



HAL
open science

Silent sources and equivalent L_p -magnetizations

Laurent Baratchart, Juliette Leblond, Masimba Nemaire

► **To cite this version:**

Laurent Baratchart, Juliette Leblond, Masimba Nemaire. Silent sources and equivalent L_p -magnetizations. 2022. hal-03915548v1

HAL Id: hal-03915548

<https://inria.hal.science/hal-03915548v1>

Preprint submitted on 29 Dec 2022 (v1), last revised 21 Dec 2023 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Silent sources and equivalent L^p -magnetizations.

Baratchart Laurent, Leblond Juliette, Nemaire Masimba

December 29, 2022

1 Introduction

sec:intro

Inverse source problems arising in static electromagnetism have various applications: for instance to medical imaging and in particular electroencephalography (EEG), where the primary cerebral current is to be estimated from measurements of the electric potential, or to geosciences where the remanent magnetization of rocks and the Earth, or other celestial bodies, is studied using magnetic field measurements in order to document their past history and understand their structure. In magnetostatic applications, one measures the magnetic field generated by a magnetization away from its support, and one seeks to reconstruct the magnetization.

Under the quasi-static assumption on Maxwell's equations [1], the scalar magnetic potential P_M generated by the magnetization M is subject to the Poisson-Laplace equation in \mathbb{R}^3 :

$$\Delta P_M = \operatorname{div} M, \tag{1} \quad \text{PLeq}$$

where $\Delta = \sum_{i=1}^3 \partial_{x_i}^2$ is the Euclidean Laplacian; the limiting condition is that P_M is zero at infinity (see Equation (34), Section 5), and the magnetic field is then given by the gradient ∇P_M of the potential. In EEG, one measures the electric potential P_M generated by the primary cerebral current M away from its support and one seeks to reconstruct that current. A common feature to both problems is that the source term on the right hand side of (1) is the divergence of a vector field: the magnetization or the primary current. Hereafter, we use the magnetostatic terminology to fix ideas, but everything applies without change in the electrostatic setting.

Such questions are severely ill-posed, due to the existence of nonzero *silent* magnetizations; *i.e.*, magnetizations that produce no field outside the body supporting them. Adding to a given magnetization a silent one yields an *equivalent* magnetization generating the same field as the original one, whence a fundamental uncertainty attaches to the solution of these inverse problems. As a consequence, further assumptions on the unknown magnetization are required to set up consistent regularising schemes. In order to derive such schemes, one needs both working and general characterizations of silent and equivalent magnetizations; this is the subject of the present paper.

In [2], silent magnetizations supported on a plane in \mathbb{R}^3 are described under weak (distributional) regularity assumptions and a similar characterization can be obtained on the 2-D sphere, only replacing Riesz transforms by their spherical analogs defined in terms of layer potentials; see Section 4. In this connection, we note that silent magnetizations of L^2 -class on the sphere have long been characterized in terms of their expansion in spherical harmonics, see for instance [3] for a discussion. Because one deals here with closed surfaces (the plane being a sphere of infinite radius), one has to distinguish between silence from one side and from both sides. On the plane or the sphere, being silent from one side is equivalent to the property that, in the so-called Hardy-Hodge decomposition of the vector field representing the magnetization, the component which is a harmonic gradient from the other side is identically zero. More generally, for a magnetization carried by a closed Lipschitz surface to be silent from one side, the double layer potential of the normal component must be equal to minus the single layer potential of the divergence of the tangential component; see [4, Thms 3.3 & 3.16] for an analysis in L^2 -classes that does carry over, using results from [5], to magnetizations of L^p -class for a restricted range of p around 2 (that depends on the surface); this characterization in fact holds for all $p \in (1, \infty)$ when the surface is C^1 -smooth.

On a closed surface, being silent from both sides is equivalent to being divergence-free, as a distribution in ambient Euclidean space. This also characterizes silent magnetizations supported on a slender set (*i.e.* a set of Lebesgue measure zero whose complement has only connected components of infinite Lebesgue measure) [6]; in this reference, magnetizations are modelled as vector-valued measures (a setting that subsumes L^p -classes). Slender sets include general open surfaces, whose complement is connected, in which case there is no notion of silence from one side.

From the previous discussion, one can get a sense of how silent mag-

netizations carried by a surface are like. In contrast, silent magnetizations supported in a volume were apparently less investigated, and the goal of this paper is to study them. Specifically, we show that silent magnetizations of L^p -class carried by a bounded open set Ω are, for $1 < p < \infty$, (the restriction to Ω of) the sum of a gradient and a divergence-free field on \mathbb{R}^n , each of which vanishes outside of Ω . Under very weak assumptions on Ω (see the precise conditions in Section 5), this amounts to say that silent magnetizations are the closure of the space generated by smooth compactly supported gradients and divergence free fields in Ω . When Ω has a bit more smoothness, for instance if its boundary is locally a Lipschitz graph, then the previous characterization tells us that silent magnetizations are the sum of a gradient vector field on Ω with zero tangential boundary component and of a divergence free field with zero normal boundary component.

As can be surmised from the above discussion, the Helmholtz-Hodge decomposition of vector fields plays a central role in our analysis. Elaborating on the previous description of silent sources, we shall characterize the magnetization of minimum L^p -norm equivalent to a given magnetization, and this gives rise to a non-classical decomposition of L^p -vector fields, as the sum of a gradient with zero tangential boundary component and a divergence-free field with zero normal boundary component, plus the duality mapping of a harmonic gradient. When $p = 2$ it coincides with the Hodge decomposition in degree 1, but unlike the latter it exists for all $p \in (1, \infty)$ on any bounded Lipschitz open set and it is nonlinear. Computing the magnetization of minimum L^2 -norm equivalent to a given one amounts to solve a Dirichlet problem for the Laplacian. When $p \neq 2$, computing an equivalent magnetization of minimum L^p -norm is more difficult.

The paper is organized as follows. In Section 2, we set up notation and recall a number of properties of Sobolev functions on Lipschitz domains and Lipschitz surfaces, as well basic facts concerning divergence-free vector fields. Section 3 is a quick review of Helmholtz decompositions and Riesz transforms, while Section 4 compiles known facts on layer potentials. In Section 5 we characterize silent vector fields on bounded open sets, and we use this in Section 6 to describe equivalent sources which are L^2 norm-minimal, as well as L^p norm-minimal for $p \neq 2$ when the open set is Lipschitz. The latter result yields a nonlinear version of the Helmholtz decomposition that exists for all $1 < p < \infty$, contrary to the classical Helmholtz decomposition that usually exists only for $p \in [\frac{3}{2} - \epsilon, 3 + \epsilon]$, where $\epsilon > 0$ depends on Ω [21, Thm 11.1].

Appendix 7 gathers some technical facts on Sobolev spaces, some of which maybe new like Theorem 8.

2 Notation and preliminaries

notation

Let \mathbb{R}^n denote the Euclidean space of dimension n . Hereafter, we assume that $n \geq 3$. We write $x = (x_1, \dots, x_n)^t$ to display the coordinates of $x \in \mathbb{R}^n$, with superscript “ t ” to mean “transpose”, and $x \cdot y$ for the scalar product of $x, y \in \mathbb{R}^n$; $|x| = x \cdot x^{1/2}$ is the Euclidean norm of x . We let $B(x, r)$ be the open ball centered at x of radius r , and $S(x, r)$ the boundary sphere. We put χ_E for the characteristic function of E , and $d(E_1, E_2)$ for the distance between $E_1, E_2 \subset \mathbb{R}^n$. We designate the set $\{x + e : e \in E\}$ by $x + E$, and for $\varepsilon > 0$ we set $E_\varepsilon := \{x \in \mathbb{R}^n : d(x, E) < \varepsilon\}$ for the ε -neighborhood of E .

2.1 Function spaces

int1

When $E \subset \mathbb{R}^n$ is Lebesgue-measurable and $1 \leq p \leq \infty$, we let $L^p(E)$ be the space of (equivalence classes of a.e. coinciding) \mathbb{R} -valued measurable functions on E whose absolute value to the p -th power is integrable, with norm $\|g\|_{L^p(E)} = (\int_E |g(y)|^p dy)^{1/p}$ (ess. sup $_E |g|$ if $p = \infty$). For $1 \leq p < \infty$ and $1/p + 1/q = 1$, the dual of $L^p(E)$ is $L^q(E)$, isometrically under the pairing $\langle f, g \rangle = \int_E fg$. We set $L^p_{loc}(E)$ to consist of functions f whose restriction $f|_K$ lies in $L^p(K)$ every compact $K \subset E$.

Given a functional space X , we write $[X]^m$ for the corresponding space of vector-fields with m components, each of which lies in X . For example, $[L^p(E)]^m$ is the space of \mathbb{R}^m -valued vector fields M on E whose components belong to $L^p(E)$, with norm

$$\|M\|_{[L^p(E)]^m} = \left(\int_E |M|^p dy \right)^{\frac{1}{p}} \quad (\text{ess. sup}_E |M| \quad \text{if } p = \infty), \quad (2) \quad \text{norm}$$

and for $1 \leq p < \infty$ the dual of $[L^p(E)]^m$ is $[L^q(E)]^m$, $1/p + 1/q = 1$, isometrically under the pairing

$$\langle F, G \rangle = \int_E F(y) \cdot G(y) dy. \quad (3) \quad \text{pairingOpq}$$

If $E \subset \mathbb{R}^n$ and $f : E \rightarrow \mathbb{R}^m$, we designate by $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ the extension of f by zero outside E . The notation stands irrespective of m, n and E , but this will cause no ambiguity.

When $\Omega \subset \mathbb{R}^n$ is open, $W^{1,p}(\Omega)$ indicates the Sobolev space of functions lying in $L^p(\Omega)$ together with their first distributional derivatives. It is a Banach space with norm

$$\|g\|_{W^{1,p}(\Omega)} = \left(\|g\|_{L^p(\Omega)}^p + \|\nabla g\|_{[L^p(\Omega)]^n}^p \right)^{1/p},$$

where $\nabla g = (\partial_1 g, \dots, \partial_n g)^t$ denotes the gradient of g and $\partial_j g$ the derivative with respect to the j -th variable. We let $W_0^{1,p}(\Omega)$ stand for the closure in $W^{1,p}(\Omega)$ of $C_c^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω . A function belongs to $W_0^{1,p}(\Omega)$ if and only if its extension by zero outside Ω belongs to $W^{1,p}(\mathbb{R}^n)$, see [7, Thm. 9.1.3]. If Ω is bounded and $\varphi \in W_0^{1,p}(\Omega)$, then the Poincaré inequality implies that

$$\|\varphi\|_{L^p(\Omega)} \leq C \|\nabla \varphi\|_{[L^p(\Omega)]^n} \quad (4) \quad \boxed{\text{Poincare}}$$

where C depends only on the diameter of Ω , see [8, Thm. 15.4.1] for a more general result. The space $W_{loc}^{1,p}(\Omega)$ is comprised of functions lying in $L_{loc}^p(\Omega)$ together with their first order derivatives; it is a Fréchet space, with seminorms the Sobolev norms on relatively compact open subsets of Ω exhausting the latter.

We put $\dot{W}^{1,p}(\Omega)$ for the quotient space, modulo constants, of distributions on Ω whose derivatives belong to $L^p(\Omega)$. Such a distribution, say ψ , necessarily lies in $W_{loc}^{1,p}(\Omega)$ [8, Sec. 1.1.2], and we write $\dot{\psi} \in \dot{W}^{1,p}(\Omega)$ for the equivalence class of ψ . Endowed with the norm $\|g\|_{\dot{W}^{1,p}(\Omega)} := \|\nabla g\|_{[L^p(\Omega)]^n}$, one can see that $\dot{W}^{1,p}(\Omega)$ is a Banach space called homogeneous Sobolev space (of index p), see [8, Sec. 1.1.13, Thm. 1]. It can be shown that $\dot{W}^{1,p}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to $\|\cdot\|_{\dot{W}^{1,p}(\mathbb{R}^n)}$ for $1 < p < \infty$, see the discussion in Section 3. Hereafter, $G^p(\Omega)$ denotes the subspace of $[L^p(\Omega)]^n$ defined by

$$G^p(\Omega) := \{\nabla \psi : \psi \in W_{loc}^{1,p}(\Omega), \nabla \psi \in [L^p(\Omega)]^n\}. \quad (5) \quad \boxed{\text{defgrad}}$$

In other words, $G^p(\Omega)$ comprises gradients of members of $\dot{W}^{1,p}(\Omega)$. We further put $G_0^p(\Omega)$ to mean the closure of $[C_c^\infty(\Omega)]^n$ in $G^p(\Omega)$:

$$G_0^p(\Omega) := \overline{\{\nabla \psi : \psi \in C^\infty(\Omega), \nabla \psi \in [C_c^\infty(\Omega)]^n\}}^{[L^p(\Omega)]^n}, \quad (6) \quad \boxed{\text{defgrad0}}$$

where $C^\infty(\Omega)$ is the space of C^∞ -smooth functions in Ω . When Ω is bounded at least, $G_0^p(\Omega)$ consists of vector fields in $[L^p(\Omega)]^n$ whose extension by zero

outside Ω is the gradient of some member of $\dot{W}^{1,p}(\mathbb{R}^n)$, see Lemma 8. We also define the space $\tilde{G}^p(\Omega) \subset G^p(\Omega)$ by

$$\tilde{G}^p(\Omega) := \{F \in G^p(\Omega) : \tilde{F} \in G^p(\mathbb{R}^n)\}. \quad (7) \quad \boxed{\text{defextgrad0}}$$

For $1 < p < \infty$, the dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,q}(\Omega)$, with $1/p + 1/q = 1$; it may be realized as the completion of $L^q(\Omega)$ with respect to the norm

$$\|\varphi\|_{W^{-1,q}(\Omega)} := \sup_{\|g\|_{W_0^{1,p}(\Omega)}=1} \left| \int_{\Omega} \varphi(y)g(y) dy \right|, \quad (8) \quad \boxed{\text{defnSne}}$$

see [9, Sec. 3.12]. Note that members of $W^{-1,q}(\Omega)$ are generally not functions but rather distributions on Ω , and that $f \mapsto \nabla f$ is continuous from $L^p(\Omega)$ to $[W^{-1,q}(\Omega)]^n$, by (4). The support of a function or a distribution T will be denoted by $\text{supp } T$.

Since every $f \in W^{1,p}(\Omega)$ *a fortiori* lies in $L^p(\Omega)$, a.e. $x \in \Omega$ is a Lebesgue point of f where $f(x) = \lim_{r \rightarrow 0} m(B(x,r))^{-1} \int_{B(x,r)} f(y)dy$, with m to indicate Lebesgue measure (one says that f is defined in the strict sense at x). But more in fact is true: when $n < p \leq \infty$, the Sobolev embedding theorem implies that f is even continuous, and when $1 < p \leq n$ it is continuous outside a set of arbitrary small $B_{1,p}$ -Bessel capacity, in particular non-Lebesgue points form a set of $B_{1,p}$ -capacity zero; also, every $f \in W^{1,p}(\Omega)$ is finely continuous outside a set of $B_{1,p}$ -capacity zero, see [7, Ch. 2] for a definition of Bessel capacities and [7, Sec. 6.2 & 6.4] for capacity properties of Lebesgue points of Sobolev functions along with other facts from nonlinear potential theory. Sets of zero $(1,p)$ -Bessel capacity are very small, in particular they have Hausdorff $H^{n-p+\varepsilon}$ -measure zero for every $\varepsilon > 0$; see [7, Sec. 5.1].

2.2 Lipschitz open sets

int2

We say that an open set $\Omega \subset \mathbb{R}^n$ is Lipschitz if its boundary $\partial\Omega$ is compact and locally isometric to the graph of a Lipschitz function; see *e.g.* [10]. More precisely, to each $x \in \partial\Omega$ there should exist an open set $U \subset \mathbb{R}^n$ containing x and a rigid transformation T of \mathbb{R}^n such that, for some open set $V \subset \mathbb{R}^{n-1}$ and some Lipschitz map $\Psi : V \rightarrow \mathbb{R}^+$, one has:

$$T(U) \cap \Omega = \{y \in \mathbb{R}^n, \quad (y_1, \dots, y_{n-1})^t \in V, \quad 0 < y_n < \Psi(y_1, \dots, y_{n-1})\}.$$

One can cover $\partial\Omega$ by finitely many (say $N \leq 1$) open sets U_j as above, with corresponding T_j , V_j and Ψ_j for $1 \leq j \leq N$. Now, if we define maps

$\phi_j : U_j \cap \partial\Omega \rightarrow V_j$ as $\phi_j := P_{n-1} \circ T_j$ where P_{n-1} is the projection on the first $(n-1)$ components, we get an atlas on $\partial\Omega$ making it a compact Lipschitz manifold. Note that the parametrizations $\phi_j^{-1} : V_j \rightarrow U_j \cap \partial\Omega \subset \mathbb{R}^n$ are themselves Lipschitz as they are given by $\phi_j^{-1} = T_j^{-1} \circ (I_{n-1} \times \Psi_j)$, where I_{n-1} is the identity map on \mathbb{R}^{n-1} . Since $\partial\Omega$ is locally the image of an open subset of \mathbb{R}^{n-1} by a Lipschitz function (e.g., $\phi_j^{-1} : V_j \rightarrow \mathbb{R}^n$), the volume measure σ coincides with the restriction to $\partial\Omega$ of Hausdorff $(n-1)$ -measure, see [11, Sec. 3.2]. Integration of functions on $\partial\Omega$ is always understood with respect to σ .

For a Lipschitz open set $\Omega \subset \mathbb{R}^n$, we say that $x \in \partial\Omega$ is singular if there is $j \in \{1, \dots, N\}$ such that $x \in U_j$ and $\phi_j^{-1} : V_j \rightarrow \mathbb{R}^n$ is not differentiable at $\phi_j(x)$. A point which is not singular is called regular. We denote the set of regular points by $\text{Reg } \partial\Omega$ and put $\text{Reg } V_j = \phi_j(\text{Reg } \partial\Omega \cap U_j)$. So defined, the set of regular points depends on the atlas, but we shall fix the latter. As ϕ_j^{-1} is Lipschitz, the set $E_j \subset V_j$ where it is not differentiable has Lebesgue measure zero, by Rademacher's theorem [12, Thm 2.2.1]. Hence, $\sigma(\phi_j(E_j)) = 0$ [12, Rem. 1.4.3], implying that singular points have σ -measure zero.

