr^{n-p}, \quad 1 < p < n, C = C(p,n), 0 < r \leq 1/2,$$

$$(2.5) \quad \frac{1}{C'} (\log \frac{1}{r})^{1-n} \leq B_{1,n}(B(x,r)) \leq C' (\log \frac{1}{r})^{1-n}, \quad 0 < r \leq r_0 < 1, C' = C'(r_0, n).$$

For $1 < p \leq n$, a set $E \subset \mathbb{R}^n$ is called *p-thin* at $x \in \mathbb{R}^n$ if

$$(2.6) \quad \int_0^1 \left(\frac{B_{1,p}(E \cap B(x,r))}{r^{n-p}} \right)^{q-1} \frac{dr}{r} < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

When $p > n$, the above definition would simply characterize $\mathbb{R}^n \setminus \overline{E}$, because then there is a constant $C = C(p,n) > 0$ such that $B_{1,p}(A) > C$ for $A \neq \emptyset$. We refer the reader to [3] for more on Bessel potentials and capacity properties of Sobolev functions, but let us stress here that sets of $B_{1,p}$ -capacity zero are very small: they are empty if $p > n$ by what precedes, and otherwise of Hausdorff $\mathcal{H}^{n-p+\varepsilon}$ -measure zero for each $\varepsilon > 0$ [3, Sec. 5.1].

A domain Ω is called an *extension domains* if there is a continuous right inverse to the restriction map $W^{1,p}(\mathbb{R}^n) \rightarrow W^{1,p}(\Omega)$; such are the so-called (ε, δ) -domains (also called locally uniform domains), for which any two points x, y with $|x - y| < \delta$ can be joined by a rectifiable arc $\gamma \subset \Omega$ with length at most $|x - y|/\varepsilon$, and such that $d(z, \partial\Omega) \geq \varepsilon|x - z|/|z - y|$ for $z \in \gamma$; see [42, Section 1.5.4] and [33] for equivalent definitions and further properties.

We put $\dot{W}^{1,p}(\Omega)$ for the quotient space, modulo constants, of distributions on Ω whose derivatives belong to $L^p(\Omega)$. Such distributions necessarily lie in $W_{loc}^{1,p}(\Omega)$ [42, Sec. 1.1.2], however $\dot{W}^{1,p}(\Omega)$ itself needs not consist of functions modulo constants when $p \geq n$ [21].

Endowed with $\|g\|_{\dot{W}^{1,p}(\Omega)} := \|\nabla g\|_{[L^p(\Omega)]^n}$, one can show that $\dot{W}^{1,p}(\Omega)$ is a Banach space called homogeneous Sobolev space of index p , see [42, Sec. 1.1.13, Thm. 1].

Hereafter, we let $G^p(\Omega)$ denote the subspace of $[L^p(\Omega)]^n$ defined by

$$(2.7) \quad G^p(\Omega) := \{\nabla\psi : \psi \in W_{loc}^{1,p}(\Omega), \nabla\psi \in [L^p(\Omega)]^n\}.$$

In other words, $G^p(\Omega)$ consists of gradients of members of $\dot{W}^{1,p}(\Omega)$. When Ω is bounded, equivalence classes of $W^{1,p}(\Omega)$ -functions are dense in $G^p(\Omega)$. Indeed, if $N > 0$ and $g \in W_{loc}^{1,p}(\Omega)$ has $\nabla g \in [L^p(\Omega)]^n$, then the function h_N which is equal to g where $|g| \leq N$ and to $(\text{sign } g)N$ at other points lies in $W^{1,p}(\Omega)$ for each $N > 0$, with $\nabla h_N = \nabla g$ on $\{|g| \leq N\}$ and $\nabla h_N = 0$ elsewhere [50, Corollary 2.1.8]. Hence, h_N converges as $N \rightarrow +\infty$ to g in $\dot{W}^{1,p}(\Omega)$, which implies the announced density. Subsequently, if Ω is a bounded extension domain then each element of $\dot{W}^{1,p}(\Omega)$ has a $W^{1,p}(\Omega)$ -representative, because a Poincaré inequality of the form (2.3) holds in this case for $\varphi \in W^{1,p}(\Omega)$ (not just $W_0^{1,p}(\Omega)$).

One can show that (equivalence classes of) smooth and compactly supported gradients are dense in $\dot{W}^{1,p}(\mathbb{R}^n)$, but it may not be so in $\dot{W}^{1,p}(\Omega)$. This is why we define $G_0^p(\Omega)$ as the closure of $[C_c^\infty(\Omega)]^n \cap G^p(\Omega)$ in $G^p(\Omega)$:

$$(2.8) \quad G_0^p(\Omega) := \overline{\{\nabla\psi : \psi \in C^\infty(\Omega), \nabla\psi \in [C_c^\infty(\Omega)]^n\}}^{[L^p(\Omega)]^n},$$

where $C^\infty(\Omega)$ is the space of C^∞ -smooth functions in Ω . When Ω is bounded, vector fields in $G_0^p(\Omega)$ are such that their extension by zero outside Ω is the gradient of some member of $\dot{W}^{1,p}(\mathbb{R}^n)$, but the converse needs not be true: Section 7.2 provides us with a counterexample. In view of this, we further define the space $\tilde{G}^p(\Omega) \subset G^p(\Omega)$ by

$$(2.9) \quad \tilde{G}^p(\Omega) := \{F \in G^p(\Omega) : \tilde{F} \in G^p(\mathbb{R}^n)\}.$$

Finding sufficient conditions on Ω for $\tilde{G}^p(\Omega)$ to coincide with $G_0^p(\Omega)$ is a variation on the spectral synthesis problem for Sobolev functions, that was apparently not much studied. In this connection, let us define:

Definition 2.1. For $p \in (1, n]$, we say that an open set $O \subset \mathbb{R}^n$ with connected components $(O_j)_{j \in J \subset \mathbb{N}}$ is p -unstretched if the following two properties hold:

- (i) O_j is non-thin at $B_{1,p}$ -quasi every point of ∂O_j for each j ;
- (ii) whenever $\overline{O_j} \cap \bigcup_{i \in I} \overline{O_i} \neq \emptyset$ for some index $j \in J$ and some subset of indices $I \subset J$, then $B_{1,p}(\overline{O_j} \cap \bigcup_{i \in I} \overline{O_i}) > 0$.

Note that when $p > n$, the conditions of Definition 2.1 are met by every open set. In Section 7, we establish the following result:

Theorem 2.2. For $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ a bounded open set with boundary of zero Lebesgue measure, it holds that $G_0^p(\Omega) \subset \tilde{G}^p(\Omega)$. If $n < p$ the converse inclusion holds, and if $1 < p \leq n$ it still holds provided that $\mathbb{R}^n \setminus \overline{\Omega}$ is unstretched.

Hence, for $G_0^p(\Omega)$ to coincide with $\tilde{G}^p(\Omega)$, it is enough when $p > n$ that $\partial\Omega$ be of Lebesgue measure zero, while if $p \leq n$ a mild additional capacity condition on the components of $\mathbb{R}^n \setminus \overline{\Omega}$ is sufficient; individual conditions ensuring that a given $G \in \tilde{G}^p(\Omega)$ lies in $G_0^p(\Omega)$ are stated in Corollary 7.2. Note that when the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$ are finite in

number with disjoint closures, the fact that it is p -unstretched amounts to say it is non-thin at quasi-every boundary point; or equivalently, that quasi-every point of $\partial\Omega$ is regular for the Dirichlet problem, say of the p -Laplacian in Ω [3, Theorem 6.3.6].

If $\partial\Omega$ is locally isometric to a continuous graph, then it has measure zero and clearly $\overline{O_j} \cap \overline{\cup_{i \in I} O_i}$ is either empty or an entire connected component of $\partial\Omega$ when $j \notin I$. Thus, since $B_{1,p}(\overline{O_j}) > 0$ for all $j \in J$ we deduce that (ii) holds in this case.

Next, taking notice that x is a density point of $\mathbb{R}^n \setminus E$ when the set E is non-thin at x [50, section 3.3], one sees that $\mathbb{R}^n \setminus \overline{\Omega}$ is thin at $x \in \partial\Omega$ when x is the vertex of a (nonempty) finite open cone included in $\mathbb{R}^n \setminus \overline{\Omega}$ (whose length and solid angle may depend on x); one then says that Ω satisfies an *exterior cone condition* at x . Altogether, we get a (far from sharp but simple) criterion for $\mathbb{R}^n \setminus \overline{\Omega}$ to be unstretched: it is enough that Ω be a bounded open set whose boundary is locally isomorphic to a continuous graph, that meets an exterior cone condition at quasi every boundary point. However, when $\partial\Omega$ is locally isometric to a continuous graph, let us stress that much weaker requirements than the exterior cone condition can ensure (i) of Definition 2.1 is met. In fact, the exterior cone condition prevents Ω from having inward-pointing boundary cusps, but such cusps are no obstacle to the non-thinness of $\mathbb{R}^n \setminus \overline{\Omega}$ at their vertex as long as their slenderness is controlled; see Proposition 7.3 in Section 7 for quantitative conditions, based on (2.4)-(2.5), implying that a cusp in a graph is not p -thin.

For $1 < p < \infty$ the dual of $W_0^{1,p}(\Omega)$ is written $W^{-1,q}(\Omega)$, $1/p + 1/q = 1$; it may be realized as the completion of $L^q(\Omega)$ with respect to the norm

$$(2.10) \quad \|\varphi\|_{W^{-1,q}(\Omega)} := \sup_{\|g\|_{W_0^{1,p}(\Omega)}=1} \left| \int_{\Omega} \varphi(y)g(y) dy \right|,$$

see [4, Sec. 3.12]. Members of $W^{-1,q}(\Omega)$ are generally not functions but distributions on Ω , and $f \mapsto \nabla f$ is continuous from $L^p(\Omega)$ to $[W^{-1,q}(\Omega)]^n$, by (2.3). The support of a function or distribution \mathfrak{D} will be denoted by $\text{supp } \mathfrak{D}$.

2.2. Lipschitz open sets and traces. We say that an open set $\Omega \subset \mathbb{R}^n$ is Lipschitz if its boundary $\partial\Omega$ is compact and locally isometric to the graph of a Lipschitz function; see e.g. [29]. That is: to each $x \in \partial\Omega$ there exists an open set $U \subset \mathbb{R}^n$ containing x and a rigid transformation T of \mathbb{R}^n such that, for some open set $V \subset \mathbb{R}^{n-1}$ and some Lipschitz map $\Psi : V \rightarrow \mathbb{R}^+$, one has:

$$T(U \cap \Omega) = \{y \in \mathbb{R}^n, \quad (y_1, \dots, y_{n-1})^t \in V, \quad 0 < y_n < \Psi(y_1, \dots, y_{n-1})\}.$$

Covering $\partial\Omega$ with finitely many open sets U_j as above associated to T_j , V_j and Ψ_j for $1 \leq j \leq N$, one can define maps $\phi_j : U_j \cap \partial\Omega \rightarrow V_j$ by $\phi_j := P_{n-1} \circ T_j$, where P_{n-1} is the projection onto the first $(n-1)$ components. This yields an atlas making $\partial\Omega$ a compact Lipschitz manifold, whose parametrizations $\phi_j^{-1} : V_j \rightarrow U_j \cap \partial\Omega \subset \mathbb{R}^n$ are themselves Lipschitz. Hence, $\partial\Omega$ is locally the image of an open subset of \mathbb{R}^{n-1} by a Lipschitz function (namely ϕ_j^{-1}), therefore the volume measure σ coincides with the restriction to $\partial\Omega$ of Hausdorff \mathcal{H}^{n-1} -measure, see [25, Sec. 3.2]. Integration of functions on $\partial\Omega$ is always understood with respect to σ .

For Ω a Lipschitz open set in \mathbb{R}^n , we say that $x \in \partial\Omega$ is singular if there is $j \in \{1, \dots, N\}$ for which $x \in U_j$ and $\phi_j^{-1} : V_j \rightarrow \mathbb{R}^n$ is not differentiable at $\phi_j(x)$. A point which is not

singular is called regular. We denote the set of regular points by $\text{Reg } \partial\Omega$ and put $\text{Reg } V_j = \phi_j(\text{Reg } \partial\Omega \cap U_j)$. Thus defined, the set of regular points depends on the atlas but we shall keep the latter fixed. As ϕ_j^{-1} is Lipschitz, the set $E_j \subset V_j$ where it is not differentiable has Lebesgue measure zero, by Rademacher's theorem [50, Thm 2.2.1]. Hence, $\sigma(\phi_j(E_j)) = 0$ [50, Rem. 1.4.3], so that singular points have σ -measure zero.

The tangent space of $\partial\Omega$ at x is the subspace $T_x\partial\Omega \subset \mathbb{R}^n$ equal to $\{0\}$ if x is singular and to $\text{Ran } D\phi_j^{-1}(\phi_j(x))$ if $x \in U_j \cap \text{Reg } \partial\Omega$, where $D\phi_j^{-1}$ denotes the total derivative of ϕ_j^{-1} . By the chain rule, the definition does not depend on j such that $x \in U_j$. At $x \in \text{Reg } \partial\Omega$, the outward pointing unit normal to $\partial\Omega$ is well-defined, and we denote it as $\nu(x)$.

