



HAL
open science

Kantorovich-Rubinstein Quasi-Metrics IV: Lenses, Quasi-Lenses and Forks

Jean Goubault-Larrecq

► **To cite this version:**

Jean Goubault-Larrecq. Kantorovich-Rubinstein Quasi-Metrics IV: Lenses, Quasi-Lenses and Forks. *Topology and its Applications*, 2023, 332, pp.108513. 10.1016/j.topol.2023.108513 . hal-03911034

HAL Id: hal-03911034

<https://inria.hal.science/hal-03911034v1>

Submitted on 22 Dec 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Kantorovich-Rubinstein Quasi-Metrics IV: Lenses, Quasi-Lenses and Forks

Jean Goubault-Larrecq

*Université Paris-Saclay, CNRS, ENS Paris-Saclay, Laboratoire Méthodes Formelles,
91190, Gif-sur-Yvette, France.*

goubault@lsv.fr

Abstract

Lenses and quasi-lenses on a space X form models of erratic non-determinism. When X is equipped with a quasi-metric d , there are natural quasi-metrics $d_{\mathcal{P}}$ and $d_{\mathcal{P}}^a$ on the space of quasi-lenses on X , which resemble the Pompeiu-Hausdorff metric (and contain it as a subcase when d is a metric), and are tightly connected to the Kantorovich-Rubinstein quasi-metrics d_{KR} and d_{KR}^a of Parts I, II and III, through an isomorphism between quasi-lenses and so-called discrete normalized forks. We show that the space of quasi-lenses on X is continuous complete, resp. algebraic complete, if X, d is itself continuous complete, resp. algebraic complete. In those cases, we also show that the $d_{\mathcal{P}}$ -Scott and $d_{\mathcal{P}}^a$ -Scott topologies coincide with the Vietoris topology. We then prove similar results on spaces of (sub)normalized forks, not necessarily discrete; those are models of mixed erratic non-determinism and probabilistic choice. For that, we need the additional assumption that the cone $\mathcal{L}X$ of lower continuous maps from X to $\overline{\mathbb{R}}_+$, with the Scott topology, has an almost open addition map (which is the case if X is locally compact and coherent, notably); we also need X to be compact in the case of normalized forks. The relevant quasi-metrics are simple extensions of the Kantorovich-Rubinstein quasi-metrics d_{KR}^a of Parts I, II and III.

Keywords: Quasi-metric, lens, quasi-lens, prevision, fork,
Kantorovich-Rubinstein quasi-metric, Vietoris topology (lower, upper)

2020 MSC: 54E99, 54H30



For the purpose of Open Access, a CC-BY public copyright licence has been applied by the authors to the present document and will be applied to all subsequent versions up to the Author Accepted Manuscript arising from this submission.

1. Introduction

A prevision on a space X is a certain kind of functional F , mapping lower semi-continuous functions $h: X \rightarrow \overline{\mathbb{R}}_+$ to an extended non-negative real number $F(h) \in \overline{\mathbb{R}}_+$ [6]. In Part I [12], Part II [13] and Part III [14], we studied the quasi-metric d_{KR}^a , defined by the formula:

$$d_{\text{KR}}^a(G, G') \stackrel{\text{def}}{=} \sup_{h \in \mathcal{L}_1^a X} \mathbf{d}_{\mathbb{R}}(G(h), G'(h)), \quad (1)$$

for any $a > 0$, where $\mathcal{L}_1^a X$ denotes the set of a -bounded 1-Lipschitz continuous maps from X to $\overline{\mathbb{R}}_+$, and $\mathbf{d}_{\mathbb{R}}(x, y) = \max(x - y, 0)$. (We use the convention that $\mathbf{d}_{\mathbb{R}}(x, y) = 0$ when $x = y$, even when $x = y = +\infty$.)

Linear previsions are in one-to-one correspondence with continuous valuations, a notion closely related to that of measures, and model probabilistic choice. In Part I, we showed that, if X, d is continuous complete (resp., algebraic complete), then the space of (sub)normalized linear previsions is, too, in the d_{KR}^a quasi-metrics.

Discrete normalized sublinear and discrete superlinear previsions are in one-to-one correspondence with hyperspaces of non-empty closed, respectively non-empty compact saturated subsets, modeling the angelic and demonic variants of non-deterministic choice. We showed similar results of preservation of continuous (resp., algebraic) completeness in Part II.

We did the same thing for sublinear previsions and superlinear previsions (not necessarily discrete) in Part III. Those model mixtures of (angelic, demonic) probabilistic and non-deterministic choice.

In this Part IV, we deal with models of erratic non-deterministic choice, through spaces of lenses and quasi-lenses, and with models of mixed, erratic non-deterministic and probabilistic choice, through so-called forks¹. We deal with the former in Section 3, with the latter in Section 4. Before that, we give some required preliminary information on the objects we deal with in Section 2. I am afraid that this section is a bit lengthy; this was written in the hope to make this paper self-contained. We will still need to invoke various other results from previous parts [13, 14] in later sections, as well as from [10, 16], and those results will be stated along the way.

¹An unpublished version of that work is available on arXiv [9]. The present paper covers Section 15.

2. Preliminaries

We refer the reader to [8, 5] for general information on topology and domain theory. The former reference contains some required information on quasi-metric space theory, for which the reader is advised to read [16], as well as [12, 13, 14], although we will recapitulate the points that we need here.

2.1. Domain Theory

A *directed complete partial order*, or *dcpo*, is a partially ordered set P in which every directed family has a supremum. A subset U of P is *Scott-open* if and only if it is upwards-closed, and every directed family whose supremum is in U intersects U . The Scott-open subsets form the *Scott topology*. A *Scott-continuous* map between two dcpos is a monotonic map that preserves directed suprema; equivalently, it is a continuous map between the two dcpos equipped with their Scott topology.

We write $\downarrow A$ for the downward closure of a subset A , and $\uparrow A$ for its upward closure. We will use similar notations in topological spaces, where the implied preordering \leq will be the *specialization ordering*: $x \leq y$ if and only if every open neighborhood of x contains y . We note that the closure of a point x is simply $\downarrow x$. The specialization preordering of a dcpo is its original ordering. A *saturated* subset of a topological space is a subset that coincides with the intersection of its open neighborhoods, or equivalent that is upwards closed in its specialization preordering.

A *retraction* of Y onto X in a given category is a pair of morphisms:

$$Y \begin{array}{c} \xrightarrow{r} \\ \xleftarrow{s} \end{array} X$$

such that $r \circ s = \text{id}_X$. The morphism s is the *section*, and r itself will often be called a retraction; then X is a *retract* of Y . A *topological retract* is a retract in the category of topological spaces and continuous maps. An *order-retract* is a retract in the category of partially ordered sets and monotonic maps. A *Scott retract* is a retract in the category of dcpos and Scott-continuous maps. We will also use a notion of 1-Lipschitz retract, which we will introduce later.

2.2. Quasi-Metrics

A *quasi-metric* on a set X is a map $d: X \times X \rightarrow \overline{\mathbb{R}}_+$ satisfying: $d(x, x) = 0$; $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality); $d(x, y) = d(y, x) = 0$ implies $x = y$. The pair X, d is then called a *quasi-metric space*. Its

specialization ordering is given by $x \leq^d y$ if and only if $d(x, y) = 0$; the name comes from the fact that \leq^d is the specialization ordering of X in several topologies, notably the open ball topology and the d -Scott topology (see below).

We will always see $\overline{\mathbb{R}}_+$ as a quasi-metric space, with quasi-metric $\mathbf{d}_{\mathbb{R}}$ (already) defined by $\mathbf{d}_{\mathbb{R}}(x, y) = \max(x - y, 0)$, with the convention that $(+\infty) - (+\infty) = 0$. Its specialization ordering is the usual ordering.

A *formal ball* on X, d is a pair $(x, r) \in X \times \mathbb{R}_+$; x is the *center* and r is the *radius*. Formal balls are organized into a poset $\mathbf{B}(X, d)$, where formal balls are ordered by $(x, r) \leq^{d^+} (y, s)$ if and only if $d(x, y) \leq r - s$ [21, 4]. There is also a quasi-metric d^+ on $\mathbf{B}(X, d)$, defined by $d^+((x, r), (y, s)) \stackrel{\text{def}}{=} \max(d(x, y) - r + s, 0)$, and \leq^{d^+} is its specialization ordering.

A quasi-metric space is *complete* if and only if $\mathbf{B}(X, d)$ is a dcpo. It is *continuous complete* if and only if $\mathbf{B}(X, d)$ is a continuous dcpo. (We will not need to know what the latter means in this paper.) There is a third, a stronger notion of completeness, being algebraic complete, which we will describe below. All three notions coincide on metric spaces.

Given a quasi-metric space X, d , there is an injective map $\eta_X: x \mapsto (x, 0)$ from X to $\mathbf{B}(X, d)$, and the *d -Scott topology* on X is the coarsest that makes it continuous. Unless specified otherwise, this is the topology we will assume for every quasi-metric space. The specialization ordering of the d -Scott topology is \leq^d . When d is a metric, the d -Scott topology coincides with the familiar open ball topology. The $\mathbf{d}_{\mathbb{R}}$ -Scott topology on $\overline{\mathbb{R}}_+$ coincides with the Scott topology of its ordering.

A quasi-metric space X, d is *standard* if and only if, for every directed family of formal balls $(x_i, r_i)_{i \in I}$, for every $s \in \mathbb{R}_+$, $(x_i, r_i)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$ if and only if $(x_i, r_i + s)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$. Writing the supremum of the former as (x, r) , we then have that $r = \inf_{i \in I} r_i$, and the supremum of the latter is $(x, r + s)$ —this holds not only for $s \in \mathbb{R}_+$, but for every $s \geq -r$. Every metric space is standard. We will be concerned with complete quasi-metric spaces only, and all complete quasi-metric spaces are standard.

A quasi-metric space X, d is *algebraic complete* if and only if it is complete and every point is the d -limit of a Cauchy net of d -finite points. We will not use that definition, and instead we will use the following. We write $B_{x,r}^d \stackrel{\text{def}}{=} \{y \in X \mid d(x, y) < r\}$ for the open ball centered at x with radius r in X . We consider open balls $B_{(x,0),\epsilon}^{d^+}$ in $\mathbf{B}(X, d)$. If those open balls are

Scott-open for every $\epsilon > 0$, then we say that x is a *center point*. For example, the center points of $\overline{\mathbb{R}}_+$ are the points of \mathbb{R}_+ , namely, all point except $+\infty$; also, every point in a metric space is a center point. When X, d is standard, the d -finite points are exactly the center points [16, Lemma 5.7]. Then, a quasi-metric space X, d is *algebraic* if and only if it is standard and for every $x \in X$, $(x, 0)$ is the supremum of some directed family of formal balls (x_i, r_i) where each x_i is a center point. For example, $\overline{\mathbb{R}}_+, d_{\mathbb{R}}$ is algebraic complete.

Refining this, a *strong basis* \mathcal{B} of a standard quasi-metric space X, d is a set of center points such that for every $x \in X$, $(x, 0)$ is the supremum of some directed family of formal balls (x_i, r_i) where each x_i is in \mathcal{B} . A standard quasi-metric space is algebraic if and only if it has a strong basis. The largest one is then the collection of all center points.

Given a strong basis \mathcal{B} of an algebraic quasi-metric space X, d , the open balls centered at points from \mathcal{B} form a base of the d -Scott topology [13, Lemma 5.19].

2.3. Lipschitz and Lipschitz continuous functions

For every $\alpha \in \mathbb{R}_+$, and for every map $f: X, d \rightarrow Y, \partial$ between quasi-metric spaces, let $\mathbf{B}^\alpha(f)$ map $(x, r) \in \mathbf{B}(X, d)$ to $(f(x), \alpha r) \in \mathbf{B}(Y, \partial)$. Then f is α -Lipschitz if and only if $\mathbf{B}^\alpha(f)$ is monotonic, and α -Lipschitz continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous [11, Definition 2.3]. The function f is α -Lipschitz if and only if $\partial(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. When both X, d and Y, ∂ are standard, $f: X \rightarrow Y$ is α -Lipschitz continuous if and only if f is both α -Lipschitz, and continuous from X with its d -Scott topology to Y with its ∂ -Scott topology [11, Proposition 2.5].

We will call *1-Lipschitz continuous retraction* any retraction in the category of standard quasi-metric spaces and 1-Lipschitz continuous maps. The main use of this notion is the following result [16, Theorem 7.9]: the continuous complete quasi-metric spaces are exactly the 1-Lipschitz continuous retracts of algebraic complete quasi-metric spaces.

A useful criterion for α -Lipschitz continuity of $\overline{\mathbb{R}}_+$ -valued maps is given by Lemma 2.7 of [11]: if X, d is standard, then $h: X, d \rightarrow \overline{\mathbb{R}}_+, d_{\mathbb{R}}$ is α -Lipschitz continuous if and only if $h': \mathbf{B}(X, d) \rightarrow \mathbb{R} \cup \{+\infty\}$, $h'(x, r) \stackrel{\text{def}}{=} h(x) - \alpha r$, is Scott-continuous.

Given any standard quasi-metric space X, d , every α -Lipschitz continuous map from X, d to $\overline{\mathbb{R}}_+$ is lower semicontinuous. We write $\mathcal{L}_\alpha(X, d)$, or $\mathcal{L}_\alpha X$ for short, for the subspace of $\mathcal{L}X$ consisting of α -Lipschitz continuous maps, with the topology induced by the Scott topology from $\mathcal{L}X$. Note that this may

or may not coincide with the Scott topology on $\mathcal{L}_\alpha(X, d)$. Let $\mathcal{L}_\infty(X, d) \stackrel{\text{def}}{=} \bigcup_{\alpha>0} \mathcal{L}_\alpha(X, d)$, also with the subspace topology from $\mathcal{L}X$.

Assuming X, d standard once again, every lower semicontinuous map $f \in \mathcal{L}X$ can be approximated as closely as one wishes by Lipschitz continuous maps. For every $\alpha \in \mathbb{R}_+$, there is a largest α -Lipschitz continuous map $f^{(\alpha)}$ pointwise below f , and the family $(f^{(\alpha)})_{\alpha \in \mathbb{R}_+}$ is a chain whose pointwise supremum is equal to f [16, Theorem 6.17].