The tangent space of $\partial\Omega$ at x is the subspace $T_x \partial\Omega \subset \mathbb{R}^n$ equal to $\{0\}$ if x is singular and to $\text{Ran } D\phi_j^{-1}(\phi_j(x))$ if $x \in U_j \cap \text{Reg } \partial\Omega$, where $D\phi_j^{-1}$ denotes the total derivative of ϕ_j^{-1} . By the chain rule, the definition does not depend on j such that $x \in U_j$. At $x \in \text{Reg } \partial\Omega$, the outward pointing unit normal to $\partial\Omega$ is well-defined, and we denote it as $\nu(x)$.

The connected components of $\partial\Omega$ are connected compact Lipschitz hypersurfaces in \mathbb{R}^n . For Γ any such hypersurface, $\mathbb{R}^n \setminus \Gamma$ has two connected components: its interior denoted by $\text{int } \Gamma$ which is bounded, and its exterior denoted by $\text{ext } \Gamma$ which is unbounded, see [13, Cor. 3.45].

The connected components of a Lipschitz open set $\Omega \subset \mathbb{R}^n$ are finite in number. Otherwise indeed, there would exist a sequence O_k of such components, $k \in \mathbb{Z}$, with $O_k \cap O_j = \emptyset$ for $k \neq j$. Then, we could construct a sequence $x_k \in O_k$ such that x_k remains at bounded distance from $\partial O_k \subset \partial\Omega$, hence x_k would be bounded and extracting a subsequence if necessary we might assume that x_k converges in \mathbb{R}^n to some y . However, this is impossible for y cannot lie in Ω since the connected components of the latter are open, nor can it lie in $\mathbb{R}^n \setminus \bar{\Omega}$, and it cannot belong to $\partial\Omega$ either because, by definition of a Lipschitz open set, each member of $\partial\Omega$ has a neighborhood whose intersection with Ω is connected. For the same reason, distinct connected components of Ω cannot have a common boundary point, hence they lie at

strictly positive distance from each other.

A Lipschitz domain is a *connected* Lipschitz open set. We record for later use an “obvious” topological lemma:

compcd

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then, $\partial\Omega$ has finitely many connected components, say $\Gamma_1, \dots, \Gamma_l$. Moreover, the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$ consist of l open sets O_1, \dots, O_l , and with a suitable ordering O_1 is the exterior of Γ_1 while O_j is the interior of Γ_j for $j \neq 1$.*

Proof. Since $\partial\Omega$ is a compact Lipschitz manifold which is locally a Lipschitz graph, each $x \in \partial\Omega$ has a neighborhood whose intersection with $\partial\Omega$ is connected. Consequently, by compactness, $\partial\Omega$ has finitely many connected components, say $\Gamma_1, \dots, \Gamma_l$, and each Γ_j is a connected compact Lipschitz hypersurface in \mathbb{R}^n . As Ω is connected by assumption, for each $j \in \{1, \dots, n\}$ either $\Omega \subset \text{int } \Gamma_j$ and then $\text{ext } \Gamma_j \subset \mathbb{R}^n \setminus \overline{\Omega}$, or else $\Omega \subset \text{ext } \Gamma_j$ and then $\text{int } \Gamma_j \subset \mathbb{R}^n \setminus \overline{\Omega}$. Since there is exactly one unbounded connected component of $\mathbb{R}^n \setminus \overline{\Omega}$, say O_1 , it must contain $\text{ext } \Gamma_j$ for all j such that $\Omega \subset \text{int } \Gamma_j$; let us enumerate these j as j_1, \dots, j_m . For $1 \leq i, k \leq m$, it holds that $\text{int } \Gamma_{j_i} \cap \text{int } \Gamma_{j_k} \neq \emptyset$ because Ω lies in this intersection, and since the Γ_j are disjoint one of these interiors is included in the other, say $\text{int } \Gamma_{j_i} \subset \text{int } \Gamma_{j_k}$. But if $j_i \neq j_k$, then $\Gamma_{j_k} \subset \text{ext } \Gamma_{j_i}$ and the latter is contained in O_1 , a contradiction. Consequently, $m = 1$ and Ω lies interior to exactly one of the Γ_j , say Γ_1 . Necessarily then, $O_1 = \text{ext } \Gamma_1$ because O_1 cannot strictly contain $\text{ext } \Gamma_1$ without containing a point of Γ_1 , which is impossible. Likewise, $\Omega \subset \text{ext } \Gamma_j$ for $j \neq 1$ and then $\text{int } \Gamma_j$ is a connected component of $\mathbb{R}^n \setminus \overline{\Omega}$. Finally, the closure of every bounded connected component of $\mathbb{R}^n \setminus \overline{\Omega}$ must meet some Γ_j , and necessarily $j \neq 1$ for each point of Γ_1 has a neighborhood included in $O_1 \cup \Omega$, by the local Lipschitz graph property. Hence, this connected component meets $\text{int } \Gamma_j$ for some $j \neq 1$, therefore it must coincide with $\text{int } \Gamma_j$. \square

2.3 Boundary traces of functions and fields

int3

When $\Omega \subset \mathbb{R}^n$ is a Lipschitz open set, $W^{1,p}(\Omega)$ coincides with the restrictions to Ω of $W^{1,p}(\mathbb{R}^n)$ -functions [14, Ch. VI, Thm 5]. Moreover, when $1 < p < \infty$, each $f \in W^{1,p}(\Omega)$ has a trace on $\partial\Omega$, say ψ , that lies in the fractional Sobolev space $W^{1-1/p,p}(\partial\Omega)$, where for $0 < s < 1$ we let

$$\|\psi\|_{W^{s,p}(\partial\Omega)} := \|\psi\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{n-1+sp}} d\sigma(x) d\sigma(y) \right)^{1/p}. \quad (9) \quad \text{defBeN}$$

This means that the restriction to $\partial\Omega$, initially defined for functions in $C_c^\infty(\mathbb{R}^n)|_\Omega$, extends to a continuous map $W^{1,p}(\Omega) \rightarrow W^{1-1/p,p}(\partial\Omega)$. Membership in $W^{1-1/p,p}(\partial\Omega)$ characterizes traces of Sobolev functions on the boundary of Lipschitz domains [15, Ch. VII, Thm 1], moreover the trace operator is surjective $W^{1,p}(\Omega) \rightarrow W^{1-1/p,p}(\partial\Omega)$ [10, Thm. 1.5.1.3]. In particular, traces on $\partial\Omega$ of $C_c^\infty(\mathbb{R}^n)$ -functions on \mathbb{R}^n are dense in $W^{1-1/p,p}(\partial\Omega)$. Also, $W_0^{1,p}(\Omega)$ coincides with the space of $W^{1,p}(\Omega)$ -functions with zero trace [16, Prop. 3.3].

For $1 < l < \infty$, we let $W^{-1/p,l}(\partial\Omega)$ indicate the fractional Sobolev space of negative order $-1/p$ and exponent l which is the dual space of $W^{1/p,l'}(\partial\Omega)$, $1/l + 1/l' = 1$; adapting the argument in [9, Sec. 3.12] as we did to get (8), $W^{-1/p,l}(\partial\Omega)$ may be realized as the completion of $L^l(\partial\Omega)$ with respect to the norm

$$\|\varphi\|_{W^{-1/p,l}(\partial\Omega)} := \sup_{\|u\|_{W^{1/p,l'}(\partial\Omega)}=1} \int_{\partial\Omega} \varphi u \, d\sigma. \quad (10) \quad \boxed{\text{defnSn}}$$

Members of $W^{-1/p,l}(\partial\Omega)$ are distributions on \mathbb{R}^n , supported on $\partial\Omega$.

Note that we defined the fractional Sobolev space $W^{s,p}(\partial\Omega)$ without resorting to the integral Sobolev space $W^{1,p}(\partial\Omega)$, that may be defined as those $f : \partial\Omega \rightarrow \mathbb{R}$ such that $f \circ \phi_j^{-1} \in W^{1,p}(V_j)$ for all j , where V_j and ϕ_j are as in Section 2.2. In fact $W^{s,p}(\partial\Omega) = [L^p(\partial\Omega), W^{1,p}(\partial\Omega)]_{s,p}$, where $[\cdot, \cdot]_{s,p}$ is the so-called real interpolation functor, see Appendix 7.1 for a brief discussion. Functions $f \in W^{1,p}(\partial\Omega)$ have a well-defined gradient $\nabla_T f \in [L^p(\partial\Omega)]^n$, valued at a.e. $x \in \Omega$ in the tangent space $T_x \partial\Omega$, see (58). Functions $\psi \in W^{1-1/p,p}(\partial\Omega)$ also have a well-defined gradient $\nabla_T \psi$, but the latter is now a $(n-2)$ -current on $\partial\Omega$ lying in the space $\mathcal{W}_1^{-1/p,p}(\partial\Omega)$, see Appendix 7.1 for details.

For Ω a Lipschitz domain and $1 < p < \infty$, consider next the subspace $\text{Div}_p(\Omega)$ of $[L^p(\Omega)]^n$ defined by

$$\text{Div}_p(\Omega) := \{u \in [L^p(\Omega)]^n : \text{div } u \in L^p(\Omega)\};$$

here $\text{div } u = \sum_{j=1}^n \partial_j u$ stands for the Euclidean (distributional) divergence of u , that we also denote sometimes as $\nabla \cdot u$. We endow Div_p with the norm

$$\|u\|_{\text{Div}_p(\Omega)} := (\|u\|_{[L^p(\Omega)]^3}^p + \|\text{div } u\|_{L^p(\Omega)}^p)^{\frac{1}{p}},$$

which makes it a Banach space. Now, from [17, Lem. 1.2.2], the normal component $u \cdot \nu$ of each $u \in \text{Div}_p$ is well-defined as a member of $W^{-1/p,p}(\partial\Omega)$,

and the divergence formula holds, with $1/p + 1/q = 1$:

$$\int_{\Omega} \nabla \varphi \cdot u = - \int_{\Omega} \varphi(y) \operatorname{div} u(y) dy + \int_{\partial \Omega} (u \cdot \nu) \varphi d\sigma, \quad \varphi \in W^{1,q}(\Omega). \quad (11) \quad \boxed{\text{divf}}$$

In this work we are interested in the following subspace of $\operatorname{Div}_p(\Omega)$:

$$\operatorname{Div}_{p,0}(\Omega) = \{u \in [L^p(\Omega)]^n : \operatorname{div} u = 0 \text{ and } u \cdot \nu = 0\}. \quad (12) \quad \boxed{\text{defdiv}}$$

It can be shown that $\operatorname{Div}_{p,0}$ is the closure, with respect to the $\operatorname{Div}_p(\Omega)$ -norm, (which coincides on $\operatorname{Div}_{p,0}$ with the $[L^p(\Omega)]^n$ -norm) of those fields in $[C_c^\infty(\Omega)]^n$ that are divergence-free, see [18, Thm III.2.3]. That is to say:

$$\operatorname{Div}_{p,0}(\Omega) = \overline{\{u \in [C_c^\infty(\Omega)]^n : \operatorname{div} u = 0\}}^{[L^p(\Omega)]^n}. \quad (13) \quad \boxed{\text{defdiv1}}$$

Below, (13) will serve as a general definition of $\operatorname{Div}_{p,0}(\Omega)$, valid even when Ω is not Lipschitz.

For $1 < p < \infty$ and $1/p + 1/q = 1$, we observe that $\operatorname{Div}_{p,0}(\Omega)$ and $G^q(\Omega)$ are mutually orthogonal spaces *via* the pairing (3), for *every* open set $\Omega \subset \mathbb{R}^n$. Indeed, $\operatorname{Div}_{p,0}(\Omega)$ and $G^q(\Omega)$ are certainly orthogonal, since smooth compactly supported divergence-free fields are orthogonal to distributional gradients. Moreover, if $F \in [L^1_{loc}(\Omega)]^n$ is orthogonal to all divergence free fields in $[C_c^\infty(\Omega)]^n$, then it is known that $F = \nabla \Psi$ where $\Psi \in W^{1,1}_{loc}(\Omega)$; see [18, Lem. III.1.1] (the subtlety here is that Ψ is globally defined, even though Ω may not be simply connected). If, moreover, F in $[L^q(\Omega)]^n$, then we get from [19, Thm 6.74] that $\Psi \in W^{1,q}_{loc}(\Omega)$, so that F belong to $G^q(\Omega)$. Thus, $G^q(\Omega) = (\operatorname{Div}_{p,0}(\Omega))^\perp$ and therefore, by the Hahn-Banach theorem, a member of $[L^p(\Omega)]^n$ which does not lie in the closed subspace $\operatorname{Div}_{p,0}(\Omega)$ cannot be orthogonal to $G^q(\Omega)$. Hence, it holds that $\operatorname{Div}_{p,0}(\Omega) = (G^q(\Omega))^\perp$, as announced.

3 The Helmholtz Decomposition

HHD

It was initially proven by Helmholtz that if $\mathbf{u} \in \mathbb{R}^3$ is a smooth vector-field that vanishes sufficiently fast at infinity, then it can be decomposed uniquely into the sum of a gradient and a curl, that is:

$$\mathbf{u} = \nabla \varphi + \nabla \times \mathbf{A}$$

where φ and \mathbf{A} are the so-called scalar and vector potentials respectively, see [20]. This type of decomposition was subsequently extended in any dimension to function spaces that are useful for partial differential equations, see the account in [18, Ch. III]. On a domain $\Omega \subset \mathbb{R}^n$, we say that $M \in [L^p(\Omega)]^n$ has a Helmholtz decomposition if there uniquely exist $\nabla\psi \in G^p(\Omega)$ and $D \in \text{Div}_{p,0}(\Omega)$ such that

$$M = D + \nabla\psi, \quad (14)$$

decomposition

where $G^p(\Omega)$ and $\text{Div}_{p,0}(\Omega)$ were defined in (5) and (13), respectively. We say that a Helmholtz decomposition holds in $[L^p(\Omega)]^n$ if, for each M from the latter, there uniquely exist $\nabla\psi \in G^p(\Omega)$ and $D \in \text{Div}_{p,0}(\Omega)$ such that (14) is valid. Note that when Ω is bounded, a decomposition like (14) is unique as soon as it exists. Because $[L^2(\Omega)]^n$ is a Hilbert space in which $G^2(\Omega)$ is the orthogonal space to $\text{Div}_{2,0}(\Omega)$, a Helmholtz decomposition holds at exponent 2 for any domain Ω . When Ω is Lipschitz, it follows from [21, Thm 11.1] that a Helmholtz decomposition exists in $[L^p(\Omega)]^n$ for $p \in [\frac{3}{2} - \epsilon, 3 + \epsilon]$, where $\epsilon > 0$ depends on Ω , and this range is in the nature of best possible for Lipschitz domains. Moreover, if Ω is either convex or C^1 -smooth, then a Helmholtz decomposition exists for all $p \in (1, \infty)$, see [22, Thm 1.3] and [21, Thm 11.1]. More about domains on which a Helmholtz decomposition holds for $1 < p < \infty$ may be found in [23].

Note that if a Helmholtz decomposition exists in $[L^p(\Omega)]^n$, then

$$\|D\|_{[L^p(\Omega)]^n} + \|\nabla\psi\|_{[L^p(\Omega)]^n} \leq C(\Omega, p) \|M\|_{[L^p(\Omega)]^n}, \quad (15)$$

nfactp

by the open mapping theorem. Hence, the Helmholtz decomposition is L^p -continuous whenever it exists. Observe also that if a Helmholtz decomposition holds in $[L^p(\Omega)]^n$ for some $p \in (1, \infty)$, then it holds in $[L^q(\Omega)]^n$ with $1/p + 1/q = 1$, by duality.

On \mathbb{R}^n , a Helmholtz decomposition exists for $1 < p < \infty$. We review this classical result below, as it is important for our purposes. The standard proof is based on Riesz transforms: for $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, its j -th Riesz transform is defined as

$$R_j(f)(x) = \lim_{\epsilon \rightarrow 0} c_n \int_{|x-y|>\epsilon} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy, \quad j \in \{1, 2, \dots, n\}, \quad (16)$$

defRj

where

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

The pointwise limit in (16) exists for a.e. $x \in \mathbb{R}^n$, moreover $R_j(f)$ is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz class of C^∞ -smooth functions on \mathbb{R}^n such that, for every pair of multi-indices (α, β) , there exists a positive constant $C_{\alpha, \beta} < \infty$ for which

$$\rho_{\alpha, \beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_\beta f(x)| \leq C_{\alpha, \beta}.$$

We shall denote by $f \mapsto \hat{f}$ the Fourier transform from $\mathcal{S}(\mathbb{R}^n)$ into itself:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{2\pi i \xi \cdot y} dy, \quad \xi \in \mathbb{R}^n$$

(we extend the notation $F \mapsto \hat{F}$ to designate the Fourier transform componentwise $[\mathcal{S}(\mathbb{R}^n)]^n \rightarrow [\mathcal{S}(\mathbb{R}^n)]^n$). For $j \in \{1, 2, \dots, n\}$, it follows from see [14, Ch. II, III] that

$$\widehat{R_j(f)}(\xi) = i \frac{\xi_j}{|\xi|} \hat{f}(\xi), \quad f \in \mathcal{S}(\mathbb{R}^n). \quad (17) \quad \boxed{\text{defRjF}}$$

Now, for $p \in (1, \infty)$, every $M \in [L^p(\mathbb{R})]^n$ has a Helmholtz decomposition:

$$M = D + \nabla \psi, \quad (18) \quad \boxed{\text{HDM1}}$$

where $D \in [L^p(\mathbb{R}^n)]^n$ is divergence-free and $\psi \in \dot{W}^{1,p}(\mathbb{R}^n)$, and in fact

$$\nabla \psi = - \left(R_j \left(\sum_{k=1}^n R_k(M_k) \right) \right)^t \quad (19) \quad \boxed{\text{gradpsi}}$$

where M_k indicates the k -th component of M . Indeed, the right-hand side of (19) lies in $L^p(\mathbb{R}^n)$ because of the L^p -boundedness of Riesz transforms. Moreover, when $M \in [L^2(\mathbb{R}^n)]^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$, it follows from (17) and the isometric character of the Fourier transform in $L^2(\mathbb{R}^n)$ that

$$\begin{aligned} & \langle R_j \left(\sum_{k=1}^n R_k(M_k) \right), \partial_l f \rangle - \langle R_l \left(\sum_{k=1}^n R_k(M_k) \right), \partial_j f \rangle \\ &= \langle \frac{\xi_j}{|\xi|^2} \sum_{k=1}^n \xi_k \hat{M}_k, 2i\pi \xi_l \hat{f} \rangle - \langle \frac{\xi_l}{|\xi|^2} \sum_{k=1}^n \xi_k \hat{M}_k, 2i\pi \xi_j \hat{f} \rangle = 0, \end{aligned} \quad (20)$$

whence the first term in (20) is zero for $M \in [L^p(\mathbb{R}^n)]^n$, by density of $L^2(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$. That is, the right hand side of (19) satisfies the (distributional)

Schwarz rule and so it is a distributional gradient, which must be the gradient of some $W_{loc}^{1,p}$ -function ψ [19, Thm 6.74]. Consequently, this right hand side belongs to $G^p(\mathbb{R}^n)$. Finally, one verifies by an argument similar to the one in (20) that $M - \nabla\psi$ is orthogonal to gradients of Schwartz-functions, and therefore is divergence-free, see [24, Sec. 10.6]. Uniqueness of the decomposition comes from the fact that no nonconstant harmonic function on \mathbb{R}^n can have a gradient in $[L^p(\Omega)]^n$.