Recall that for Γ a connected compact Lipschitz hypersurface, $\mathbb{R}^n \setminus \Gamma$ has two connected components: its interior denoted by $\text{int } \Gamma$ which is bounded, and its exterior denoted by $\text{ext } \Gamma$ which is unbounded, see [32, Cor. 3.45]. We record for later use an "obvious" topological lemma, a proof of which may be found in [10, Lemma 1].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then, $\partial\Omega$ has finitely many connected components, say $\Gamma_1, \dots, \Gamma_l$, each of which is a compact connected Lipschitz hypersurface. Moreover, the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$ consist of l open sets O_1, \dots, O_l , and with a suitable ordering O_1 is the exterior of Γ_1 while O_j is the interior of Γ_j for $j \neq 1$.*

For $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz open set, $G^p(\Omega)$ consists of gradients of functions in $W^{1,p}(\Omega)$ and the latter space coincides with the restrictions $(W^{1,p}(\mathbb{R}^n))|_{\Omega}$, since Lipschitz domains are *extension domains* [46, Ch. VI, Thm 5]. Besides, for $1 < p < \infty$, each $f \in W^{1,p}(\Omega)$ has a trace on $\partial\Omega$ that lies in the fractional Sobolev space $W^{1-1/p,p}(\partial\Omega)$, where for $0 < s < 1$ we let

$$(2.11) \quad \|\psi\|_{W^{s,p}(\partial\Omega)} := \|\psi\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{n-1+sp}} d\sigma(x) d\sigma(y) \right)^{1/p}.$$

In other words, the trace on $\partial\Omega$, initially defined by continuity for functions in $C_c^\infty(\mathbb{R}^n)|_{\Omega}$, extends to a continuous map $W^{1,p}(\Omega) \rightarrow W^{1-1/p,p}(\partial\Omega)$. Moreover, the trace operator is surjective $W^{1,p}(\Omega) \rightarrow W^{1-1/p,p}(\partial\Omega)$ [29, Thm. 1.5.1.3], in particular traces on $\partial\Omega$ of $C_c^\infty(\mathbb{R}^n)$ -functions on \mathbb{R}^n are dense in $W^{1-1/p,p}(\partial\Omega)$. Also, $W_0^{1,p}(\Omega)$ coincides with the space of $W^{1,p}(\Omega)$ -functions with zero trace, see for example [37, Prop. 3.3].

For $1 < q, l < \infty$, we let $W^{-1/q,l}(\partial\Omega)$ denote the fractional Sobolev space of negative order $-1/q$ and exponent l , dual to $W^{1/q,l'}(\partial\Omega)$, $1/l + 1/l' = 1$. Adapting the argument of [4, Sec. 3.12] as we did to get (2.10), $W^{-1/q,l}(\partial\Omega)$ may be realized as the completion of $L^l(\partial\Omega)$ with respect to the norm

$$(2.12) \quad \|\varphi\|_{W^{-1/q,l}(\partial\Omega)} := \sup_{\|u\|_{W^{1/q,l'}(\partial\Omega)}=1} \int_{\partial\Omega} \varphi u d\sigma.$$

Members of $W^{-1/q,l}(\partial\Omega)$ are distributions on \mathbb{R}^n supported on $\partial\Omega$, and when $q = \frac{p}{p-1}$ they form the dual of the trace space $W^{1-1/p,p}(\partial\Omega)$ of $W^{1,p}(\Omega)$.

We defined the fractional Sobolev space $W^{s,p}(\partial\Omega)$ through (2.11), without resorting to the integral Sobolev space $W^{1,p}(\partial\Omega)$ of those $f : \partial\Omega \rightarrow \mathbb{R}$ such that $f \circ \phi_j^{-1} \in W^{1,p}(V_j)$ for all j , where V_j and ϕ_j are as before. In fact, $W^{s,p}(\partial\Omega) = [L^p(\partial\Omega), W^{1,p}(\partial\Omega)]_s$ where $[\cdot, \cdot]_s$ is

the so-called real interpolation functor; this follows from the analogous result on \mathbb{R}^{n-1} [14, Ch. 4, Cor. 4.13] and [23, Lem. 1.1] which allows one to localize the statement in the charts (U_j, ϕ_j) , using a Lipschitz partition of unity.

Functions $f \in W^{1,p}(\partial\Omega)$ have a well-defined gradient $\nabla_T f \in [L^p(\partial\Omega)]^n$, valued at a.e. $x \in \partial\Omega$ in the tangent space $T_x \partial\Omega$, and given on $U_j \cap \partial\Omega$ by

$$(2.13) \quad \nabla_T f \circ \phi_j^{-1} = D\phi_j^{-1}((D\phi_j^{-1})^t D\phi_j^{-1})^{-1} \nabla(f \circ \phi_j^{-1}).$$

Functions in $W^{1-1/p,p}(\partial\Omega)$ also have a well-defined gradient, but the latter is a $(n-2)$ -current on $\partial\Omega$ rather than a vector field: it lies in the space $\mathcal{W}_1^{-1/p,p}(\partial\Omega)$ which is dual to the interpolation space of $(n-2)$ -forms:

$$(2.14) \quad \mathcal{W}_{n-2}^{1/p,p/(p-1)}(\partial\Omega) := [\mathcal{L}_{n-2}^{p/(p-1)}(\partial\Omega), \mathcal{W}_{n-2}^{1,p/(p-1)}(\partial\Omega)]_{1/p},$$

where $\mathcal{L}_{n-2}^{p/(p-1)}(\partial\Omega)$ consists of $(n-2)$ -forms of $L^{p/(p-1)}$ -class on $\partial\Omega$ and $\mathcal{W}_{n-2}^{1,p/(p-1)}$ is the subspace of those form having distributional exterior derivative in $\mathcal{L}_{n-1}^{p/(p-1)}(\partial\Omega)$; details can be found in [44, Section 1.9.1].

2.3. Divergence-free fields. For $\Omega \subset \mathbb{R}^n$ an open set and $1 < p < \infty$, define $\text{Div}_p(\Omega)$ by

$$\text{Div}_p(\Omega) := \{u \in [L^p(\Omega)]^n : \text{div } u \in L^p(\Omega)\};$$

here $\text{div } u = \sum_{j=1}^n \partial_j u_j$ is the (distributional) Euclidean divergence of u . We endow Div_p with the following norm that turns it into a Banach space:

$$\|u\|_{\text{Div}_p(\Omega)} := (\|u\|_{[L^p(\Omega)]^n}^p + \|\text{div } u\|_{L^p(\Omega)}^p)^{\frac{1}{p}}.$$

We shall deal with the closure in $\text{Div}_p(\Omega)$ of divergence-free fields in $[C_c^\infty(\Omega)]^n$, denoted as $\text{Div}_{p,0}(\Omega)$. That is, we formally put

$$(2.15) \quad \text{Div}_{p,0}(\Omega) = \overline{\{u \in [C_c^\infty(\Omega)]^n : \text{div } u = 0\}}^{[L^p(\Omega)]^n}.$$

The next lemma is crucial to our purposes.

Lemma 2.4. *For $1 < p < \infty$ and $1/p + 1/q = 1$, each of $\text{Div}_{p,0}(\Omega)$ and $G^q(\Omega)$ is the orthogonal space to the other under the pairing (2.2).*

Proof. Clearly, $\text{Div}_{p,0}(\Omega)$ and $G^q(\Omega)$ are orthogonal, since the divergence operator is dual to the gradient operator. Moreover, if $F \in [L_{loc}^1(\Omega)]^n$ is orthogonal to all divergence free fields in $[C_c^\infty(\Omega)]^n$, then it is known that $F = \nabla \Psi$ with $\Psi \in W_{loc}^{1,1}(\Omega)$; see [26, Lem. III.1.1] (the subtlety is that Ψ is globally defined, even though Ω may not be simply connected). If moreover F in $[L^q(\Omega)]^n$, we get from [20, Thm 6.74] that $\Psi \in W_{loc}^{1,q}(\Omega)$, whence $F \in G^q(\Omega)$. Thus, $G^q(\Omega) = (\text{Div}_{p,0}(\Omega))^\perp$ and therefore, by the Hahn-Banach theorem, a member of $[L^p(\Omega)]^n$ not lying in the closed subspace $\text{Div}_{p,0}(\Omega)$ cannot be orthogonal to $G^q(\Omega)$, implying that $\text{Div}_{p,0}(\Omega) = (G^q(\Omega))^\perp$. \blacksquare

Clearly, $D \in \text{Div}_{p,0}(\Omega)$ implies that its extension by zero \tilde{D} is divergence-free on \mathbb{R}^n . The converse may not hold, but it does when gradients of restrictions $(C_c^\infty(\mathbb{R}^n))|_\Omega$ are dense in $G^q(\Omega)$ with $1/p + 1/q = 1$. Indeed, letting $G = (\nabla g)|_\Omega$ with $g \in C_c^\infty(\mathbb{R}^n)$, we get that $\langle G, D \rangle_\Omega = \langle \nabla g, \tilde{D} \rangle_{\mathbb{R}^n} = 0$ if \tilde{D} is divergence-free, and when such G are dense in $G^q(\Omega)$ the Hahn-Banach theorem entails that $D \in (G^q(\Omega))^\perp = \text{Div}_{p,0}(\Omega)$. Consequently, since equivalence classes of $W^{1,q}(\Omega)$ -functions are dense in $G^q(\Omega)$ when Ω is bounded (see discussion after (2.7)), any sufficient condition for $(C_c^\infty(\mathbb{R}^n))|_\Omega$ to be dense in $W^{1,q}(\Omega)$ automatically yields in this case a sufficient condition for $\text{Div}_{p,0}(\Omega)$ to consist exactly of fields whose extension by zero is divergence-free. For example, it is enough for this to hold that Ω be bounded and satisfies the segment condition: to each $x \in \partial\Omega$ there should exist a ball $B(x, r)$ and a nonzero $y_x \in \mathbb{R}^n$ such that $z + ty_x \in \Omega$ for $0 < t < 1$, whenever $z \in \bar{\Omega} \cap B(x, r)$ [4, Theorem 3.22]. Clearly, the segment condition is fulfilled when $\partial\Omega$ is locally isometric to a continuous graph

Another condition, more involved but interesting, can be expressed in terms of hyperbolic geometry: recall that a domain Ω is *quasiconvex* if, for some constant $C \geq 1$, any pair of points $z_1, z_2 \in \Omega$ can be connected by a rectifiable curve $\gamma \subset \Omega$ whose length is less than $C|z_1 - z_2|$. Recall also the *quasihyperbolic distance* $\text{dist}_{qh}(z_1, z_2)$ between z_1 and z_2 defined as $\inf_\gamma \int_\gamma d(z, \partial\Omega)^{-1} d\mathcal{H}^1(z)$, where the infimum is taken over all rectifiable curves $\gamma \subset \Omega$ connecting z_1 and z_2 and \mathcal{H}^1 stands for 1-dimensional Hausdorff measure. A domain is called *Gromov hyperbolic* if for some $\delta > 0$, all $x, y, z \in \Omega$ and any corresponding quasihyperbolic geodesics $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$, it holds that $\text{dist}_{qh}(w, \gamma_{x,z} \cup \gamma_{yz}) \leq \delta$ for all $w \in \gamma_{xy}$. Now, the previous discussion and [39, Theorem 1.3] give us the following result.

Lemma 2.5. *For $\Omega \subset \mathbb{R}^n$ a bounded domain and $p \in (1, \infty)$, a sufficient condition for $\text{Div}_{p,0}(\Omega)$ to consist exactly of those fields whose extension by zero is divergence-free on \mathbb{R}^n is that Ω satisfies the segment condition; the same conclusion holds if Ω is both quasiconvex and Gromov hyperbolic.*

In fact, Gromov hyperbolicity implies that $W^{1,\infty}(\Omega)$ is dense in $W^{1,q}(\Omega)$ and quasiconvexity implies that functions in $W^{1,\infty}(\Omega)$ are globally Lipschitz on Ω , hence they can be extended to compactly supported Lipschitz functions on \mathbb{R}^n , which implies the density of $(C_c^\infty(\mathbb{R}^n))|_\Omega$ in $W^{1,q}(\Omega)$ by mollification. Recall that a homeomorphism of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *quasiconformal* if $\sup\{|f(x) - f(y)| : |x - y| \leq r\} / \inf\{|f(x) - f(y)| : |x - y| \geq r\}$ is bounded independently of $x \in \mathbb{R}^n$ and $r > 0$. It is worth pointing out that quasiconformal images of uniform domains are Gromov hyperbolic (see [16]), so that Lemma 2.5 indeed qualifies many domains. We refer to [39] for further details and more amenable characterizations of Gromov hyperbolicity.

When Ω is Lipschitz, the normal component $u \cdot \nu$ of $u \in \text{Div}_p$ on $\partial\Omega$ is well-defined in $W^{-1/p,p}(\partial\Omega)$, and the divergence formula holds in that, for $1/p + 1/q = 1$,

$$(2.16) \quad \int_\Omega \nabla \varphi \cdot u = - \int_\Omega \varphi(y) \text{div} u(y) dy + \int_{\partial\Omega} (u \cdot \nu) \varphi d\sigma, \quad \varphi \in W^{1,q}(\Omega).$$

In this case, it can be shown [26, Thm III.2.3] that

$$(2.17) \quad \text{Div}_{p,0}(\Omega) = \{u \in [L^p(\Omega)]^n : \text{div} u = 0 \text{ and } u \cdot \nu = 0\}.$$

More general Ω for which such a Green formula holds include domains of finite perimeter satisfying the segment condition, see [50, Section 5.4].