The spaces of Lipschitz-continuous maps are closed under the usual operations, provided that X, d is standard, see Proposition 6.7 of [16] or Proposition 2.8 of [11]. Notably, if $f \in \mathcal{L}_\alpha X$ and $g \in \mathcal{L}_\beta X$, then $f + g$ is in $\mathcal{L}_{\alpha+\beta} X$, and $\min(f, g)$ is in $\mathcal{L}_{\max(\alpha, \beta)} X$.

We also write $\mathcal{L}_\alpha^a(X, d)$, or $\mathcal{L}_\alpha^a X$, for the subspace of $\mathcal{L}_\alpha X$ of those α -Lipschitz continuous maps from X, d to $\overline{\mathbb{R}}_+$ that take their values in $[0, \alpha a]$. We let $\mathcal{L}_\infty^b(X, d) \stackrel{\text{def}}{=} \bigcup_{\alpha>0} \mathcal{L}_\alpha^a(X, d)$ be the subspace of all bounded maps in $\mathcal{L}_\infty(X, d)$.

2.4. Previsions

We write $\mathcal{L}X$ for the space of lower semicontinuous maps from a space X to $\overline{\mathbb{R}}_+$. Elements of $\mathcal{L}X$ are ordered pointwise, and $\mathcal{L}X$ is topologized with the Scott topology of that ordering.

A *prevision* on a topological space X is a Scott-continuous map F from $\mathcal{L}X$ to $\overline{\mathbb{R}}_+$ that is *positively homogeneous*, namely: $F(\alpha h) = \alpha F(h)$ for all $\alpha \in \mathbb{R}_+$, $h \in \mathcal{L}X$. It is:

- *sublinear* if $F(h + h') \leq F(h) + F(h')$ holds for all $h, h' \in \mathcal{L}X$,
- *superlinear* if $F(h + h') \geq F(h) + F(h')$ holds,
- *linear* if it is both sublinear and superlinear,
- *subnormalized* if $F(\alpha \mathbf{1} + h) \leq \alpha + F(h)$ holds, where $\mathbf{1}$ is the constant 1 map,
- *normalized* if $F(\alpha \mathbf{1} + h) = \alpha + F(h)$ holds,
- *discrete* if $F(f \circ h) = f(F(h))$ for every $h \in \mathcal{L}X$ and every strict $f \in \mathcal{L}\overline{\mathbb{R}}_+$ —a map f is strict if and only if $f(0) = 0$.

The previsions of the form $h \mapsto \int_{x \in X} h(x) d\nu$, where ν is a continuous valuation on X , are all linear, and those are the only linear previsions. Continuous valuations are a close cousin to the notion of measure, and coincide with them on large classes of spaces [3]; this includes all continuous complete quasi-metric spaces in their d -Scott topologies. We write $\mathbf{V}X$ for the space of linear previsions on X , $\mathbf{V}_{\leq 1}X$ for the space of subnormalized linear previsions on X , \mathbf{V}_1X for the space of normalized linear previsions on X .

The space $\mathbf{P}X$ of all previsions on X is given the *weak topology*, defined as the coarsest topology such that $F \mapsto F(h)$ is lower semicontinuous for every $h \in \mathcal{L}X$. It has subbasic open sets of the form $[h > a] \stackrel{\text{def}}{=} \{F \in \mathbf{P}X \mid F(h) > a\}$, for each $h \in \mathcal{L}X$ and $a \in \mathbb{R}_+$. All the subspaces of previsions we will consider, including $\mathbf{V}X$, $\mathbf{V}_{\leq 1}X$, and \mathbf{V}_1X , will have the corresponding subspace topology, which we again call the weak topology, and whose subbasic open sets will again be written as $[h > a]$.

Defining $\mathbf{P}f$ for every continuous map f by $\mathbf{P}f(F)(h) \stackrel{\text{def}}{=} F(h \circ f)$ turns \mathbf{P} into an endofunctor on the category of topological spaces.

The *Hoare hyperspace* \mathcal{H}_0X (resp., $\mathcal{H}X$) is the space of all (resp., non-empty) closed subsets of X , with the *lower Vietoris topology*. That topology is generated by subbasic open sets $\diamond U = \{C \in \mathcal{H}_0X \mid C \cap U \neq \emptyset\}$, where U ranges over the open subsets of X . The map $C \mapsto F^C$, where $F^C(h) \stackrel{\text{def}}{=} \sup_{x \in C} h(x)$ for every $h \in \mathcal{L}X$, is a homeomorphism of \mathcal{H}_0X (resp., $\mathcal{H}X$) onto the space of discrete sublinear (resp., and normalized) previsions on X with the weak topology [13, Lemma 3.4]. This is the object part of a functor defined on morphisms by $\mathcal{H}_0(f)(C) \stackrel{\text{def}}{=} cl(f[C])$, where cl is closure and $f[C]$ is the image of C under f .

Symmetrically, the *Smyth hyperspace* $\mathcal{Q}X$ is the space of all non-empty compact saturated subsets of X , with the *upper Vietoris topology*, generated by basic open sets $\square U \stackrel{\text{def}}{=} \{Q \in \mathcal{Q}X \mid Q \subseteq U\}$, where U ranges over the open subsets of X . When X is sober, the map $Q \mapsto F_Q$, where $F_Q(h) \stackrel{\text{def}}{=} \min_{x \in Q} h(x)$, is a homeomorphism of $\mathcal{Q}X$ onto the space of normalized discrete superlinear previsions on X with the weak topology [13, Lemma 3.6]. This is the object part of a functor defined on morphisms by $\mathcal{Q}(f)(Q) \stackrel{\text{def}}{=} \uparrow f[Q]$, where we recall that \uparrow denotes upward closure in the specialization preordering of the target space.

2.5. Quasi-metrics on spaces of previsions

We have already defined the quasi-metric d_{KR}^a in Equation (1). Now that we have defined $\mathcal{L}_1^a X$ formally, this makes sense. The main results of Part III are as follows. In all cases, the *naive supremum* (G, r) of a directed family of formal balls $(G_i, r_i)_{i \in I}$ (with respect to the d_{KR}^a quasi-metric) is defined by $r \stackrel{\text{def}}{=} \inf_{i \in I} r_i$, and:

$$G(h) \stackrel{\text{def}}{=} \sup_{i \in I} (G_i(h) + \alpha r - \alpha r_i) \quad (2)$$

for every $\alpha > 0$ and every $h \in \mathcal{L}_\alpha^a X$, and $r \stackrel{\text{def}}{=} \inf_{i \in I} r_i$. This defines G on $\mathcal{L}_\infty^b X = \bigcup_{\alpha > 0} \mathcal{L}_\alpha^a X$, and one can check that the definition does not actually depend on the chosen α such that h is in $\mathcal{L}_\alpha^a X$.

Note that nothing imposes that G be a prevision. In fact, G is not even defined on the whole of $\mathcal{L}X$. However, and if X, d is standard, there is at most one way to extend (2) to a continuous function from $\mathcal{L}X$ to $\overline{\mathbb{R}}_+$, and if that happens, then that extension, which we again write as G , is a prevision. We will state that more precisely in Theorem 2.4. Meanwhile, we have the following three completeness results, one for linear previsions, one for sublinear previsions, and the final one for superlinear previsions.

The first result synthesizes Theorems 4.15, 8.5, 9.5, 5.7, 6.1, Corollary 8.3 and Corollary 8.6 of Part I [12]. A *simple valuation* is a finite linear combination $\sum_{i=1} a_i \delta_{x_i}$, where each a_i is in \mathbb{R}_+ , and δ_x is the linear prevision $h \mapsto h(x)$. A simple valuation is *supported on* a set \mathcal{B} if and only if every x_i is in \mathcal{B} .

Theorem 2.1 (Completeness for linear previsions [12]). *For every continuous complete quasi-metric space X, d , the spaces $\mathbf{V}_{\leq 1} X$ and $\mathbf{V}_1 X$ are continuous complete with respect to the d_{KR}^a quasi-metric ($a > 0$). Directed suprema of formal balls are computed as naive suprema. The d_{KR}^a -Scott topology coincides with the weak topology.*

If X, d is algebraic complete, then $\mathbf{V}_{\leq 1} X$ and $\mathbf{V}_1 X$ are algebraic complete under d_{KR}^a . The simple (sub)normalized valuations supported on \mathcal{B} form a strong basis, for any given strong basis \mathcal{B} of X, d .

$\mathbf{V}_{\leq 1}, d_{KR}^a$ and \mathbf{V}_1, d_{KR}^a define endofunctors on the category of continuous complete quasi-metric spaces and 1-Lipschitz continuous maps, whose action on morphisms is that of \mathbf{P} : for every 1-Lipschitz continuous map $f: X, d \rightarrow Y, \partial$ between continuous complete quasi-metric spaces, $\mathbf{P}f$ is 1-Lipschitz continuous from $\mathbf{V}_{\leq 1} X, d_{KR}^a$ to $\mathbf{V}_{\leq 1} Y, \partial_{\mathcal{H}}^a$ and from $\mathbf{V}_1 X, d_{KR}^a$ to $\mathbf{V}_1 Y, \partial_{\mathcal{H}}^a$.

The next result synthesizes Theorem 3.15, Lemma 3.16, Theorem 3.17, and Corollary 3.19 of Part III [14].

Theorem 2.2 (Completeness for sublinear previsions [14]). *Let X, d be a continuous complete quasi-metric space, and $a > 0$.*

The space of all subnormalized (resp., normalized) sublinear previsions on X with the d_{KR}^a quasi-metric is continuous complete. The d_{KR}^a -Scott topology coincides with the weak topology. Directed suprema of formal balls are computed as naive suprema.

If X, d is algebraic complete, with a strong basis \mathcal{B} , then the space of all subnormalized (resp., normalized) sublinear previsions on X with the d_{KR}^a quasi-metric is algebraic complete. The sublinear previsions of the form $h \mapsto \max_{i=1}^m \sum_{j=1}^{n_i} a_{ij}h(x_{ij})$, with $m \geq 1$, $\sum_{j=1}^{n_i} a_{ij} \leq 1$ (resp., $= 1$) for every i , and where each x_{ij} is in \mathcal{B} , are center points, and form a strong basis.

There is an endofunctor on the category of continuous (resp., algebraic) complete quasi-metric spaces and 1-Lipschitz continuous maps, which sends every object X, d to the space of (sub)normalized sublinear previsions on X with the d_{KR}^a -Scott topology ($a > 0$), and every 1-Lipschitz continuous map f to $\mathbf{P}f$.

The next result summarizes Theorems 4.6, 4.7, and Corollary 4.9 of Part III [14].

Theorem 2.3 (Completeness for superlinear previsions [14]). *Let X, d be a continuous complete quasi-metric space, and $a > 0$.*

The space of all subnormalized (resp., normalized) superlinear previsions on X with the d_{KR}^a quasi-metric is continuous complete. The d_{KR}^a -Scott topology coincides with the weak topology. Directed suprema of formal balls are computed as naive suprema.

If X, d is algebraic complete, with a strong basis \mathcal{B} , then the space of all subnormalized (resp., normalized) superlinear previsions on X with the d_{KR}^a quasi-metric is algebraic complete. The superlinear previsions of the form $h \mapsto \min_{i=1}^m \sum_{j=1}^{n_i} a_{ij}h(x_{ij})$, with $m \geq 1$, $\sum_{j=1}^{n_i} a_{ij} \leq 1$ (resp., $= 1$) for every i , and where each x_{ij} is in \mathcal{B} , are center points, and form a strong basis.

There is an endofunctor on the category of continuous (resp., algebraic) complete quasi-metric spaces and 1-Lipschitz continuous maps, which sends every object X, d to the space of (sub)normalized superlinear previsions on X with the d_{KR}^a -Scott topology ($a > 0$), and every 1-Lipschitz continuous map f to $\mathbf{P}f$.

The variant d_{KR} is defined by the following formula, where $\mathcal{L}_1^a X$ is replaced by $\mathcal{L}_1 X$.

$$d_{\text{KR}}(G, G') \stackrel{\text{def}}{=} \sup_{h \in \mathcal{L}_1 X} \mathbf{d}_{\mathbb{R}}(G(h), G'(h)). \quad (3)$$

This is a more natural quasi-metric on spaces of discrete previsions, which were studied in Part II. In that case, we overload the notion of *naive supremum*, by reusing (2), where now h ranges over $\mathcal{L}_\alpha(X, d)$, $\alpha > 0$, not $\mathcal{L}_\alpha^a(X, d)$.

The importance of naive suprema is due to the following results, which are Proposition 4.7, 4.15 of Part II [13], read through the lens of Propositions 4.5 and 4.14 of Part II respectively.

Theorem 2.4 ([13]). *Let X, d be a standard quasi-metric space, and let $(G_i, r_i)_{i \in I}$ be any directed family of formal balls on $\mathbf{P}X$ with the d_{KR} (resp., d_{KR}^a) quasi-metric, and let it have (G, r) as naive supremum.*

If G is continuous from $\mathcal{L}_\infty X$ (resp., $\mathcal{L}_\infty^b X$) to $\overline{\mathbb{R}}_+$ with its Scott topology, then G is the restriction to $\mathcal{L}_\infty X$ (resp., $\mathcal{L}_\infty^b X$) of a unique prevision on X . That prevision maps every $h \in \mathcal{L}X$ to $\sup_{\alpha \in \mathbb{R}_+} G(h^{(\alpha)})$. Additionally, and equating G with that unique prevision, (G, r) is the supremum of $(G_i, r_i)_{i \in I}$.

If G_i is sublinear, resp. superlinear, resp. linear, resp. subnormalized, resp. normalized, resp. discrete, then so is G .

For discrete normalized superlinear previsions, the homeomorphism $Q \mapsto F_Q$ allows us to transport the d_{KR}^a and d_{KR} quasi-metrics to quasi-metrics $d_{\mathcal{Q}}^a$ and $d_{\mathcal{Q}}$ on $\mathcal{Q}X$, namely: $d_{\mathcal{Q}}(Q, Q') \stackrel{\text{def}}{=} d_{\text{KR}}(F_Q, F_{Q'})$ [13, Definition 6.1] and $d_{\mathcal{Q}}^a(Q, Q') \stackrel{\text{def}}{=} d_{\text{KR}}^a(F_Q, F_{Q'})$. If X, d is standard, then we have the Pompeiu-Hausdorff-like formula $d_{\mathcal{Q}}(Q, Q') = \sup_{x' \in Q'} \inf_{x \in Q} d(x, x')$ [13, Lemma 6.3], and $d_{\mathcal{Q}}^a(Q, Q') = \min(d_{\mathcal{Q}}(Q, Q'), a)$ [13, Remark 6.4].