The mechanism behind formula (19) is made transparent by the following, formal observation. Let ω_n denote the volume of the unit sphere in \mathbb{R}^n , and R_2 indicate the Riesz kernel of order 2:

$$R_2(x) := \frac{1}{(n-2)\omega_n|x|^{n-2}}. \quad (21) \quad \boxed{\text{defR2}}$$

If we put $U(x) = -R_2 * \text{div}M$ for the harmonic potential of $\text{div}M$, with “ $*$ ” to denote convolution, then $\Delta U = \text{div}M$ where $\Delta := \sum_{j=1}^n \partial_j^2$ is the Euclidean Laplacian, whence $D := M - \nabla U$ is divergence-free and so $M = \nabla U + D$ is the Helmholtz decomposition, provided that $\nabla U \in [L^p(\mathbb{R}^n)]^n$. As the Fourier transform of $\text{div}M$ is $-2i\pi \sum_{k=1}^n \xi_k \hat{M}_k(\xi)$ while $\hat{R}_2(\xi) = (2\pi|\xi|)^{-2}$ [14, Ch. V, Lem. 1], one has $\hat{U}(\xi) = i(2\pi)^{-1}|\xi|^{-2} \sum_{k=1}^n \xi_k \hat{M}_k(\xi)$ and therefore ∇U has Fourier transform $(\xi_1, \dots, \xi_n)^t |\xi|^{-2} \sum_{k=1}^n \xi_k \hat{M}_k(\xi)$, which is equivalent to (19) in view of (17). What precedes suggests that $\psi = -R_2 * \text{div}M$ is a natural candidate in (18), and the lemma below gives a rigorous argument to this effect when $M \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for some $q \in (1, n)$. However, we trade $R_2 * \text{div}M$ for $\nabla R_2 * M$ (a formal integration by parts), as it will serve our purposes.

psiHD **Lemma 2.** *For $M \in [L^p(\mathbb{R}^n)]^n \cap [L^q(\mathbb{R}^n)]^n$ with $1 < p < \infty$ and $1 < q < n$, let us define*

$$\Psi_M(x) := \frac{1}{\omega_n} \int_{\mathbb{R}^n} M(y) \cdot \frac{(x-y)}{|x-y|^n} dy, \quad x \in \mathbb{R}^n. \quad (22) \quad \boxed{\text{defMP}}$$

Then:

- (i) *the integral (22) converges absolutely for a.e. x and $M \mapsto \Psi_M$ is continuous from $[L^q(\mathbb{R}^n)]^n$ into $L^{nq/(n-q)}(\mathbb{R}^n)$;*
- (ii) *$\nabla \Psi_M \in L^p(\mathbb{R}^n)$, and decomposition (18) holds with $\psi = \Psi_M$.*

Proof. Since $|x/|x|^n| \leq |x|^{1-n}$, assertion (i) follows from properties of Riesz potentials, see [14, Ch. V, Thm. 1]. We claim that it is enough to prove (ii) when $M \in [\mathcal{S}(\mathbb{R}^n)]^n$. Indeed, if $M \in [L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)]^n$ then there is a sequence $F_n \in [\mathcal{S}(\mathbb{R}^n)]^n$ converging to M both in $[L^p(\mathbb{R}^n)]^n$ and in $[L^q(\mathbb{R}^n)]^n$, by mollification. Hence, if $M = D + \nabla\psi$ and $F_n = D_n + \nabla P_{F_n}$ are the Helmholtz decompositions of M and F_n respectively, we know from (15) that $\lim_n \nabla P_{F_n} = \nabla\psi$ in $[L^p(\mathbb{R}^n)]^n$ and from (i) that $\lim_n P_{F_n} = \Psi_M$ in $L^{nq/(n-q)}(\mathbb{R}^n)$. Because the latter limit implies that ∇P_{F_n} converges to $\nabla\Psi_M$ as a distribution, and since L^p -convergence implies convergence in the distributional sense, we conclude that $\nabla\Psi_M = \nabla\psi$, thereby proving the claim.

We now show that (ii) holds when $M \in [\mathcal{S}(\mathbb{R}^n)]^n$. In this case, the integral (22) converges absolutely for every x , and using Fubini's theorem and integration by parts one checks that $\Psi_M = -R_2 * \operatorname{div}M$, with R_2 as in (21). Hence, $\Delta\Psi_M = \operatorname{div}M$ and, by the discussion before the lemma, it remains to prove that $\nabla\Psi_M \in [L^p(\mathbb{R}^n)]^n$. Note that if $M \in [\mathcal{S}(\mathbb{R}^n)]^n$, then

$$\hat{\Psi}_M(\xi) = \frac{i}{2\pi|\xi|^2} \sum_{k=1}^n \xi_k \hat{M}_k(\xi) \quad (23) \quad \boxed{\text{FTU}}$$

as a tempered distribution, by [14, Ch. V, Lem. 1]. Let $\mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ consist of functions whose Fourier transform vanishes at 0; *i.e.*, functions in $\mathcal{S}(\mathbb{R}^n)$ with zero mean on \mathbb{R}^n . Let further $\Sigma(\mathbb{R}^n) \subset \mathcal{S}_0(\mathbb{R}^n)$ comprise those f such that \hat{f} vanishes in a neighborhood of the origin. For $M \in [\Sigma(\mathbb{R}^n)]^n$, it is clear from (23) that $\hat{\Psi}_M \in \mathcal{S}(\mathbb{R}^n)$, hence also $\Psi_M \in \mathcal{S}(\mathbb{R}^n)$ and *a fortiori* $\nabla\Psi_M \in [L^p(\mathbb{R}^n)]^n$, as desired. The conclusion in fact extends to every $M \in [\mathcal{S}_0(\mathbb{R}^n)]^n$, because $\Sigma(\mathbb{R}^n)$ is $L^p \cap L^q$ -dense in $\mathcal{S}_0(\mathbb{R}^n)$ (see Lemma 7) and we may resort to a limiting argument resembling the one we used to reduce the proof to the case where $M \in [\mathcal{S}(\mathbb{R}^n)]^n$. Thus, in order to prove that $\nabla\Psi_M \in [L^p(\mathbb{R}^n)]^n$ for all $M \in \mathcal{S}^n$, it is enough to show this is true when $M = hv$ for any $v \in \mathbb{R}^n$ and some particular $h \in \mathcal{S} \setminus \mathcal{S}_0$, because the space of such functions complements $[\mathcal{S}_0]^n$ in $[\mathcal{S}]^n$. Since the function $R_2 * h$ is locally bounded and for $k \in \{1, \dots, n\}$ the function $x \mapsto R_2(x - y)$ is absolutely continuous on each line $L_k := \{x : x_j = c_j, j \neq k\}$ except when $c_j = y_j$ for all $j \neq k$, we get on differentiating under the integral sign that $P_{hv} = -\sum_{k=1}^n v_k \partial_k (R_2 * h)$. So, we are left to check there exists $h \in \mathcal{S} \setminus \mathcal{S}_0$ such that $R_2 * h$ has all its second derivatives in $L^p(\mathbb{R}^n)$. Lemma 8 provides us with such a h , thereby concluding the proof.

□

As alluded to earlier, given a bounded open domain $\Omega \subset \mathbb{R}^n$, a Helmholtz decomposition does not always exist in $[L^p(\Omega)]^n$ for $1 < p < \infty$, even if Ω is Lipschitz. In Section 6, we will prove there is a related, three-term decomposition that exists in $[L^p(\Omega)]$ for all $1 < p < \infty$ as soon as Ω is Lipschitz. We state the result here.

OurDecompTheo

Theorem 1. *Let Ω be a bounded Lipschitz open set and $M \in [L^p(\Omega)]^n$ a nonzero vector field with $1 < p < \infty$. Then, there exists uniquely $\nabla\psi \in G_0^p(\Omega)$ and $D \in \text{Div}_{p,0}(\Omega)$, together with a harmonic function h in Ω meeting $\int_{\Gamma_i} (\partial\phi/\partial n) d\mathcal{H}^{n-1} = 0$ for each connected component Γ_i of $\partial\Omega$, such that*

$$M = \nabla\psi + D + |\nabla h|^{q-2} \nabla h. \quad (24)$$

OurDecomp

4 Double and Single layer potentials

PotentialsSection

In this section we assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with connected boundary. Hence, $\mathbb{R}^n \setminus \partial\Omega$ has two connected components, namely $\Omega = \text{int } \partial\Omega$ and $\Omega^- := \text{ext } \partial\Omega$.

The double and single layer potentials of a function ψ defined on $\partial\Omega$, whose smoothness will be made precise later on, are defined by

$$\mathcal{K}\psi(x) = -\frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega, \quad (25)$$

defDL

and

$$\mathcal{S}\psi(x) = -\frac{1}{(n-2)\omega_n} \int_{\partial\Omega} \psi(y) \frac{1}{|x-y|^{n-2}} d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega, \quad (26)$$

respectively. We call ψ the density of the double or single layer potential, moreover $\mathcal{K}\psi$ and $\mathcal{S}\psi$ are harmonic in $\mathbb{R}^n \setminus \partial\Omega$. For $x \in \partial\Omega$, the double layer potential is defined as the singular integral

$$\begin{aligned} K\psi(x) &= p.v. - \frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) d\sigma(y) \\ &= \lim_{\epsilon \rightarrow 0} -\frac{1}{\omega_n} \int_{|x-y| > \epsilon} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) d\sigma(y). \end{aligned} \quad (27)$$

limDLP

As soon as $\psi \in L^p(\partial\Omega)$ for some $p \in (1, \infty)$, it is well-known that $K\psi$ exists a.e. in $\partial\Omega$ and belongs to $L^p(\partial\Omega)$. Furthermore, for almost every $y \in \partial\Omega$, $\mathcal{K}\psi(x)$ converges to $(\frac{1}{2}Id + K)\psi(y)$ (resp. to $(-\frac{1}{2}Id + K)\psi(y)$), as $x \rightarrow y$ non-tangentially in Ω (resp. Ω^-), see [25, Thm 1.10] for a precise statement and references. It follows that the non-tangential limits on $\partial\Omega$ of the double layer potential from inside and outside, differ by the density ψ of the potential. For appropriate range of exponents, the double layer potential on Ω is a famous tool to solve the Dirichlet problem for the Laplace equation, which is to find $w : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \Delta w &= 0 \text{ in } \Omega, \\ w &= g \text{ on } \partial\Omega. \end{aligned} \tag{28} \quad \boxed{\text{DirichletP}}$$

In fact, $(\frac{1}{2}Id + K) : W^{1/q,p}(\partial\Omega) \rightarrow W^{1/q,p}(\partial\Omega)$ is invertible for $p \in [\frac{3}{2}, 3]$ and $1/p + 1/q = 1$, see [21, Thm 8.1]. Hence, the solution to (28) when $g \in W^{1/q,p}(\partial\Omega)$ with $p \in [\frac{3}{2}, 3]$ is given by $w = \mathcal{K}(\frac{1}{2}Id + K)^{-1}g$ and belongs to $W^{1,p}(\Omega)$. Here, the boundary condition in (28) is satisfied both as a Sobolev trace and as a non-tangential limit a.e. Likewise, the double layer potential on Ω^- is a tool to solve the exterior Dirichlet problem, which is to find $w : \Omega^- \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \Delta w &= 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega} \cup \{\infty\}, \\ w &= g \text{ on } \partial\Omega; \end{aligned} \tag{29} \quad \boxed{\text{eq:DPout}}$$

here, as $n \geq 3$, harmonicity at infinity means that $\lim_{|x| \rightarrow \infty} w(x) = 0$ [26, Thm 4.8]. In fact, $(-\frac{1}{2}Id + K) : W^{1/q,p}(\partial\Omega)/\langle 1 \rangle \rightarrow W^{1/q,p}(\partial\Omega)/\langle 1 \rangle$ is invertible for $p \in [\frac{3}{2}, 3]$, where the quotient by $\langle 1 \rangle$ means “modulo constants”, see [21, Thm 8.1]. Hence, $w = \mathcal{K}(-\frac{1}{2}Id + K)^{-1}g$ will solve the exterior Dirichlet problem up to a constant when $g \in W^{1/q,p}(\partial\Omega)$ and $p \in [\frac{3}{2}, 3]$, with $w|_{\Omega^- \cap B(0,R)} \in W^{1,p}(\Omega^- \cap B(0,R))$ for all $R > 0$ and $\nabla w \in [L^p(\Omega^-)]^n$. To account for constant boundary conditions, observe that the exterior Dirichlet problem with constant data on $\partial\Omega$ can be solved using the single layer potential of the Newtonian equilibrium measure of Ω [27, Ch. IV, Sec. 5, §20] (the latter has L^2 density with respect to σ after [28, Cor. to Thm 3], so that its single layer potential has gradient in $[L^p(\Omega^-)]^n$). Altogether, the exterior Dirichlet problem with data $g \in W^{1/q,p}(\partial\Omega)$ with $p \in [\frac{3}{2}, 3]$ can be solved using a combination of double and single layer potentials.

Unlike the double layer potential, the single layer potential is continuous across $\partial\Omega$, though its normal derivative is not. Note that the gradient of the

single layer potential is given by

$$\nabla \mathcal{S}\psi(x) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{(x-y)}{|x-y|^n} \psi(y) d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega. \quad (30)$$

For $1 < p < \infty$ and $\psi \in L^p(\partial\Omega)$, it holds for a.e. $y \in \partial\Omega$ that $\nu(y) \cdot \nabla \mathcal{S}\psi(x)$ converges to $-(\frac{1}{2}Id - K^*)\psi(y)$ (resp. $(\frac{1}{2}Id + K^*)\psi(y)$) as $x \rightarrow y$ non-tangentially in Ω (resp. Ω^-), where K^* operates on $L^q(\partial\Omega)$ for $1 < q < \infty$ and is the adjoint of K :

$$K^*\psi(x) = p.v. \frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(x) d\sigma(y), \quad (31) \quad \boxed{\text{defadK}}$$

see [25, Thm 1.11] for a statement and further references.

For appropriate range of exponents, the single layer potential on Ω allows one to solve the Neumann problem for the Laplace equation:

$$\begin{aligned} \Delta w &= 0 \text{ in } \Omega \\ \nabla w \cdot \nu &= g \text{ on } \partial\Omega, \end{aligned} \quad (32) \quad \boxed{\text{No}}$$

where the boundary condition in (32) is meant to satisfy the divergence formula; *i.e.* (11) holds when u gets replaced by ∇w and $(u \cdot \nu)$ by g . Likewise, the single layer potential on Ω^- can be used to solve the exterior Neumann problem:

$$\begin{aligned} \Delta w &= 0 \text{ in } \mathbb{R}^n \setminus \bar{\Omega} \cup \{\infty\}, \\ \nabla w \cdot \nu &= g \text{ on } \partial\Omega. \end{aligned} \quad (33) \quad \boxed{\text{N}}$$

More precisely, it follows from [21, Thm 8.1] that $(\pm \frac{1}{2}Id + K^*)$ extends to an invertible map $\tilde{W}^{-1/p,p}(\partial\Omega) \rightarrow \tilde{W}^{-1/p,p}(\partial\Omega)$ for $p \in [\frac{3}{2}, 3]$, where we have set

$$\tilde{W}^{-\frac{1}{p},p}(\partial\Omega) := \{f \in W^{-\frac{1}{p},p}(\partial\Omega) : \int_{\partial\Omega} f(y) d\sigma(y) = 0\},$$

Thus, by [21, Thm 9.2], the solution to (32) (resp. (33)) can be written as $w = \mathcal{S}(\mp \frac{1}{2}Id + K^*)^{-1}g$, up to an additive constant, under the (necessary) condition that $\int_{\partial\Omega} g(y) d\sigma(y) = 0$. Moreover, w belongs to $W^{1,p}(\Omega)$ (resp. $w|_{\Omega^- \cap B(0,R)} \in W^{1,p}(\Omega^- \cap B(0,R))$) for all $R > 0$ and $\nabla w \in [L^p(\Omega^-)]^n$.

5 Silent magnetic sources

SSC

Let us represent a magnetization carried by a bounded domain $\Omega \subset \mathbb{R}^3$ as a vector-field $M \in [L^p(\Omega)]^3$, with $1 < p < \infty$. Under the quasi-static assumption on Maxwell's equations, it is known (see [1]) that the scalar magnetic potential P_M generated by M is related to the latter by

$$\Delta P_M = \nabla \cdot M \tag{34} \quad \text{Poisson}$$

where P_M is zero at infinity. It follows from (34) and the vanishing of P_M at infinity (recall Ω is bounded by assumption) that

$$P_M(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\nabla \cdot M(y)}{|x-y|} dy \tag{35}$$

that can be rewritten, since Ω is bounded, as

$$P_M(x) = \frac{1}{4\pi} \int_{\Omega} M(y) \cdot \frac{(x-y)}{|x-y|^3} dy. \tag{36} \quad \text{expopot}$$

A magnetization is said to be Ω -silent, or a silent source in $\mathbb{R}^3 \setminus \Omega$, if it generates the zero field there. Although the main Physical interest attaches to dimension 3, nothing gets simpler if we restrict to this setting, and perspective is gained if we extend the terminology to any dimension strictly greater than 2 by making the following, formal definition:

dfn:sc

Definition 1. For $\Omega \subset \mathbb{R}^n$ a bounded open set, we say that $M \in [L^p(\Omega)]^n$ is Ω -silent if

$$\nabla P_M(x) = \nabla_x \left(\int_{\Omega} M(y) \cdot \frac{(x-y)}{|x-y|^n} dy \right) = 0, \quad a.e. x \in \mathbb{R}^n \setminus \Omega. \tag{37} \quad \text{expopotn}$$

Note that P_M is harmonic in $\mathbb{R}^n \setminus \bar{\Omega}$. In particular, if $\partial\Omega$ has Lebesgue measure 0 then Definition 1 says that M is Ω -silent if and only if P_M is locally constant in $\mathbb{R}^n \setminus \bar{\Omega}$.