2.4. Regularity for the zero extension. Recall that Ω is termed regular for the zero extension in L^p if it is both div-regular and grad-regular for this extension. That is, every divergence-free (resp. gradient) field in $L^p(\Omega, \mathbb{R}^n)$ which is extendable by zero lies in the closure of smooth divergence-free (resp. gradient) fields compactly supported in Ω . Gathering results discussed in Sections 2.1 and 2.3, we obtain:

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega$ of zero measure, and $p \in (1, \infty)$ with $1/p + 1/q = 1$. If $(C_c^\infty(\mathbb{R}^n))|_\Omega$ is dense in $W^{1,q}(\Omega)$, and if moreover $\mathbb{R}^n \setminus \bar{\Omega}$ is unstretched in case $p \leq n$, then Ω is regular for the zero extension in L^p . Sufficient conditions for this to hold include the following two.*

- (i) $\partial\Omega$ is locally isomorphic to a continuous graph, and the latter meets an exterior cone condition at quasi every boundary point if $p \leq n$; in the latter case, it is enough more generally that quasi every inward-pointing cusp (if any) satisfies the quantitative conditions of Proposition 7.3.
- (ii) The connected components Ω_i of Ω have disjoint closures, each Ω_i is a quasiconformal image of a uniform domain, and if $p \leq n$ then $\mathbb{R}^n \setminus \bar{\Omega}_i$ has finitely many components for each i , each of which is non p -thin at quasi every boundary point.

Proof. The first assertion follows from Theorem 2.2 and the discussion after the proof of Lemma 2.4. The sufficiency of (i) is a consequence of the discussion after Theorem 2.2, together with Lemma 2.4, and the fact that an open set whose boundary is locally isometric to a continuous graph satisfies the segment condition. Finally, (ii) follows from Lemma 2.4 and the discussion thereafter, taking into account that if Ω has finitely many connected components with disjoint closures, then it is regular for the zero extension in L^p if and only if each component does. ■

For example, we deduce from (ii) that the quasiconformal image of a ball is regular for the zero extension in L^n ; it is so because quasiconformal maps preserve n -thinness [41].

3. Helmholtz -Hodge Decomposition. On a domain $\Omega \subset \mathbb{R}^n$, we say that $M \in [L^p(\Omega)]^n$ has a Helmholtz decomposition if there exist uniquely $\nabla\psi \in G^p(\Omega)$ and $D \in \text{Div}_{p,0}(\Omega)$ for which

$$(3.1) \quad M = D + \nabla\psi,$$

where $G^p(\Omega)$ and $\text{Div}_{p,0}(\Omega)$ were defined in (2.7) and (2.15). We say that a Helmholtz decomposition holds in $[L^p(\Omega)]^n$, or holds on Ω with index p if, for each M in this space, a decomposition of the form (3.1) uniquely exists. When it is the case, then

$$(3.2) \quad \|D\|_{[L^p(\Omega)]^n} + \|\nabla\psi\|_{[L^p(\Omega)]^n} \leq C(\Omega, p) \|M\|_{[L^p(\Omega)]^n},$$

by the open mapping theorem. Observe also that if a Helmholtz decomposition holds in $[L^p(\Omega)]^n$ for some $p \in (1, \infty)$, then it holds in $[L^q(\Omega)]^n$ where $1/p + 1/q = 1$, by duality.

Because $[L^2(\Omega)]^n$ is a Hilbert space and $G^2(\Omega)$ is the orthogonal space to $\text{Div}_{2,0}(\Omega)$, a Helmholtz decomposition holds when $p = 2$ on any domain Ω . When Ω is Lipschitz, it follows from [23, Thm 11.1] that a Helmholtz-Hodge decomposition exists in $[L^p(\Omega)]^n$ for $p \in [\frac{3}{2} - \epsilon, 3 + \epsilon]$, where $\epsilon > 0$ depends on Ω , and this range is in the nature of best possible

for Lipschitz domains. If Ω is either convex or C^1 -smooth, then a Helmholtz decomposition exists for all $p \in (1, \infty)$, see [27, Thm 1.3] and [23, Thm 11.1]. More on such issues can be found in [15].

On \mathbb{R}^n , a Helmholtz decomposition exists for $1 < p < \infty$, and it is somewhat explicit. The standard approach uses Riesz transforms: when $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, its j -th Riesz transform is defined for $1 \leq j \leq n$ as

$$(3.3) \quad R_j(f)(x) = \lim_{\epsilon \rightarrow 0} c_n \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy, \quad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

The pointwise limit in (3.3) exists for a.e. $x \in \mathbb{R}^n$, and $R_j : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is a bounded operator for $1 < p < \infty$ [46, Ch. II, III]. Moreover, if we let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of C^∞ -smooth functions on \mathbb{R}^n decaying faster than every negative power of $|x|$ at infinity together with all its derivatives, while denoting by \hat{T} the Fourier transform of a tempered distribution T obtained by duality from the Fourier transform $f \mapsto \hat{f}$ from $\mathcal{S}(\mathbb{R}^n)$ into itself:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{2\pi i \xi \cdot y} dy, \quad \xi \in \mathbb{R}^n,$$

it follows from see [46, Ch. II, III] that

$$(3.4) \quad \widehat{R_j(f)}(\xi) = i \frac{\xi_j}{|\xi|} \hat{f}(\xi), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Now, for $p \in (1, \infty)$, each $M \in [L^p(\mathbb{R})]^n$ has a Helmholtz decomposition

$$(3.5) \quad M = D + \nabla \psi$$

where $\psi \in \dot{W}^{1,p}(\mathbb{R}^n)$ while $D \in [L^p(\mathbb{R}^n)]^n$ is divergence-free, and in fact

$$(3.6) \quad \nabla \psi = -\left(R_j \left(\sum_{k=1}^n R_k(M_k)\right)\right)^t$$

where M_k indicates the k -th component of M , see [35, Sec. 10.6]. Uniqueness of the decomposition comes from the fact that no nonconstant harmonic function on \mathbb{R}^n can have a gradient in $[L^p(\Omega)]^n$, as follows easily from the mean value property and Liouville's theorem.

The next lemma recasts the function ψ in (3.5) as a scalar magnetic potential, under mild assumptions. Recall the Riesz kernel of order 2 in \mathbb{R}^n :

$$(3.7) \quad \mathfrak{R}_2(x) := \frac{1}{(n-2)\omega_n |x|^{n-2}},$$

with ω_n to denote the volume of the unit sphere.

Lemma 3.1. *For $M \in [L^p(\mathbb{R}^n)]^n \cap [L^q(\mathbb{R}^n)]^n$ with $1 < p < \infty$ and $1 < q < n$, let us define*

$$(3.8) \quad \Psi_M(x) := \frac{1}{\omega_n} \int_{\mathbb{R}^n} M(y) \cdot \frac{(x-y)}{|x-y|^n} dy, \quad x \in \mathbb{R}^n.$$

Then:

- (i) the integral (3.8) converges absolutely for a.e. x and $\Psi_M \in L^{nq/(n-q)}(\mathbb{R}^n)$,
(ii) $\nabla\Psi_M \in L^p(\mathbb{R}^n)$, and decomposition (3.5) holds with $\psi = \Psi_M$.

Proof. Since $|x/|x|^n| = |x|^{1-n}$, assertion (i) follows from properties of Riesz potentials [46, Ch. V, Thm. 1]. We claim that it is enough to prove (ii) when $M \in [\mathcal{S}(\mathbb{R}^n)]^n$; for suppose we did, and let $M \in [L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)]^n$. Some sequence $F_k \in [\mathcal{S}(\mathbb{R}^n)]^n$ converges to M both in $[L^p(\mathbb{R}^n)]^n$ and $[L^q(\mathbb{R}^n)]^n$, by truncation and mollification. If $M = D + \nabla\psi$ and $F_k = D_k + \nabla P_{F_k}$ are Helmholtz decompositions of M and F_k respectively, we get from (3.2) that $\lim_k \nabla P_{F_k} = \nabla\psi$ in $[L^p(\mathbb{R}^n)]^n$, and from [46, Ch. V, Thm. 1] that $\lim_k P_{F_k} = \Psi_M$ in $L^{nq/(n-q)}(\mathbb{R}^n)$. Hence $\nabla\Psi_M = \nabla\psi$ since both are the distributional limit of ∇P_{F_k} , thereby proving the claim.

We now show that (ii) holds when $M \in [\mathcal{S}(\mathbb{R}^n)]^n$. In this case, (3.8) converges absolutely for every x , and integrating by parts one can check that

$$(3.9) \quad \Psi_M = -\mathfrak{R}_2 * \operatorname{div} M,$$

with \mathfrak{R}_2 as in (3.7). As the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ converts ∂_j into multiplication by $-2i\pi\xi_j$, it follows from (3.9) and [46, Ch. V, Lem. 1] that Ψ_M , viewed as a tempered distribution, has Fourier transform the function

$$(3.10) \quad \hat{\Psi}_M(\xi) = \frac{i}{2\pi|\xi|^2} \sum_{k=1}^n \xi_k \hat{M}_k(\xi).$$

Hence, $\nabla\Psi_M$, viewed as a vector of tempered distributions, has Fourier transform (we extend the definition componentwise):

$$\widehat{\nabla\Psi_M} = \frac{(\xi_1, \dots, \xi_n)^t}{|\xi|^2} \sum_{k=1}^n \xi_k \hat{M}_k(\xi),$$

which is equivalent, in view of (3.4), to (3.6) with ψ replaced by Ψ_M . ■

When $M \in [L^p(\mathbb{R}^n)]^n$ for some $p \geq n$ but $M \notin [L^q(\mathbb{R}^n)]^n$ when $1 < q < n$, Lemma 3.1 does not apply and ψ in (3.5) may not be a function, only a member of $\dot{W}^{1,p}(\mathbb{R}^n)$. In this case, the lemma still allows one to describe Ψ by a limiting process:

Lemma 3.2. *Let $M \in [L^p(\mathbb{R}^n)]^n$ with $1 < p < \infty$ and (3.5) its Helmholtz-Hodge decomposition. Let further M_k be a sequence in $[L^p(\mathbb{R}^n)]^n \cap [L^q(\mathbb{R}^n)]^n$, for some $q \in (1, n)$, that converges to M in $[L^p(\mathbb{R}^n)]^n$. Then the sequence Ψ_{M_k} , as defined by (3.8), is such that $\dot{\Psi}_{M_k}$ converges to ψ in $\dot{W}^{1,p}(\mathbb{R}^n)$.*

Proof. This is obvious from Lemma 3.1 and the continuity of Helmholtz-Hodge decompositions, see (3.2). ■

As mentioned earlier, a Helmholtz decomposition needs not exist in $[L^p(\Omega)]^n$ for $1 < p < \infty$, even if the open set Ω is bounded and Lipschitz. As a substitute, we shall prove in Section 6 that a related, but nonlinear three-terms decomposition exists for all $p \in (1, \infty)$ when Ω is regular for the zero extension. We state the result here for the ease of comparison.

Theorem 3.3. *For Ω a bounded open set which is regular for the zero extension in L^p and $M \in [L^p(\Omega)]^n$ with $1 < p < \infty$, there exist uniquely $\nabla\psi \in G_0^p(\Omega)$ with $\psi \in W^{1,p}(\Omega)$, $D \in \text{Div}_{p,0}(\Omega)$ and a harmonic function h on Ω with $\nabla h \in G^q(\Omega)$, $1/p + 1/q = 1$, such that*

$$(3.11) \quad M = \nabla\psi + D + |\nabla h|^{q-2} \nabla h.$$

4. Double and Single layer potentials. In this section we assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with connected boundary. Hence, $\mathbb{R}^n \setminus \partial\Omega$ has two connected components, namely $\Omega = \text{int } \partial\Omega$ and $\Omega^- := \text{ext } \partial\Omega$.

The double and single layer potentials of a function ψ defined on $\partial\Omega$, whose smoothness will be made precise later on, are defined by

$$(4.1) \quad \mathcal{K}\psi(x) = -\frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega,$$

and

$$(4.2) \quad \mathcal{S}\psi(x) = -\frac{1}{(n-2)\omega_n} \int_{\partial\Omega} \psi(y) \frac{1}{|x-y|^{n-2}} d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega,$$

respectively. We call ψ the density of the double or single layer potential, moreover $\mathcal{K}\psi$ and $\mathcal{S}\psi$ are harmonic in $\mathbb{R}^n \setminus \partial\Omega$. For $x \in \partial\Omega$, the double layer potential is defined as the singular integral

$$(4.3) \quad \begin{aligned} K\psi(x) &= p.v. - \frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) d\sigma(y) \\ &= \lim_{\epsilon \rightarrow 0} -\frac{1}{\omega_n} \int_{|x-y| > \epsilon} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) d\sigma(y). \end{aligned}$$

As soon as $\psi \in L^p(\partial\Omega)$ for some $p \in (1, \infty)$, it is well-known that $K\psi$ exists a.e. in $\partial\Omega$ and belongs to $L^p(\partial\Omega)$. Furthermore, for a. e. $y \in \partial\Omega$, the function $\mathcal{K}\psi(x)$ converges to $(\frac{1}{2}Id + K)\psi(y)$ (resp. to $(-\frac{1}{2}Id + K)\psi(y)$) as $x \rightarrow y$ non-tangentially in Ω (resp. Ω^-), see [47, Thm 1.10] for a precise statement and references. The non-tangential limits on $\partial\Omega$ of the double layer potential, from inside and outside, differ by the density ψ of the potential. For appropriate range of exponents, the double layer potential on Ω is a famous tool to solve the Dirichlet problem for the Laplace equation. In fact, $(\frac{1}{2}Id + K) : W^{1/q,p}(\partial\Omega) \rightarrow W^{1/q,p}(\partial\Omega)$ is invertible for $p \in [\frac{3}{2}, 3]$ and $1/p + 1/q = 1$; and for solving the exterior Dirichlet problem, one uses that $(-\frac{1}{2}Id + K) : W^{1/q,p}(\partial\Omega)/\langle 1 \rangle \rightarrow W^{1/q,p}(\partial\Omega)/\langle 1 \rangle$ is invertible for $p \in [\frac{3}{2}, 3]$, where the quotient by $\langle 1 \rangle$ means “modulo constants”, see [23, Thm 8.1].