We will also need to know that, when X, d is standard, the specialization ordering of $d_{\mathcal{Q}}$ is reverse inclusion, namely that $d_{\mathcal{Q}}(Q, Q') = 0$ if and only if $Q \supseteq Q'$ [13, Lemma 6.6]; and similarly with $d_{\mathcal{Q}}^a$ [13, Remark 6.7] We will also rely on the following results, which one can find as Lemma 6.3, 6.5, Remark 6.4, Proposition 6.9, Remark 6.10, Theorems 6.23, 6.26, 6.19 Remark 6.20, Corollary 6.24 and Lemma 6.22 of Part II [13]. The fact that $Q \mapsto F_Q$ is an isometry is by definition of $d_{\mathcal{Q}}$ and $d_{\mathcal{Q}}^a$. Note that every continuous complete quasi-metric space X, d is sober in its d -Scott topology [16, Proposition 4.1].

Theorem 2.5 (Completeness for the Smyth hyperspace [13]). *For every quasi-metric space X, d , the map $Q \mapsto F_Q$ is an isometry between $\mathcal{Q}X, d_{\mathcal{Q}}$*

(resp., $d_{\mathcal{Q}}^a$, $a > 0$) and the space of discrete normalized superlinear previsions on X with the d_{KR} (resp., d_{KR}^a) quasi-metric.

For every standard quasi-metric space X, d , for all $Q, Q' \in \mathcal{Q}X$, $d_{\mathcal{Q}}(Q, Q') = \sup_{x' \in Q'} \min_{x \in Q} d(x, x')$, and $d_{\mathcal{Q}}^a(Q, Q') = \min(a, d_{\mathcal{Q}}(Q, Q'))$.

If X is also sober in its d -Scott topology, then $\mathcal{Q}X$ is complete under both the $d_{\mathcal{Q}}$ and $d_{\mathcal{Q}}^a$ quasi-metrics. Through the $Q \mapsto F_Q$ isometry, suprema of directed families of formal balls are computed as naive suprema.

If X, d is continuous complete, then $\mathcal{Q}X$ is continuous complete under both $d_{\mathcal{Q}}$ and $d_{\mathcal{Q}}^a$. The $d_{\mathcal{Q}}$ -Scott, $d_{\mathcal{Q}}^a$ -Scott, and upper Vietoris topologies all coincide.

If X, d is algebraic complete, then so is $\mathcal{Q}X$ under both $d_{\mathcal{Q}}$ and $d_{\mathcal{Q}}^a$. The compact saturated sets of the form $\uparrow E$, where E is any finite non-empty set of points of \mathcal{B} form a strong basis, where \mathcal{B} is any given strong basis of X, d .

$\mathcal{Q}, d_{\mathcal{Q}}$ defines an endofunctor on the category of continuous (resp., algebraic) complete quasi-metric spaces and 1-Lipschitz continuous maps, and similarly with $d_{\mathcal{Q}}^a$ instead of $d_{\mathcal{Q}}$: for every 1-Lipschitz continuous map $f: X, d \rightarrow Y, \partial$ between continuous complete quasi-metric spaces, $\mathcal{Q}f$ is a 1-Lipschitz continuous map from $\mathcal{Q}X, d_{\mathcal{Q}}$ to $\mathcal{Q}Y, \partial_{\mathcal{Q}}$ and from $\mathcal{Q}X, d_{\mathcal{Q}}^a$ to $\mathcal{Q}Y, \partial_{\mathcal{Q}}^a$. Moreover, for every $Q \in \mathcal{Q}X$, $F_{\mathcal{Q}f(C)} = \mathbf{P}f(F_Q)$.

On the Hoare hyperspace \mathcal{H}_0X (where X has the d -Scott topology), the natural definition of quasi-metric is $d_{\mathcal{H}}(C, C') \stackrel{\text{def}}{=} \sup_{x \in C} d(x, C')$, where the supremum of an empty set is taken to be 0 [13, Definition 5.3]. One would expect $d(x, C')$ to be defined as $\inf_{y \in C'} d(x, y)$ (where the infimum of an empty set is $+\infty$), and while this holds in several important cases, the actual definition is as the supremum of those $r \in \mathbb{R}_+$ such that $(x, r) \in \widehat{U}$, where U is the complement of C and \widehat{U} is the largest Scott-open subset of $\mathbf{B}(X, d)$ such that $\eta_X^{-1}(\widehat{U}) = U$. We will not need to understand this at that level of detail, except for the following fact: $d(x, C')$ is equal to $\inf_{y \in C'} d(x, y)$ if x is a center point and X, d is standard [13, Fact 5.2]. This will be required in Remark 3.5 and in the proof of Theorem 3.9.

We will need to know that, given that X, d is standard, the specialization ordering of $d_{\mathcal{H}}$ is inclusion, namely that $d_{\mathcal{H}}(C, C') = 0$ if and only if $C \subseteq C'$ [13, Lemma 5.7, item 2]; similarly with $d_{\mathcal{H}}^a$, defined by $d_{\mathcal{H}}^a(C, C') \stackrel{\text{def}}{=} \min(a, d_{\mathcal{H}}(C, C'))$. We will also require the following results, which summarize Proposition 5.8, Remark 5.9, Theorem 5.25, Proposition 5.12, Remark 5.13, Theorem 5.30, Theorem 5.22, Corollary 5.26, and Lemma 5.24 of Part II [13].

Theorem 2.6 (Completeness for the Hoare hyperspaces [13]). *For every standard quasi-metric space X, d , the same map $C \mapsto F^C$ is an isometry between $\mathcal{H}_0X, d_{\mathcal{H}}$ (resp., $d_{\mathcal{H}}^a$, $a > 0$) and the space of discrete sublinear previsions on X with the d_{KR} (resp., d_{KR}^a) quasi-metric.*

If X, d is continuous complete, then \mathcal{H}_0X and $\mathcal{H}X$ are continuous complete under the $d_{\mathcal{H}}$ and $d_{\mathcal{H}}^a$ quasi-metrics. Through the $C \mapsto F^C$ isometry, suprema of directed families of formal balls are computed as naive suprema. The $d_{\mathcal{H}}$ -Scott, $d_{\mathcal{H}}^a$ -Scott, and lower Vietoris topologies all coincide.

If X, d is algebraic complete, then so are \mathcal{H}_0X and $\mathcal{H}X$ under both $d_{\mathcal{H}}$ and $d_{\mathcal{H}}^a$. The closed sets of the form $\downarrow E$, where E is any finite (and non-empty in the case of $\mathcal{H}X$) set of points of \mathcal{B} form a strong basis, where \mathcal{B} is any given strong basis of X, d .

$\mathcal{H}_0, d_{\mathcal{H}}$ defines an endofunctor on the category of continuous (resp., algebraic) complete quasi-metric spaces and 1-Lipschitz continuous maps, and similarly with \mathcal{H} instead of \mathcal{H}_0 or with $d_{\mathcal{H}}^a$ instead of $d_{\mathcal{H}}$: for every 1-Lipschitz continuous map $f: X, d \rightarrow Y, \partial$ between continuous complete quasi-metric spaces, \mathcal{H}_0f is a 1-Lipschitz continuous map from $\mathcal{H}_0X, d_{\mathcal{H}}$ to $\mathcal{H}_0Y, \partial_{\mathcal{H}}$ and similarly with \mathcal{H} instead of \mathcal{H}_0 or with $d_{\mathcal{H}}^a, \partial_{\mathcal{H}}^a$ instead of $d_{\mathcal{H}}, \partial_{\mathcal{H}}$. Moreover, for every $C \in \mathcal{H}_0X$, $F^{\mathcal{H}_0f(C)} = \mathbf{P}f(F^C)$.

2.6. Lenses, Quasi-Lenses, and Forks

Mixing angelic and demonic non-deterministic choice yields *erratic* non-deterministic choice, modeled by the *Plotkin powerdomain*. There are several inequivalent definitions [1, Section 6.2]. A popular one is as the space of *lenses* on X . A lens L is an intersection $Q \cap C$ of a compact saturated set Q and a closed set C , where Q intersects C . For every lens L , one can write L canonically as $Q \cap C$, by taking $Q = \uparrow L$ and $C = cl(L)$. Lenses are ordered by the so-called topological Egli-Milner ordering, which we will define below.

The corresponding *Plotkin hyperspace* is the space of lenses with the *Vietoris topology*, generated by subsets $\square U \stackrel{\text{def}}{=} \{L \mid L \subseteq U\}$ and $\diamond U \stackrel{\text{def}}{=} \{L \mid L \cap U \neq \emptyset\}$, where U ranges over the open subsets of X . The specialization ordering of that space is the *topological Egli-Milner ordering*, in which L is less than or equal to L' if and only if $cl(L) \subseteq cl(L')$ and $\uparrow L \supseteq \uparrow L'$.

An elegant alternative to lenses is given by Heckmann's *continuous A-valuations* [18]. Those are Scott-continuous maps $\alpha: \mathcal{O}X \rightarrow \mathbf{A}$ such that $\alpha(\emptyset) = 0$, $\alpha(X) = 1$, and for all opens U, V , $\alpha(U) = 0$ implies $\alpha(U \cup V) = \alpha(V)$, and $\alpha(U) = 1$ implies $\alpha(U \cap V) = \alpha(V)$. Here \mathbf{A} is the poset $\{0, \mathbf{M}, 1\}$ ordered by $0 < \mathbf{M} < 1$. The *Vietoris topology* on the space of continuous

\mathbf{A} -valuations is generated by the subsets $\square U$ of continuous \mathbf{A} -valuations such that $\alpha(U) = 1$, and the subsets $\diamond U$ of continuous \mathbf{A} -valuations such that $\alpha(U) \neq 0$, where U ranges over the open subsets of X .

It was shown in [7, Fact 5.2] that, provided X is sober, the space of \mathbf{A} -valuations is homeomorphic to a space of so-called *quasi-lenses*, namely, pairs (Q, C) of a compact saturated set Q and a closed set C such that Q intersects C , $Q \subseteq \uparrow(Q \cap C)$ (whence $Q = \uparrow(Q \cap C)$), and for every open neighborhood U of Q , $C \subseteq cl(U \cap C)$. Quasi-lenses originate from Heckmann's work [17, Theorem 9.6].

The space of quasi-lenses is topologized by a topology that we shall again call the *Vietoris topology*, generated by subsets $\square U \stackrel{\text{def}}{=} \{(Q, C) \mid Q \subseteq U\}$ and $\diamond U \stackrel{\text{def}}{=} \{(Q, C) \mid C \cap U \neq \emptyset\}$. The homeomorphism maps (Q, C) to α where $\alpha(U)$ is equal to 1 if $Q \subseteq U$, to 0 if C does not intersect U , and to \mathbf{M} otherwise.

The space of quasi-lenses is homeomorphic to the space of lenses when X is weakly Hausdorff, through the map $L \mapsto (\uparrow L, cl(L))$ [15, Theorem 6.3]; in particular when X is stably compact, generalizing Proposition 5.3 of [7], or when X is Hausdorff, generalizing Theorem 5.1 of [18]. Without weak Hausdorffness, the space of quasi-lenses is in general larger [15, Example 6.6]. We note that, in case X is not only stably compact but also Hausdorff, i.e., when X is compact Hausdorff, then the lenses are just the compact subsets of X . It should therefore come as no surprise that we shall equip our space of (quasi-)lenses with a quasi-metric resembling the Pompeiu-Hausdorff metric (also known as the Hausdorff metric).

We will see that quasi-lenses and continuous \mathbf{A} -valuations are a special case of the notion of a *fork* [6]. A fork on X is by definition a pair (F^-, F^+) of a superlinear prevision F^- and a sublinear prevision F^+ satisfying *Walley's condition* [6]:

$$F^-(h + h') \leq F^-(h) + F^+(h') \leq F^+(h + h')$$

for all $h, h' \in \mathcal{L}X$. This condition was independently discovered by Keimel and Plotkin, who call such a pair *medial* [20].

By taking $h' = 0$, or $h = 0$, Walley's condition in particular implies that $F^- \leq F^+$. A fork (F^-, F^+) is normalized, subnormalized, or discrete if and only if both F^- and F^+ are.

We shall call *weak topology* on the space of forks the topology induced by the inclusion in the product of the spaces of superlinear and sublinear

previsions, each equipped with their weak topology. Explicitly, it is generated by subbasic open sets of the form:

$$[h > a]_- \stackrel{\text{def}}{=} \{(F^-, F^+) \mid F^-(h) > a\} \quad (4)$$

$$[h > a]_+ \stackrel{\text{def}}{=} \{(F^-, F^+) \mid F^+(h) > a\}, \quad (5)$$

where h ranges over $\mathcal{L}X$ and $a \in \mathbb{R}_+$. The specialization quasi-ordering of the weak topology is given by $(F^-, F^+) \leq (F'^-, F'^+)$ if and only if $F^- \leq F'^-$ and $F^+ \leq F'^+$.

On the quasi-metric side, we define the following quasi-metrics on spaces of forks.

Definition 2.7 ($d_{\mathbf{KR}}$, $d_{\mathbf{KR}}^a$ on forks). *Let X, d be a quasi-metric space. The $d_{\mathbf{KR}}$ quasi-metric on the space of forks on X is defined by:*

$$d_{\mathbf{KR}}((F^-, F^+), (F'^-, F'^+)) \stackrel{\text{def}}{=} \max(d_{\mathbf{KR}}(F^-, F'^-), d_{\mathbf{KR}}(F^+, F'^+)),$$

and similarly for the $d_{\mathbf{KR}}^a$ quasi-metric, $a > 0$.

Those are quasi-metrics indeed.

Lemma 2.8. *Let X, d be a standard quasi-metric space. For all forks (F^-, F^+) and (F'^-, F'^+) on X , the following are equivalent: (a) $(F^-, F^+) \leq (F'^-, F'^+)$; (b) $d_{\mathbf{KR}}((F^-, F^+), (F'^-, F'^+)) = 0$; (c) $d_{\mathbf{KR}}^a((F^-, F^+), (F'^-, F'^+)) = 0$, for any $a > 0$. \square*

PROOF. This follows from the fact that, for any two previsions F and F' on a standard quasi-metric spaces, $F \leq F'$ if and only if $d_{\mathbf{KR}}(F, F') = 0$ if and only if $d_{\mathbf{KR}}^a(F, F') = 0$ [12, Lemma 3.7]. \square

3. The Plotkin Hyperspace

To start with, we give substance to our earlier claim that quasi-lenses (and continuous \mathbf{A} -valuations) are a special case of the notion of a fork.