For $\Omega \subset \mathbb{R}^n$ a bounded open set and V a vector field in $[L^p(\Omega)]^m$, recall the notation \tilde{V} for its extension by zero to the whole of \mathbb{R}^n . Note that P_M defined in (36) coincides with $\Psi_{\tilde{M}}$ given by (22). As a consequence, Lemma 2 provides us at once with a criterion for M to be Ω -silent:

thms

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. A field $M \in [L^p(\Omega)]^n$ is Ω -silent if and only if the Helmholtz decomposition of \widetilde{M} on \mathbb{R}^n , say $\widetilde{M} = \nabla\psi + D$, is such that both $\nabla\psi$ and D are zero a.e. on $\mathbb{R}^n \setminus \Omega$.*

Proof. Since Ω is bounded, \widetilde{M} lies in $[L^q(\mathbb{R}^n)]^n$ for $1 < q \leq p$. Hence, in the Helmholtz decomposition $\widetilde{M} = \nabla\psi + D$, we may choose $\psi = \nabla\Psi_{\widetilde{M}}$, by Lemma 2. Thus, as $P_M = \Psi_{\widetilde{M}}$, the gradient term in the Helmholtz decomposition of \widetilde{M} is ∇P_M , and therefore \widetilde{M} is silent if and only if this gradient term is zero a.e. on $\mathbb{R}^n \setminus \Omega$, by (37). Because \widetilde{M} is zero on $\mathbb{R}^n \setminus \Omega$, this happens if and only if D is zero a.e. on $\mathbb{R}^n \setminus \Omega$, as desired. \square

Note that Theorem 2 and Lemma 2 together imply that if M is Ω -silent it is the sum of a silent gradient and a silent divergence free fields.

When Ω is a union of positively separated pieces, the question whether M is Ω -silent reduces to the corresponding question on each piece. This we record as a complement to Theorem 2, see *e.g.* [29] and [30, Cor. 3.5] for related, somewhat less general results.

ccr

Lemma 3. *Let $M \in [L^p(\Omega)]^n$ for some $p \in (1, \infty)$ with $\Omega = \cup_{j=1}^l \Omega_j$, where $\Omega_j \subset \mathbb{R}^n$ is a bounded open set whose boundary has zero measure and $d(\Omega_j, \Omega_k) > 0$ for $j \neq k$. Then, M is Ω -silent if and only if the restriction $M|_{\Omega_j}$ is Ω_j -silent for each $j \in \{1, \dots, l\}$.*

Proof. Let us put $O_1 := \cup_{j \neq 1} \Omega_j$. For each $i \in \{1, \dots, n\}$, we have that $\partial_{x_i} P_{M|_{\Omega_1}}$ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega_1}$ and $\partial_{x_i} P_{M|_{O_1}}$ is harmonic in $\mathbb{R}^n \setminus \overline{O_1}$, moreover $\partial_{x_i} P_{M|_{\Omega_1}} + \partial_{x_i} P_{M|_{O_1}} = 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$. Because $\overline{\Omega_1}$ and $\overline{O_1}$ are positively separated, it entails that $\partial_{x_i} P_{M|_{\Omega_1}}$ extends harmonically to the whole of \mathbb{R}^n , and since it vanishes at infinity by inspection of (37) it must vanish identically, thanks to Liouville's theorem. Therefore $P_{M|_{\Omega_1}}$ is locally constant on $\mathbb{R}^n \setminus \overline{\Omega_1}$, as wanted. \square

In the case where Ω is Lipschitz, Theorem 2 makes contact with the Helmholtz decomposition in $[L^p(\Omega)]^n$. We begin by making some comments and introducing some notation. If Ω is a bounded Lipschitz domain of arbitrary topology with boundary $\partial\Omega$ and we let

$$\mathbb{R}_{\partial\Omega} = \text{span}\{\chi_\omega : \omega \text{ is a connected component of } \partial\Omega\},$$

we shall denote by $W_{\mathbb{R}_{\partial\Omega}}^{1,p}(\Omega)$ the space of all members of $W^{1,p}(\Omega)$ that have trace in $\mathbb{R}_{\partial\Omega}$; that is: a member of $W_{\mathbb{R}_{\partial\Omega}}^{1,p}(\Omega)$ has constant trace on each connected component of $\partial\Omega$ (the constant may depend on the component).

indzHH

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded lipschitz open set, and $M \in [L^p(\Omega)]^n$ with $p \in (1, \infty)$. Then, M is Ω -silent if and only if $M = D_0 + \nabla\psi_0$, where $D_0 \in \text{Div}_{p,0}(\Omega)$ and $\psi_0 \in W_{\mathbb{R}\partial\Omega}^{1,p}(\Omega)$.*

Proof. Let $\Omega^{(1)} \dots, \Omega^{(L)}$ be the connected components of Ω . We adapt the notation of Lemma 1 by writing $\Gamma_1^{(i)}, \dots, \Gamma_{l_i}^{(i)}$ for the connected components of $\partial\Omega^{(i)}$ and $O_1^{(i)}, \dots, O_{l_i}^{(i)}$ for the components of $\mathbb{R}^n \setminus \overline{\Omega^{(i)}}$, with $O_1^{(i)} = \text{ext } \Gamma_1^{(i)}$ and $O_j^{(i)} = \text{int } \Gamma_j^{(i)}$ for $j \neq 1$. Assume first that $M = D_0 + \nabla\psi_0$, with D_0 and ψ_0 as in the statement. Because D_0 has zero normal component on $\partial\Omega$ by assumption, one can see from (12) that \tilde{D}_0 is divergence free on \mathbb{R}^n , as a distribution. Besides, as we suppose that ψ_0 is locally constant on $\partial\Omega$, to each $\Gamma_j^{(i)}$ there is a constant $c_{i,j}$ such that $(\psi_0)|_{\Gamma_j^{(i)}} = c_{i,j}$ for $i \in \{1, \dots, N\}$ and $1 \leq j \leq l_i$. Replacing ψ_0 by $\psi_0 - c_{i,1}$ on $\Omega^{(i)}$, we may assume that $c_{i,1} = 0$ for all i and then one sees from Lemma 1 that the function ψ_1 , equal to ψ_0 on Ω and to $c_{i,j}$ on $O_j^{(i)}$, lies in $W^{1,p}(\mathbb{R}^n)$ with $\nabla\psi_1 = \nabla\psi_0$ on Ω and $\nabla\psi_1 = 0$ outside Ω . Thus, $\tilde{M} = \tilde{D}_0 + \nabla\psi_1$ is the Helmholtz decomposition of \tilde{M} on \mathbb{R}^n , and both terms of the decomposition vanish outside Ω , as desired.

Conversely, assume that in the Helmholtz decomposition $\tilde{M} = D + \nabla\psi$ on $[L^p(\mathbb{R}^n)]^n$, the summands D and $\nabla\psi$ vanish a.e. outside Ω . Then, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have that $\int_\Omega D \cdot \nabla\varphi = \int_{\mathbb{R}^n} D \cdot \nabla\varphi = 0$ and therefore $\int_{\partial\Omega} (D \cdot \nu) \varphi d\sigma = 0$, by (11). Since traces of C_c^∞ -functions are dense in $W^{1-1/p,p}(\partial\Omega)$, we conclude that $D \cdot \nu = 0$ in $W^{-1/p,p}(\partial\Omega)$ so that the restriction $D|_\Omega$ lies in $\text{Div}_{p,0}(\Omega)$. Moreover, as $\nabla\psi = 0$ a.e. on $\mathbb{R}^n \setminus \overline{\Omega}$, the function ψ is constant on each $O_j^{(i)}$ and so it has constant trace on each $\Gamma_j^{(i)}$. Hence, we can put $D_0 = D|_\Omega$ and $\psi_0 = \psi|_\Omega$. \square

More generally, we have the following result.

SilentSources

Theorem 4. *Let $M \in [L^p(\Omega)]^n$ for some $p \in (1, \infty)$, with $\Omega \subset \mathbb{R}^n$ a bounded finitely connected open set such that $\partial\Omega$ has zero measure and the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$ are positively separated. Then, M is Ω -silent if and only if $M = D + \nabla\psi$ where \tilde{D} is divergence-free and $\psi \in W^{1,p}(\Omega)$ is such that $\nabla\psi \in G_0^p(\Omega)$.*

Proof. This follows at once from Lemma 3 together with Theorems 2 and 8. \square

Theorem 4 raises the issue as to whether, under very mild conditions on Ω like those in that theorem, the divergence free character of \tilde{D} implies that

$D \in \text{Div}_{p,0}(\Omega)$; a positive answer would yield a generalization of Theorem 3 to very rough domains.

6 Equivalent Sources

EqS

In this section, we study Ω -equivalent sources; *i.e.*, magnetisations producing the same field outside Ω . That is: $M_1, M_2 \in [L^p(\Omega)]^n$ are Ω -equivalent if and only if $M_1 - M_2$ is an Ω -silent source. The following result, dealing with the case where Ω is Lipschitz, is a straightforward consequence of Theorem 3.

Equiva

Corollary 1. *Let Ω be bounded Lipschitz and $p \in [\frac{3}{2}, 3]$. For $M_1, M_2 \in [L^p(\Omega)]^n$, write the Helmholtz decompositions as $M_1 = D_1 + \nabla\psi_1$ and $M_2 = D_2 + \nabla\psi_2$, with $D_1, D_2 \in \text{Div}_{p,0}(\Omega)$ and $\psi_1, \psi_2 \in W^{1,p}(\Omega)$. Then, M_1 and M_2 are Ω -equivalent if and only if the traces of ψ_1 and ψ_2 differ by a member of $\mathbb{R}_{\partial\Omega}$; that is, if they differ by constants on each connected component of $\partial\Omega$.*

We stressed earlier that recovering a magnetisation $M \in [L^p(\Omega)]^n$ from the knowledge of the potential or its field outside Ω is not unique because of the existence of Ω -equivalent vector-fields. We are now interested in norm-minimising Ω -equivalent sources. Namely, given $M \in [L^p(\Omega)]^n$, we want to find a silent source M_S that satisfies the following

$$M_S = \arg \min_{M_0 \in S_p} \|M - M_0\|_{[L^p(\Omega)]^n}, \quad (38)$$

SetupforDual

where S_p is the space of Ω -silent sources in $[L^p(\Omega)]^n$. It then follows that $M - M_S$ is the norm-minimising source Ω -equivalent to M . That M_S exists follows at once from the next lemma; uniqueness is ensured by strict convexity of the L^p -norm.

K_pclosed

Lemma 4. *S_p is closed in $[L^p(\Omega)]^n$ for $p \in (1, \infty)$.*

Proof. It is obvious from Definition 1 that S_p is a weakly closed subspace of $[L^p(\mathbb{R}^n)]^n$, hence it is closed by Mazur's theorem. \square

When $p = 2$, the Helmholtz decomposition is orthogonal [17, Lem. 2.5.1] and therefore, given $M \in [L^2(\Omega)]^n$ with Helmholtz decomposition $M = D + \nabla\psi$, the magnetisation of minimum-norm Ω -equivalent to M is the same as the one Ω -equivalent to $\nabla\psi$; it is so because a $\text{Div}_{p,0}(\Omega)$ -field is a Ω -silent source. Moreover, by the same reason, the magnetisation of minimum-norm

Ω -equivalent to $\nabla\psi$ is the gradient of a Sobolev function and therefore, the norm-minimising vector field Ω -equivalent to M is $\nabla\psi - \nabla u$, where ∇u is the projection of $\nabla\psi$ on $\tilde{G}^2(\Omega)$, defined in (7). The latter is not so easy to characterise for general open sets: in this connection, we shall find it convenient to make the following definition.

defunstretched

Definition 2. For $p \in (1, \infty)$, we say that the complement of a bounded open set $\Omega \subset \mathbb{R}^n$ is p -unstretched if $\partial\Omega$ has zero Lebesgue measure and if, letting $\{O_j\}_{j \in J}$ denote the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$, the following two properties hold:

- (i) O_j is non-thin at $B_{1,p}$ -quasi every point of ∂O_j for all j ;
- (ii) whenever $\bar{O}_j \cap \overline{\cup_{i \in I} O_i} \neq \emptyset$ for some index $j \in J$ and some subset of indices $I \subset J$ then $B_{1,p}(\bar{O}_j \cap \overline{\cup_{i \in I} O_i}) > 0$.

Note that when $p > n$, conditions (i) and (ii) above are always met and so $\mathbb{R}^n \setminus \Omega$ is unstretched if and only if $\partial\Omega$ has Lebesgue measure zero. Observe also that $\mathbb{R}^n \setminus \Omega$ is unstretched as soon as $\mathbb{R}^n \setminus \bar{\Omega}$ has finitely many connected components satisfying the segment condition. In particular, this is the case when Ω is Lipschitz.

When $\mathbb{R}^n \setminus \Omega$ is unstretched, we can characterise $[L^2(\Omega)]^n$ -norm minimising Ω -equivalent sources as solutions to a boundary-value problem:

NM-2g

Theorem 5. Let Ω be a bounded open set and $M \in [L^2(\Omega)]^n$. If we write $M = D + \nabla\psi \in [L^2(\Omega)]^n$ for the Helmholtz decomposition, then the unique norm-minimising vector field Ω -equivalent to M is $\nabla\psi - \nabla u$, where ∇u is the projection of $\nabla\psi$ on $\tilde{G}^2(\Omega)$. Moreover, $\psi_{nm} := \psi - u$ is a $W^{1,2}(\Omega)$ -solution to the following boundary value problem:

$$\begin{aligned} \Delta\psi_{nm} &= 0 \text{ in } \Omega, \\ \nabla\psi_{nm} - \nabla\psi &\in \tilde{G}^2(\Omega), \\ \int_{\Gamma} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} &= 0 \text{ for all compact } C^\infty\text{-hypersurfaces } \Gamma \subset \Omega, \end{aligned} \tag{39}$$

DirichletProblem

where \mathcal{H}^{n-1} indicates $(n-1)$ -Hausdorff measure. Conversely, if $\mathbb{R}^n \setminus \Omega$ is unstretched, then ψ_{nm} is the unique solution to (39), up to an additive constant.

Proof. As mentioned before the theorem, the norm-minimising vector field Ω -equivalent to M is $\nabla\psi - \nabla u$, where ∇u is the projection of $\nabla\psi$ on $\tilde{G}^2(\Omega)$. Putting $\psi_{nm} := \psi - u$, we get that $\nabla\psi_{nm}$ is orthogonal to $\tilde{G}^2(\Omega)$ and in particular to $\nabla\varphi$ for all $\varphi \in C_c^\infty(\Omega)$. It implies that ψ_{nm} is harmonic. Moreover, if $\Gamma \subset \Omega$ is a compact C^∞ -hypersurface, we can find another smooth compact hypersurface $\Gamma_1 \subset \Omega$ with $\Gamma_1 \cap \Gamma = \emptyset$, close enough to Γ that the open “shell” S between Γ and Γ_1 is contained in Ω (using for instance a tubular neighborhood of Γ , see [31, Ch. 2, Sec. 3]). Assume for definiteness that S lies inside Γ (if it lies outside the argument is similar). We can find a function $g \in C^\infty(\mathbb{R}^n)$ which is equal to 1 on $\text{ext } \Gamma$ and to zero on a neighborhood of $\Gamma_1 \cup \text{int } \Gamma_1$. Then, $\nabla g \in G_0^2(\Omega) \subset \tilde{G}_2$ so that $\int \nabla\psi_{nm} \cdot \nabla g dm = 0$, and by the divergence formula the latter is equal to $\int_\Gamma \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1}$. Hence, ψ_{nm} satisfies (39). Conversely, assume that (39) holds and pick $\nabla g \in G_0^2(\Omega)$. Let further $K \subset \Omega$ be the support of $|\nabla g|$. Since K is compact, there is $h \in C^\infty(\mathbb{R}^n)$ such that $h \geq 0$ and K is the zero set of h ; this follows easily from a combination of [14, Chapter VI, Theorem 2] and [32, Theorem I]. Replacing h with h^2 we may assume that $h \geq 0$, and redefining $h(x)$ for $|x|$ large we can arrange things so that $h \rightarrow 1$ at infinity. Putting $L := h^{-1}(\varepsilon)$ for ε a sufficiently small regular value of h (almost every positive number is a regular value by Sard’s theorem), we find that L is a finite union of smooth compact hypersurfaces $\Gamma_1, \dots, \Gamma_N$ included in Ω , and we can find $\Gamma_{i_1}, \dots, \Gamma_{i_k}$, each of which lies exterior to the others, with $K \subset \cup_\ell \text{int } \Gamma_{i_\ell}$; moreover, if for some i_ℓ it holds that $\text{int } \Gamma_{i_\ell} \not\subset \Omega$, there are $\Gamma_{j_1}, \dots, \Gamma_{j_m}$, each of which lies interior to $\text{int } \Gamma_{i_\ell}$, such that $K \subset \cap_k \text{ext } \Gamma_{j_k}$ (recall that $h > \eta > 0$ on $\mathbb{R}^n \setminus \Omega$). By construction g is equal to a constant c_j on Γ_j , and therefore $\int_\Omega \nabla\psi_{nm} \cdot \nabla g dm = \sum_j c_j \int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} = 0$, by (39). Thus, $\nabla\psi_{nm}$ is orthogonal to $G_0^2(\Omega)$, and by Theorem 8 the latter coincides with $\tilde{G}^2(\Omega)$ when $\mathbb{R}^n \setminus \Omega$ is unstretched. We now see from the second equation in (39) that $\nabla\psi_{nm} = \nabla\psi - \nabla u$ where ∇u is the projection of $\nabla\psi$ on $\tilde{G}^2(\Omega)$, as desired. \square

When Ω is Lipschitz, Theorem 5 yields a fairly explicit characterization of norm-minimising Ω -equivalent sources in $[L^2(\Omega)]^n$. By Lemma 3, it is enough to consider the case where Ω is connected:

NM-2 **Theorem 6.** *Let Ω be a bounded Lipschitz domain and $\Gamma_1, \dots, \Gamma_l$ the connected components of $\partial\Omega$. Let further $M \in [L^2(\Omega)]^n$ and write the Helmholtz decomposition as $M = D + \nabla\psi \in [L^2(\Omega)]^n$. Then, the norm-minimising*

vector field Ω -equivalent to M is $\nabla\psi_{nm}$, where $\psi_{nm} = u + \sum_{j=1}^l c_j\omega(\Gamma_j)$ with u the $W^{1,2}(\Omega)$ solution to the Dirichlet problem:

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= \psi \text{ on } \partial\Omega, \end{aligned} \tag{40} \quad \boxed{\text{DirichletProblem}}$$

and $\omega(\Gamma_j)$ the harmonic measure of Γ_j , while the vector $(c_1, \dots, c_l)^t \in \mathbb{R}^l$ is determined, up to a multiple of $(1, \dots, 1)^t$ (which will only alter ψ_{nm} by a constant and therefore respect $\nabla\psi_{nm}$) by the property that $\int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} = 0$ for $1 \leq i \leq l$.