Unlike the double layer potential, the single layer potential is continuous a.e. across $\partial\Omega$, though its normal derivative is not. Note that the gradient of the single layer potential is given by

$$(4.4) \quad \nabla\mathcal{S}\psi(x) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{(x-y)}{|x-y|^n} \psi(y) d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega.$$

For $1 < p < \infty$ and $\psi \in L^p(\partial\Omega)$, it holds for a.e. $y \in \partial\Omega$ that $\nu(y) \cdot \nabla \mathcal{S}\psi(x)$ converges to $-(\frac{1}{2}Id - K^*)\psi(y)$ (resp. $(\frac{1}{2}Id + K^*)\psi(y)$) as $x \rightarrow y \in \partial\Omega$ non-tangentially in Ω (resp. Ω^-), where K^* is the adjoint of K :

$$(4.5) \quad K^*\psi(x) = p.v. \frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(x) d\sigma(y);$$

see [47, Thm 1.11] for a statement and further references.

For appropriate range of exponents, the single layer potential on Ω allows one to solve the Neumann problem for the Laplace equation. More precisely, it follows from [23, Thm 8.1] that $(\pm\frac{1}{2}Id + K^*)$ extends to an invertible map $\tilde{W}^{-1/p,p}(\partial\Omega) \rightarrow \tilde{W}^{-1/p,p}(\partial\Omega)$ for $p \in [\frac{3}{2}, 3]$, where we have set

$$(4.6) \quad \tilde{W}^{-\frac{1}{p},p}(\partial\Omega) := \{f \in W^{-\frac{1}{p},p}(\partial\Omega) : \int_{\partial\Omega} f(y) d\sigma(y) = 0\};$$

here, the integral in (4.6) is just a notation for f acting on the constant function 1 on $\partial\Omega$. Then the Neumann problem of finding a harmonic function in $W^{1,p}(\Omega)$ with normal derivative $g \in \tilde{W}^{-1/p,p}(\partial\Omega)$ has a unique solution, up to an additive constant, given by the single layer potential $\mathcal{S}f$, where $f \in \tilde{W}^{-1/p,p}(\partial\Omega)$ is such that $(-\frac{1}{2}Id + K^*)f = g$, see [23, Thm 9.2]

5. Silent magnetic sources. Let us represent a magnetisation carried by a bounded domain $\Omega \subset \mathbb{R}^3$ as a vector-field $M \in [L^p(\Omega)]^3$, with $1 < p < \infty$. Under the quasi-static assumption on Maxwell's equations, it is known (see [36]) that the scalar magnetic potential P_M generated by M is related to the latter by

$$(5.1) \quad \Delta P_M = \nabla \cdot M$$

where P_M is zero at infinity. It follows from Lemma 3.1 (with $n = 3$) that $\Psi_{\tilde{M}}$, as defined in (3.8), satisfies $\tilde{M} = \nabla \Psi_{\tilde{M}} + D$ where D is divergence-free, hence $\Delta \Psi_{\tilde{M}} = \nabla \cdot M$. Since $\Psi_{\tilde{M}}$ vanishes at infinity (recall Ω is bounded by assumption), we get that $P_M = \Psi_{\tilde{M}}$:

$$(5.2) \quad P_M(x) = \frac{1}{4\pi} \int_{\Omega} M(y) \cdot \frac{(x-y)}{|x-y|^3} dy.$$

A magnetisation is said to be Ω -silent, or a silent source in $\mathbb{R}^3 \setminus \Omega$ if it generates the zero field there. Although the main Physical interest attaches to dimension 3, perspective is gained if we extend the terminology to any dimension strictly greater than 2 by making the following, formal definition:

Definition 5.1. For $\Omega \subset \mathbb{R}^n$ a bounded open set, we say that $M \in [L^p(\Omega)]^n$ is Ω -silent if

$$(5.3) \quad \nabla_x P_M(x) = \nabla_x \left(\frac{1}{\omega_n} \int_{\Omega} M(y) \cdot \frac{(x-y)}{|x-y|^n} dy \right) = 0, \quad a.e. x \in \mathbb{R}^n \setminus \Omega,$$

where ∇_x indicates the gradient with respect to the x variable and $P_M(x)$ is defined by the expression in between parentheses in the right hand side.

In particular, if $\partial\Omega$ has Lebesgue measure 0 then M is Ω -silent if and only if the function P_M , which is harmonic in $\mathbb{R}^n \setminus \bar{\Omega}$ by (5.3), is in fact locally constant there.

For $\Omega \subset \mathbb{R}^n$ a bounded open set and V a vector field in $[L^p(\Omega)]^m$, recall the notation \tilde{V} for its extension by zero to the whole of \mathbb{R}^n . As the potential P_M defined by (5.3) coincides with the function $\Psi_{\tilde{M}}$ from (3.8), Lemma 3.1 easily yields a characterization of silent magnetizations:

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. A field $M \in [L^p(\Omega)]^n$ is Ω -silent if, and only if the Helmholtz decomposition $\tilde{M} = \nabla\psi + D$ of \tilde{M} on \mathbb{R}^n is such that both $\nabla\psi$ and D are zero a.e. on $\mathbb{R}^n \setminus \Omega$.*

Proof. Since Ω is bounded, \tilde{M} lies in $[L^q(\mathbb{R}^n)]^n$ for $1 < q \leq p$. Hence, in the Helmholtz decomposition of \tilde{M} , we may choose $\psi = \Psi_{\tilde{M}} = P_M$ by Lemma 3.1. Thus, \tilde{M} is silent if and only if the gradient term in that decomposition is zero a.e. on $\mathbb{R}^n \setminus \Omega$, by (5.3). Because \tilde{M} is zero on $\mathbb{R}^n \setminus \Omega$, this happens if and only if D is zero a.e. on $\mathbb{R}^n \setminus \Omega$, as desired. ■

Remark 5.3. Even if Ω is unbounded, P_M can still be defined as a member of $\dot{W}^{1,p}(\mathbb{R}^n)$ (a quotient space of distributions) and Definition 5.1 makes good sense. Theorem 5.2 continues to hold as stated, but we shall not pursue this.

When Ω is a finite collection of positively separated pieces, the question whether M is Ω -silent reduces to the corresponding question on each piece. This we record as a complement to Theorem 5.2, see [24] and [8, Cor. 3.5] for related results.

Corollary 5.4. *Let $\Omega = \cup_{j=1}^l \Omega_j$, where $\Omega_j \subset \mathbb{R}^n$ is a bounded open set whose boundary has zero measure and $d(\Omega_j, \Omega_k) > 0$ for $j \neq k$. For $M \in [L^p(\Omega)]^n$, $1 < p < \infty$, the field M is Ω -silent if and only if the restriction $M|_{\Omega_j}$ is Ω_j -silent for $j \in \{1, \dots, l\}$.*

Proof. Let M be Ω -silent, and put for simplicity $M_i := M|_{\Omega_j}$. Theorem 5.2 implies that $\nabla P_{\tilde{M}_i} = -\sum_{j \neq i} \nabla P_{\tilde{M}_j}$ on $\mathbb{R}^n \setminus \Omega$ for each i , and since the Ω_ℓ are at positive distance from each other $\nabla P_{\tilde{M}_i}$ extends to a harmonic gradient in $L^p(\mathbb{R}^n)$. Of necessity such a gradient is identically zero, as wanted.

Remark 5.5. If $p > n$, then the assumption that $d(\Omega_j, \Omega_k) > 0$ in Corollary 5.4 can be weakened to $B_{1,2}(\bar{\Omega}_j \cap \overline{\cup_{k \neq j} \Omega_k}) = 0$, which no longer entails that the Ω_k are finitely many even though Ω is bounded. It is so because then $P_{\tilde{M}_i}$ is locally bounded since it is continuous by the Sobolev embedding theorem (it is in fact globally bounded for it vanishes at infinity), while compact sets of $B_{1,2}$ -capacity zero are of Riesz \mathfrak{R}_2 -capacity zero [3, Section 2.2], and therefore they are removable for bounded harmonic functions [5, Corollary 5.2.3].

When Ω is Lipschitz, the characterization of silent sources given in Theorem 5.2 can be recast in terms of their boundary behaviour. To state this result, we make an additional piece of notation by putting

$$\mathbb{R}_{\partial\Omega} = \text{span}\{\chi_\omega : \omega \text{ is a connected component of } \partial\Omega\},$$

and we denote by $W_{\mathbb{R}_{\partial\Omega}}^{1,p}(\Omega)$ the space of functions in $W^{1,p}(\Omega)$ having trace in $\mathbb{R}_{\partial\Omega}$. In other words, members of $W_{\mathbb{R}_{\partial\Omega}}^{1,p}(\Omega)$ are locally constant on $\partial\Omega$.

Corollary 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz open set, and $M \in [L^p(\Omega)]^n$ for some $p \in (1, \infty)$. Then, M is Ω -silent if and only if $M = D_0 + \nabla\psi_0$, where $D_0 \in \text{Div}_{p,0}(\Omega)$ and $\psi_0 \in W_{\mathbb{R}\partial\Omega}^{1,p}(\Omega)$. Equivalently, D_0 is divergence-free in Ω with zero normal boundary component $D_0 \cdot \nu \in W^{-1/p,p}(\partial\Omega)$, while $\nabla\psi_0$ is a gradient in $G^p(\Omega)$ with zero tangential boundary component $\nabla_T\psi_0 \in \mathcal{W}_1^{-1/p,p}(\partial\Omega)$.*

Proof. In view of Corollary 5.4, we can assume that Ω is connected. Then, with the notation of Lemma 2.3, functions in $C^\infty(\Omega)$ whose gradient lies in $[C_c^\infty(\Omega)]^n$ are exactly the smooth functions on \mathbb{R}^n that are locally constant in a neighborhood of each \bar{O}_i . Now, since Ω is regular for the zero extension in L^p by Theorem 2.6 (i), it follows from Theorem 5.2 that M is silent if and only if $M = D_0 + G_0$ where $D_0 \in \text{Div}_{p,0}(\Omega)$ and $G_0 \in G_0^p(\Omega)$. The result is now a consequence of (2.17) and the discussion about (2.14).

Though we focus in this paper on vector fields of L^p -class with $1 < p < \infty$, it is natural to ask about the endpoint cases $p = 1, \infty$. The characterization of silent sources given in Theorem 5.2 goes through when the components of \widetilde{M} lie the real Hardy space $\mathfrak{h}^1(\mathbb{R}^n)$ or in the space of functions with bounded variation $BMO(\mathbb{R}^n)$, because these spaces are stable under Riesz transforms. However, the issue as to which functions in $\mathfrak{h}^1(\Omega)$ or $BMO(\Omega)$ can be extended by zero and still yield a function in $\mathfrak{h}^1(\mathbb{R}^n)$ or $BMO(\mathbb{R}^n)$ raises difficulties. In another connection, the case where M is a \mathbb{R}^n -valued *measure* is especially interesting as it is related to sparse recovery of magnetizations, see [13]. When Ω is a bounded open set with $\partial\Omega$ of zero measure, it is still true that M is silent if and only if the summands of its Helmholtz decomposition are supported on $\bar{\Omega}$, as distributions. However, the Riesz transforms of a measure are no longer measures unless the initial measure is a $\mathfrak{h}^1(\mathbb{R}^n)$ -function [46, Chapter VII, Section 3.2, Corollary 1], therefore cancellation must occur between these summands in general, and so they cannot be parametrized independently. Hence, while the sum of the gradient of a function of bounded variation which is constant on $\partial\Omega$ and of a divergence free vector measure carried by Ω is silent, there are other silent measures that cannot be decomposed this way.

6. Equivalent Sources. In this section, we study Ω -equivalent sources; *i.e.*, magnetisations producing the same field outside Ω . In other words, $M_1, M_2 \in [L^p(\Omega)]^n$ are Ω -equivalent if and only if $M_1 - M_2$ is an Ω -silent source. We stressed earlier that the question of recovering a magnetisation $M \in [L^p(\Omega)]^n$ from the knowledge of its potential or field outside Ω never has a unique answer, because there are vector fields Ω -equivalent to a given one. We are now interested in norm-minimising Ω -equivalent sources. Namely, given $M \in [L^p(\Omega)]^n$, we want to find the silent source M_S such that

$$(6.1) \quad M_S = \arg \min_{M_0 \in S_p} \|M - M_0\|_{[L^p(\Omega)]^n},$$

where S_p is the space of Ω -silent sources in $[L^p(\Omega)]^n$. Clearly then, $M - M_S$ is the norm-minimising source Ω -equivalent to M . That M_S exists follows at once from the next lemma, and then its uniqueness is ensured by strict convexity.