Proposition 3.1. *Let X be a sober space. There is a homeomorphism between the space of quasi-lenses on X , with the Vietoris topology, and the space of discrete normalized forks on X , with the weak topology, which maps the quasi-lens (Q, C) to (F_Q, F^C) .*

PROOF. The map $(Q, C) \mapsto (F_Q, F^C)$ sends quasi-lenses to discrete normalized forks. Let us fix a quasi-lens (Q, C) . Then F^C is a discrete normalized sublinear prevision, and F_Q is a discrete normalized superlinear prevision. In order to show that (F_Q, F^C) is a discrete normalized fork, it remains to verify Walley's condition.

We claim that $F_Q(h) = \inf_{x \in Q \cap C} h(x)$ for every $h \in \mathcal{L}X$. Indeed, $F_Q(h) = \inf_{x \in Q} h(x) \leq \inf_{x \in Q \cap C} h(x)$ follows from the fact that Q contains $Q \cap C$, and in the reverse direction, we use the equality $Q = \uparrow(Q \cap C)$ to show that for every $x \in Q$, there is a $y \in Q \cap C$ such that $h(x) \geq h(y)$.

For all $h, h' \in \mathcal{L}X$, it follows that $F_Q(h + h') = \inf_{x \in Q \cap C} (h(x) + h'(x)) \leq \inf_{x \in Q \cap C} (h(x) + \sup_{x' \in Q \cap C} h'(x')) \leq \inf_{x \in Q \cap C} (h(x) + \sup_{x' \in C} h'(x')) = F_Q(h) + F^C(h')$.

In order to show that $F_Q(h) + F^C(h') \leq F^C(h + h')$, we assume the contrary: there is a real number t such that $F_Q(h) + F^C(h') > t \geq F^C(h + h')$. The second inequality implies that $F^C(h')$, which is less than or equal to $F^C(h + h')$, is a (non-negative) real number $a < +\infty$. The first inequality then implies that $F_Q(h) > t - a$. It follows that there must be a real number b such that $F_Q(h) > b > t - a$. Since $F_Q(h) > b$, every element x of Q is such that $h(x) > b$. Therefore Q is included in the open set $U = h^{-1}(]b, +\infty])$. We know that $C \subseteq cl(U \cap C)$, and this implies that $F^C(h') = \sup_{x \in C} h'(x) \leq \sup_{x \in cl(U \cap C)} h'(x)$. Since $b > t - a$ and $a = F^C(h')$, there is a point in $cl(U \cap C)$ whose image by h' is strictly larger than $t - b$, in other words $cl(U \cap C)$ intersects $h'^{-1}(]t - b, +\infty])$. Since the latter is open, it also intersects $U \cap C$, so there is a point x in $U \cap C$ such that $h'(x) > t - b$. Because x is in U , $h(x) > b$, so $h(x) + h'(x) > t$, contradicting the inequality $t \geq F^C(h + h')$.

Hence if (Q, C) is a quasi-lens, then (F_Q, F^C) is a fork, which is necessarily discrete and normalized.

$(Q, C) \mapsto (F_Q, F^C)$ is a bijection. Since $Q \mapsto F_Q$ and $C \mapsto F^C$ are homeomorphisms, they are in particular injective. We claim that the map $(Q, C) \mapsto (F_Q, F^C)$ is surjective. In other words, let us assume that (F_Q, F^C) is a (discrete, normalized) fork. We need to show that (Q, C) is a quasi-lens. We have seen that, in particular, $F_Q \leq F^C$. Hence $F_Q(\chi_U) \leq F^C(\chi_U)$ where U is the complement of C . That implies that $F_Q(\chi_U) = 0$, i.e., Q is not included in U , so that Q intersects C .

Let us use the inequality $F_Q(h + h') \leq F_Q(h) + F^C(h')$ on $h' \stackrel{\text{def}}{=} \chi_U$, where U is the complement of C again. Then $F^C(h') = 0$, so $\inf_{y \in Q} (h(y) + \chi_U(y)) \leq \inf_{y \in Q} h(y)$. In other words, $\min(\inf_{y \in Q \cap U} (h(y) + 1), \inf_{y \in Q \cap C} h(y)) \leq \inf_{y \in Q} h(y)$,

or equivalently, one of the two numbers $\inf_{y \in Q \cap U} (h(y) + 1)$, $\inf_{y \in Q \cap C} h(y)$ is less than or equal to $\inf_{y \in Q} h(y)$. That cannot be the first one, which is larger than or equal to $1 + \inf_{y \in Q \cap U} h(y) \geq 1 + \inf_{y \in Q} h(y)$, so: (*) $\inf_{y \in Q \cap C} h(y) \leq \inf_{y \in Q} h(y)$. Now, for every $x \in Q$, the downward closure $\downarrow x$ of x in the specialization ordering of X is closed (this is really the closure of x); hence its complement V is open. Let us take $h \stackrel{\text{def}}{=} \chi_V$. Then $\inf_{y \in Q} h(y) \leq h(x) = 0$, therefore $\inf_{y \in Q \cap C} h(y) = 0$ by (*). Since h only takes the values 0 or 1 (or since $Q \cap C$ is compact), there is a point y in $Q \cap C$ such that $h(y) = 0$, i.e., such that $y \leq x$. Therefore Q is included in $\uparrow(Q \cap C)$.

Finally, let U be any open neighborhood of Q . We use the inequality $F_Q(h) + F^C(h') \leq F^C(h + h')$ on $h \stackrel{\text{def}}{=} \chi_U$, and we obtain that $1 + F^C(h') \leq F^C(h + h')$. For every open subset V that intersects C , taking $h' \stackrel{\text{def}}{=} \chi_V$ we obtain that $2 \leq F^C(\chi_U + \chi_V) = \sup_{x \in C} (\chi_U(x) + \chi_V(x))$. Since $\chi_U(x) + \chi_V(x)$ can only take the values 0, 1, or 2, there must be a point x in C such that $\chi_U(x) + \chi_V(x) = 2$. Such a point must be in both U and V . We have shown that every open subset V that intersects C also intersects $U \cap C$. Letting V be the complement of $cl(U \cap C)$, it follows that C cannot intersect V , hence must be included in $cl(U \cap C)$.

The map $(Q, C) \mapsto (F_Q, F^C)$ is a homeomorphism. Now that we know that this map is bijective, we simply recall that $Q \mapsto F_Q$, resp. $C \mapsto F^C$, are homeomorphisms themselves, between $\mathcal{Q}X$ (resp., $\mathcal{H}X$) and the space of normalized discrete superlinear (resp. sublinear) previsions on X . \square

Through the bijection of Proposition 3.1, the d_{KR} and d_{KR}^a quasi-metrics transports to quasi-metrics on spaces of quasi-lenses.

Definition 3.2 ($d_{\mathcal{P}}, d_{\mathcal{P}}^a$). *Let X, d be a quasi-metric space. For any two quasi-lenses (Q, C) and (Q', C') , let:*

$$\begin{aligned} d_{\mathcal{P}}((Q, C), (Q', C')) &\stackrel{\text{def}}{=} d_{KR}((F_Q, F^C), (F_{Q'}, F^{C'})) \\ d_{\mathcal{P}}^a((Q, C), (Q', C')) &\stackrel{\text{def}}{=} d_{KR}^a((F_Q, F^C), (F_{Q'}, F^{C'})) \quad \text{for every } a > 0. \end{aligned}$$

By the first parts of Theorem 2.5 and 2.6, we obtain the following characterization of those orderings.

Lemma 3.3. *Let X, d be a standard quasi-metric space. Then:*

$$\begin{aligned} d_{\mathcal{P}}((Q, C), (Q', C')) &= \max(d_{\mathcal{Q}}(Q, Q'), d_{\mathcal{H}}(C, C')) \\ d_{\mathcal{P}}^a((Q, C), (Q', C')) &= \max(d_{\mathcal{Q}}^a(Q, Q'), d_{\mathcal{H}}^a(C, C')) \quad \text{for every } a > 0. \end{aligned}$$

Together with Proposition 3.1, this implies the following.

Corollary 3.4. *Let X, d be a sober standard quasi-metric space. The bijection $(Q, C) \mapsto (F_Q, F^C)$ is an isometry between the space of quasi-lenses on X with the $d_{\mathcal{P}}$ quasi-metric, and the space of discrete normalized forks on X with the d_{KR} quasi-metric. \square*

Remark 3.5 (The Pompeiu-Hausdorff quasi-metric). *Let X, d be a standard quasi-metric space. If we equate a lens L with the quasi-lens $(\uparrow L, cl(L))$, then we can reinterpret Lemma 3.3 by saying that $d_{\mathcal{P}}(L, L') = \max(d_{\mathcal{Q}}(\uparrow L, \uparrow L'), d_{\mathcal{H}}(cl(L), cl(L')))$.*

We have $d_{\mathcal{Q}}(\uparrow L, \uparrow L') = \sup_{x' \in \uparrow L'} \inf_{x \in \uparrow L} d(x, x')$ by the Pompeiu-Hausdorff-like formula for $d_{\mathcal{Q}}$, and this is also equal to $\sup_{x' \in L'} \inf_{x \in L} d(x, x')$, since d is monotonic in its first argument and antitonic in its second argument. We also have $d_{\mathcal{H}}(cl(L), cl(L')) = \sup_{x \in cl(L)} d(x, cl(L'))$ by definition, and one checks easily that this is equal to $\sup_{x \in L} d(x, cl(L'))$, using the fact that $d(_, cl(L'))$ is 1-Lipschitz continuous [13, Fact 5.1], hence continuous.

If x is a center point, then $d(x, cl(L')) = \inf_{x' \in cl(L')} d(x, x')$ [13, Fact 5.2]. Also the map $-d(x, _)$ is Scott-continuous, because the inverse image of $]t, +\infty]$ is the open ball $B_{x, < -t}^d$ (meaning the empty set if $t \leq 0$), so $\inf_{x' \in cl(L')} d(x, x') = -\sup_{x' \in cl(L')} -d(x, x') = -\sup_{x' \in L'} -d(x, x') = \inf_{x' \in L'} d(x, x')$.

All that implies that, on a standard quasi-metric space X, d where every point is a center point, the $d_{\mathcal{P}}$ quasi-metric coincides with the familiar Pompeiu-Hausdorff quasi-metric:

$$d_{\mathcal{P}}(L, L') = \max\left(\sup_{x \in L} \inf_{x' \in L'} d(x, x'), \sup_{x' \in L'} \inf_{x \in L} d(x, x')\right).$$

This is the case, notably, if X, d is Smyth-complete, a stronger notion of completeness than the one we use here; we will only remark that a complete quasi-metric space is Smyth-complete if and only if all its points are center points, a result stated as such in [16, Proposition 5.17] and originally due to Ali-Akbari, Honari, Pourmahdian, and Rezaei [2].

This is also the case if X, d is a metric space, since every metric space is standard and every point in a metric space is a center point. In that case, $d_{\mathcal{P}}$ is exactly the Pompeiu-Hausdorff metric on the space of non-empty compact subsets of X .

Lemma 3.6. *Let X, d be a sober, standard quasi-metric space. For every directed family of formal balls $((F_i^-, F_i^+), r_i)_{i \in I}$ on the space of forks with*

the $d_{\mathcal{P}}$ (resp., $d_{\mathcal{P}}^a$, $a > 0$) quasi-metric, if $(F_i^-, r_i)_{i \in I}$ has a naive supremum (F^-, r) where F^- is continuous, and if $(F_i^+, r_i)_{i \in I}$ has a naive supremum (F^+, r) where F^+ is continuous, then (F^-, F^+) is a fork.

PROOF. We deal with $d_{\mathcal{P}}$ first, and we will state what changes have to be made to the proof in the case of $d_{\mathcal{P}}^a$ at the end of the argument.

We note that, because the quasi-metric on forks is defined as a maximum, the two families $(F_i^-, r_i)_{i \in I}$ and $(F_i^+, r_i)_{i \in I}$ are directed, so taking their naive suprema makes sense. Explicitly, let us define $i \preceq j$ if and only if $((F_i^-, F_i^+), r_i) \leq^{d_{\text{KR}}^+} ((F_j^-, F_j^+), r_j)$. Then $i \preceq j$ implies that $d_{\text{KR}}((F_i^-, F_i^+), (F_j^-, F_j^+)) \leq r_i - r_j$, hence $d_{\text{KR}}(F_i^-, F_j^-) \leq r_i - r_j$ and $d_{\text{KR}}(F_i^+, F_j^+) \leq r_i - r_j$, so that $(F_i^-, r_i) \leq^{d_{\text{KR}}^+} (F_j^-, r_j)$ and $(F_i^+, r_i) \leq^{d_{\text{KR}}^+} (F_j^+, r_j)$. For all $i, j \in I$, there is a $k \in I$ such that $i, j \preceq k$, and that implies that $(F_i^-, r_i), (F_j^-, r_j) \leq^{d_{\text{KR}}^+} (F_k^-, r_k)$, and similarly with $+$ instead of $-$.

We note also that the naive suprema (F^-, r) and (F^+, r) must have the same radius $r = \inf_{i \in I} r_i$, by definition.

By Theorem 2.4, F^- is a superlinear prevision and F^+ is a sublinear prevision. We must prove Walley's condition.

For all $h \in \mathcal{L}_{\alpha}(X, d)$ and $h' \in \mathcal{L}_{\beta}(X, d)$, for all $\alpha, \beta > 0$, we have:

$$F_i^-(h + h') \leq F_i^-(h) + F_i^+(h') \leq F_i^+(h + h')$$

for every $i \in I$, since (F_i^-, F_i^+) is a fork. Therefore:

$$\begin{aligned} F_i^-(h + h') + (\alpha + \beta)r - (\alpha + \beta)r_i &\leq (F_i^-(h) + \alpha r - \alpha r_i) + (F_i^+(h') + \beta r - \beta r_i) \\ &\leq F_i^+(h + h') + (\alpha + \beta)r - (\alpha + \beta)r_i \end{aligned}$$

We note that $h + h'$ is $(\alpha + \beta)$ -Lipschitz continuous. By taking (directed) suprema over $i \in I$, and noting that addition is Scott-continuous on $\overline{\mathbb{R}}_+$, we obtain that:

$$F^-(h + h') \leq F^-(h) + F^+(h') \leq F^+(h + h')$$

for all $h \in \mathcal{L}_{\alpha}(X, d)$ and $h' \in \mathcal{L}_{\beta}(X, d)$.