Proof. As Lipschitz open sets have unstretched complement, Theorem 5 and Corollary 1 imply that the norm-minimising vector field Ω -equivalent to M is $\nabla\psi_{nm}$ with $\psi_{nm} = u + \sum_{j=1}^l c_j\omega(\Gamma_j)$, where $u \in W^{1,2}(\Omega)$ solves (40) and $\omega(\Gamma_j)$ is the harmonic measure of Γ_j , while the c_j are real numbers. Using the notation of Lemma 1 and appealing to the Whitney and Sard theorems as in the proof of Theorem 5 to construct non-negative functions $h_j \in C^\infty(\mathbb{R}^n)$ whose zero set is $\overline{\text{ext}O_1}$ for $j = 1$ and $\overline{\text{int}O_j}$ for $j \neq 1$, we construct smooth compact hypersurfaces $\Sigma_j \subset \Omega$ such that the shell between Γ_j and Σ_j is included in Ω . Since $\int_{\Sigma_j} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} = 0$ by (40), we get from the Green formula that $\int_{\Gamma_j} \frac{\partial\psi_{nm}}{\partial n} d\mathcal{H}^{n-1} = 0$ for $1 \leq j \leq l$. Hence, it holds for $j \in \{1, \dots, l\}$ that

$$- \int_{\Gamma_j} \frac{\partial u}{\partial n} d\mathcal{H}^{n-1} = \sum_{\ell=1}^l c_\ell \int_{\Gamma_j} \frac{\partial\omega(\Gamma_\ell)}{\partial n} d\mathcal{H}^{n-1}. \tag{41} \quad \boxed{\text{detsyscl}}$$

To see that (41) determines (c_1, \dots, c_l) up to a multiple of $(1, \dots, 1)$, observe that if we put $v = \sum_{\ell} a_\ell\omega(\Gamma_\ell)$ then

$$\int_{\Omega} \nabla v \cdot \nabla v \, dm = \sum_{j=1}^l \int_{\Gamma_j} v \frac{\partial v}{\partial n} d\mathcal{H}^{n-1} = \sum_{j,\ell=1}^l a_j a_\ell \int_{\Gamma_j} \frac{\partial\omega(\Gamma_\ell)}{\partial n} d\mathcal{H}^{n-1},$$

so that the quadratic form on \mathbb{R}^l whose matrix has (j, ℓ) -entry $\int_{\Gamma_j} \frac{\partial\omega(\Gamma_\ell)}{\partial n} d\mathcal{H}^{n-1}$ is non-negative with kernel the multiples of $(1, \dots, 1)^t$ (corresponding to a constant v). □

When $p \neq 2$, we shall not be able to characterize Ω -equivalent sources of minimum L^p -norm in such an explicit manner. However, when $\mathbb{R}^n \setminus \Omega$ is Lipschitz at least, the problem can be approached *via* duality. For this, recall that S_p is a closed subspace of $[L^p(\Omega)]^n$ by Lemma 4, and let $S_p^\perp \subset [L^q(\Omega)]^n$, $1/p + 1/q = 1$, be the space of annihilators of S_p ; that is: the space of all $\Phi \in [L^q(\Omega)]^n$ such that $\langle M_0, \Phi \rangle = 0$ for all $M_0 \in S_p$.

LAdet

Lemma 5. *Let Ω be a bounded Lipschitz open set, and $p \in (1, \infty)$ with $1/p + 1/q = 1$. Then, S_p^\perp consists of those vector-fields $\nabla\phi \in [L^q(\Omega)]^n$ such that $\phi \in W^{1,q}(\Omega)$ is harmonic in Ω and $\langle \nabla\phi \cdot \nu, \chi \rangle = 0$ for all $\chi \in \mathbb{R}_{\partial\Omega}$; in other words, $\int_{\Gamma_i} (\partial\phi/\partial n) d\mathcal{H}^{n-1} = 0$ for each connected component Γ_i of $\partial\Omega$.*

Proof. By Lemma 3, we may assume that Ω is connected. Let $\Phi \in [L^q(\Omega)]^n$ be such that $\langle M_0, \Phi \rangle = 0$ for all $M_0 \in S_p$. By Theorem 3, it means that $\langle \Phi, \nabla\psi_0 + D_0 \rangle = 0$ for $\psi_0 \in W_{\mathbb{R}_{\partial\Omega}}^{1,p}(\Omega)$ and $D_0 \in \text{Div}_{p,0}$. In particular Φ is orthogonal to $\text{Div}_{p,0}$ whence it is a gradient, say $\Phi = \nabla\phi$ with $\phi \in W^{1,q}(\Omega)$. Moreover, as $\nabla\phi = \Phi$ is orthogonal to all gradients of functions in $C_c^\infty(\Omega) \subset W_{\mathbb{R}_{\partial\Omega}}^{1,p}(\Omega)$, it follows that $\nabla\phi$ is divergence-free as a distribution; *i.e.*, ϕ is harmonic in Ω . Using the notation of Lemma 1 so that the O_j are the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$ and the Γ_j are the connected components of $\partial\Omega$ for $1 \leq j \leq l$, we can argue as in the proof of Theorem 6 and construct for each j a closed smooth hypersurface $\Sigma_j \subset \Omega$ such that the shell between Γ_j and Σ_j is included in Ω . Then, we can find $h_j \in C^\infty(\mathbb{R}^n)$ such that $h_j = 1$ on $\overline{O_j}$ and zero outside a neighborhood of Γ_j containing Σ_j . Then $h_j \in W_{\mathbb{R}_{\partial\Omega}}^{1,p}(\Omega)$, and we get from the divergence formula applied to $h_j \nabla\phi$ on the shell between Σ_j and Γ_j that $\int_{\Gamma_j} \frac{\partial\phi}{\partial n} d\mathcal{H}^{n-1} = 0$ for $1 \leq j \leq l$. This proves that the elements of S_p^\perp are of the announced type, and the previous arguments are easily reverted to yield the converse. \square

Remark 1. *Whenever Ω is such that $G_0^p(\Omega)$, coincides with $\tilde{G}^p(\Omega)$ and $\text{Div}_{p,0}$ coincides with divergence-free fields in $[L^p(\mathbb{R}^n)]^n$ that vanish off Ω , the reasoning in Lemma 5 is easily adapted to yield that S_p^\perp consists of those $\nabla\Phi \in [L^q(\Omega)]^n$ such that Φ is a harmonic function in Ω and $\int_\Gamma \frac{\partial\Phi}{\partial n} d\mathcal{H}^{n-1} = 0$ for all closed C^∞ -hypersurfaces $\Gamma \subset \Omega$. In particular, if we knew that $\text{Div}_{p,0}$ coincides with divergence-free fields in $[L^p(\mathbb{R}^n)]^n$ vanishing off Ω when $\mathbb{R}^n \setminus \Omega$ is unstretched, then the above characterisation of S_p^\perp would hold under very general assumptions.*

We now return to our investigation of norm-minimising Ω -equivalent

sources. It ensues from the Hahn-Banach theorem [36, Thm 7.2] that

$$\begin{aligned} \inf_{M_0 \in S_p} \|M - M_0\|_{[L^p(\Omega)]^n} &= \sup_{\|\Phi\|_{[L^q(\Omega)]^n} \leq 1, \Phi \in S_p^\perp} |\langle M, \Phi \rangle| \\ &= \sup_{\|\Phi\|_{[L^q(\Omega)]^n} = 1, \Phi \in S_p^\perp} |\langle M, \Phi \rangle|, \end{aligned} \quad (42) \quad \boxed{\text{EquivDualProblem}}$$

where the second equality in (42) is obvious. In view of Lemma 5, it follows that when Ω is Lipschitz:

$$\inf_{M_0 \in S_p} \|M - M_0\|_{[L^p(\Omega)]^n} = \sup_{\|\nabla\phi\|_{[L^q(\Omega)]^n} = 1, \Delta\phi = 0 \text{ in } \Omega, \int_{\Gamma_i} (\partial\phi/\partial n) d\mathcal{H}^{n-1} = 0} |\langle M, \nabla\phi \rangle|, \quad (43) \quad \boxed{\text{specmneL}}$$

with $\Gamma_1, \dots, \Gamma_l$ to denote the connected component Γ_i of $\partial\Omega$. Still assuming that Ω is Lipschitz, the existence of a harmonic function $\phi_S \in W^{1,q}(\Omega)$ that maximises (43) is a byproduct of Poincaré's inequality and the closedness of S_p^\perp , while uniqueness of ϕ_S when $M \not\equiv 0$ is an elementary consequence of strict convexity. Further, observing that the infimum is attained in (43) and letting M_S be as in (38), we see from (43) that ϕ_S satisfies:

$$\langle M - M_S, \nabla\phi_S \rangle = \|M - M_S\|_{[L^p(\Omega)]^n}, \quad (44) \quad \boxed{\text{DualEquality}}$$

and the case of equality in Hölder's inequality shows that

$$\nabla\phi_S = \frac{|M - M_S|^{p-2}(M - M_S)}{\|M - M_S\|_{[L^p(\Omega)]^n}^{p-1}}. \quad (45) \quad \boxed{\text{HarmonicCharofMin}}$$

Thus, inverting the duality mapping in (45), it follows when Ω is a bounded Lipschitz open set that given $M \in [L^p(\Omega)]^n$, the norm-minimising Ω -equivalent source M_S is given by

$$M - M_S = \langle M, \nabla\phi_S \rangle |\nabla\phi_S|^{q-2} \nabla\phi_S, \quad (46) \quad \boxed{\text{NM-p}}$$

where ϕ_S is the unique harmonic function in Ω with gradient of unit L^q -norm and zero flux across each connected component of $\partial\Omega$. Note that when $\partial\Omega$ is connected, the latter condition is automatic. Altogether, we proved the following result that generalises Theorem 6:

GeneralResult

Theorem 7. *Let Ω be a bounded Lipschitz open set, and $\Gamma_1, \dots, \Gamma_l$ the connected components of $\partial\Omega$. Given a vector-field $0 \neq M \in [L^p(\Omega)]^n$, $1 < p < \infty$, the norm-minimising Ω -equivalent vector-field is given by*

$$\langle M, \nabla\phi_S \rangle |\nabla\phi_S|^{q-2} \nabla\phi_S$$

where ϕ_S is, up to an additive constant, the unique harmonic function on Ω with $\|\nabla\phi_S\|_{[L^q(\Omega)]^n} = 1$ and $\int_{\Gamma_i} (\partial\phi_S/\partial n) d\mathcal{H}^{n-1} = 0$ for $1 \leq i \leq l$ that satisfies:

$$\langle M, \nabla\phi_S \rangle = \max_{\|\nabla\phi\|_{[L^q(\Omega)]^n}=1, \Delta\phi=0 \text{ in } \Omega, \int_{\Gamma_i} (\partial\phi/\partial n) d\mathcal{H}^{n-1}=0, 1 \leq i \leq l} \langle M, \nabla\phi \rangle. \quad (47) \quad \text{OptProb}$$

When $\partial\Omega$ is connected, (47) may be replaced by

$$\langle M, \nabla\phi_S \rangle = \max_{\|\nabla\phi\|_{[L^q(\Omega)]^n}=1, \Delta\phi=0 \text{ in } \Omega} \langle M, \nabla\phi \rangle. \quad (48) \quad \text{OptProbL}$$

We are now in position to prove Theorem 1.

Proof of Theorem 1. It follows from (38) and the theorem above that the norm-minimising source Ω -equivalent to M can be written uniquely in the form

$$\langle M, \nabla\phi_S \rangle |\nabla\phi_S|^{q-2} \nabla\phi_S = M - \nabla\phi_0 - D_0,$$

where ϕ_S is harmonic with $\nabla\phi_S \in [L^q(\Omega)]^n$ and $\int_{\Gamma_i} (\partial\phi_S/\partial n) d\mathcal{H}^{n-1} = 0$ for $1 \leq i \leq l$, while $\nabla\phi_0 \in G_0^p(\Omega)$ and $D_0 \in \text{Div}_{p,0}(\Omega)$. Letting $h = \langle M, \nabla\phi_S \rangle^{\frac{1}{q-1}} \phi_S$ together with $\psi = \phi_0$ and $D = D_0$, we get the decomposition (24). \square

7 Appendix

appendixg

apf

7.1 More on Sobolev functions

This section complements Section 2, and we stick to the notation introduced there. In particular, \tilde{F} stands for the extension by zero of F , initially defined on $\Omega \subset \mathbb{R}^n$, to the whole of \mathbb{R}^n ; recall also the spaces $G^p(\Omega)$ and $G_0^p(\Omega)$ introduced in (5) and (6), as well as the space $\tilde{G}^p(\Omega) \subset G^p(\Omega)$ defined by (7). Theorem 2 and Lemma 2 imply that $\tilde{G}^p(\Omega)$ consists exactly of the

magnetic fields, generated by a magnetization in $L^p(\Omega)$, that vanish outside Ω ; *i.e.*, the fields of silent magnetizations of L^p -class.

Unlike $G^p(\Omega)$ or $G_0^p(\Omega)$, the space $\tilde{G}^p(\Omega)$ only depends on Ω modulo those $x \in \partial\Omega$ having a neighbourhood V_x such that $V_x \setminus \Omega$ has zero Lebesgue measure; for convenience, we say that such points are *Lebesgue isolated* in $\mathbb{R}^n \setminus \Omega$. The union of Ω and of the *Lebesgue-isolated* points in $\mathbb{R}^n \setminus \Omega$ is a bounded open set $\tilde{\Omega} \supset \Omega$ such that $\tilde{G}^p(\Omega) = \tilde{G}^p(\tilde{\Omega})$ and therefore, when studying $\tilde{G}^p(\Omega)$, we may as well assume that $\mathbb{R}^n \setminus \Omega$ has no Lebesgue-isolated points. This hypothesis typically eliminates “cracks” from Ω .

We shall see that in fact $\tilde{G}^p(\Omega)$ coincides with $G_0^p(\Omega)$ when $\partial\Omega$ has Lebesgue measure zero and $\mathbb{R}^n \setminus \Omega$ has no Lebesgue-isolated points, provided that the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$ are positively separated, meaning that the distance between any two of them is bigger than a strictly positive constant independent of the components; this is a consequence of Theorem 8 below. Of course, as Ω is bounded, the separation condition entails that the connected components are finite in number, and then separation reduces to the property that their closures do not meet. The separation condition cannot be dispensed with in general, as the example in Section 7.3 shows. More generally, in Theorem 8, we handle the situation where Ω is a bounded open set such that $\partial\Omega$ has Lebesgue measure zero and the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$ may only accumulate in a “thick” manner; the case of an arbitrary bounded Ω will not be considered here. We begin with a lemma:

VB **Lemma 6.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set whose boundary $\partial\Omega$ has zero Lebesgue measure. Let further $\{O_j\}_{j \in J}$ designate the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$. For $p \in (1, \infty)$, a vector field F lies in $\tilde{G}^p(\Omega)$ if, and only if \tilde{F} is of the form ∇f for some (necessarily unique) $f \in W^{1,p}(\mathbb{R}^n)$ which is constant on each O_j .*

Proof. By definition, every $F \in G^p(\mathbb{R}^n)$ is of the form ∇f with $f \in W_{loc}^{1,p}(\mathbb{R}^n)$. If moreover $F = 0$ on $\mathbb{R}^n \setminus \Omega$, then f is constant in each O_j ; conversely, since $\partial\Omega$ has Lebesgue measure 0, it is equivalent to say that F vanishes on $\mathbb{R}^n \setminus \Omega$ and that f is constant on each O_j . Let O_0 designate, without loss of generality, the unbounded connected component of $\mathbb{R}^n \setminus \bar{\Omega}$; the latter uniquely exists since Ω is bounded. Adding a constant to f if necessary, we may assume that it vanishes on O_0 , and then $f \in W^{1,p}(\mathbb{R}^n)$. \square

Note: if $f \in W^{1,p}(\mathbb{R}^n)$ is such that $\nabla f \in \tilde{G}^p(\Omega)$, necessarily f vanishes in the unbounded component of $\mathbb{R}^n \setminus \bar{\Omega}$, for it is constant there and must lie

in $L^p(\mathbb{R}^n)$.

EPC

Theorem 8. For $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^n$ a bounded open set with boundary $\partial\Omega$ of zero Lebesgue measure, it holds that $G_0^p(\Omega) \subset \widetilde{G}^p(\Omega)$. If $n < p$, the converse inclusion holds; namely, $G_0^p(\Omega) = \widetilde{G}^p(\Omega)$. More generally, if we let $\{O_j\}_{j \in J}$ denote the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$ and we assume in addition that O_j is non-thin at $B_{1,p}$ -quasi every point of ∂O_j for all j , and if whenever $\overline{O_j} \cap \overline{\cup_{i \in I} O_i} \neq \emptyset$ for some index $j \in J$ and some subset of indices $I \subset J$ then $B_{1,p}(\overline{O_j} \cap \overline{\cup_{i \in I} O_i}) > 0$ (these conditions are automatically satisfied if $n < p$), it holds that $G_0^p(\Omega) = \widetilde{G}^p(\Omega)$.