Lemma 6.1. *S_p is closed in $[L^p(\Omega)]^n$ for $p \in (1, \infty)$.*

Proof. It is obvious from Definition 5.1 that S_p is a weakly closed subspace of $[L^p(\mathbb{R}^n)]^n$, hence it is closed by Mazur's theorem. ■

When $p = 2$, one has $M_S = M - P_{S_2}M$ where P_{S_2} indicates the orthogonal projection onto S_2 in $[L^2(\Omega)]^n$. By Theorem 5.2, the space S_2 is the sum of \tilde{G}_2 and the space \tilde{D}_2 comprising those $D \in [L^2(\Omega)]^n$ such that \tilde{D} is divergence-free on \mathbb{R}^n , the sum being obviously orthogonal. Moreover, the Helmholtz decomposition exists and one can write $M = D + \nabla\psi$ with $D \in \text{Div}_{2,0}(\Omega)$ and $G \in G^2(\Omega)$. Since \tilde{D} is divergence-free, D is silent by Theorem 5.2 again, so that $M_S = G_S = G - P_{S_2}G$. The latter is not easy to characterize for general open sets, but more can be said if Ω is div-regular, because in this case G is orthogonal to the space \tilde{D}_2 which is but $\text{Div}_{2,0}(\Omega)$. It entails that $G - P_{S_2}G = G - P_{\tilde{G}_2}G$, and the latter can be recast as a solution to a certain boundary value problem, which turns out to be unique if Ω is also grad-regular:

Theorem 6.2. *Let Ω be a bounded open set which is div-regular for the zero extension in L^2 , and let $M \in [L^2(\Omega)]^n$. If we write $M = D + \nabla\psi \in [L^2(\Omega)]^n$ for the Helmholtz decomposition with $D \in \text{Div}_{2,0}(\Omega)$ and $\psi \in L^{2n/(n-2)}(\Omega)$ (see Lemma 3.1), then the norm-minimising vector field in $[L^2(\Omega)]^n$ which is Ω -equivalent to M is $\nabla\psi - \nabla u$, where ∇u is the projection of $\nabla\psi$ on $\tilde{G}^2(\Omega)$. Also, $\psi_{nm} := \psi - u$ is a $W^{1,2}(\Omega)$ -solution to the boundary value problem:*

$$(6.2) \quad \begin{aligned} \Delta\psi_{nm} &= 0 \text{ in } \Omega, \\ \nabla\psi_{nm} - \nabla\psi &\in \tilde{G}^2(\Omega), \\ \int_{\Gamma} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} &= 0 \text{ for all } C^\infty\text{-hypersurfaces } \Gamma \subset \Omega, \end{aligned}$$

where \mathcal{H}^{n-1} indicates $(n-1)$ -Hausdorff measure. If moreover Ω is regular for the zero extension in L^2 , then ψ_{nm} is the unique solution to (6.2), up to an additive constant.

Proof. As pointed out before the theorem, the norm-minimising vector field Ω -equivalent to M is $\nabla\psi - \nabla u$, where ∇u is the projection of $\nabla\psi$ on $\tilde{G}^2(\Omega)$. Putting $\psi_{nm} := \psi - u$, we get that $\nabla\psi_{nm}$ is orthogonal to $\tilde{G}^2(\Omega)$ and in particular to $\nabla\varphi$ for all $\varphi \in C_c^\infty(\Omega)$. It implies that ψ_{nm} is harmonic. Moreover, if $\Gamma \subset \Omega$ is a C^∞ -hypersurface, we can find another smooth hypersurface $\Gamma_1 \subset \Omega$ with $\Gamma_1 \cap \Gamma = \emptyset$, close enough to Γ that the open ‘‘shell’’ S between Γ and Γ_1 is contained in Ω (using for instance a tubular neighborhood of Γ , see [30, Ch. 2, Sec. 3]). Assume for definiteness that S lies interior to Γ (if it lies exterior to Γ the argument is similar). We can find a function $g \in C^\infty(\mathbb{R}^n)$ which is equal to 1 on $\text{ext}\Gamma$ and to zero on a neighborhood of $\Gamma_1 \cup \text{int}\Gamma_1$. Then, $\nabla g \in G_0^2(\Omega) \subset \tilde{G}_2$ so that $\int \nabla\psi_{nm} \cdot \nabla g dm = 0$, and by the divergence formula the latter is equal to $\int_{\Gamma} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1}$. Hence, ψ_{nm} satisfies (6.2). Conversely, assume that (6.2) holds and pick $\nabla g \in G_0^2(\Omega)$. Let further $K \subset \Omega$ be the support of $|\nabla g|$. Since K is compact, there is $h \in C^\infty(\mathbb{R}^n)$ such that K is the zero set of h ; this follows easily from a combination of [46, Chapter VI, Theorem 2] and [49, Theorem I]. Replacing h with h^2 we may assume that $h \geq 0$, and putting $L := h^{-1}(\varepsilon)$ for ε a sufficiently small regular value of h (almost every positive number is a regular value by Sard’s theorem), we find that L is a finite union of smooth compact hypersurfaces $\Gamma_1, \dots, \Gamma_N$ included in Ω such that $K \subset \cup_j \text{int}\Gamma_j$. By construction g is equal to a constant c_j on Γ_j , and therefore $\int_{\Omega} \nabla\psi_{nm} \cdot \nabla g dm = \sum_j c_j \int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} = 0$, by (6.2). Thus, $\nabla\psi_{nm}$ is orthogonal to $G_0^2(\Omega)$, and by definition the latter coincides with $\tilde{G}^2(\Omega)$ when Ω is regular for the zero extension in

L^2 . We now see from the second equation in (6.2) that $\nabla\psi_{nm} = \nabla\psi - \nabla u$ where ∇u is the projection of $\nabla\psi$ on $\tilde{G}^2(\Omega)$, as desired. \blacksquare

When Ω is Lipschitz, Theorem 6.2 yields a fairly explicit characterization of norm-minimising Ω -equivalent sources in $[L^2(\Omega)]^n$. By Lemma 5.4, it is enough to consider the case where Ω is connected. Recall that a domain Ω is said to be *regular for the Dirichlet problem* if to each continuous function φ on $\partial\Omega$, there is a harmonic function h_φ on Ω which is continuous on $\bar{\Omega}$ and such that $(h_\varphi)|_{\partial\Omega} = \varphi$. The harmonic measure ω_ξ at $\xi \in \Omega$ is then the Borel probability measure on $\partial\Omega$ such that $h_\varphi(x) = \int \varphi d\omega_x$ for all such φ , see [5, Chapter 6]. If Ω is Lipschitz then it is regular for the Dirichlet problem, its harmonic measure ω_x is absolutely continuous with respect to σ for each $x \in \Omega$, and for every Borel set $E \subset \partial\Omega$ the function $x \mapsto \omega_x(E)$ is bounded harmonic in Ω with nontangential limit 1 a.e. on E and 0 a.e. on $\partial\Omega \setminus E$ [19].

Corollary 6.3. *Let Ω be a bounded Lipschitz domain and $\Gamma_1, \dots, \Gamma_l$ the connected components of $\partial\Omega$. Let further $M \in [L^2(\Omega)]^n$ and write the Helmholtz decomposition of M as $D + \nabla\psi$. Then, the norm-minimising vector field Ω -equivalent to M is $\nabla\psi_{nm}$, where $\psi_{nm}(x) = u(x) + \sum_{j=1}^l c_j \omega_x(\Gamma_j)$ with u the $W^{1,2}(\Omega)$ solution to the Dirichlet problem:*

$$(6.3) \quad \begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= \psi \text{ on } \partial\Omega, \end{aligned}$$

and $\omega_x(\Gamma_j)$ the harmonic measure of Γ_j at $x \in \Omega$, while the vector $(c_1, \dots, c_l)^t \in \mathbb{R}^l$ is determined, up to a multiple of $(1, \dots, 1)^t$ (which will only alter ψ_{nm} by a constant and therefore will respect $\nabla\psi_{nm}$), by the property that $\int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} = 0$ for $1 \leq j \leq l$.

Proof. As Lipschitz open sets are regular for the zero extension (in every L^p), and since $W_{\mathbb{R}\partial\Omega}^{1,2}(\Omega) = \text{span}\{x \mapsto \omega_x(\Gamma_j)\}$ because $x \mapsto \omega_x(\Gamma_j)$ is the unique harmonic function in $W^{1,2}(\Omega)$ which is 1 on Γ_j and zero on Γ_k for $k \neq j$ (it lies in $W^{1,2}(\Omega)$ by, say [47, Corollary 3.5] and uniqueness of a $W^{1,2}$ -solution to the Dirichlet problem on a Lipschitz domain is standard), Theorem 6.2 and Corollary 5.6 together imply that the norm-minimising vector field Ω -equivalent to M is $\nabla\psi_{nm}$ with $\psi_{nm}(x) = u(x) + \sum_{j=1}^l c_j \omega_x(\Gamma_j)$, where $u \in W^{1,2}(\Omega)$ solves (6.3) and the c_j are real numbers. Using the notation of Lemma 2.3 and proceeding as in the proof of Theorem 6.2 to construct non-negative functions $h_j \in C^\infty(\mathbb{R}^n)$ whose zero set is $\text{ext } \bar{O}_1$ for $j = 1$ and $\text{int } \bar{O}_j$ for $j \neq 1$, we construct smooth hypersurfaces $\Sigma_j \subset \Omega$ such that the shell between Γ_j and Σ_j is included in Ω . Since $\int_{\Sigma_j} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} = 0$ by (6.2), we get from the Green formula that $\int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} = 0$ for $1 \leq j \leq l$. Hence, it holds for $j \in \{1, \dots, l\}$ that

$$(6.4) \quad - \int_{\Gamma_j} \frac{\partial u}{\partial n} d\mathcal{H}^{n-1} = \sum_{\ell=1}^l c_\ell \int_{\Gamma_j} \frac{\partial\omega(\Gamma_\ell)}{\partial n} d\mathcal{H}^{n-1}.$$

To see that (6.4) determines (c_1, \dots, c_l) up to a multiple of $(1, \dots, 1)$, observe that if we put $v = \sum_{\ell} a_\ell \omega(\Gamma_\ell)$ then

$$\int_{\Omega} \nabla v \cdot \nabla v \, dm = \sum_{j=1}^l \int_{\Gamma_j} v \frac{\partial v}{\partial n} d\mathcal{H}^{n-1} = \sum_{j,\ell=1}^l a_j a_\ell \int_{\Gamma_j} \frac{\partial\omega(\Gamma_\ell)}{\partial n} d\mathcal{H}^{n-1}, \quad \blacksquare$$

so that the quadratic form on \mathbb{R}^l whose matrix has (j, ℓ) -entry $\int_{\Gamma_j} \frac{\partial \omega(\Gamma_\ell)}{\partial n} d\mathcal{H}^{n-1}$ is non-negative with kernel the multiples of $(1, \dots, 1)^t$ (that correspond to a constant v).

The case $p \in (1, \infty)$ but $p \neq 2$ is more involved, since now $M_S = P_{S_p}^{\{p\}} M$, where $P_{S_p}^{\{p\}}$ indicates the projection of best approximation onto S_p in $[L^p(\Omega)]^n$ which is convex but non-linear. Let S_p^\perp be the space orthogonal to S_p under the pairing (2.2), that is the set of all $\Phi \in [L^q(\Omega)]^n$ such that $\langle M_0, \Phi \rangle = 0$ for all $M_0 \in S_p$.