We recall that for every $h \in \mathcal{L}X$, $h^{(\alpha)}$ is the largest α -Lipschitz continuous map below h , and that $(h^{(\alpha)})_{\alpha \in \mathbb{R}_+}$ is a chain whose supremum is h ; also, $F^-(h) = \sup_{\alpha \in \mathbb{R}_+} F^+(h^{(\alpha)})$, and similarly with F^+ , by Theorem 2.4.

For all $h, h' \in \mathcal{L}X$, for every $\alpha > 0$, we have that $F^-(h^{(\alpha)} + h'^{(\alpha)}) \leq F^-(h^{(\alpha)}) + F^+(h'^{(\alpha)}) \leq F^-(h) + F^+(h')$, hence, taking suprema over α ,

$F^-(h + h') \leq F^-(h) + F^+(h)$. Similarly, $F^-(h^{(\alpha)}) + F^+(h'^{(\alpha)}) \leq F^+(h^{(\alpha)} + h'^{(\alpha)}) \leq F^+(h + h')$, hence, taking suprema over α , $F^-(h) + F^+(h') \leq F^+(h + h')$.

In the case of $d_{\mathcal{P}}^a$, the same arguments apply, except for the replacement of $\mathcal{L}_\alpha(X, d)$ by $\mathcal{L}_\alpha^a(X, d)$, and for the replacement of $h^{(\alpha)}$ by $\min(a\alpha, h^{(\alpha)})$ in the last paragraph. \square

Proposition 3.7 (Completeness, Plotkin hyperspace). *Let X, d be a continuous complete quasi-metric space. The space of quasi-lenses on X , with the d_{KR} (resp., d_{KR}^a , $a > 0$) quasi-metric, is complete.*

The supremum of a directed family of formal balls $((Q_i, C_i), r_i)_{i \in I}$ is $((Q, C), r)$ where $r = \inf_{i \in I} r_i$, (F_Q, r) is the naive supremum of $(F_{Q_i}, r_i)_{i \in I}$, and (F^C, r) is the naive supremum of $(F^{C_i}, r_i)_{i \in I}$. In particular, (Q, r) is the supremum of $(Q_i, r_i)_{i \in I}$ and (C, r) is the supremum of $(C_i, r_i)_{i \in I}$.

PROOF. We note that every continuous complete quasi-metric space is sober [16, Proposition 4.1].

Let $((Q_i, C_i), r_i)_{i \in I}$ be a directed family of formal balls on the space of quasi-lenses. The families $(Q_i, r_i)_{i \in I}$ and $(C_i, r_i)_{i \in I}$ are also directed, because $d_{\mathcal{P}}$ is defined as a maximum. We also obtain a directed family of formal balls $((F_{Q_i}, F^{C_i}), r_i)$ on the space of discrete normalized forks, by Proposition 3.1. By Theorem 2.6 and Theorem 2.5, the supremum (C, r) of $(C_i, r_i)_{i \in I}$ exists and is a naive supremum, and the supremum (Q, r) (with the same r) of $(Q_i, r_i)_{i \in I}$ exists and is a naive supremum. By Lemma 3.6, (F_Q, F^C) is a fork, and certainly it is discrete and normalized.

In order to show that $((Q, C), r)$ is the desired supremum, we observe that the upper bounds $((Q', C'), r')$ of $((Q_i, C_i), r_i)_{i \in I}$ are exactly those formal balls such that (Q', r') is an upper bound of $(Q_i, r_i)_{i \in I}$ and (C', r') is an upper bound of $(C_i, r_i)_{i \in I}$. This is again because our d_{KR} quasi-metric on forks (or the $d_{\mathcal{P}}$ quasi-metric on quasi-lenses) is defined as a maximum, and similarly for d_{KR}^a and $d_{\mathcal{P}}^a$. Hence $((Q, C), r)$ is such an upper bound, and if $((Q', C'), r')$ is any other upper bound, then $(Q, r) \leq^{d_{\mathcal{P}}^+} (Q', r')$ and $(C, r) \leq^{d_{\mathcal{P}}^+} (C', r')$, so $((Q, C), r) \leq^{d_{\mathcal{P}}^+} ((Q', C'), r')$. \square

Remark 3.8. *For every complete metric space X, d , Proposition 3.7 specializes to the well-known fact that the space of non-empty compact subsets of X , with the Pompeiu-Hausdorff metric, is complete.*

Let us write $\langle E \rangle$ for the *order-convex hull* of the set E . This is defined as $\downarrow E \cap \uparrow E$. When E is finite and non-empty, $\downarrow E$ is closed and intersects $\uparrow E$, so $\langle E \rangle$ is a lens. Therefore $(\uparrow \langle E \rangle, cl(\langle E \rangle))$ is a quasi-lens. We observe that $\uparrow \langle E \rangle = \uparrow E$, and that $cl(\langle E \rangle) = \downarrow E$. (Only the latter needs an argument: $\downarrow E$ is the closure of E since E is finite, and as $E \subseteq \langle E \rangle$, $\downarrow E \subseteq cl(\langle E \rangle)$; in the reverse direction, $\langle E \rangle \subseteq \downarrow E$, and since $\downarrow E$ is closed, it must also contain the closure of $\langle E \rangle$.) It follows that $(\uparrow E, \downarrow E)$ is a quasi-lens.

Theorem 3.9 (Algebraicity of Plotkin hyperspaces). *Let X, d be an algebraic complete quasi-metric space, with strong basis \mathcal{B} . The space of quasi-lenses on X, d , with the $d_{\mathcal{P}}$ (resp., $d_{\mathcal{P}}^a$, $a > 0$) quasi-metric, is algebraic complete.*

Every quasi-lens of the form $(\uparrow E, \downarrow E)$, where E is a finite non-empty set of center points, is a center point in the space of quasi-lenses, and they form a strong basis, even if we restrict E to be a subset of \mathcal{B} .

The $d_{\mathcal{P}}$ -Scott topology, the $d_{\mathcal{P}}^a$ -Scott topology, and the Vietoris topology coincide.

PROOF. We deal with $d_{\mathcal{P}}$ only. We will only indicate the occasional changes one has to do in the $d_{\mathcal{P}}^a$ case between square brackets. We will rely on the following results.

(I) [13, Lemma 5.20] Given a standard quasi-metric space Y, ∂ , a set \mathcal{B}_0 of center points of Y, ∂ , and a topology τ on Y such that:

- (i) every open ball centered at any point of \mathcal{B}_0 is τ -open,
- (ii) for every $\epsilon > 0$, for every τ -open neighborhood U of z , there is an open ball $B_{y, < r}^{\partial}$ included in U and which contains z , for some $y \in \mathcal{B}_0$ and $0 < r \leq \epsilon$,
- (iii) for all $y, z \in Y$, if $y \leq z$ in the specialization preordering \leq of τ , then $y \leq^{\partial} z$, namely $\partial(y, z) = 0$,

then Y, ∂ is algebraic, \mathcal{B}_0 is a strong basis, and τ coincides with the ∂ -Scott topology.

(II) [13, Lemma 5.21] If X, d is standard, and $C_0 \stackrel{\text{def}}{=} \downarrow \{x_1, \dots, x_n\}$, where $n \geq 1$ and each x_i is a center point of X, d , then for every $r > 0$, $B_{C_0, < r}^{d_{\mathcal{H}}} = \bigcap_{i=1}^n \diamond B_{x_i, < r}^d$ in $\mathcal{H}X$. [Similarly with $d_{\mathcal{H}}^a$, provided that $r \leq a$.]

- (III) [13, Lemma 6.18] If X, d is standard, and $Q_0 \stackrel{\text{def}}{=} \uparrow\{x_1, \dots, x_n\}$, where $n \geq 1$ and each x_i is a center point of X, d , then for every $r > 0$, $B_{Q_0, <r}^{d_Q} = \square(\bigcup_{i=1}^n B_{x_i, <r}^d)$. [Similarly with d_Q^a , provided that $r \leq a$.]
- (IV) [13, Lemma 5.19] If X, d is standard algebraic, with a strong basis \mathcal{B} , then for every $\epsilon > 0$, the open balls $B_{x, <r}^d$ with $x \in \mathcal{B}$ and $0 < r < \epsilon$ form a base of the d -Scott topology on X .
- (V) [13, Fact 5.2] If X, d is standard, then for every center point x of X , for every closed subset C of X in its d -Scott topology, $d(x, C) = \inf_{y \in C} d(x, y)$. (We have already used this result earlier in this paper.) We will use the following consequence: for every $r > 0$, $d(x, C) < r$ if and only if C intersects $B_{x, <r}^d$.
- (VI) [13, Lemma 6.16] If X, d is standard algebraic, with a strong basis \mathcal{B} , then for every compact subset K of X , for every open neighborhood U of K , and for every $\epsilon > 0$, there are finitely many points x_1, \dots, x_n in \mathcal{B} and radii $r_1, \dots, r_n < \epsilon$ such that $K \subseteq \bigcup_{j=1}^n B_{x_j, <r_j}^d \subseteq U$.
- (VII) [13, Lemma 6.17] If X, d is standard, and if Q is compact saturated and included in $\bigcup_{j=1}^m B_{x_j, <r}^d$, where x_1, \dots, x_m are center points and $r > 0$, then for some $\epsilon > 0$ with $\epsilon < r$, Q is included in $\bigcup_{j=1}^m B_{x_j, <r-\epsilon}^d$.

We apply (I) on the set of quasi-lenses on X for Y . We let ∂ be $d_{\mathcal{P}}$, resp. $d_{\mathcal{P}}^a$, \mathcal{B}_0 be the set of quasi-lenses ($\uparrow E, \downarrow E$), where E ranges over the finite non-empty subsets of \mathcal{B} , and τ be the Vietoris topology. The Theorem will follow immediately from (I), provided that we verify its own assumptions (i), (ii), and (iii).

For disambiguation purposes, we write $\square_{\mathcal{P}}U$ for the set of lenses (Q, C) such that $Q \subseteq U$, reserving the notation $\square U$ for the set of non-empty compact saturated subsets Q of X such that $Q \subseteq U$. We use a similar convention with $\diamond_{\mathcal{P}}U$ and $\diamond U$.

(i) Let $E \stackrel{\text{def}}{=} \{x_1, \dots, x_n\}$ where $n \geq 1$ and x_1, \dots, x_n are in \mathcal{B} , $Q_0 \stackrel{\text{def}}{=} \uparrow E$, and $C_0 \stackrel{\text{def}}{=} \downarrow E$. Using Lemma 3.3, the open ball $B_{(Q_0, C_0), <r}^{d_{\mathcal{P}}}$ is the collection of quasi-lenses (Q, C) such that $d_Q(Q_0, Q) < r$ and $d_{\mathcal{H}}(C_0, C) < r$, namely such that $Q \in \square B_{Q_0, <r}^{d_Q}$ and $C \in \square B_{C_0, <r}^{d_{\mathcal{H}}}$. Equivalently, $Q \in \square(\bigcup_{i=1}^n B_{x_i, <r}^d)$ and $C \in \bigcap_{i=1}^n \diamond B_{x_i, <r}^d$, using (II) and (III). Hence $B_{(Q_0, C_0), <r}^{d_{\mathcal{P}}}$ is equal to the Vietoris open set $\square_{\mathcal{P}}(\bigcup_{i=1}^n B_{x_i, <r}^d) \cap \bigcap_{i=1}^n \diamond_{\mathcal{P}} B_{x_i, <r}^d$. [This still holds with $d_{\mathcal{P}}^a$

instead of $d_{\mathcal{P}}$, provided that $r \leq a$; otherwise $B_{(Q_0, C_0), < r}^{d_{\mathcal{P}}}$ is the whole space of quasi-lenses.]

(ii) Let $\epsilon > 0$, (Q, C) be a quasi-lens, and \mathcal{U} be a Vietoris open neighborhood of (Q, C) . By definition, there are open subset U, U_1, \dots, U_m of X such that $(Q, C) \in \square_{\mathcal{P}} U \cap \bigcap_{i=1}^m \diamond_{\mathcal{P}} U_i \subseteq \mathcal{U}$.

For every i , $1 \leq i \leq m$, C intersects U_i . Since $Q \subseteq U$, by definition of quasi-lenses C is included in $cl(U \cap C)$, so $cl(U \cap C)$ intersects U_i . It follows that $U \cap C$ intersects U_i , so that C is in $\diamond(U \cap U_i)$. Hence (Q, C) is in the open set $\square_{\mathcal{P}} U \cap \bigcap_{i=1}^m \diamond_{\mathcal{P}} (U \cap U_i)$. Replacing U_i by $U \cap U_i$ if necessary, we will therefore assume that every U_i is included in U .

For every i , $1 \leq i \leq m$, U_i intersects C . Using (IV), there is an open ball $B_{x'_i, < r'_i}^d$ included in U_i that intersects C , with $x'_i \in \mathcal{B}$ and $0 < r'_i < \epsilon$. By (V), $d(x'_i, C) < r'_i$. Let ϵ_1 be any positive number such that $d(x'_i, C) < r'_i - \epsilon_1$ for every i , $1 \leq i \leq m$. Using (V) again, this means that $B_{x'_i, < r'_i - \epsilon_1}^d$ intersects C .

Since Q is compact and C is closed, $Q \cap C$ is compact. Using (VI), there are finitely many points $y'_1, \dots, y'_p \in \mathcal{B}$ and radii $s'_1, \dots, s'_p < \epsilon$ such that $Q \cap C \subseteq \bigcup_{j=1}^p B_{y'_j, < s'_j}^d \subseteq U$. By (VII), there is a positive number ϵ_2 such that $\epsilon_2 < s'_j$ for every j , $1 \leq j \leq p$, and such that $Q \cap C \subseteq \bigcup_{j=1}^p B_{y'_j, < s'_j - \epsilon_2}^d \subseteq U$.

Let $r \stackrel{\text{def}}{=} \min(\epsilon, \epsilon_1, \epsilon_2)$. [In the case of $d_{\mathcal{P}}^a$, $r \stackrel{\text{def}}{=} \min(a, \epsilon, \epsilon_1, \epsilon_2)$.]