Proof. Without loss of generality, we assume that O_0 is the unbounded component of $\mathbb{R}^n \setminus \overline{\Omega}$. Pick $\varphi \in C^\infty(\Omega)$ with $\nabla\varphi \in [C_c(\Omega)]^n$, and let the U_i enumerate, for $i \in I \subset \mathbb{N}$, the connected components of $\mathbb{R}^n \setminus \text{supp } \nabla\varphi$, with U_0 to denote the unbounded component. The $U_i \cap \Omega$ are nonempty open sets partitioning $\Omega \setminus \text{supp } \nabla\varphi$. Assume first that Ω is connected, and define $B_i := \cup_{k \neq i} U_k \cup \Omega$ for $i \in I$. Clearly, B_i is a connected open set and $U_i \cup B_i = \mathbb{R}^n$. We contend that $U_i \cap B_i$ is connected. Indeed, as \mathbb{R}^n is simply connected, the last portion of the Mayer-Vietoris sequence in homology [13, Sec. 2.2] yields (with H_l to indicate the l -th homology group):

$$0 = H_1(U_i \cup B_i) \rightarrow H_0(U_i \cap B_i) \rightarrow H_0(U_i) \oplus H_0(B_i) \rightarrow H_0(U_i \cup B_i) \rightarrow 0$$

from which it follows, since $H_0(U_i) = H_0(B_i) = H_0(U_i \cup B_i) = \mathbb{Z}$ (because U_i , B_i and $U_i \cup B_i$ are connected), that $H_0(U_i \cap B_i) = \mathbb{Z}$ as well. This proves our contention, and so $U_i \cap \Omega = U_i \cap B_i$ is in turn connected. Hence, φ is constant on $U_i \cap \Omega$ because $\nabla\varphi = 0$ there, say $\varphi|_{U_i \cap \Omega} \equiv c_i$. If we define Φ to be $\varphi - c_0$ on Ω and $c_i - c_0$ on U_i , it is readily checked that Φ is well-defined in $C_c^\infty(\mathbb{R}^n)$ with $\nabla\Phi = \widetilde{\nabla}\varphi$; in fact Φ is zero on U_0 , in particular it vanishes outside every ball containing Ω . To recap, we showed that if Ω is connected and $\varphi \in C^\infty(\Omega)$ with $\nabla\varphi \in [C_c^\infty(\Omega)]^n$, then there is $\Phi \in C_c^\infty(\mathbb{R}^n)$ such that $\nabla\Phi = \widetilde{\nabla}\varphi$. We claim that the same holds even when Ω is not connected. Indeed, let $\{\Omega_l\}_{l \in L}$ denote the connected components of Ω , where $L = \{1, 2, \dots\}$ is an initial segment of natural numbers (finite or infinite). If $\varphi \in C^\infty(\Omega)$ is such that $\nabla\varphi \in [C_c^\infty(\Omega)]^n$, then $\text{supp } \nabla\varphi$ can meet only finitely many Ω_l , say $\Omega_1, \dots, \Omega_N$. Thus, φ is constant on Ω_l for $l > N$, and we may as well assume it is zero there because this does not change $\nabla\varphi$. For $1 \leq l \leq N$, the function $\varphi_l := \varphi|_{\Omega_l}$ lies in $C^\infty(\Omega_l)$ and $\nabla\varphi_l = (\nabla\varphi)|_{\Omega_l}$ is compactly supported in Ω_l . Hence, by the first part of the proof, there is

$\Phi_l \in C_c^\infty(\mathbb{R}^n)$ such that $\nabla \Phi_l = \widetilde{\nabla \varphi_l}$, and so $\Phi := \sum_{1 \leq l \leq N} \Phi_l$ lies in $C_c^\infty(\mathbb{R}^n)$. By construction $\nabla \Phi = \sum_{1 \leq l \leq N} \widetilde{\nabla \varphi_l}$ is zero outside $\cup_{1 \leq l \leq N} \Omega_l$ and coincides with $\nabla \varphi_l$ on Ω_l , so that $\nabla \Phi = \widetilde{\nabla \varphi}$ as claimed.

Next, consider $F \in G_0^p(\Omega)$ and let φ_k be a sequence in $C^\infty(\Omega)$ such that $\nabla \varphi_k \in [C_c^\infty(\Omega)]^n$ with $\nabla \varphi_k \rightarrow F$ in $[L^p(\Omega)]^n$. Let further Φ_k be a sequence in $C_c^\infty(\mathbb{R}^n)$ with $\nabla \Phi_k = \widetilde{\nabla \varphi_k}$ and such that Φ_k is zero on O_0 ; such a sequence exists by what precedes. Applying (4) on a ball containing Ω , we deduce that Φ_k converges in $W^{1,p}(\mathbb{R}^n)$, as $k \rightarrow \infty$, to some f such that $\nabla f = \widetilde{F}$. Hence, $F \in \widetilde{G}^p(\Omega)$ so that $G_0^p(\Omega) \subset \widetilde{G}^p(\Omega)$.

Conversely, let $F \in \widetilde{G}^p(\Omega)$ so that, according to Lemma 6, $\widetilde{F} = \nabla f$ where $f \in W^{1,p}(\mathbb{R}^n)$ assumes a constant value $c_j(f)$ on O_j , for every $j \in J$. Note, since $c_0(f) = 0$ while O_0 is non-thin at quasi every point of ∂O_0 by assumption, that $f(x) = 0$ for $B_{1,p}$ -quasi every $x \in \overline{O}_0$: it is so because f is continuous outside a set of arbitrary small $B_{1,p}$ -capacity [7, Proposition 6.1.2]. Suppose for a while that $f \geq 0$ and fix $\delta > 0$, together with a sequence $\{\delta_i\}_{i \in \mathbb{N}}$ of strictly positive numbers such that $\sum_i \delta_i = \delta$. Pick $\varepsilon > 0$ and then $\varepsilon' \in (0, \varepsilon]$ small enough that $\int_{\{0 \leq f \leq \varepsilon'\}} |\nabla f|^p dm \leq \varepsilon^p$; since $\|\nabla f\|_{L^p(\mathbb{R}^n)} < \infty$, such a ε' exists, by the monotone convergence theorem. Let $I_0 \subset J$ be the set of indices i for which $c_i(f) \geq \varepsilon'$, and define $K_0 := \overline{\cup_{i \in I_0} O_i}$. Of necessity $K_0 \cap \overline{O}_0 = \emptyset$, otherwise $B_{1,p}(K_0 \cap \overline{O}_0) > 0$ by assumption and, since f is continuous outside a set of arbitrary small $B_{1,p}$ -capacity, it would imply that $f \geq \varepsilon' > 0$ on a set $E \subset \partial O_0$ with $B_{1,p}(E) > 0$, contradicting that $f(x) = 0$ for $B_{1,p}$ -quasi every $x \in \overline{O}_0$. Hence, by [7, Theorem 9.1.3], there is a C^∞ function $\eta_0 : \mathbb{R}^n \rightarrow [0, 1]$ with $\eta_0 = 1$ on K_0 and $\eta_0 = 0$ on \overline{O}_0 such that $\|f - \eta f\|_{W^{1,p}(\mathbb{R}^n)} \leq \varepsilon'$. Now, if we let $f_0 := \max\{\eta f - \varepsilon', 0\}$, we find that $\nabla f_0 \in \widetilde{G}^p(\Omega)$ with $c_i(f_0) = c_i(f) - \varepsilon'$ for $i \in I_0$ and $c_i(f_0) = 0$ for $i \notin I_0$. Moreover, it follows from [7, Theorem 3.3.1] that $\nabla f_0 = \nabla(\eta f)$ a.e. on $\{x : \eta(x)f(x) > \varepsilon'\}$ and $\nabla f_0 = 0$ a.e. on $\{x : \eta(x)f(x) \leq \varepsilon'\}$. Therefore,

$$\|\eta f - f_0\|_{W^{1,p}(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |\eta f - f_0|^p dm + \int_{\mathbb{R}^n} |\nabla(\eta f) - \nabla f_0|^p dm \quad (49)$$

$$\leq (\varepsilon')^p m(\Omega) + \int_{\{0 \leq \eta f \leq \varepsilon'\}} |\nabla(\eta f)|^p dm \quad (50)$$

$$\leq (\varepsilon')^p m(\Omega) + \left(\varepsilon' + \left(\int_{\{0 \leq f \leq \varepsilon'\}} |\nabla f|^p dm \right)^{1/p} \right)^p \quad (51)$$

$$\leq (\varepsilon')^p m(\Omega) + (\varepsilon' + \varepsilon)^p. \quad (52)$$

Altogether, as $\varepsilon' \leq \varepsilon$, we get from the triangle inequality that

$$\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)} \leq (1 + (m(\Omega) + 2^p)^{1/p}) \varepsilon \quad (53) \quad \boxed{\text{maj e 0}}$$

which is arbitrary small with ε . In the general case where f is not signed, we write $f = f^+ - f^-$ with $f^+ := \max\{f, 0\}$ and $f^- = (-f)^+$ and remark that ∇f^\pm satisfies the same assumptions as \tilde{F} [7, Thm. 3.3.1]. So, we can apply what precedes to f^+ and f^- to obtain functions $f_{0,+}$ and $f_{0,-}$; we then put $f_0 := f_{0,+} - f_{0,-}$. To recap, we constructed $f_0 \in W^{1,p}(\mathbb{R}^n)$ with $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)}$ arbitrary small, such that f_0 is constant on a neighbourhood of \bar{O}_0 and on each O_j ; in particular, we can make $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)} < \delta_0$. Note also that if f was constant on a neighbourhood of \bar{O}_j for some j , so is f_0 on the same neighbourhood (possibly with a different constant, though). We now proceed inductively: to complete the next step, pick $x_1 \in O_1$ and consider the inversion with center x_1 given by $\mathcal{I}_{x_1}(x) := (x - x_1)/|x - x_1|^2$. It is a smooth involution of the ‘‘sphere’’ $\mathbb{R}^n \cup \{\infty\} \sim \mathbb{S}^n$ that maps x_1 to ∞ , with Jacobian determinant $-|x - x_1|^{-2n}$ at $x \neq x_1$; in fact, the Jacobian matrix is unitarily conjugate to $\text{diag}\{-|x - x_1|^{-2}, |x - x_1|^{-2}, \dots, |x - x_1|^{-2}\}$, see [26, Proof of Theorem 4.2] (the unitary matrix depends on x , though). Clearly, $\Omega_1 := \mathcal{I}_{x_1}(\Omega)$ is a bounded open set, and the $U_i := \mathcal{I}_{x_1}(O_j)$ are the connected components of $\mathbb{R}^n \setminus \bar{\Omega}_1$, with U_1 being the unbounded component. From the change of variable formula and the definition of $B_{1,p}$ -capacity, one checks that \mathcal{I}_{x_1} preserves sets of $B_{1,p}$ -capacity zero in $\mathbb{R}^n \setminus \{x_1, \infty\}$, consequently $B_{1,p}(\bar{U}_j \cap \overline{\cup_{i \in I} U_i}) > 0$ whenever $\bar{U}_j \cap \overline{\cup_{i \in I} U_i} \neq \emptyset$; also, by definition of thinness [7, Definition 6.3.7], \mathcal{I}_{x_1} preserves thinness of a set at any $x \neq x_1$. Thus, U_j is non-thin at $B_{1,p}$ -quasi every point of ∂U_j for each j . Furthermore, if U is an open set with compact closure in $\mathbb{R}^n \setminus \{x_1\}$ and we put $m_1 := \inf_{x \in U} |x - x_1|$, we get from the change of variable formula again that for $g \in W^{1,p}(U)$:

$$\|g\|_{W^{1,p}(U)} \leq \max\{m_1^{-2n/p}, m_1^{-2(n-p)/p}\} \|g \circ \mathcal{I}_{x_1}\|_{W^{1,p}(\mathcal{I}_{x_1}(U))}. \quad (54)$$

transSobn

As the function $h_1 := f_0 \circ \mathcal{I}_{x_1} - c_1(f_0)$ lies in $W^{1,p}(\mathbb{R}^n)$ and is constant on U_j for each j as well as constant on a neighborhood of \bar{U}_0 , we can argue as we did to construct f_0 from f , replacing Ω by Ω_1 and f by h_1 . This provides us with a function g_1 that can be made arbitrary close to h_1 in $W^{1,p}(\mathbb{R}^n)$ while it is constant on each U_j and on a neighborhood of both \bar{U}_0 and \bar{U}_1 . If λ_0 denotes the constant value assumed by g_1 on U_0 , we get since $h_1 = -c_1(f_0)$ on U_0 that $|\lambda_0 + c_1(f_0)| m_1^{1/p}(U_0) \leq \|h_1 - g_1\|_{L^p(U_0)}$, whence $|\lambda_0 + c_1(f_0)|$ can be made arbitrary small with $\|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)}$. Thus, letting $f_1 := (g_1 \circ \mathcal{I}_{x_1} - \lambda_0)$,

we get in view of (54) since $f_0 = f_1 = 0$ on O_0 :

$$\|f_0 - f_1\|_{W^{1,p}(\mathbb{R}^n)} = \|(h_1 - g_1) \circ \mathcal{I}_{x_1} + c_1(f_0) + \lambda_0\|_{W^{1,p}(\mathbb{R}^n \setminus U_1)} \quad (55)$$

$$\leq C \|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)} + |\lambda_0 + c_1(f_0)| m^{1/p}(\mathbb{R}^n \setminus U_1) \quad (56)$$

$$\leq C' \|h_1 - g_1\|_{W^{1,p}(\mathbb{R}^n)}, \quad (57)$$

where C and C' are geometric constants depending on Ω and our choice of $x_1 \in O_1$. One deduces from (55) that $\|f_0 - f_1\|_{W^{1,p}(\mathbb{R}^n)}$ can be made arbitrary small, in particular smaller than δ_1 . Iterating the argument, we construct a sequence of functions f_k , $0 \leq k$, such that $\|f - f_0\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta_0$ and $\|f_k - f_{k+1}\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta_{k+1}$, with f_ℓ constant on a neighborhood V_k of \bar{O}_k for $\ell \geq k$. Since $\sum_k \delta_k = \delta < \infty$, the sequence f_k is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$ that converges to some \tilde{f} ; clearly $\nabla \tilde{f}$ is compactly supported in Ω , and $\|f - \tilde{f}\|_{W^{1,p}(\mathbb{R}^n)} \leq \delta$. By mollification, we can now construct a $\varphi \in C^\infty(\mathbb{R}^n)$ which is constant on a neighborhood of O_j for each j (the constant being 0 when $j = 0$), and such that $\|f - \varphi\|_{W^{1,p}(\mathbb{R}^n)} \leq 2\delta$. Since δ was arbitrary, we find that $F \in G_0^p(\Omega)$, as desired. \square

EPCc

Corollary 2. *Let $p \in (1, n]$ and $\Omega \subset \mathbb{R}^n$ a bounded open set with boundary $\partial\Omega$ of zero Lebesgue measure. Denote by $\{O_j\}_{j \in J}$ the connected components of $\mathbb{R}^n \setminus \bar{\Omega}$, and assume that O_j is non-thin at $B_{1,p}$ -quasi every point of ∂O_j for all j . Let further $F \in \tilde{G}^p(\Omega)$ so that, according to Lemma 6, one can write $\tilde{F} = \nabla f$ for some unique $f \in W^{1,p}(\mathbb{R}^n)$ which is constant on each O_j ; say, $f \equiv c_j(f)$ there. Assume that $\lim_{k \rightarrow \infty} c_{j_k}(f) = c_j(f)$ whenever the sequence $\{j_k\} \subset J$ satisfies $\lim_{k \rightarrow \infty} \text{dist}(O_{j_k}, O_j) = 0$ for some $j \in J$. Then, $F \in G_0^p(\Omega)$.*

Note: the hypothesis entails that $c_i(f) = c_j(f)$ when $\bar{O}_i \cap \bar{O}_j \neq \emptyset$.

Proof. We proceed as in the proof of the second assertion of Theorem 8: defining K_0 in the same way, the assumptions on f ensure that $K_0 \cap \bar{O}_0 = \emptyset$ which allows us to complete the first step of the induction. The analog of f_0 thus obtained has $c_j(f_0) = c_j(f) \mp \varepsilon'$ when $\pm c_j(f) \geq \varepsilon'$ and $c_j(f_0) = 0$ otherwise, hence it satisfies the same assumptions as f . The induction can now be carried out as before. \square

Let $\Omega \subset \mathbb{R}^n$ be open. For $f \in W^{1,p}(\Omega)$ and $g \in W_0^{1,q}(\Omega)$, it holds for $1 \leq i \leq n$ that $\int_\Omega \partial_{x_i} f g = - \int_\Omega \partial_{x_i} g f$, by absolute continuity on a.e. line of

Sobolev functions [12, Thm. 2.1.4]. Thus, $\|\nabla f\|_{[W^{-1,p}(\Omega)]^n} \leq n^{1/p}\|f\|_{L^p(\Omega)}$, and since $W^{1,p}(\Omega)$ is dense in $L^p(\Omega)$ it follows that $f \mapsto \nabla f$ extends to a continuous map from $L^p(\Omega)$ into $[W^{-1,p}(\Omega)]^n$.

When Ω is a Lipschitz open set and $(U_j \cap \partial\Omega, \phi_j)$ an atlas for $\partial\Omega$ as in Section 2.2, the Sobolev space $W^{1,p}(\partial\Omega)$ comprises those $f : \partial\Omega \rightarrow \mathbb{R}$ such that $f \circ \phi_j^{-1} \in W^{1,p}(V_j)$ for all j , with $V_j = \phi_j(U_j \cap \partial\Omega)$. The definition does not depend on the atlas, since Lipschitz changes of variables preserve Sobolev functions [12, Thm 2.2.2]. The tangential gradient $\nabla_T f \in [L^p(\partial\Omega)]^n$ is given on $U_j \cap \partial\Omega$ by

$$\nabla_T f \circ \phi_j^{-1} = D\phi_j^{-1}((D\phi_j^{-1})^t D\phi_j^{-1})^{-1} \nabla(f \circ \phi_j^{-1}); \quad (58) \quad \boxed{\text{gradtan}}$$

note that the definitions agree a.e. on $U_{j_1} \cap U_{j_2}$ and that $\nabla_T f(x) \in T_x \partial\Omega$ for σ -a.e. $x \in \partial\Omega$. For a.e. $x \in U_j \cap \partial\Omega$ and each $X \in T_x \partial\Omega$, we have $\nabla_T f(x) \cdot X = df(x)(X)$ where the differential $df(x)$ of f at x is the 1-form given by $df(x)(X) = \nabla(f \circ \phi_j^{-1})(\phi_j(x)) \cdot D\phi_j(x)(X)$. One sees that $W^{1,p}(\partial\Omega)$ is a Banach space for the norm

$$\|f\|_{W^{1,p}(\partial\Omega)} = \left(\|f\|_{L^p(\partial\Omega)}^p + \|\nabla_T f\|_{[L^p(\partial\Omega)]^n}^p \right)^{1/p} \\ (\max\{\|f\|_{L^\infty(\partial\Omega)}, \|\nabla_T f\|_{[L^\infty(\partial\Omega)]^n}\} \quad \text{if } p = \infty)$$

which is equivalent to $\sum_{j=1}^N \|f \circ \phi_j^{-1}\|_{W^{1,p}(V_j)}$. Observe that $W^{1,\infty}(\partial\Omega)$ identifies with Lipschitz functions on $\partial\Omega$.