Lemma 6.4. *For Ω a bounded open domain and $p \in (1, \infty)$, each element of S_p^\perp is a harmonic gradient on Ω . If moreover Ω is regular for the zero extension in L^p , then S_p^\perp consists exactly of harmonic gradients; that is, of vector-fields $\nabla \phi \in [L^q(\Omega)]^n$ such that ϕ is a harmonic function on Ω .*

Proof. Let $\Phi \in [L^q(\Omega)]^n$ such that $\langle M, \Phi \rangle = 0$ for all $M \in S_p$. By Theorem 5.2, it follows that $\langle \Phi, \nabla \psi + D \rangle = 0$ for all $\nabla \psi \in G^p \Omega$ and D such that \tilde{D} is divergence-free on \mathbb{R}^n . In particular, $\langle \Phi, D_0 \rangle = 0$ for all $D_0 \in \text{Div}_{p,0}(\Omega)$ whence $\Phi \in G^q(\Omega)$ by Lemma 2.4, say $\Phi = \nabla \phi$ with $\phi \in W_{loc}^{1,q}(\Omega)$. Similarly, $\langle \Phi, \nabla \psi_0 \rangle = \langle \nabla \phi, \nabla \psi_0 \rangle = 0$ for all $\nabla \psi_0 \in [C_c^\infty(\Omega)]^n$. Hence, $\nabla \phi$ must be divergence-free as a distribution, implying that ϕ is harmonic in Ω . Therefore, members of S_p^\perp are indeed harmonic gradients. Conversely, when Ω is regular for the zero extension in L^p , S_p consists exactly of the sum $\text{Div}_{p,0}(\Omega) + G_0^p(\Omega)$ by Theorem 5.2, and the previous argument is easily reversed to yield that harmonic gradients in $[L^q(\Omega)]^n$ lie in S_p^\perp . ■

We assume in the rest of this section that Ω is bounded and regular for the zero extension in L^p . Hence, we get from Lemma 6.4 and the Hahn-Banach theorem [22, Thm 7.2] that

$$(6.5) \quad \begin{aligned} \inf_{M_0 \in S_p} \|M - M_0\|_{[L^p(\Omega)]^n} &= \max_{\|\Phi\|_{[L^q(\Omega)]^n} \leq 1, \Phi \in S_p^\perp} |\langle M, \Phi \rangle| \\ &= \max_{\|\nabla \phi\|_{[L^q(\Omega)]^n} = 1, \Delta \phi = 0 \text{ in } \Omega} |\langle M, \nabla \phi \rangle|. \end{aligned}$$

The existence of a harmonic function ϕ_S on Ω with $\|\nabla \phi_S\|_{[L^q(\Omega)]^n} = 1$ that maximises (6.5) comes from the fact that any sequence of harmonic gradients of unit norm in $[L^q(\Omega)]^n$ has a subsequence that converges locally uniformly to a harmonic gradient whose $[L^q(\Omega)]^n$ -norm is at most 1. This fact follows at once from the mean-value property, Harnack's theorem and the weak compactness of balls in $[L^q(\Omega)]^n$. Besides, M_S and ϕ_S satisfy the identity

$$(6.6) \quad \langle M - M_S, \nabla \phi_S \rangle = \|M - M_S\|_{[L^p(\Omega)]^n},$$

therefore a simple argument based on the case of equality in Hlder's inequality shows that

$$(6.7) \quad \nabla \phi_S = \frac{|M - M_S|^{p-2} (M - M_S)}{\|M - M_S\|_{[L^p(\Omega)]^n}^{p-1}},$$

from which uniqueness of ϕ_S follows. Thus, if $M \in [L^p(\Omega)]^n$ then the norm-minimising source Ω -equivalent to M is given by

$$(6.8) \quad M - M_S = \langle M, \nabla \phi_S \rangle |\nabla \phi_S|^{q-2} \nabla \phi_S.$$

Altogether, we proved the following theorem that generalises the first part of Theorem 6.2 to $p \neq 2$:

Theorem 6.5. *Given $M \in [L^p(\Omega)]^n$, $1 < p < \infty$, the norm-minimising vector field Ω -equivalent to M is given by*

$$\langle M, \nabla \phi_S \rangle |\nabla \phi_S|^{q-2} \nabla \phi_S, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where ϕ_S is, up to an additive constant, the unique harmonic function on Ω whose gradient has unit $[L^q(\Omega)]^n$ -norm and that satisfies

$$(6.9) \quad \langle M, \nabla \phi_S \rangle = \max_{\|\nabla \phi\|_{[L^q(\Omega)]^n} = 1, \Delta \phi = 0 \text{ in } \Omega} \langle M, \nabla \phi \rangle.$$

We are now in a position to prove Theorem 3.3.

Proof of Theorem 3.3. It follows from (6.1) and Theorem 6.5 that given any $M \in [L^p(\Omega)]^n$, $1 < p < \infty$, the norm-minimising source Ω -equivalent to M can be written in the form

$$\langle M, \nabla \phi_S \rangle |\nabla \phi_S|^{q-2} \nabla \phi_S = M - \nabla \psi_0 - D_0,$$

where ϕ_S is harmonic and satisfies $\nabla \phi_S \in [L^q(\Omega)]^n$, while $\psi_0 \in W^{1,p}(\Omega)$ with $\nabla \psi_0 \in G_0^p(\Omega)$, and $D_0 \in \text{Div}_{p,0}(\Omega)$. Moreover, ϕ_S is unique as well as $\nabla \psi_0 + D_0$. Letting $h = \langle M, \nabla \phi_S \rangle^{\frac{1}{q-1}} \phi_S$ together with $\psi = \psi_0$ and $D = D_0$, we get decomposition (3.11). To establish uniqueness, it remains to show that $\nabla \psi_0 + D_0$ determines $\nabla \psi_0$ and D_0 uniquely. However, this is clear since $\widetilde{\nabla \psi_0}$ and \widetilde{D}_0 are respectively a gradient and a divergence-free vector field in $[L^p(\mathbb{R}^n)]^n$, so we can appeal to uniqueness of the Helmholtz decomposition there. \blacksquare

If one considers now, instead of (6.1), the following two extremal problems:

$$(6.10) \quad D_S = \arg \min_{D \in \text{Div}_{p,0}(\Omega)} \|M - D\|_{[L^p(\Omega)]^n}, \quad G_S = \arg \min_{G \in G_0^p(\Omega)} \|M - G\|_{[L^p(\Omega)]^n},$$

and if one observes that $(\text{Div}_{p,0}(\Omega))^\perp = G^q(\Omega)$ and $(G_0^p(\Omega))^\perp = \{D \in \text{Div}_q(\Omega) : \text{div } D = 0\}$, one gets on reasoning as in the proofs of Theorems 6.5 and 3.3 the following result.

Corollary 6.6. *For Ω a bounded open set and $M \in [L^p(\Omega)]^n$ with $1 < p < \infty$, there exist uniquely $D \in \text{Div}_{p,0}(\Omega)$ and $\nabla \psi \in G^p(\Omega)$ with $\psi \in W_{loc}^{1,p}(\Omega)$ such that*

$$(6.11) \quad M = D + |\nabla \psi|^{q-2} \nabla \psi, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Likewise, there exist uniquely $G \in G_0^p(\Omega)$ and $D \in \text{Div}_q(\Omega)$ with $\text{div } D = 0$ such that

$$(6.12) \quad M = G + |D|^{q-2} D.$$

When $M \in \text{Div}_p(\Omega)$, taking the divergence on both sides of 6.11 yields that ψ the solution, in a rather general setting, of the Neumann problem $\Delta_q \psi = \text{div } M$ with boundary condition $|\nabla \psi|^{q-2} \nabla \psi \cdot \nu = M \cdot \nu$, where Δ_q is the q -Laplacian. In another connection, if in addition Ω has a Helmholtz decomposition in $[L^q(\Omega)]^n$ (hence also in $[L^p(\Omega)]^n$), one can put $D = \text{curl } H$ for some $h \in \text{Div}_{p,0}(\Omega) \cap [W^{1,q}(\Omega)]^n$ in (6.12); for this, define $L := \mathfrak{R}_2 \star \widetilde{D}$ so that $-\Delta L = \widetilde{D}$

componentwise, and select for h the $\text{Div}_{q,0}(\Omega)$ -component of the Helmholtz decomposition of $(\text{curl } L)|_{\Omega}$. Then, taking the curl in (6.12) and putting $a(x, v) := |v|^{q-2}v$, together with $f := \text{curl } M$ and $g := M \cdot \nu$, we find that h is a solution to the system of equations referenced as [43, Equations (2.2)], studied there on Lipschitz domains in connection with certain nonlinear Maxwell systems. We refer to [44] for an approach to problem (6.1) on Lipschitz domains, based on alternating solutions to the problems in (6.10).

To conclude this section, let us recast (6.1) as a boundary value problem for the maximizing function ϕ in (6.9), in the case where Ω is a Lipschitz domain and $p \in [3/2, 3]$. Then, the harmonic function $\phi \in W^{1,q}(\Omega)$ has a normal derivative $\nabla\phi \cdot \nu \in W^{-1/q,q}(\partial\Omega)$, and it can be expressed up to an additive constant as a single layer potential: $\phi = \mathcal{S}f$ where f lies in $\tilde{W}^{-\frac{1}{q},q}(\partial\Omega)$ defined in (4.6) and meets $(-\frac{1}{2}Id + K^*)f = \nabla\phi \cdot \nu$, see Section 4. Now, it follows from (4.4) that

$$(6.13) \quad \langle M, \nabla \mathcal{S}f \rangle = - \int_{\partial\Omega} P_M f \, d\sigma.$$

Indeed, (6.13) follows immediately from Fubini's theorem when M is compactly supported in Ω and the general case follows by a limiting argument, letting M_k be a sequence of compactly supported vector fields in Ω converging to M in $[L^p(\Omega)]^n$, and observing that $P_{M_k} = \Psi_{\tilde{M}_k}$ converges to P_M in $W^{1,p}(\Omega)$ by Lemma 3.1 and its proof so that the trace $(P_k)|_{\partial\Omega}$ converges to $P|_{\partial\Omega}$ in $W^{1/q,p}(\partial\Omega)$. Hence, (6.9) can be rewritten as

$$(6.14) \quad \langle M, \nabla \phi_S \rangle = \max_{f \in \tilde{W}^{-\frac{1}{q},q}(\partial\Omega), \|\nabla \mathcal{S}f\|_{[L^q(\Omega)]^n} = 1} \int_{\partial\Omega} P_M f \, d\sigma.$$

By the Hahn-Banach theorem $\tilde{W}^{-\frac{1}{q},q}(\partial\Omega)$ is dual to the quotient space $W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}$; hereafter, we put \dot{g} for the equivalence class of $g \in W^{\frac{1}{q},p}(\partial\Omega)$. So, if we renorm $\tilde{W}^{-\frac{1}{q},q}(\partial\Omega)$ with

$$\|f\|_{\tilde{W}^{-\frac{1}{q},q}(\partial\Omega)}^{\{1\}} := \|\nabla \mathcal{S}f\|_{[L^q(\Omega)]^n},$$

and endow $W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}$ with the dual norm:

$$\begin{aligned} \|\dot{u}\|_{W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}}^{\{1\}} &:= \sup_{f \in \tilde{W}^{-\frac{1}{q},q}(\partial\Omega), \|\nabla \mathcal{S}f\|_{[L^q(\Omega)]^n} = 1} \int_{\partial\Omega} u f \, d\sigma \\ &= \sup_{\|f\|_{\tilde{W}^{-\frac{1}{q},q}(\partial\Omega)}^{\{1\}} = 1} \langle u, f \rangle, \end{aligned}$$

we find that $\phi_S = \mathcal{S}f$ with f a constant multiple of the image of $(\dot{P}_M)|_{\partial\Omega} \in W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R}$ under the *duality mapping* $W^{\frac{1}{q},p}(\partial\Omega)/\mathbb{R} \rightarrow \tilde{W}^{-\frac{1}{q},q}(\partial\Omega)$, which is the unique isometric homeomorphism from a uniformly convex space onto its dual. We refer the reader to [17, Chapter 3] and [38] for more on duality mappings, and to [44] for a sketch of numerical approaches to (6.1) based on (6.14) and the construction of frames for $W^{-\frac{1}{q},q}(\partial\Omega)$ introduced in [40].

7. Appendix.

7.1. More on Sobolev functions. What follows complements Section 2, and we stick to the notation introduced there. In particular, \tilde{F} stands for the extension by zero of F , initially defined on $\Omega \subset \mathbb{R}^n$, to the whole of \mathbb{R}^n ; recall also the spaces $G^p(\Omega)$ and $G_0^p(\Omega)$ introduced in (2.7) and (2.8), as well as the space $\tilde{G}^p(\Omega) \subset G^p(\Omega)$ defined by (2.9). Theorem 5.2 and Lemma 3.1 imply that $\tilde{G}^p(\Omega)$ consists exactly of the fields generated in $[L^p(\Omega)]^n$ by Ω -silent magnetisations.

We shall see that in fact $\tilde{G}^p(\Omega)$ coincides with $G_0^p(\Omega)$ when $\partial\Omega$ has Lebesgue measure zero, and the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$ accumulate only in a “thick” manner (or do not accumulate at all). This is a consequence of Theorem 2.2 whose proof will be given momentarily, and the thickness condition cannot be dispensed with in general, as the example in Section 7.2 shows. The case of an arbitrary bounded open set Ω will not be considered here. We begin with a lemma:

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set whose boundary $\partial\Omega$ has zero Lebesgue measure. Let further $\{O_j\}_{j \in J}$ designate the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$. For $p \in (1, \infty)$, a vector field F lies in $\tilde{G}^p(\Omega)$ if, and only if \tilde{F} is of the form ∇f for some (necessarily unique) $f \in W^{1,p}(\mathbb{R}^n)$ which is constant on each O_j .*

Proof. By definition, every $F \in G^p(\mathbb{R}^n)$ is of the form ∇f with $f \in W_{loc}^{1,p}(\mathbb{R}^n)$. If moreover $F = 0$ on $\mathbb{R}^n \setminus \Omega$, then f is constant in each O_j ; conversely, since $\partial\Omega$ has Lebesgue measure 0, the field F vanishes on $\mathbb{R}^n \setminus \Omega$ if and only if f is constant on each O_j . Let O_0 designate, without loss of generality, the unbounded connected component of $\mathbb{R}^n \setminus \bar{\Omega}$. Adding a constant to f if necessary, we can ensure that it vanishes on O_0 , and then $f \in W^{1,p}(\mathbb{R}^n)$. ■

Note: if $f \in W^{1,p}(\mathbb{R}^n)$ is such that $\nabla f \in \tilde{G}^p(\Omega)$, necessarily f vanishes in the unbounded component of $\mathbb{R}^n \setminus \bar{\Omega}$, for it is constant there and must lie in $L^p(\mathbb{R}^n)$.