We use (IV) a second time. For every i , $1 \leq i \leq m$, $B_{x'_i, < r'_i - \epsilon_1}^d$ intersects C , so there is an open ball $B_{x_i, < r_i}^d$ included in $B_{x'_i, < r'_i - \epsilon_1}^d$ that intersects C , with $x_i \in \mathcal{B}$ and $0 < r_i < r$. Now the larger open ball $B_{x_i, < r}^d$ also intersects C , and we claim that it is included in U_i . For every $z \in B_{x_i, < r}^d$, $d(x_i, z) < r$, so $d(x'_i, z) \leq d(x'_i, x_i) + d(x_i, z) < r'_i - \epsilon_1 + r \leq r'_i - \epsilon_1 + \epsilon_1 = r'_i$; this shows that z is in $B_{x'_i, < r'_i}^d$, hence in U_i .

We now use (VI) a second time. Since $Q \cap C \subseteq \bigcup_{j=1}^p B_{y'_j, < s'_j - \epsilon_2}^d$, there are finitely many points $y_1, \dots, y_q \in \mathcal{B}$ and radii $s_1, \dots, s_q < r$ such that $Q \cap C \subseteq \bigcup_{k=1}^q B_{y_k, < s_k}^d \subseteq \bigcup_{j=1}^p B_{y'_j, < s'_j - \epsilon_2}^d$. We may assume that every term $B_{y_k, < s_k}^d$ in the union $\bigcup_{k=1}^q B_{y_k, < s_k}^d$ intersects $Q \cap C$: any term of that form that does not intersect $Q \cap C$ can safely be removed from the union. In particular, $B_{y_k, < s_k}^d$ intersects C for every k , $1 \leq k \leq q$. It follows that the larger open ball $B_{y_k, < r}^d$ also intersects C .

Since $Q \cap C$ is included in $\bigcup_{k=1}^q B_{y_k, < s_k}^d$, it is included in the larger set $\bigcup_{k=1}^q B_{y_k, < r}^d$. We claim that the latter is included in U . For every element z of $\bigcup_{k=1}^q B_{y_k, < r}^d$, there is an index k , $1 \leq k \leq q$, such that $d(y_k, z) < r$. Since y_k is

in $\bigcup_{j=1}^p B_{y'_j, < s'_j - \epsilon_2}^d$, there is an index j , $1 \leq j \leq p$, such that $d(y'_j, y_k) < s'_j - \epsilon_2$. Then $d(y'_j, z) \leq d(y'_j, y_k) + d(y_k, z) < s'_j - \epsilon_2 + r \leq s'_j - \epsilon_2 + \epsilon_2 = s'_j$, so z is in $B_{y'_j, < s'_j}^d$, hence in U .

In summary, we have found finitely many points x_1, \dots, x_m and y_1, \dots, y_q in \mathcal{B} and a number r such that $0 < r \leq \epsilon$ [$0 < r \leq \min(\epsilon, a)$ in the case of $d_{\mathcal{P}}^a$] and: (a) $B_{x_i, < r}^d$ intersects C for each i , $1 \leq i \leq m$, (b) $B_{x_i, < r}^d$ is included in U_i for each i , $1 \leq i \leq m$, (c) $B_{y_k, < r}^d$ intersects C for each k , $1 \leq k \leq q$; (d) $Q \cap C$ is included in $\bigcup_{k=1}^q B_{y_k, < r}^d$, and (e) $\bigcup_{k=1}^q B_{y_k, < r}^d$ is included in U .

Let $E \stackrel{\text{def}}{=} \{x_1, \dots, x_m, y_1, \dots, y_q\}$. This is a finite subset of \mathcal{B} . By (d) and since $Q \cap C$ is non-empty, Q is different from 0, so E is non-empty. Using (d) again, $Q \cap C$ is included in $\bigcup_{x \in E} B_{x, < r}^d$, and therefore Q also is included in $\bigcup_{x \in E} B_{x, < r}^d$, since $Q \subseteq \uparrow(Q \cap C)$ by definition of quasi-lenses, and since open sets such as $\bigcup_{x \in E} B_{x, < r}^d$ are upwards-closed. For every $x \in E$, $B_{x, < r}^d$ intersects C , by (a) and (c). Therefore (Q, C) is in $\square_{\mathcal{P}}(\bigcup_{x \in E} B_{x, < r}^d) \cap \bigcap_{x \in E} \diamond_{\mathcal{P}} B_{x, < r}^d$, and we have noticed in (i) that the latter set is just $B_{(Q_0, C_0), < r}^{d_{\mathcal{P}}}$, where $Q_0 \stackrel{\text{def}}{=} \uparrow E$ and $C_0 \stackrel{\text{def}}{=} \downarrow E$. [In the case of $d_{\mathcal{P}}^a$, this is also equal to $B_{(Q_0, C_0), < r}^{d_{\mathcal{P}}^a}$, since $r \leq a$.]

It remains to show that $B_{(Q_0, C_0), < r}^{d_{\mathcal{P}}}$ is included in \mathcal{U} . Let (Q', C') be any element of $B_{(Q_0, C_0), < r}^{d_{\mathcal{P}}} = \square_{\mathcal{P}}(\bigcup_{x \in E} B_{x, < r}^d) \cap \bigcap_{x \in E} \diamond_{\mathcal{P}} B_{x, < r}^d$. From (b), (e), and the fact that each U_i is included in U , we obtain that $\bigcup_{x \in E} B_{x, < r}^d \subseteq U$. Hence, and since $Q' \subseteq \bigcup_{x \in E} B_{x, < r}^d$, Q' is included in U . Next, using (V), for every $x \in E$, $d(x, C') < r$. In particular, for every i , $1 \leq i \leq m$, $d(x_i, C') < r$. By (V), C' intersects $B_{x_i, < r}^d$. Since the latter is included in U_i by (b), C' intersects U_i . It follows that (Q', C') is in $\square_{\mathcal{P}} U \cap \bigcap_{i=1}^n \diamond_{\mathcal{P}} U_i$, hence in the larger set \mathcal{U} .

(iii) We claim that the specialization ordering of the Vietoris topology is $\supseteq \times \subseteq$. For any two quasi-lenses (Q, C) and (Q', C') , if $Q \supseteq Q'$ and $C \subseteq C'$, then clearly any basic open subset $\square_{\mathcal{P}} U \cap \bigcap_{i=1}^n \diamond_{\mathcal{P}} U_i$ that contains (Q, C) also contains (Q', C') . Conversely, let us assume that (Q, C) is smaller than or equal to (Q', C') in the specialization ordering of the Vietoris topology. For every open neighborhood U of Q , (Q, C) is in $\square_{\mathcal{P}} U$, hence also (Q', C') , so U contains Q' . The intersection of all those open neighborhoods is Q , since Q is saturated, and contains Q' . In order to show that $C \subseteq C'$, let U' be the complement of C' . Since $(Q, C) \in \diamond_{\mathcal{P}} U'$ implies $(Q', C') \in \diamond_{\mathcal{P}} U'$, and since the latter is false, C does not intersect U' , meaning that C is included in C' .

Now, if (Q, C) and (Q', C') are any two lenses such that $Q \supseteq Q'$ and $C \subseteq C'$, then $d_{\mathcal{P}}((Q, C), (Q', C')) = \max(d_{\mathcal{Q}}(Q, Q'), d_{\mathcal{H}}(C, C'))$ by Lemma 3.3, and this is simply 0, since \supseteq is the specialization ordering of $d_{\mathcal{Q}}$ and since \subseteq is the specialization ordering of $d_{\mathcal{H}}$. [Similarly with $d_{\mathcal{P}}^a, d_{\mathcal{Q}}^a, d_{\mathcal{H}}^a.$] \square

As for the Hoare and Smyth hyperspaces [13], we reduce the study of continuity to algebraicity.

Lemma 3.10. *Let X, d and Y, ∂ be two continuous complete quasi-metric spaces, and $f: X, d \mapsto Y, \partial$ be a 1-Lipschitz continuous map. The map $\mathcal{P}l f$ defined by $\mathcal{P}l f(Q, C) \stackrel{\text{def}}{=} (\mathcal{Q}f(Q), \mathcal{H}_0f(C)) = (\uparrow f[Q], cl(f[C]))$ is a 1-Lipschitz continuous map from the space of quasi-lenses on X with the $d_{\mathcal{P}}$ (resp., $d_{\mathcal{P}}^a, a > 0$) quasi-metric to the space of quasi-lenses on Y with the $\partial_{\mathcal{P}}$ (resp., $\partial_{\mathcal{P}}^a$) quasi-metric.*

PROOF. Once again, we only deal with $d_{\mathcal{P}}$ and $\partial_{\mathcal{P}}$. We need to check that $\mathcal{P}l f(Q, C)$ is a quasi-lens. Let $Q' \stackrel{\text{def}}{=} \uparrow f[Q]$, $C' \stackrel{\text{def}}{=} cl(f[C])$. We rely on Proposition 3.1: we know that (F_Q, F^C) satisfies Walley's condition, and we show that (Q', C') is a quasi-lens by showing that $(F_{Q'}, F^{C'})$ satisfies Walley's condition, too. By the last parts of Theorem 2.6 and of Theorem 2.5, $F^{C'} = \mathbf{P}f(F^C)$ and $F_{Q'} = \mathbf{P}f(F_Q)$. Now $F_{Q'}(k + k') = \mathbf{P}f(F_Q)(k + k') = F_Q((k + k') \circ f) \leq F_Q(k \circ f) + F^C(k' \circ f)$ (by Walley's condition, left inequality, on (F_Q, F^C)) $= \mathbf{P}f(F_Q)(k) + \mathbf{P}f(F^C)(k') = F_{Q'}(k) + F^{C'}(k')$. The other part of Walley's condition is proved similarly.

By Theorem 2.6 and of Theorem 2.5, $\mathcal{Q}f$ and \mathcal{H}_0f are 1-Lipschitz continuous. In particular, they are 1-Lipschitz, and then Lemma 3.3 immediately implies that $\mathcal{P}l f$ is 1-Lipschitz, too.

Given a directed family of formal balls $((Q_i, C_i), r_i)_{i \in I}$ on quasi-lenses, we have seen in Proposition 3.7 that its supremum $((Q, C), r)$ is characterized by $r = \inf_{i \in I} r_i$, (Q, r) is the supremum of $(Q_i, r_i)_{i \in I}$ in $\mathbf{B}(\mathcal{Q}X, d_{\mathcal{Q}})$ and (C, r) is the supremum of $(C_i, r_i)_{i \in I}$ in $\mathbf{B}(\mathcal{H}X, d_{\mathcal{H}})$. Since $\mathcal{Q}f$ and \mathcal{H}_0f are 1-Lipschitz continuous, $(\mathcal{Q}f(Q), r)$ is the supremum of $(\mathcal{Q}f(Q_i), r_i)_{i \in I}$ and $(\mathcal{H}_0f(C), r)$ is the supremum of $(\mathcal{H}_0f(C_i), r_i)_{i \in I}$. By Proposition 3.7, the supremum of $(\mathcal{P}l f(Q_i, C_i), r_i)_{i \in I}$ is $((\mathcal{Q}f(Q), \mathcal{H}_0f(C)), r) = (\mathcal{P}l f(Q, C), r)$. Therefore $\mathcal{P}l f$ is 1-Lipschitz continuous. \square

Let X, d be a continuous complete quasi-metric space. We recall from Section 2.3 that X, d is a 1-Lipschitz continuous retract of some algebraic complete quasi-metric space Y, ∂ . Let $r: Y \rightarrow X$ be the retraction and $s: X \rightarrow Y$ be the section. By Lemma 3.10, $\mathcal{P}l r$ and $\mathcal{P}l s$ are also 1-Lipschitz

continuous, and clearly $\mathcal{P}l r \circ \mathcal{P}l s = \text{id}$, so the space of quasi-lenses on X with the $d_{\mathcal{P}}$ quasi-metric is a 1-Lipschitz continuous retract of that on Y , with the $\partial_{\mathcal{P}}$ quasi-metric. Theorem 3.9 states that the latter is algebraic complete, whence the following.

Theorem 3.11 (Continuity for the Plotkin hyperspace). *Let X, d be a continuous complete quasi-metric space. The space of quasi-lenses on X with the $d_{\mathcal{P}}$ (resp., $d_{\mathcal{P}}^a$, $a > 0$) quasi-metric is continuous complete. \square*

3.1. The Vietoris Topology

Theorem 3.12 ($d_{\mathcal{P}}$ quasi-metrizes the Vietoris topology). *Let X, d be a continuous complete quasi-metric space. The $d_{\mathcal{P}}$ -Scott topology and the $d_{\mathcal{P}}^a$ -Scott topology ($a > 0$) coincide with the Vietoris topology on the space of quasi-lenses on X .*

PROOF. X, d is the 1-Lipschitz continuous retract of an algebraic complete quasi-metric space Y, ∂ . Let us call $s: X \rightarrow Y$ the section and $r: Y \rightarrow X$ the retraction. Using Corollary 3.4, we confuse quasi-lenses with discrete normalized forks. Then $\mathbf{P}s$ and $\mathbf{P}r$ form a 1-Lipschitz continuous section-retraction pair by Lemma 3.10, and in particular $\mathbf{P}s$ is an embedding of the space of quasi-lenses on X into the space of quasi-lenses on Y , with their $d_{\mathcal{P}}$ -Scott and $\partial_{\mathcal{P}}$ -Scott topologies, as well as with their $d_{\mathcal{P}}^a$ -Scott and $\partial_{\mathcal{P}}^a$ -Scott topologies. However, s and r are also just continuous, since they are 1-Lipschitz continuous, so $\mathbf{P}s$ and $\mathbf{P}r$ also form a section-retraction pair between the same spaces, this time with their weak topologies (as spaces of forks). Indeed, \mathbf{P} is an endofunctor on the category of topological spaces. However, since Y, ∂ is algebraic complete, the weak topology on $\mathbf{P}Y$ coincides with its $d_{\mathcal{P}}$ -Scott topology and with its $d_{\mathcal{P}}^a$ -Scott topology. Then $\mathbf{P}s$ is a topological embedding of the space of quasi-lenses on X (equivalently, discrete normalized forks), either with its weak topology, with its $d_{\mathcal{P}}$ -Scott topology, or with its $d_{\mathcal{P}}^a$ -Scott topology into the same space. Hence the three topologies coincide. \square

4. Forks

Just as one would expect by analogy with spaces of sublinear and super-linear previsions [14], the space of (sub)normalized forks on a space X arises as a retract of the Plotkin hyperspace on its space of (sub)normalized linear previsions.