When $1 < p < \infty$ and $1/p + 1/q = 1$, the dual space of $W^{1,p}(\partial\Omega)$, denoted by $W^{-1,q}(\partial\Omega)$, can be realized as the completion of $L^q(\partial\Omega)$ for the norm

$$\|\varphi\|_{W^{-1,q}(\partial\Omega)} := \sup_{\|g\|_{W^{1,p}(\partial\Omega)}=1} \int_{\partial\Omega} \varphi g \, d\sigma \quad (59) \quad \boxed{\text{defnSnex}}$$

by the same argument that leads to $(W_0^{1,p}(\Omega))^* = W^{-1,q}(\Omega)$.

Forms on $\partial\Omega$ proceed as in the smooth case: for $1 \leq k \leq n-1$, a k -form is a map $x \mapsto \omega(x)$ where $\omega(x)$ is an alternating k -linear map on $T_x \partial\Omega$ (a 0-form is simply a function). Its local representative in the chart $\phi_j : U_j \cap \partial\Omega \rightarrow V_j$ is the k -form $(\phi_j^{-1})^*(\omega)$ on V_j which the pullback of ω under ϕ_j^{-1} :

$$(\phi_j^{-1})^*(\omega)(y) := \omega(\phi_j^{-1}(y)) \circ (D\phi_j^{-1}(y) \times \cdots \times D\phi_j^{-1}(y)), \quad y \in \text{Reg } V_j. \quad (60) \quad \boxed{\text{replocc}}$$

Rearranging (60), we get an expression of the form

$$(\phi_j^{-1})^*(\omega)(y) = \sum_{i_1 < i_2, \dots, < i_k} a_{i_1, \dots, i_k}^{\{\phi_j\}}(y) \, dy_{i_1} \wedge \cdots \wedge dy_{i_k}, \quad y \in \text{Reg } V_j,$$

where the coefficients $a_{i_1, \dots, i_k}^{\{\phi_j\}}$ are functions on V_j that transform naturally under changes of coordinates. We identify forms that agree a.e. and say that ω is of L^p -class if the $a_{i_1, \dots, i_k}^{\{\phi_j\}}$ lie in $L^p(V_j)$ for each j . In this case we write $\omega \in F_k^p(\partial\Omega)$ and

$$\|\omega\|_{F_k^p(\partial\Omega)} := \sum_{j=1}^N \sum_{i_1 < i_2 < \dots < i_k} \|a_{i_1, \dots, i_k}^{\{\phi_j\}}\|_{L^p(V_j)}.$$

A change of atlas yields an equivalent norm. The image under ϕ_j of the restriction $\sigma|_{U_j}$ is the measure on V_j , absolutely continuous with respect to Lebesgue measure, with differential $(1 + |\nabla\Psi_j(y)|^2)^{1/2} dy$. Since $\Psi_j \in L^\infty(V_j)$, it follows that $f \in L^p(\partial\Omega)$ if and only if it is of L^p -class as a 0-form, and $\|f\|_{L^p(\partial\Omega)}$ is equivalent to $\|f\|_{F_0^p(\partial\Omega)}$.

Integrating $(n-1)$ -forms on $\partial\Omega$ goes as in the smooth case on an oriented Riemannian manifold [46, Sec. 4.10] (note that $\partial\Omega$ is oriented by construction). That is: for ω a $(n-1)$ -form of L^1 -class on $\partial\Omega$ and (φ_j) a Lipschitz partition of unity subordinated to the $U_j \cap \partial\Omega$, if one writes

$$(\phi_j^{-1})^*(\varphi_j\omega)(y) = a(y)dy_1 \wedge \dots \wedge y_{n-1}$$

then

$$\int_{\partial\Omega} \omega = \sum_{j=1}^N \int_{V_j} a_j(y)dy.$$

The latter is independent from the atlas and the partition of unity, thanks to the change of variable formula which is valid for Lipschitz reparametrizations. In particular, if we define on $U_j \cap \partial\Omega$ the $(n-1)$ -form

$$\omega^j(x) \left(\sum_{i=1}^{n-1} \lambda_i \partial_{y_i} \phi_j^{-1} \right) := \lambda_1 \lambda_2 \dots \lambda_{n-1} J_k(\phi_j(x)),$$

where J_k is the square root of the sum of the squares of the $(n-1) \times (n-1)$ minors of the Jacobian matrix $D\phi_j^{-1}$, then $\omega^{j_1} = \omega^{j_2}$ a.e. on $U_{j_1} \cap U_{j_2} \cap \partial\Omega$ and the $(n-1)$ -form ω_{vol} on $\partial\Omega$ whose restriction to $U_j \cap \partial\Omega$ is ω^j (the so-called volume form) satisfies $\int_{\partial\Omega} f d\sigma = \int_{\partial\Omega} f \omega_{vol}$ for every $f \in L^1(\partial\Omega)$.

Let us define $\mathcal{W}_k^{1,p}(\partial\Omega)$ to consist of k -forms $\omega \in F_k^p(\partial\Omega)$ for which there exists a $(k+1)$ -form $d\omega \in F_{k+1}^p(\partial\Omega)$ with the property that, for each $j \in$

$\{1, \dots, N\}$,

$$\int_{V_j} (\phi_j^{-1})^*(\omega) \wedge d\mu_j = (-1)^{k+1} \int_{V_j} (\phi_j^{-1})^*(d\omega) \wedge \mu_j$$

whenever μ_j is a C^∞ -smooth $(n - k - 1)$ -form compactly supported on V_j . Here, the exterior derivative $d\mu_j$ is the usual one for smooth forms. Though not obvious at first glance, this definition is consistent, for if $d\omega$ exists then $(\phi_{j_1} \circ \phi_{j_2}^{-1})^*((\phi_{j_1}^{-1})^*(\omega)) \in \mathcal{W}_k^{1,p}(\phi_{j_2}(U_{j_1} \cap U_{j_2} \cap \partial\Omega))$ and on $\phi_{j_2}(U_{j_1} \cap U_{j_2} \cap \partial\Omega)$ it holds a.e. that

$$d\left((\phi_{j_1} \circ \phi_{j_2}^{-1})^*((\phi_{j_1}^{-1})^*(\omega))\right) = (\phi_{j_1} \circ \phi_{j_2}^{-1})^*((\phi_{j_1}^{-1})^*(d\omega)),$$

see [47, Thm 2.2]. We endow $\mathcal{W}_k^{1,p}(\partial\Omega)$ with the norm

$$\|\omega\|_{\mathcal{W}_k^{1,p}(\partial\Omega)} = (\|\omega\|_{F_k^p(\partial\Omega)}^p + \|d\omega\|_{F_{k+1}^p(\partial\Omega)}^p)^{1/p}.$$

It is easy to see that $f \in W^{1,p}(\partial\Omega)$ if and only if $f \in \mathcal{W}_0^{1,p}(\partial\Omega)$ and that

$$(\phi_j^{-1})^*(df) = \sum_i \partial_{y_i}(f \circ \phi_j^{-1}) dy_i.$$

We let $\mathcal{W}_k^{-1,q}(\partial\Omega)$ be the completion of $F_k^q(\partial\Omega)$ for the norm

$$\|\omega\|_{\mathcal{W}_k^{-1,q}(\partial\Omega)} := \sup_{\|\mu\|_{\mathcal{W}_{n-1-k}^{1,p}(\partial\Omega)}=1} \int_{\partial\Omega} \omega \wedge \mu,$$

so that $\mathcal{W}_k^{-1,q}(\partial\Omega)$ consists of linear forms on $\mathcal{W}_{n-1-k}^{1,p}$; *i.e.*, of $(n - 1 - k)$ -currents on $\partial\Omega$. When $f \in W^{1,p}(\partial\Omega)$ and $\mu \in \mathcal{W}_{n-2}^{1,q}(\partial\Omega)$, it holds that $\int_{\partial\Omega} df \wedge \mu = -\int_{\partial\Omega} f d\mu$, as can be checked in local coordinates from the absolute continuity on a.e. line of Sobolev functions [12, Thm 2.1.4]. Thus, $\|df\|_{\mathcal{W}_1^{-1,p}(\partial\Omega)} \leq C\|f\|_{L^p(\Omega)}$, and since $W^{1,p}(\partial\Omega)$ is dense in $L^p(\partial\Omega)$ it follows that $f \mapsto df$ extends to a continuous map from $L^p(\partial\Omega)$ into $\mathcal{W}_1^{-1,p}(\partial\Omega)$.

Since the gradient defines a continuous map $\nabla : L^p(V_j) \rightarrow [W^{-1,p}(V_j)]^{n-1}$ as pointed out at the beginning of this section, (58) implies on using a Lipschitz partition of unity subordinated to the U_j that $\nabla_T : L^p(\partial\Omega) \rightarrow \mathcal{W}_1^{-1,p}(\partial\Omega)$ is continuous.

It is known that $W^{s,p}(\partial\Omega) = [L^p(\partial\Omega), W^{1,p}(\partial\Omega)]_{s,p}$, where $[\cdot, \cdot]_{s,p}$ is the so-called real interpolation functor, see [48, Ch.4] for a definition of the latter. This fact follows from the analogous result on \mathbb{R}^{n-1} [48, Ch. 4, Cor. 4.13] and [21, Lem. 1.1] which allows one to localize the statement in the charts (U_j, ϕ_j) , using a Lipschitz partition of unity on $\partial\Omega$. Hence, by duality [49, Thm 3.7.1], we get that $W^{-1/p,p}(\partial\Omega) \sim [L^p(\partial\Omega), W^{-1,p}(\partial\Omega)]_{1/p,p} = [W^{-1,p}(\partial\Omega), L^p(\partial\Omega)]_{1-1/p,p}$ with equivalence of norms, and since taking the gradient maps $L^p(\partial\Omega)$ into $[W^{-1,p}(\partial\Omega)]^n$ and $W^{1,p}(\partial\Omega)$ into $[L^p(\partial\Omega)]^n$ continuously, we see by interpolation upon setting $s = 1 - 1/p$ that the tangential gradient $\nabla_T \psi$ of $\psi \in W^{1-1/p,p}(\partial\Omega)$ exists as a member of $\mathcal{W}_1^{-1/p,p}(\partial\Omega)$.

7.2 Auxiliary lemmas

app
densS0

Lemma 7. *To every $f \in \mathcal{S}$ on \mathbb{R}^n , there is a sequence $\phi_n \in \mathcal{S}$ such that*

- $\lim_{n \rightarrow \infty} \phi_n = f$ in $L^p(\mathbb{R}^n)$ for all $p \in [1, \infty)$,
- to each n there is a nonempty neighborhood V_n of 0 with the property that $\hat{\phi}_n(\xi) = \hat{f}(0)$ for $\xi \in V_n$.

Proof. We adapt the proof of [50, Lem. 9.2.]. Pick $g \in \mathcal{S}$ such that $\hat{g} = 1$ in some ball $B(0, r)$, $r > 0$. For $\lambda > 0$, put $g_\lambda(x) := \lambda^{-n}g(x/\lambda)$ and define

$$h_\lambda(x) := \hat{f}(0)g_\lambda(x) - f * g_\lambda(x), \quad x \in \mathbb{R}^n. \quad (61) \quad \text{defhl}$$

Clearly, $h_\lambda \in \mathcal{S}$, and since $\hat{g}_\lambda = 1$ in some neighborhood V_λ of 0 we see from (61) that $\hat{h}_\lambda(\xi) = \hat{f}(0) - \hat{f}(\xi)$ for $\xi \in V_\lambda$. Thus, letting $\phi_\lambda := f + h_\lambda$, we get that $\hat{\phi}_\lambda(\xi) = \hat{f}(0)$ for $\xi \in V_\lambda$. It remains to show that $\lim_{\lambda \rightarrow +\infty} h_\lambda = 0$ in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, for then ϕ_n will satisfy our requirements. Now, since $|f|$ is summable because $f \in \mathcal{S}$, we get by convexity that

$$|h_\lambda(x)|^p = \left| \int_{\mathbb{R}^n} f(y)(g_\lambda(x) - g_\lambda(x-y))dy \right|^p \leq \int_{\mathbb{R}^n} |f(y)| |g_\lambda(x) - g_\lambda(x-y)|^p dy.$$

Therefore, by Fubini's theorem and the change of variable $x = \lambda z$,

$$\|h_\lambda\|_{L^p(\mathbb{R}^n)}^p \leq \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(z) - g(z-y/\lambda)|^p dz \right) dy. \quad (62) \quad \text{FubS}$$

The inner integral in (62) is at most $2\|g\|_{L^p(\mathbb{R}^n)}^p$ and it tends to zero for fixed y as $\lambda \rightarrow +\infty$, by the continuity of argument translation in $L^p(\mathbb{R}^n)$. Hence, the right hand side of (62) goes to zero when $\lambda \rightarrow +\infty$, by dominated convergence. \square

calcpa

Lemma 8. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be C^∞ -smooth and supported in $(0, 1)$. Then, $h(x) := \varphi(|x|^n)$ lies in $\mathcal{S}(\mathbb{R}^n)$ and its Newton potential

$$N(x) := -\frac{1}{(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{\varphi(|y|^n)}{|x-y|^{n-2}} dy, \quad x \in \mathbb{R}^n,$$

is a C^∞ -smooth function with gradient given by

$$\nabla N(x) = -\frac{x}{|x|^n} \frac{\Phi(|x|^n)}{n}, \quad (63) \quad \text{fpot}$$

where $\Phi : [0, \infty) \rightarrow \mathbb{R}$ is the indefinite integral of φ satisfying $\Phi(0) = 0$. Moreover, the second derivatives $\partial_{i,j}^2 N$ lie in $L^p(\mathbb{R}^n)$ for $1 \leq i, j \leq n$.

Proof. Since φ is C^∞ -smooth and vanishes in a neighborhood of 0 and ∞ , it is clear that $h \in \mathcal{S}$. Integrating in polar coordinates, we get

$$N(x) = \frac{1}{(n-2)\omega_n} \int_0^1 r^{n-1} \varphi(r^n) dr \int_{S(0,1)} \frac{d\sigma(\zeta)}{|x-r\zeta|^{n-2}}.$$

When $|x| > r$ the mean value property for harmonic functions yields that

$$\frac{1}{\omega_n} \int_{S(0,1)} \frac{d\sigma(\zeta)}{|x-r\zeta|^{n-2}} = \frac{1}{|x|^{n-2}},$$

and when $|x| \leq r$ it holds that

$$\frac{1}{\omega_n} \int_{S(0,1)} \frac{d\sigma(\zeta)}{|x-r\zeta|^{n-2}} = \frac{1}{r^{n-2}\omega_n} \int_{S(0,1)} \frac{d\sigma(\zeta)}{|x/r-\zeta|^{n-2}} = \frac{1}{r^{n-2}} \quad (64) \quad \text{eqNmc}$$

because σ/ω_n is the Newtonian equilibrium measure of $\overline{B(0,1)}$ which is a regular set, whence (64) is independent of $x \in \overline{B(0,1)}$, see [27, Ch. II, Sec. 13]. Altogether, we obtain:

$$N(x) = \frac{1}{n-2} \int_0^{\min\{|x|,1\}} r^{n-1} \varphi(r^n) \frac{dr}{|x|^{n-2}} + \frac{1}{n-2} \int_{\min\{|x|,1\}}^1 r \varphi(r^n) dr,$$

implying that

$$N(x) = \frac{1}{n(n-2)|x|^{n-2}} \Phi(|x|^n) + \frac{1}{(n-2)} \int_{|x|}^1 r \varphi(r^n) dr, \quad (65) \quad \text{vN}$$

where the integral is interpreted as zero for $|x| \geq 1$. Since φ and Φ vanish in a neighborhood of 0 while φ also vanishes in a neighborhood of 1, one can see that $N \in C^\infty(\mathbb{R}^n)$. Differentiating (65) we get that

$$\nabla N(x) = -\frac{x\Phi(|x|^n)}{n|x|^n} + \frac{1}{n-2} \frac{x}{|x|} (|x|\varphi(|x|^n) - |x|\varphi(|x|^n)),$$

which is (63). Differentiating once more gives us:

$$\partial_{i,j}^2 N(x) = \frac{\Phi(|x|^n)}{n} \left(-\frac{\delta_{i,j}}{|x|^n} + n \frac{x_i x_j}{|x|^{n+2}} \right) - \frac{x_i x_j}{|x|^2} \varphi(|x|^n),$$

and using that φ is compactly supported while Φ is bounded and vanishes in a neighborhood of 0, one verifies that $\partial_{i,j}^2 N \in L^p(\mathbb{R}^n)$. \square

7.3 Example of Sobolev function on a thin set

exampleeffile

We begin by considering the set $E \subset \mathbb{R}^3$ that we define as

$$E = \{(x, y, z) : z \geq 2 - e^{-(x^2+y^2)^\alpha}\} \cap B(0, \sqrt{2})$$

with $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^3$ be the set

$$\Omega = (B(0, \sqrt{2}) \setminus E) \setminus \overline{B(0, 1)},$$

we let $O_0 = (\mathbb{R}^3 \setminus \overline{\Omega}) \setminus B(0, 1)$ and $O_1 = B(0, 1)$. We will call Ω_0 the intersection of Ω and the closed cone with vertex at $(0, 0, 0)$ that passes through the points (x, y, z) on $S(0, \sqrt{2})$ that satisfy $x^2 + y^2 + z^2 = x^2 + y^2 + (2 - e^{-(x^2+y^2)^\alpha})^2$. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be defined as follows