We turn to the proof of Theorem 2.2. The statement to be established is the following.

For $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ a bounded open set with boundary $\partial\Omega$ of zero Lebesgue measure, it holds that $G_0^p(\Omega) \subset \tilde{G}^p(\Omega)$. If $n < p$, the converse inclusion holds; namely, $G_0^p(\Omega) = \tilde{G}^p(\Omega)$. More generally, if we let $\{O_j\}_{j \in J}$ denote the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$ and we assume in addition that O_j is non-thin at $B_{1,p}$ -quasi every point of ∂O_j for all j , and if whenever $\bar{O}_j \cap \bar{\cup_{i \in I} O_i} \neq \emptyset$ for some index $j \in J$ and some subset of indices $I \subset J \setminus \{j\}$ then $B_{1,p}(\bar{O}_j \cap \bar{\cup_{i \in I} O_i}) > 0$ (these conditions are automatically satisfied if $n < p$), it holds that $G_0^p(\Omega) = \tilde{G}^p(\Omega)$.

Proof of Theorem 2.2. Without loss of generality, we assume that O_0 is the unbounded component of $\mathbb{R}^n \setminus \bar{\Omega}$. Pick $\varphi \in C^\infty(\Omega)$ with $\nabla\varphi \in [C_c(\Omega)]^n$, and let the U_i enumerate, for $i \in I \subset \mathbb{N}$, the connected components of $\mathbb{R}^n \setminus \text{supp } \nabla\varphi$, with U_0 to denote the unbounded component. The $U_i \cap \Omega$ are nonempty open sets partitioning $\Omega \setminus \text{supp } \nabla\varphi$. Assume first that Ω is connected, and define $B_i := \cup_{k \neq i} U_k \cup \Omega$ for $i \in I$. Clearly, B_i is a connected open set and $U_i \cup B_i = \mathbb{R}^n$. We contend that $U_i \cap B_i$ is connected. Indeed, as \mathbb{R}^n is simply connected, the last portion of the Mayer-Vietoris sequence in homology [32, Sec. 2.2] yields (with H_l to indicate the l -th homology group):

$$0 = H_1(U_i \cup B_i) \rightarrow H_0(U_i \cap B_i) \rightarrow H_0(U_i) \oplus H_0(B_i) \rightarrow H_0(U_i \cup B_i) \rightarrow 0$$

from which it follows, since $H_0(U_i) = H_0(B_i) = H_0(U_i \cup B_i) = \mathbb{Z}$ (because U_i, B_i and $U_i \cup B_i$ are connected), that $H_0(U_i \cap B_i) = \mathbb{Z}$ as well. *This proves our contention*, and so $U_i \cap \Omega = U_i \cap B_i$ is in turn connected. Hence, φ is constant on $U_i \cap \Omega$ because $\nabla\varphi = 0$ there, say $\varphi|_{U_i \cap \Omega} \equiv c_i$. If we define Φ to be $\varphi - c_0$ on Ω and $c_i - c_0$ on U_i , it is readily checked that Φ is well-defined in $C_c^\infty(\mathbb{R}^n)$ with $\nabla\Phi = \widetilde{\nabla\varphi}$; in fact Φ is zero on U_0 , in particular it vanishes outside every ball containing Ω . To recap, we showed that if Ω is connected and $\varphi \in C^\infty(\Omega)$ with $\nabla\varphi \in [C_c^\infty(\Omega)]^n$, then there is $\Phi \in C_c^\infty(\mathbb{R}^n)$ such that $\nabla\Phi = \widetilde{\nabla\varphi}$. *We claim* that the same holds even when Ω is not connected. Indeed, let $\{\Omega_l\}_{l \in L}$ denote the connected components of Ω , where $L = \{1, 2, \dots\}$ is an initial segment of natural numbers (finite or infinite). If $\varphi \in C^\infty(\Omega)$ is such that $\nabla\varphi \in [C_c^\infty(\Omega)]^n$, then $\text{supp } \nabla\varphi$ can meet only finitely many Ω_l , say $\Omega_1, \dots, \Omega_N$. Thus, φ is constant on Ω_l for $l > N$, and we may as well assume it is zero there because this does not change $\nabla\varphi$. For $1 \leq l \leq N$, the function $\varphi_l := \varphi|_{\Omega_l}$ lies in $C^\infty(\Omega_l)$ and $\nabla\varphi_l = (\nabla\varphi)|_{\Omega_l}$ is compactly supported in Ω_l . Hence, by the first part of the proof, there is $\Phi_l \in C_c^\infty(\mathbb{R}^n)$ such that $\nabla\Phi_l = \widetilde{\nabla\varphi_l}$, and so $\Phi := \sum_{1 \leq l \leq N} \Phi_l$ lies in $C_c^\infty(\mathbb{R}^n)$. By construction $\nabla\Phi = \sum_{1 \leq l \leq N} \widetilde{\nabla\varphi_l}$ is zero outside $\cup_{1 \leq l \leq N} \Omega_l$ and coincides with $\nabla\varphi_l$ on Ω_l , so that $\nabla\Phi = \widetilde{\nabla\varphi}$ as claimed.

Next, consider $F \in G_0^p(\Omega)$ and let φ_k be a sequence in $C^\infty(\Omega)$ such that $\nabla\varphi_k \in [C_c^\infty(\Omega)]^n$ with $\nabla\varphi_k \rightarrow F$ in $[L^p(\Omega)]^n$. Let further Φ_k be a sequence in $C_c^\infty(\mathbb{R}^n)$ with $\nabla\Phi_k = \widetilde{\nabla\varphi_k}$ and such that Φ_k is zero on O_0 ; such a sequence exists by what precedes. Applying (2.3) on a ball containing Ω , we deduce that Φ_k converges in $W^{1,p}(\mathbb{R}^n)$, as $k \rightarrow \infty$, to some f such that $\nabla f = \widetilde{F}$. Hence, $F \in \widetilde{G}^p(\Omega)$ so that $G_0^p(\Omega) \subset \widetilde{G}^p(\Omega)$.

Conversely, let $F \in \widetilde{G}^p(\Omega)$ so that, according to Lemma 7.1, $\widetilde{F} = \nabla f$ where $f \in W^{1,p}(\mathbb{R}^n)$ assumes a constant value $c_j(f)$ on O_j , for every $j \in J$. Note, since $c_0(f) = 0$ while O_0 is non-thin at quasi every point of ∂O_0 by assumption, that $f(x) = 0$ for $B_{1,p}$ -quasi every $x \in \overline{O_0}$: it is so because f is continuous outside a set of arbitrary small $B_{1,p}$ -capacity [3, Proposition 6.1.2]. Suppose for a while that $f \geq 0$ and fix $\delta > 0$, together with a sequence $\{\delta_i\}_{i \in \mathbb{N}}$ of strictly positive numbers such that $\sum_i \delta_i = \delta$. Pick $\varepsilon > 0$ and then $\varepsilon' \in (0, \varepsilon]$ small enough that $\int_{\{0 \leq f \leq \varepsilon'\}} |\nabla f|^p dm \leq \varepsilon^p$; since $\|\nabla f\|_{L^p(\mathbb{R}^n)} < \infty$, such a ε' exists, by the monotone convergence theorem. Let $I_0 \subset J$ be the set of indices i for which $c_i(f) \geq \varepsilon'$, and define $K_0 := \overline{\cup_{i \in I_0} O_i}$. Of necessity $K_0 \cap \overline{O_0} = \emptyset$, otherwise $B_{1,p}(K_0 \cap \overline{O_0}) > 0$ by assumption and, since f is continuous outside a set of arbitrary small $B_{1,p}$ -capacity, it would imply that $f \geq \varepsilon' > 0$ on a set $E \subset \partial O_0$ with $B_{1,p}(E) > 0$, contradicting that $f(x) = 0$ for $B_{1,p}$ -quasi every $x \in \overline{O_0}$. Hence, by [3, Theorem 9.1.3], there is a C^∞ function $\eta_0 : \mathbb{R}^n \rightarrow [0, 1]$ with $\eta_0 = 1$ on K_0 and $\eta_0 = 0$ on $\overline{O_0}$ such that $\|f - \eta_0 f\|_{W^{1,p}(\mathbb{R}^n)} \leq \varepsilon'$. Now, if we let $f_0 := \max\{\eta_0 f - \varepsilon', 0\}$, we find that $\nabla f_0 \in \widetilde{G}^p(\Omega)$ with $c_i(f_0) = c_i(f) - \varepsilon'$ for $i \in I_0$ and $c_i(f_0) = 0$ for $i \notin I_0$. Moreover, it follows from [3, Theorem 3.3.1] that $\nabla f_0 = \nabla(\eta_0 f)$ a.e. on $\{x : \eta_0(x)f(x) > \varepsilon'\}$

and $\nabla f_0 = 0$ a.e. on $\{x : \eta(x)f(x) \leq \varepsilon'\}$. Therefore,

$$(7.1) \quad \|\eta f - f_0\|_{W^{1,p}(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |\eta f - f_0|^p dm + \int_{\mathbb{R}^n} |\nabla(\eta f) - \nabla f_0|^p dm$$

$$(7.2) \quad \leq (\varepsilon')^p m(\Omega) + \int_{\{0 \leq \eta f \leq \varepsilon'\}} |\nabla(\eta f)|^p dm$$

$$(7.3) \quad \leq (\varepsilon')^p m(\Omega) + \left(\varepsilon' + \left(\int_{\{0 \leq f \leq \varepsilon'\}} |\nabla f|^p dm \right)^{1/p} \right)^p$$

$$(7.4) \quad \leq (\varepsilon')^p m(\Omega) + (\varepsilon' + \varepsilon)^p.$$

Altogether, as $\varepsilon' \leq \varepsilon$, we get from the triangle inequality that

$$(7.5) \quad \|f - f_0\|_{W^{1,p}(\mathbb{R}^n)} \leq \left(1 + (m(\Omega) + 2^p)^{1/p}\right) \varepsilon$$

which is arbitrary small with ε . In the general case where f is not signed, we write $f = f^+ - f^-$ with $f^+ := \max\{f, 0\}$ and $f^- = (-f)^+$ and remark that ∇f^\pm satisfies the same assumptions as \tilde{F} [3, Thm. 3.3.1]. So, we can apply what precedes to f^+ and f^- to obtain functions $f_{0,+}$ and $f_{0,-}$; we then put $f_0 := f_{0,+} - f_{0,-}$. To recap, we constructed $f_0 \in W^{1,p}(\mathbb{R}^n)$ with $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)}$ arbitrary small, such that f_0 is constant on a neighbourhood of \overline{O}_0 and on each O_j ; in particular, we can make $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)} < \delta_0$. Note also that if f was constant on a neighbourhood of \overline{O}_j for some j , so is f_0 on the same neighbourhood (possibly with a different constant, though). We now proceed inductively: to complete the next step, pick $x_1 \in O_1$ and consider the inversion with center x_1 given by $\mathcal{I}_{x_1}(x) := (x - x_1)/|x - x_1|^2$. It is a smooth involution of the ‘‘sphere’’ $\mathbb{R}^n \cup \{\infty\} \sim \mathbb{S}^n$ that maps x_1 to ∞ , with Jacobian determinant $-|x - x_1|^{-2n}$ at $x \neq x_1$; in fact, the Jacobian matrix is unitarily conjugate to $\text{diag}\{-|x - x_1|^{-2}, |x - x_1|^{-2}, \dots, |x - x_1|^{-2}\}$, see [6, Proof of Theorem 4.2] (the unitary matrix depends on x , though). Clearly, $\Omega_1 := \mathcal{I}_{x_1}(\Omega)$ is a bounded open set, and the $U_i := \mathcal{I}_{x_1}(O_j)$ are the connected components of $\mathbb{R}^n \setminus \overline{\Omega}_1$, with U_1 being the unbounded component. From the change of variable formula and the definition of $B_{1,p}$ -capacity, one checks that \mathcal{I}_{x_1} preserves sets of $B_{1,p}$ -capacity zero in $\mathbb{R}^n \setminus \{x_1, \infty\}$, consequently $B_{1,p}(\overline{U}_j \cap \overline{\cup_{i \in I} U_i}) > 0$ whenever $\overline{U}_j \cap \overline{\cup_{i \in I} U_i} \neq \emptyset$; also, by definition of thinness [3, Definition 6.3.7], \mathcal{I}_{x_1} preserves thinness of a set at any $x \neq x_1$. Thus, U_j is non-thin at $B_{1,p}$ -quasi every point of ∂U_j for each j . Furthermore, if U is an open set with compact closure in $\mathbb{R}^n \setminus \{x_1\}$ and we put $m_1 := \inf_{x \in U} |x - x_1|$, we get from the change of variable formula again that for $g \in W^{1,p}(U)$:

$$(7.6) \quad \|g\|_{W^{1,p}(U)} \leq \max\{m_1^{-2n/p}, m_1^{-2(n-p)/p}\} \|g \circ \mathcal{I}_{x_1}\|_{W^{1,p}(\mathcal{I}_{x_1}(U))}.$$