We recall that $\mathbf{V}X$ is the space of linear previsions on X , $\mathbf{V}_{\leq 1}X$ is its subspace of subnormalized linear previsions, and \mathbf{V}_1X is the subspace of normalized linear previsions on X , all with the weak topology. We will in general use the notation $\mathbf{V}_{\bullet}X$, where \bullet can be nothing, “ ≤ 1 ”, or “1”.

For every quasi-lens $(\mathcal{Q}, \mathcal{C})$ on $\mathbf{V}_{\bullet}X$, let $r_{\text{ADP}}(\mathcal{Q}, \mathcal{C})$ denote the pair $(r_{\text{DP}}(\mathcal{Q}), r_{\text{AP}}(\mathcal{C}))$, where for every $h \in \mathcal{L}X$, $r_{\text{DP}}(\mathcal{Q})(h) \stackrel{\text{def}}{=} \inf_{G \in \mathcal{Q}} G(h)$ and $r_{\text{AP}}(\mathcal{C})(h) \stackrel{\text{def}}{=} \sup_{G \in \mathcal{C}} G(h)$.

In the reverse direction, for every (sub)normalized fork (F^-, F^+) on X , let $s_{\text{ADP}}^{\bullet}(F^-, F^+)$ be $(s_{\text{DP}}^{\bullet}(F^-), s_{\text{AP}}^{\bullet}(F^+))$, where s_{DP}^{\bullet} and s_{AP}^{\bullet} are defined as follows. When \bullet is nothing, s_{DP} maps every superlinear prevision F to the set of all linear previsions $G \geq F$, and s_{AP} maps every sublinear prevision F to the set of all linear previsions $G \leq F$. When \bullet is “ ≤ 1 ”, $s_{\text{DP}}^{\leq 1}$ maps every subnormalized superlinear prevision F to the set of all subnormalized linear previsions $G \geq F$, and $s_{\text{AP}}^{\leq 1}$ maps every subnormalized sublinear prevision to the set of all subnormalized linear previsions $G \leq F$; $s_{\text{DP}}^1(F)$ and $s_{\text{AP}}^1(F)$ are defined similarly, with “normalized” replacing “subnormalized” throughout.

The maps r_{ADP} and s_{ADP}^{\bullet} are close cousins of the eponymous maps of [10, Definition 3.23]. The main difference is that lenses were used instead of quasi-lenses there.

Following [19, Definition 4.8], we say that $\mathcal{L}X$ is *locally convex* if and only if every element $h \in \mathcal{L}X$ has a neighborhood basis of open convex neighborhoods. By [3, Lemma 13.6], this holds if X is LCS-complete space, notably if X has the d -Scott topology of a quasi-metric such that X, d is continuous complete [3, Theorem 4.1].

Following [19, Definition 4.6], we say that addition is *almost open* on $\mathcal{L}X$ if and only if, for any two open subsets U and V of $\mathcal{L}X$, the upward closure $\uparrow(U + V)$ is open. This happens notably when X is core-compact and core-coherent [10, Lemma 3.24]; for a definition of core-coherence, see Definition 5.2.18 of [8]; every locally compact, coherent space is core-compact and core-coherent, where coherence means that the intersection of two compact saturated sets is compact [8, Lemma 5.2.24].

The following is a variant of Proposition 3.32 and Proposition 4.8 of [10], using quasi-lenses instead of lenses. We say that a quasi-lens (Q, C) is *convex* if and only if both Q and C are.

Proposition 4.1. *Let X be a space such that $\mathcal{L}X$ is locally convex and has an almost open addition map, and let \bullet be nothing, “ ≤ 1 ”, or “1”. Additionally, if \bullet is “1”, we assume that X is compact.*

The map r_{ADP} forms a retraction of the space of quasi-lenses on $\mathbf{V}_\bullet X$ with the Vietoris topology onto the space of forks (resp., of subnormalized forks if \bullet is “ ≤ 1 ”, of normalized forks if \bullet is “1”) with the weak topology, with associated section s_{ADP}^\bullet .

This retraction restricts to a homeomorphism between the subspace of convex quasi-lenses on $\mathbf{V}_\bullet X$ and the space of forks (resp., of subnormalized forks if \bullet is “ ≤ 1 ”, of normalized forks if \bullet is “1”) with the weak topology.

PROOF. First, r_{DP} forms a topological retraction of $\mathcal{Q}(\mathbf{V}_\bullet X)$ onto the space of superlinear previsions on X (with the required (sub)normalization requirement, depending on \bullet) with the weak topology, and s_{DP}^\bullet is the associated section. We also know that r_{AP} forms a topological retraction of $\mathcal{H}(\mathbf{V}_\bullet X)$ onto the space of sublinear previsions on X (again with the required (sub)normalization requirement) with the weak topology, and that s_{AP}^\bullet is the associated section. This was already used in Part III [14], and stems from Proposition 3.22 of [10] for r_{DP} and s_{DP}^\bullet , and from Corollary 3.12 of [10] for r_{AP} and s_{AP}^\bullet . The latter result is where the local convexity of $\mathcal{L}X$ is required.

The map r_{ADP} takes its values in a space of forks. For every quasi-lens $(\mathcal{Q}, \mathcal{C})$ on $\mathbf{V}_\bullet X$, we need to check that $r_{\text{ADP}}(\mathcal{Q}, \mathcal{C})$ is a fork, and the only thing that remains to be checked is Walley’s condition.

For every $h \in \mathcal{L}X$, let $h^\perp: \mathbf{V}_\bullet X \rightarrow \overline{\mathbb{R}}_+$ map G to $G(h)$. This is a lower semicontinuous map, since $h^{\perp^{-1}}([a, +\infty]) = [h > a]$ for every $a \in \mathbb{R}_+$. Also, for all $h, h' \in \mathcal{L}X$, $(h + h')^\perp = h^\perp + h'^\perp$, because every G in $\mathbf{V}_\bullet X$ is linear. We note that, for every quasi-lens $(\mathcal{Q}, \mathcal{C})$ on $\mathbf{V}_\bullet X$, for every $h \in \mathcal{L}X$, $r_{\text{DP}}(\mathcal{Q})(h) = \inf_{G \in \mathcal{Q}} G(h) = \inf_{G \in \mathcal{Q}} h^\perp(G) = F_{\mathcal{Q}}(h^\perp)$, and similarly $r_{\text{AP}}(\mathcal{C})(h) = F^{\mathcal{C}}(h^\perp)$. By Proposition 3.1, $(F_{\mathcal{Q}}, F^{\mathcal{C}})$ is a (discrete) normalized fork, so for all $h, h' \in \mathcal{L}X$,

$$\begin{aligned} r_{\text{DP}}(\mathcal{Q})(h + h') &= F_{\mathcal{Q}}((h + h')^\perp) = F_{\mathcal{Q}}(h^\perp + h'^\perp) \\ &\leq F_{\mathcal{Q}}(h^\perp) + F^{\mathcal{C}}(h'^\perp) && \text{(Walley’s condition)} \\ &= r_{\text{DP}}(\mathcal{Q})(h) + r_{\text{AP}}(\mathcal{C})(h'), \end{aligned}$$

and

$$\begin{aligned} r_{\text{DP}}(\mathcal{Q})(h) + r_{\text{AP}}(\mathcal{C})(h') &= F_{\mathcal{Q}}(h^\perp) + F^{\mathcal{C}}(h'^\perp) \\ &\leq F^{\mathcal{C}}(h^\perp + h'^\perp) && \text{(Walley’s condition)} \\ &= F^{\mathcal{C}}((h + h')^\perp) = r_{\text{AP}}(h + h'). \end{aligned}$$

The map r_{ADP} is continuous. We recall that r_{DP} and r_{AP} are both continuous, being part of a retraction-section pair. Since the weak topology on spaces of forks is induced by the product topology on the product of the spaces of superlinear and sublinear previsions, it suffices to show that the maps $(\mathcal{Q}, \mathcal{C}) \mapsto r_{\text{DP}}(\mathcal{Q})$ and $(\mathcal{Q}, \mathcal{C}) \mapsto r_{\text{AP}}(\mathcal{C})$ are continuous in order to establish that r_{ADP} is continuous. In turn, this follows from the fact that the projection maps $(\mathcal{Q}, \mathcal{C}) \mapsto \mathcal{Q}$ and $(\mathcal{Q}, \mathcal{C}) \mapsto \mathcal{C}$ are continuous, which is clear since the inverse image of any basic open set $\square U$ by the first map is $\square_{\mathcal{P}} U$ and the inverse image of any subbasic open set $\diamond U$ by the second map is $\diamond_{\mathcal{P}} U$.

The map s_{ADP}^{\bullet} takes its values in the space of quasi-lenses. Let (F^-, F^+) be an arbitrary fork on X (resp. subnormalized, or normalized, depending on \bullet), and let $(\mathcal{Q}, \mathcal{C}) \stackrel{\text{def}}{=} s_{\text{ADP}}^{\bullet}(F^-, F^+)$. We already know that \mathcal{Q} is a non-empty compact saturated subset of $\mathbf{V}_{\bullet} X$, and that \mathcal{C} is a non-empty closed subset of $\mathbf{V}_{\bullet} X$.

We use Lemma 3.28 of [10], which says that, for every $G' \in \mathbf{V}_{\bullet} X$ such that $F^- \leq G'$, there is a $G \in \mathbf{V}_{\bullet} X$ such that $F^- \leq G \leq F^+$ and $G \leq G'$. (This requires no condition on the space X .) By definition of \mathcal{Q} and \mathcal{C} , this can be rephrased as: for every $G' \in \mathcal{Q}$, there is an element $G \in \mathcal{Q} \cap \mathcal{C}$ such that $G \leq G'$. In other words, $\mathcal{Q} \subseteq \uparrow(\mathcal{Q} \cap \mathcal{C})$.

Lemma 3.29 of [10] states that, given that $\mathcal{L}X$ is locally convex and has an almost open addition map, and that X is compact if \bullet is “1”, then for every $G' \in \mathbf{V}_{\bullet} X$ such that $G' \leq F^+$, there is a $G \in \mathbf{V}_{\bullet} X$ such that $F^- \leq G \leq F^+$ and $G' \leq G$. This means that for every $G' \in \mathcal{C}$, G' is in $\downarrow(\mathcal{Q} \cap \mathcal{C})$. Therefore $\mathcal{C} \subseteq \downarrow(\mathcal{Q} \cap \mathcal{C}) \subseteq cl(\mathcal{Q} \cap \mathcal{C})$. In particular, for every open neighborhood \mathcal{U} of \mathcal{Q} , \mathcal{C} is included in $cl(\mathcal{Q} \cap \mathcal{U})$, so $(\mathcal{Q}, \mathcal{C})$ is a quasi-lens.

The map s_{ADP}^{\bullet} is continuous. This follows from the fact that s_{DP}^{\bullet} and s_{AP}^{\bullet} are continuous.

$r_{\text{ADP}} \circ s_{\text{ADP}}^{\bullet}$ is the identity map. This follows since $r_{\text{DP}} \circ s_{\text{DP}}^{\bullet}$ and $r_{\text{AP}} \circ s_{\text{AP}}^{\bullet}$ are both identity maps.

$s_{\text{ADP}}^{\bullet} \circ r_{\text{ADP}}$ is the identity map on the space of convex quasi-lenses. It suffices to observe that $s_{\text{DP}}^{\bullet} \circ r_{\text{DP}}$ is the identity map on the space of convex non-empty compact saturated subsets of X , and that $s_{\text{AP}}^{\bullet} \circ r_{\text{AP}}$ is the identity map on the space of convex non-empty closed subsets of X . Those are Propositions 4.5 and 4.3 of [10], respectively; the second one requires that \mathcal{L} be locally convex, while the first one requires nothing from X . \square

Remark 4.2. *Under the assumption that $\mathcal{L}X$ is locally convex and has an almost open addition map (and that X is compact in the case where \bullet is “1”), then the composition of the homeomorphism of Proposition 4.1 with the homeomorphism of [10, Proposition 4.8] yields a homeomorphism from the space of convex lenses to the space of convex quasi-lenses on $\mathbf{V}\bullet X$.*

Theorem 4.3 (Continuity for forks). *Let X, d be a continuous complete quasi-metric space, $a > 0$, and \bullet be “ ≤ 1 ” or “1”. Let us also assume that $\mathcal{L}X$ has an almost open addition map, and that X is compact in the case where \bullet is “1”.*

The space of all subnormalized (if \bullet is “ ≤ 1 ”, normalized if \bullet is “1”) forks on X with the d_{KR}^a quasi-metric is continuous complete.

It arises as a 1-Lipschitz continuous retract of the space of quasi-lenses over $\mathbf{V}\bullet X$ with the $(d_{KR}^a)_{\mathcal{P}}$ quasi-metric through r_{ADP} and s_{ADP}^{\bullet} . That retraction cuts down to an isometry between the space of (sub)normalized forks on X and the space of convex quasi-lenses over $\mathbf{V}\bullet X$, with the $(d_{KR}^a)_{\mathcal{P}}$ quasi-metric.

The supremum of a directed family of formal balls $((F_i^-, F_i^+), r_i)_{i \in I}$ is $((F^-, F^+), r)$ where $r = \inf_{i \in I} r_i$, F^+ is the naive supremum of $(F_i^+, r_i)_{i \in I}$, and F^- is the naive supremum of $(F_i^-, r_i)_{i \in I}$. In particular, (F^+, r) is the supremum of $(F_i^+, r_i)_{i \in I}$ in the space of (sub)normalized sublinear previsions and (F^-, r) is the supremum of $(F_i^-, r_i)_{i \in I}$ in the space of (sub)normalized superlinear previsions.

PROOF. We use Theorem 2.2 and Theorem 2.3 without any further reference.

Given a directed family of formal balls $((F_i^-, F_i^+), r_i)_{i \in I}$, $(F_i^-, r_i)_{i \in I}$ is a directed family of formal balls on the space of (sub)normalized superlinear previsions on X with the d_{KR}^a quasi-metric, and $(F_i^+, r_i)_{i \in I}$ is a directed family of formal balls on the space of (sub)normalized sublinear previsions on X with the d_{KR}^a quasi-metric. Let (F^-, r) be the (naive) supremum of the former and (F^+, r) be the (naive) supremum of the latter. Lemma 3.6 states that (F^-, F^+) is a fork, provided we check that X is sober. We recall that every continuous complete quasi-metric space is sober [16, Proposition 4.1]. One easily checks that (F^-, F^+) is the supremum of $((F_i^-, F_i^+), r_i)_{i \in I}$.