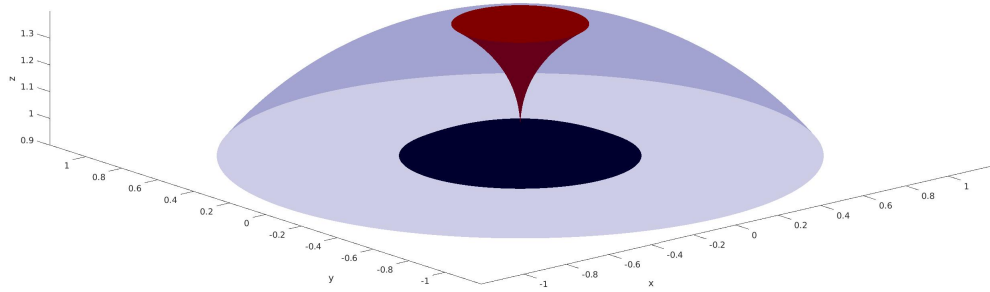
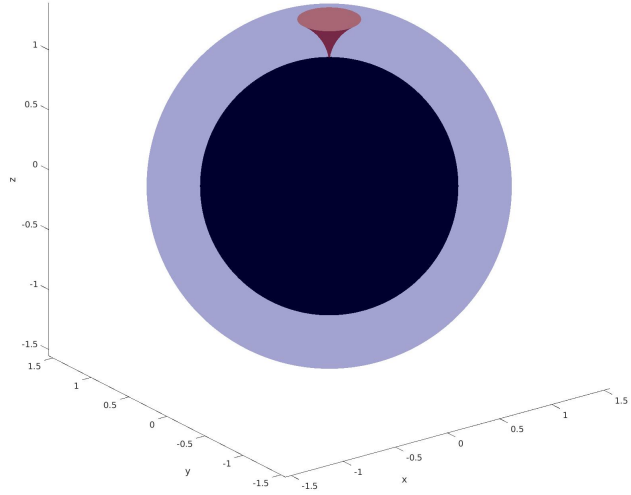


Figure 1: The domain Ω is contained in the grey part, $\alpha = \frac{1}{4}$. The subset of O_0 in red thin at $(0, 0, 1)$. Below is close up of the region around $(0, 0, 1)$

fig1

$$f(x, y, z) = \begin{cases} 1 & \text{on } \overline{O_0} \setminus \{(0, 0, 1)\} \\ x^2 + y^2 + z^2 - 1 & \text{on } \overline{\Omega} \setminus \overline{\Omega_0} \\ \frac{x^2 + y^2 + z^2 - 1}{x^2 + y^2 + (2 - e^{-(x^2+y^2)^\alpha})^2 - 1} & \text{on } \Omega_0 \\ 0 & \text{on } \overline{O_1} \end{cases}.$$

We claim that $f \in W^{1,2}(\mathbb{R}^3)$ for some α . We note that the gradient of f on Ω_0 is given by

$$\nabla f = \begin{pmatrix} \frac{2x}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1} - \frac{4\alpha x(x^2+y^2)^{\alpha-1}e^{-(x^2+y^2)^\alpha}(2-e^{-(x^2+y^2)^\alpha})+2x}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2(x^2+y^2+z^2-1)^{-1}} \\ \frac{2y}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1} - \frac{4\alpha y(x^2+y^2)^{\alpha-1}e^{-(x^2+y^2)^\alpha}(2-e^{-(x^2+y^2)^\alpha})+2y}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2(x^2+y^2+z^2-1)^{-1}} \\ \frac{2z}{x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1} \end{pmatrix}.$$

To show that $f \in W^{1,2}(\mathbb{R}^3)$ we need only show that the following integrals associated with the L^2 -norm of the partial derivatives are finite for some α . In what follows we need to recall the following series representations

$$\begin{aligned} 2 - e^{-r^{2\alpha}} &= 1 + r^{2\alpha} - \frac{r^{4\alpha}}{2} + \frac{r^{6\alpha}}{6} - \frac{r^{8\alpha}}{24} + \frac{r^{10\alpha}}{120} + O(r^{12\alpha}) \\ \sqrt{1 - r^2} &= 1 - \frac{r^2}{2} - \frac{r^4}{8} - \frac{r^6}{16} + O(r^8) \end{aligned} \quad (66) \quad \boxed{\text{seriesexpansions}}$$

We begin with the following integrals associated with the partial derivatives with respect to x and y :

$$\begin{aligned} &\int_{(x^2+y^2) < R^2} \int_{z=\sqrt{1-(x^2+y^2)}}^{z=2-e^{-(x^2+y^2)^\alpha}} \frac{4(x^2+y^2)}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2} dz dx dy \\ &= 4 \int_0^{2\pi} \int_0^R \frac{r^2(2-e^{-r^{2\alpha}}-\sqrt{1-r^2})}{(r^2+(2-e^{-r^{2\alpha}})^2-1)^2} r dr d\theta \\ &= 4 \int_0^{2\pi} \int_0^R \frac{r^2}{(2-e^{-r^{2\alpha}}-\sqrt{1-r^2})(2-e^{-r^{2\alpha}}+\sqrt{1-r^2})^2} r dr d\theta, \end{aligned}$$

using the series expansions in (66) that near $r = 0$ the integrand is dominated by $r^{3-2\alpha}$, hence if $3 - 2\alpha > -1$ the integral is finite, which is necessarily true

for $0 < \alpha < 1$. To finish the discussion for integrals associated with the partial derivatives with respect to x and y we look at

$$\begin{aligned}
& \int_{(x^2+y^2)<R^2} \int_{z=\sqrt{1-(x^2+y^2)}}^{z=2-e^{-(x^2+y^2)^\alpha}} \frac{(4\alpha(x^2+y^2)^{\alpha-1}e^{-(x^2+y^2)^\alpha}(2-e^{-(x^2+y^2)^\alpha})+2)^2}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^4} \\
& \quad (x^2+y^2)(x^2+y^2+z^2-1)^2 dz dx dy \\
&= \int_0^{2\pi} \int_0^R \frac{r^2(4\alpha r^{2\alpha-2}e^{-r^{2\alpha}}(2-e^{-r^{2\alpha}})+2)^2}{(r^2+(2-e^{-r^{2\alpha}})^2-1)^4} \\
& \quad \left[(2-e^{-r^{2\alpha}}-\sqrt{1-r^2})(r^4-2r^2+1)+ \right. \\
& \quad \left. \frac{(2-e^{-r^{2\alpha}})^3-(\sqrt{1-r^2})^3}{3}(2r^2-2)+\frac{(2-e^{-r^{2\alpha}})^5-(\sqrt{1-r^2})^5}{5} \right] r dr d\theta,
\end{aligned}$$

using the series expansions in (66) we have that

$$\begin{aligned}
& \left[(2-e^{-r^{2\alpha}}-\sqrt{1-r^2})(r^4-2r^2+1)+ \right. \\
& \quad \left. \frac{(2-e^{-r^{2\alpha}})^3-(\sqrt{1-r^2})^3}{3}(2r^2-2)+\frac{(2-e^{-r^{2\alpha}})^5-(\sqrt{1-r^2})^5}{5} \right]
\end{aligned}$$

is dominated by $r^{6\alpha}$ near $r=0$. We can show that the integrand near $r=0$ is dominated by $r^{6\alpha-1}$ hence integrable for $0 < \alpha < 1$. Using the fact that for $a, b \in \mathbb{R}$, $(a+b)^2 \leq 2(a^2+b^2)$ and the above integrals we have that the partial derivatives with respect to x and y are in $L^2(\mathbb{R}^3)$. Finally the integral associated with the partial derivative with respect to z :

$$\begin{aligned}
& \int_{(x^2+y^2)<R^2} \int_{z=\sqrt{1-(x^2+y^2)}}^{z=2-e^{-(x^2+y^2)^\alpha}} \frac{4z^2}{(x^2+y^2+(2-e^{-(x^2+y^2)^\alpha})^2-1)^2} dz dx dy \\
&= \frac{4}{3} \int_0^{2\pi} \int_0^R \frac{(2-e^{-r^{2\alpha}})^3-(\sqrt{1-r^2})^3}{(r^2+(2-e^{-r^{2\alpha}})^2-1)^2} r dr d\theta,
\end{aligned}$$

using the series expansions in (66) the integrand is dominated by $r^{1-2\alpha}$ near $r=0$ hence for $0 < \alpha < 1$ the integral is finite. Hence we have shown that for any α with $0 < \alpha < 1$, $f \in W^{1,2}(\mathbb{R}^3)$ and is not constant on $\mathbb{R}^3 \setminus \Omega$.

It remains only to show that the set O_0 is thin at $(0,0,1)$. We use [51, Thm 7.2.5] to show that the set E_0 such that $(x, y, z) \in E_0$ with $0 < z < 1 - \varepsilon(\alpha)$

for some $\varepsilon(\alpha) > 0$ satisfy $\sqrt{(x^2 + y^2)} < \ln\left(\frac{1}{1-z}\right)^{\frac{1}{2\alpha}}$ is thin at $(0,0,0)$. By letting

$$f(t) = \ln\left(\frac{1}{1-t}\right)^{\frac{1}{2\alpha}} \quad \text{for } 0 < t < 1 - \varepsilon(\alpha)$$

then continuously and boundedly extending f in $1 - \varepsilon(\alpha) \leq t \leq 1$ we study the following integral

$$\int_0^1 \frac{1}{t\left(1 + \ln^+\left(\frac{t}{f(t)}\right)\right)} dt.$$

We can choose $\varepsilon(\alpha)$ to be such that $\frac{t}{f(t)} > 1$ in $0 < t < 1 - \varepsilon(\alpha)$, such an $\varepsilon(\alpha)$ exists since $\frac{t}{f(t)} \rightarrow \infty$ as $t \rightarrow 0$. Further we can show that for $t \ln\left(\frac{t}{f(t)}\right) \rightarrow 0$ as $t \rightarrow 0$. It thus follows that

$$\ln\left(t\left(1 + \ln^+\left(\frac{t}{f(t)}\right)\right)\right) > \ln(t), \quad \text{for all } 0 < t < 1 - \varepsilon(\alpha).$$

Since $\ln(t) < 0$ in $0 < t < 1 - \varepsilon(\alpha)$ it follows that we can find ρ with $0 < \rho < 1$ such that

$$\ln\left(t\left(1 + \ln^+\left(\frac{t}{f(t)}\right)\right)\right) > \rho \ln(t) > \ln(t), \quad \text{for all } 0 < t < 1 - \varepsilon(\alpha).$$

Hence, for $0 < t < 1 - \varepsilon(\alpha)$ the integrand is dominated by $\frac{1}{t^\rho}$ for $0 < \rho < 1$ and the integral is finite implying that E_0 is thin at $(0,0,0)$. By noting that $(x, y, z) \in O_0$ with $\sqrt{(x^2 + y^2)} < 1$ and $1 < z < 2 - \varepsilon$ is just $(0,0,1) + E_0$ we conclude that O_0 is thin at $(0,0,1)$.

References

Jac1975

- [1] J. D. Jackson. *Classical electrodynamics*. John Wiley & Sons, Inc., New York-London-Sydney, second edition, 1975.

Baratchart_2012

- [2] L. Baratchart, D. P. Hardin, E. E. Lima, E. B. Saff, and B. P. Weiss. Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions. *Inverse Problems*, 29(1):015004, 2012.

- Gerhards** [3] C. Gerhards. On the unique reconstruction of induced spherical magnetizations. *Inverse Problems*, 32, 2016.
- BGK** [4] L. Baratchart, C. Gerhards, and A. Kegeles. Decomposition of L^2 -vector fields on Lipschitz surfaces: characterization via null-spaces of the scalar potential. arxiv.org/pdf/2009.05337.pdf, 2020.
- BDQ** [5] L. Baratchart, T. Qian, and D. Pei. Hardy-Hodge decomposition of vector fields on compact Lipschitz hypersurfaces. hal.inria.fr/hal-02936934, 2020.
- BGHNS-2020** [6] L. Baratchart, C. Villalobos-Guillén, D. Hardin, M. Northington, and E. B. Saff. Inverse potential problems for divergence of measures with total variation regularization. *Foundations of Computational Mathematics*, 20:1273–1307, 2020.
- AdamsHedberg** [7] D. R. Adams and L. I. Hedberg. *Function spaces and potential theory*. Grundlehren der mathematischen Wissenschaften. Springer, 1996.
- Mazy'a** [8] V. Maz'ya. *Sobolev spaces*, volume 342 of *Grundlehren der mathematischen wissenschaften*. Springer, 2d edition, 2011.
- Adams** [9] R. A. Adams. *Sobolev spaces*. Academic Press New York, 1975.
- Grisvard** [10] P. Grisvard. *Elliptic problems in nonsmooth domains*. Pitman, 1985.
- Federer** [11] H. Federer. *Geometric measure theory*. Springer, 1996.
- Ziemer** [12] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.
- Hatcher** [13] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.
- Stein** [14] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions (PMS-30)*. Princeton University Press, 1970.
- JoWal** [15] A. Jonsson and H. Wallin. *Function spaces on subsets of \mathbb{R}^n* . Harwood Academic Publishers, 1984.
- JKinhom** [16] D. A. Jerison and C. E. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *Journal of Functional Analysis*, 130:161–219, 1995.

- Sohr** [17] H. Sohr. *The Navier-Stokes Equations: An Elementary Functional Analytic Approach*. Birkhäuser Advanced Texts Basler Lehrbücher. Springer Basel, 2001.
- Galdi** [18] G. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems*. Springer Monographs in Mathematics. Springer New York, 2011.
- Demengel** [19] F. Demengel, G. Demengel, and R. Ern . *Functional Spaces for the Theory of Elliptic Partial Differential Equations*. Universitext. Springer, 2012.
- BNPB** [20] H. Bhatia, G. Norgard, V. Pascucci, and P. T. Bremer. The Helmholtz-Hodge decomposition—a survey. *IEEE Trans. on Visualization and Computer Graphics*, 19(8), 2013.
- Fabes** [21] O. Fabes, O. Mendez, and M. Mitrea. Boundary Layers on Sobolev–Besov Spaces and Poisson’s Equation for the Laplacian in Lipschitz Domains. *Journal of Functional Analysis*, 159(2):323 – 368, 1998.
- GengShen** [22] J. Geng and Z. Shen. The Neumann problem and Helmholtz decomposition in convex domains. *Journal of Functional Analysis*, 259:2147–2164, 2010.
- Bogovskii** [23] M. Bogovskii. Decomposition of $L^p(\Omega; \mathbb{R}^n)$ into the direct sum of subspaces of solenoidal and potential vector fields. *Doklady Mathematics*, 33:161–165, 03 1986.
- Martin** [24] T. Iwaniec and G. Martin. *Geometric Function Theory and Non-linear Analysis*. Oxford mathematical monographs. Oxford University Press, 2001.
- VERCHOTA** [25] G. Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains. *Journal of Functional Analysis*, 59(3):572 – 611, 1984.
- Axler** [26] S. Axler, P. Bourdon, and W. Ramey. *Harmonic Function Theory*. Graduate Texts in Mathematics. Springer New York, 2006.
- Landkof** [27] N. S. Landkof. *Foundations of modern potential theory*. Number 180 in Grundlehren der Mat. Wiss. Springer, 1972.

- Dahlberg** [28] B. E. Dahlberg. Estimates of harmonic measure. *Ark. Rat. Mec. Anal.*, 65:278–288, 1977.
- FdeG** [29] K. Fabian and L. V. de Groot. A uniqueness theorem for tomography-assisted potential-field inversion. *Geophysical Journal International*, 216:760–766, 2019.
- BGKM** [30] L. Baratchart, C. Gerhards, A. Kegeles, and P. Menzel. Unique reconstruction of simple magnetizations from their magnetic potential. in preparation, 2021.
- GP** [31] V. Guillemin and A. Pollack. *Differential topology*. Prentice-Hall, 1974.
- Whitney** [32] Hassler Whitney. Analytic extensions of differentiable functions defined in closed sets. *Transactions of the Amer. Math. Soc.*, pages 63–89, 1934.
- Jost** [33] J. Jost. *Partial Differential Equations*. Graduate Texts in Mathematics. Springer New York, 2002.
- Clarkson** [34] J. A. Clarkson. Uniformly convex spaces. *Transactions of the American Mathematical Society*, 40(3):396–414, 1936.
- Beauzamy** [35] B. Beauzamy. *Introduction to Banach spaces and their geometry*. Number no. 86; no. 1985 in Notas de matemática. North-Holland, 1985.
- Duren** [36] P. L. Duren. *Theory of H^p Spaces*. Academic Press, 1970.
- Chidume** [37] C. Chidume. *Geometric Properties of Banach Spaces and Nonlinear Iterations*. Lecture Notes in Mathematics. Springer London, 2008.
- Alber** [38] A. Kartsatos(editor). *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*. Lecture Notes in Pure and Applied Mathematics. Taylor & Francis, 1996.
- Petru** [39] G. Kyriazis and P. Petrushev. On the construction of frames for spaces of distributions. *Journal of Functional Analysis*, 257:2159–2187, 2009.
- Miranda** [40] F. Miranda, J. Rodrigues, and L. Santos. A class of stationary non-linear Maxwell systems. *Mathematical Models and Methods in Applied Sciences*, 19:1883–1905, 10 2009.

- Amrouche** [41] C. Amrouche and N. E. H. Seloula. Lp-Theory for vector potentials and Sobolev's inequalities for vector fields: Application to the Stokes equations with pressure boundary conditions. *Mathematical Models and Methods in Applied Sciences*, 23(01):37–92, 2013.
- Lions1969** [42] J.-L. Lions. *Quelques méthodes de résolution de problèmes aux limites non linéaires*. Dunod, 1969.
- Brezis** [43] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer New York, 2010.
- Frames** [44] L. Borup, R. Gribonval, and M. Nielsen. Tight wavelet frames in Lebesgue and Sobolev spaces. *Journal of function spaces and applications*, 2, 01 2004.
- Showalter** [45] R.E. Showalter and American Mathematical Society. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Mathematical surveys and monographs. American Mathematical Society, 1997.
- Warner** [46] F. Warner. *Foundations of differential manifolds and Lie groups*. Springer, 1983.
- GKS** [47] V. M. Gol'dshtein, V. I. Kuz'minov, and I. A. Shvedov. Differential forms on Lipschitz manifolds. *Siberian Mathematical Journal*, 23(2):151–161, 1982.
- BeSha** [48] C. Bennett and R. Sharpley. *Interpolation of Operators*. Academic Press, 1988.
- BeLof** [49] J. Bergh and J. Löfström. *Interpolation Spaces*. Springer, 1976.
- RudinFA** [50] W. Rudin. *Functional Analysis*. McGraw-Hill, 1973.
- Armitage** [51] D.H. Armitage and S.J. Gardiner. *Classical Potential Theory*. Springer Monographs in Mathematics. Springer London, 2000.