As the function $h_1 := f_0 \circ \mathcal{I}_{x_1} - c_1(f_0)$ lies in $W^{1,p}(\mathbb{R}^n)$ and is constant on U_j for each j as well as constant on a neighborhood of \overline{U}_0 , we can argue as we did to construct f_0 from f , replacing Ω by Ω_1 and f by h_1 . This provides us with a function g_1 that can be made arbitrary close to h_1 in $W^{1,p}(\mathbb{R}^n)$ while it is constant on each U_j and on a neighborhood of both \overline{U}_0 and \overline{U}_1 . If λ_0 denotes the constant value assumed by g_1 on U_0 , we get since $h_1 = -c_1(f_0)$ on U_0 that $|\lambda_0 + c_1(f_0)| m_1^{1/p}(U_0) \leq \|h_1 - g_1\|_{L^p(U_0)}$, whence $|\lambda_0 + c_1(f_0)|$ can be made arbitrary small with

$\|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)}$. Thus, letting $f_1 := g_1 \circ \mathcal{I}_{x_1} - \lambda_0$, we get in view of (7.6) since $f_0 = f_1 = 0$ on O_0 :

$$(7.7) \quad \|f_0 - f_1\|_{W^{1,p}(\mathbb{R}^n)} = \|(h_1 - g_1) \circ \mathcal{I}_{x_1} + c_1(f_0) + \lambda_0\|_{W^{1,p}(\mathbb{R}^n \setminus U_1)}$$

$$(7.8) \quad \leq C \|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)} + |\lambda_0 + c_1(f_0)| m^{1/p}(\mathbb{R}^n \setminus U_1)$$

$$(7.9) \quad \leq C' \|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)},$$

where C and C' are geometric constants depending on Ω and our choice of $x_1 \in O_1$. One deduces from (7.7) that $\|f_0 - f_1\|_{W^{1,p}(\mathbb{R}^n)}$ can be made arbitrary small, in particular smaller than δ_1 . Iterating the argument, we construct a sequence of functions f_k , $0 \leq k$, such that $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta_0$ and $\|f_k - f_{k+1}\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta_{k+1}$, with f_ℓ constant on a neighborhood V_k of \bar{O}_k for $\ell \geq k$. Since $\sum_k \delta_k = \delta < \infty$, the sequence f_k is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$ that converges to some \tilde{f} ; clearly $\nabla \tilde{f}$ is compactly supported in Ω , and $\|f - \tilde{f}\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta$. By mollification, we can now construct a $\varphi \in C^\infty(\mathbb{R}^n)$ which is constant on a neighborhood of O_j for each j (the constant being 0 when $j = 0$), and such that $\|f - \varphi\|_{W^{1,p}(\mathbb{R}^n)} \leq 2\delta$. Since δ was arbitrary, we find that $F \in G_0^p(\Omega)$, as desired. \blacksquare

Corollary 7.2. *Let $p \in (1, n]$ and $\Omega \subset \mathbb{R}^n$ a bounded open set with boundary $\partial\Omega$ of zero Lebesgue measure. Denote by $\{O_j\}_{j \in J}$ the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$, and assume that O_j is non-thin at $B_{1,p}$ -quasi every point of ∂O_j for all j . Let further $F \in \tilde{G}^p(\Omega)$ so that, according to Lemma 7.1, one can write $\tilde{F} = \nabla f$ for some unique $f \in W^{1,p}(\mathbb{R}^n)$ which is constant on each O_j ; say, $f \equiv c_j(f)$ there. Assume that $\lim_{k \rightarrow \infty} c_{j_k}(f) = c_j(f)$ whenever the sequence $\{j_k\} \subset J$ satisfies $\lim_{k \rightarrow \infty} \text{dist}(O_{j_k}, O_j) = 0$ for some $j \in J$. Then, $F \in G_0^p(\Omega)$.*

Note: the hypothesis entails that $c_i(f) = c_j(f)$ when $\bar{O}_i \cap \bar{O}_j \neq \emptyset$.

Proof. We proceed as in the proof of the second assertion of Theorem 2.2: defining K_0 in the same way, the assumptions on f ensure that $K_0 \cap \bar{O}_0 = \emptyset$ which allows us to complete the first step of the induction. The analog of f_0 thus obtained has $c_j(f_0) = c_j(f) \mp \varepsilon'$ when $\pm c_j(f) \geq \varepsilon'$ and $c_j(f_0) = 0$ otherwise, hence it satisfies the same assumptions as f . The induction can now be carried out as before.

Recall Definition 2.1 of unstretched open sets: the assumptions of Theorem 2.2 mean that Ω is bounded, that $m(\partial\Omega) = 0$ and that $\mathbb{R}^n \setminus \bar{\Omega}$ is unstretched. Since $\partial\Omega$ has zero measure if it is locally isometric to a graph (by Fubini's theorem), and since it is clear in this case that $\bar{O}_j \cap \overline{\cup_{i \in I} O_i} = \emptyset$ for each $j \notin I$, the next proposition yields a sufficient condition for Ω to meet the assumptions of Theorem 2.2. For convenience, we shall write $x = (x', x_n)^t$ for $x \in \mathbb{R}^n$, with $x' := (x_1, \dots, x_{n-1})^t \in \mathbb{R}^{n-1}$.

Proposition 7.3. *Let $1 < p \leq n$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set whose boundary $\partial\Omega$ is locally isometric to a continuous graph. Pick a point on $\partial\Omega$ and assume, after an appropriate rigid motion, that this point lies at the origin and that for some $r_0 > 0$ and $h > 0$:*

$$\Omega \cap (B(0, r_0) \times (-h, h)) = \{(x', x_n), |x'| < r_0, x_n < \Psi(x')\},$$

where $B(0, r_0) \subset \mathbb{R}^{n-1}$ and $\Psi : B(0, r_0) \rightarrow \mathbb{R}$ is a continuous function with $\Psi(0) = 0$. Then, $\mathbb{R}^n \setminus \bar{\Omega}$ is not p -thin at 0 provided that $|y'| > \rho(|\Psi(y')|)$ for $y' \in B(0, r_0)$, where $\rho : [0, r_0) \rightarrow \mathbb{R}^+$

is a continuous increasing function satisfying $\rho(0) = 0$ and

$$(7.10) \quad \int_0^{r_0} \left(r^{p-n} \int_0^r \rho(y)^{n-p-1} dy \right)^{1/(p-1)} \frac{dr}{r} = +\infty \quad \text{for } 1 < p < n,$$

$$(7.11) \quad \int_0^{r_0} \left(\int_0^r \left(\log \frac{1}{\rho(y)} \right)^{1-n} \frac{dy}{\rho(y)} \right)^{1/(n-1)} \frac{dr}{r} = +\infty \quad \text{for } p = n.$$

If this holds at quasi every point of $\partial\Omega$ then $\mathbb{R}^n \setminus \bar{\Omega}$ is p -unstretched.

Proof. The assumptions imply that $\mathbb{R}^n \setminus \bar{\Omega}$ contains the set of revolution $E := \{x = (x', x_n)^t, 0 < x_n < h, |x'| < \rho(x_n)\}$, therefore it is enough to show that E is non-thin at 0. Approximating E from above (in the sense of inclusion) by a stack of disjoint truncated circular cylinders along the x_n -axis with section $B(0, \rho(y)) \subset \mathbb{R}^{n-1}$ and length δy , and making use of (2.4) or (2.5) and [3, Proposition 5.1.4 & Section 5.6.2] along with (a localized version of) [2, Theorem 4] to show such a cylinder has $B_{1,p}$ -capacity of the order of $\rho(y)^{n-p-1} \delta y$ ($(\log \frac{1}{\rho(y)})^{1-p} \delta y / \rho(y)$ if $p = n$), we get in view of (2.6) on letting $\delta y \rightarrow 0$ that E is not p -thin at 0 if (7.10) or (7.11) holds. ■

7.2. Example of Sobolev function on a thin set. We begin by considering the set $E \subset \mathbb{R}^3$ that we define as

$$E = \{(x, y, z) : z \geq 2 - e^{-(x^2+y^2)^\alpha}\} \cap B(0, \sqrt{2})$$

with $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^3$ be the set

$$\Omega = (B(0, \sqrt{2}) \setminus E) \setminus \overline{B(0, 1)},$$

we let $O_0 = (\mathbb{R}^3 \setminus \bar{\Omega}) \setminus B(0, 1)$ and $O_1 = B(0, 1)$. We will call Ω_0 the intersection of Ω and the closed cone with vertex at $(0, 0, 0)$ that passes through the points (x, y, z) on $S(0, \sqrt{2})$ that satisfy $x^2 + y^2 + z^2 = x^2 + y^2 + (2 - e^{-(x^2+y^2)^\alpha})^2$. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined as follows



Figure 1. The domain Ω is contained in the grey part, $\alpha = \frac{1}{4}$. The subset of O_0 in red thin at $(0, 0, 1)$ and a close up view of the region around $(0, 0, 1)$

$$f(x, y, z) = \begin{cases} 1 & \text{on } \overline{O_0} \setminus \{(0, 0, 1)\} \\ x^2 + y^2 + z^2 - 1 & \text{on } \overline{\Omega} \setminus \overline{\Omega_0} \\ \frac{x^2 + y^2 + z^2 - 1}{x^2 + y^2 + (2 - e^{-(x^2+y^2)^\alpha})^2 - 1} & \text{on } \Omega_0 \\ 0 & \text{on } \overline{O_1} \end{cases}.$$

We claim that $f \in W^{1,2}(\mathbb{R}^3)$ for some α . We note that the gradient of f on Ω_0 is given by

$$\nabla f = \begin{pmatrix} \frac{2x}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1} - \frac{4\alpha x(x^2+y^2)^{\alpha-1}e^{-(x^2+y^2)^\alpha}(2-e^{-(x^2+y^2)^\alpha})+2x}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2(x^2+y^2+z^2-1)^{-1}} \\ \frac{2y}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1} - \frac{4\alpha y(x^2+y^2)^{\alpha-1}e^{-(x^2+y^2)^\alpha}(2-e^{-(x^2+y^2)^\alpha})+2y}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2(x^2+y^2+z^2-1)^{-1}} \\ \frac{2z}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1} \end{pmatrix}.$$

It can be shown that the L^2 -norm of the partial derivatives are finite of $0 < \alpha < 1$, see for example the computation in [44, Section 1.9.3].

It remains only to show that the set O_0 is thin at $(0,0,1)$. We use [5, Thm 7.2.5] to show that the set E_0 such that $(x, y, z) \in E_0$ with $0 < z < 1 - \varepsilon(\alpha)$ for some $\varepsilon(\alpha) > 0$ satisfy $\sqrt{(x^2 + y^2)} < \ln(\frac{1}{1-z})^{\frac{1}{2\alpha}}$ is thin at $(0,0,0)$, see for example the computation is [44, Section 1.9.3].

7.3. Silent sources in EEG. Consider a finite sequence of nested bounded connected Lipschitz domains in \mathbb{R}^3 with connected boundary, say $\Omega_0 \subset \overline{\Omega_0} \subset \Omega_1 \subset \overline{\Omega_1} \subset \dots \subset \Omega_\ell \subset \overline{\Omega_\ell}$. In applications to EEG, we would set $\ell = 3$ and Ω_0 is the encephalon, $O_1 := \Omega_1 \setminus \Omega_0$ the cerebrospinal fluid, $O_2 := \Omega_2 \setminus \Omega_1$ the skull and $O_3 := \Omega_3 \setminus \Omega_2$ the scalp. We also write Γ_j , $0 \leq j \leq \ell$ for the boundary of Ω_j . Let σ be a bounded function on \mathbb{R}^3 , supported on $\overline{\Omega_\ell}$ and assuming the constant value $\sigma_j > 0$ on O_j . Let J be a \mathbb{R}^3 -valued distribution supported on $\overline{\Omega_0}$, and P_j satisfy

$$(7.12) \quad \operatorname{div}(\sigma \nabla P_j) = \operatorname{div} J.$$

Lemma 7.4. *For $p \in (1, \infty)$, if $J \in L^p(\mathbb{R}^n)$ is carried by Ω_0 and P_j is a solution to 7.12 in $W_{loc}^{1,p}(\mathbb{R}^n)$, then P_j vanishes in $\mathbb{R}^3 \setminus \overline{\Omega_\ell}$ if and only if the Newton potential of $\operatorname{div} J$ vanishes a.e. in $\mathbb{R}^3 \setminus \Omega_0$.*

Proof. By a theorem of De Giorgi and Nash [1, Theorem 2.108] P_j is (Hölder) continuous, and clearly it is harmonic in $\mathbb{R}^n \setminus \cup_j \Gamma_j$ for $j > 0$. Moreover, at the interface between two layers of constant conductivity σ_1 and σ_2 , its normal derivatives from each side in the outward-pointing direction satisfy $\sigma_1 \frac{\partial P_j}{\partial n_1} = \sigma_2 \frac{\partial P_j}{\partial n_2}$. Now, if P_j vanishes in $\mathbb{R}^n \setminus \overline{\Omega_\ell}$ then its normal derivatives on Γ_ℓ vanish from each side, hence P_j is harmonic across Γ_ℓ and therefore zero in $\mathbb{R}^n \setminus \overline{\Omega_{\ell-1}}$. Continuing in this fashion, we find that P_j vanishes in $\mathbb{R}^n \setminus \overline{\Omega_0}$, hence a.e. in $\mathbb{R}^n \setminus \Omega_0$.

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