That characterization of directed suprema, together with the analogous characterization of directed suprema of formal balls on the space of quasi-lenses (Proposition 3.7) and the fact that r_{AP} , r_{DP} , s_{AP} , s_{DP} are 1-Lipschitz continuous, shows that r_{ADP} and s_{ADP} are 1-Lipschitz continuous. Note that we apply Proposition 3.7 to the space of quasi-lenses over $\mathbf{V}\bullet X$, d_{KR}^a , and that is legitimate since $\mathbf{V}\bullet X, d_{KR}^a$ is continuous complete, by Theorem 2.1.

We deal with the case of r_{ADP} to give the idea of the proof. Let $((Q_i, C_i), r_i)_{i \in I}$ be a directed family of formal balls on the space of quasi-lenses of $\mathbf{V}_\bullet X$, with supremum $((Q, C), r)$. In particular, $r = \inf_{i \in I} r_i$, (Q, r) is the supremum of the directed family $(Q_i, r_i)_{i \in I}$ and (C, r) is the supremum of the directed family $(C_i, r_i)_{i \in I}$. Since r_{DP} and r_{AP} are 1-Lipschitz continuous, $(r_{\text{DP}}(Q), r)$ is the supremum of $(r_{\text{DP}}(Q_i), r_i)_{i \in I}$ and $(r_{\text{AP}}(C), r)$ is the supremum of $(r_{\text{AP}}(C_i), r_i)_{i \in I}$. We have just seen that this implies that $(r_{\text{ADP}}(Q, C), r) = ((r_{\text{DP}}(Q), r_{\text{AP}}(C)), r)$ is the supremum of $((r_{\text{DP}}(Q_i), r_{\text{AP}}(C_i)), r_i)_{i \in I}$, that is, of $(r_{\text{ADP}}(Q_i, C_i), r_i)_{i \in I}$.

We have $r_{\text{ADP}}(s_{\text{ADP}}(F^-, F^+)) = (r_{\text{DP}}(s_{\text{DP}}(F^-)), r_{\text{AP}}(s_{\text{AP}}(F^+))) = (F^-, F^+)$, so r_{ADP} and s_{ADP} form a 1-Lipschitz continuous retraction. That cuts down to an isometry when we restrict ourselves to convex quasi-lenses, because r_{DP} , s_{DP} and r_{AP} , s_{AP} form isometries when restricted to convex non-empty compact saturated sets and convex non-empty closed sets respectively, by the final part of Proposition 4.1.

Since $\mathbf{V}_\bullet X, d_{\text{KR}}^a$ is continuous complete, Theorem 3.11 tells us that the space of quasi-lenses over $\mathbf{V}_\bullet X$, with the $(d_{\text{KR}}^a)_{\mathcal{P}}$ quasi-metric, is continuous complete. Every 1-Lipschitz continuous retract of a continuous complete quasi-metric space is continuous complete [16, Theorem 7.1], therefore the space of (sub)normalized forks over X with the d_{KR}^a quasi-metric is continuous complete. \square

Theorem 4.4 (Algebraicity for forks). *Let X, d be an algebraic complete quasi-metric space, with a strong basis \mathcal{B} . Let $a > 0$, and \bullet be “ ≤ 1 ” or “1”. Let us also assume that $\mathcal{L}X$ has an almost open addition map, and that X is compact in the case where \bullet is “1”.*

The space of all subnormalized (resp., normalized) forks on X with the d_{KR}^a quasi-metric is algebraic complete.

All the forks of the form (F^-, F^+) where $F^-(h) = \min_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$ and $F^+(h) = \max_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$, with $m \geq 1$, $\sum_{j=1}^{n_i} a_{ij} \leq 1$ (resp., = 1) for every i , and where each x_{ij} is a center point, are center points, and they form a strong basis, even when each x_{ij} is constrained to lie in \mathcal{B} .

PROOF. We start with the second statement. Let (F_0^-, F_0^+) where $F_0^-(h) \stackrel{\text{def}}{=} \min_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$ and $F_0^+(h) \stackrel{\text{def}}{=} \max_{i=1}^m \sum_{j=1}^{n_i} a_{ij} h(x_{ij})$, with $m \geq 1$, $\sum_{j=1}^{n_i} a_{ij} \leq 1$ (resp., = 1) for every i , and where each x_{ij} is a center point. In order to show that (F_0^-, F_0^+) is a center point, we consider the open

ball $B_{((F_0^-, F_0^+), 0) < \epsilon}^{d_{\text{KR}}^{a+}}$, and we claim that it is Scott-open. For every directed family of formal balls $((F_i^-, F_i^+), r_i)_{i \in I}$, with supremum $((F^-, F^+), r)$ inside $B_{((F_0^-, F_0^+), 0) < \epsilon}^{d_{\text{KR}}^{a+}}$, we have $r = \inf_{i \in I} r_i$, $d_{\text{KR}}^a(F_0^-, F^-) < \epsilon - r$, and $d_{\text{KR}}^a(F_0^+, F^+) < \epsilon - r$. Also, by Theorem 4.3, (F_0^-, r) is the supremum of the directed family $(F_i^-, r_i)_{i \in I}$. Since F_0^- is a center point, $B_{(F_0^-, 0) < \epsilon}^{d_{\text{KR}}^{a+}}$ is Scott-open, and contains (F_0^-, r) , hence contains some (F_i^-, r_i) ; similarly, $B_{(F_0^+, r) < \epsilon - r}^{d_{\text{KR}}^{a+}}$ contains some (F_i^+, r_i) , and we can take the same i , by directedness. It follows that $d_{\text{KR}}^a(F_0^-, F_i^-) < \epsilon - r_i$ and $d_{\text{KR}}^a(F_0^+, F_i^+) < \epsilon - r_i$. Hence $((F_i^-, F_i^+), r_i)$ is in $B_{((F_0^-, F_0^+), 0) < \epsilon}^{d_{\text{KR}}^{a+}}$.

Finally, we show that the forks (F_0^-, F_0^+) as above, where $x_{ij} \in \mathcal{B}$, form a strong basis. By Theorem 2.1, the (sub)normalized simple valuations $\sum_{i=1}^n a_i \delta_{x_i}$ where $x_i \in \mathcal{B}$ form a strong basis of $\mathbf{V}_\bullet X$. Hence the forks of the form $(\uparrow E, \downarrow E)$, where E is a finite non-empty set of such simple valuations, form a strong basis of the space of quasi-lenses over $\mathbf{V}_\bullet X$, by Theorem 3.9.

Lemma 3.13 of [14] states that any 1-Lipschitz continuous retract Z of an algebraic complete quasi-metric space Y, ∂ with strong basis \mathcal{B} is itself continuous complete, and that for every point z of Z , $(z, 0)$ is the supremum of a directed family of formal balls whose centers are images of points of \mathcal{B} by the retraction. Applying this to the 1-Lipschitz continuous retraction r_{ADP} (Theorem 4.3), we obtain that for every (sub)normalized fork (F^-, F^+) , $((F^-, F^+), 0)$ is the supremum of a directed family of formal balls with centers of the form $r_{\text{ADP}}(\uparrow E, \downarrow E)$, where E is as above. We check easily that $r_{\text{ADP}}(\uparrow E, \downarrow E)$ is the fork (F_0^-, F_0^+) given at the outset, and this allows us to conclude. \square

Lemma 4.5. *Let X, d and Y, ∂ be two continuous complete quasi-metric spaces, and $f: X, d \mapsto Y, \partial$ be a 1-Lipschitz continuous map. Let $a > 0$. Let us also assume that $\mathcal{L}X$ and $\mathcal{L}Y$ have almost open addition maps, and that X and Y are compact in the case where \bullet is “1”. The map $(F^-, F^+) \mapsto (\mathbf{P}f(F^-), \mathbf{P}f(F^+))$ is 1-Lipschitz continuous from the space of normalized, resp. subnormalized forks on X to the space of normalized, resp. subnormalized forks on Y , with the d_{KR}^a and ∂_{KR}^a quasi-metrics.*

PROOF. By the last parts of Theorem 2.2 and 2.3, $\mathbf{P}f(F^-)$ is a normalized, resp. subnormalized superlinear prevision, and $\mathbf{P}f(F^+)$ is a normalized, resp.

subnormalized sublinear prevision. Moreover, the maps $F^- \mapsto \mathbf{P}f(F^-)$ and $F^+ \mapsto \mathbf{P}f(F^+)$ are 1-Lipschitz continuous.

Walley's condition is immediate. For all $k, k' \in \mathcal{L}Y$,

$$\begin{aligned} \mathbf{P}f(F^-)(k + k') &= F^-(k \circ f + k' \circ f) \\ &\leq F^-(k \circ f) + F^+(k' \circ f) = \mathbf{P}f(F^-)(k) + \mathbf{P}f(F^+)(k'), \end{aligned}$$

and

$$\begin{aligned} \mathbf{P}f(F^-)(k) + \mathbf{P}f(F^+)(k') &= F^-(k \circ f) + F^+(k' \circ f) \\ &\leq F^+(k \circ f + k' \circ f) = \mathbf{P}f(F^+)(k + k'). \end{aligned}$$

By definition of d_{KR}^a as a maximum (Definition 2.7), and since $\mathbf{P}f$ is 1-Lipschitz, the map $(F^-, F^+) \mapsto (\mathbf{P}f(F^-), \mathbf{P}f(F^+))$ is 1-Lipschitz. For every directed family $((F_i^-, F_i^+), r_i)_{i \in I}$ of formal balls on (sub)normalized forks, let $((F^-, F^+), r)$ be its supremum, as given in the last part of Theorem 4.3: (F^-, r) is the supremum of the directed family $(F_i^-, r_i)_{i \in I}$ in the space of formal balls over the space of (sub)normalized superlinear previsions, and (F^+, r) is the supremum of $(F_i^+, r_i)_{i \in I}$. By the 1-Lipschitz continuity of $\mathbf{P}f$ on spaces of sublinear and superlinear (sub)normalized previsions, $(\mathbf{P}f(F^-), r)$ is the supremum of $(\mathbf{P}f(F_i^-), r_i)_{i \in I}$, and $(\mathbf{P}f(F^+), r)$ is the supremum of $(\mathbf{P}f(F_i^+), r_i)_{i \in I}$, so $((\mathbf{P}f(F^-), \mathbf{P}f(F^+)), r)$ is the supremum of $((\mathbf{P}f(F_i^-), \mathbf{P}f(F_i^+)), r_i)_{i \in I}$, by the last part of Theorem 4.3 again. \square

With Theorem 4.4 and Theorem 4.3, we obtain the following.

Corollary 4.6. *There is a functor from the category of continuous (resp., algebraic) complete quasi-metric spaces X such that $\mathcal{L}X$ has an almost open addition map, and 1-Lipschitz continuous maps, to the category of continuous (resp., algebraic) complete quasi-metric spaces, which sends every object X, d to the space of subnormalized forks on X with the d_{KR}^a -Scott quasi-metric ($a > 0$), and every 1-Lipschitz continuous map f to the map $(F^-, F^+) \mapsto (\mathbf{P}f(F^-), \mathbf{P}f(F^+))$.*

There is a functor from the category of compact continuous (resp., algebraic) complete quasi-metric spaces X such that $\mathcal{L}X$ has an almost open addition map, and 1-Lipschitz continuous maps, to the category of compact continuous (resp., algebraic) complete quasi-metric spaces, which sends every object X, d to the space of normalized forks on X with the d_{KR}^a -Scott quasi-metric ($a > 0$), and every 1-Lipschitz continuous map f to the map $(F^-, F^+) \mapsto (\mathbf{P}f(F^-), \mathbf{P}f(F^+))$. \square

Lemma 4.7. *Let $f: X \rightarrow Y$ be a continuous map between topological spaces. The map $(F^-, F^+) \mapsto (\mathbf{P}f(F^-), \mathbf{P}f(F^+))$ is continuous from the space of (sub)normalized forks on X to the space of (sub)normalized forks on Y , both spaces being equipped with the weak topology.*

PROOF. The inverse image of the subbasic open set $[k > a]_- = \{(H^-, H^+) \mid H^-(k) > a\}$ is $\{(F^-, F^+) \mid \mathbf{P}f(F^-)(k) > a\} = \{(F^-, F^+) \mid F^-(k \circ f) > a\} = [k \circ f > a]_-$; similarly, the inverse image of $[k > a]_+$ is $[k \circ f > a]_+$. \square

As in the proof of Proposition 3.9, let us write $\square_{\mathcal{P}}U$ for the set of lenses (Q, C) such that $Q \subseteq U$, reserving the notation $\square U$ for the set of non-empty compact saturated subsets Q of X such that $Q \subseteq U$. We use a similar convention with $\diamond_{\mathcal{P}}U$ and $\diamond U$.

Proposition 4.8 (d_{KR}^a quasi-metrizes the weak topology, forks). *Let X, d be a continuous complete quasi-metric space. Let us also assume that $\mathcal{L}X$ has an almost open addition map, and that X is compact in the case where \bullet is “1”. The d_{KR}^a -Scott topology coincides with the weak topology on the space of (sub)normalized forks on X .*

PROOF. Let Y denote the space of (sub)normalized forks on X , and Z denote the space of quasi-lenses on $\mathbf{V}_{\bullet}X$. By Theorem 2.1, the d_{KR}^a -Scott topology coincides with the weak topology on $\mathbf{V}_{\bullet}X$, hence we may take either topology in order to define the quasi-lenses on $\mathbf{V}_{\bullet}X$, hence the elements of Z . Also, $\mathbf{V}_{\bullet}X, d_{\text{KR}}^a$ is continuous complete. This allows us to apply Theorem 3.12 and deduce that the $(d_{\text{KR}}^a)_{\mathcal{P}}$ -Scott and Vietoris topologies coincide on Z .

The pair of maps $r_{\text{ADP}}, s_{\text{ADP}}^{\bullet}$ defines a topological retraction of Z (with either topology) onto Y , with its weak topology, by Proposition 4.1. By Theorem 4.3, it also defines a 1-Lipschitz continuous retraction of $Z, (d_{\text{KR}}^a)_{\mathcal{P}}$ onto Y, d_{KR}^a . Since the two spaces are continuous complete, hence standard, this defines a topological retraction of Z onto Y , with its d_{KR}^a -Scott topology. This implies that s_{ADP}^{\bullet} defines a topological embedding of Y , either with its weak topology or with its d_{KR}^a -Scott topology, into Z , hence that the two topologies on Y coincide. \square

Remark 4.9. *We recall that every locally compact, coherent space X is such that addition is almost open on $\mathcal{L}X$. We may summarize some of our findings on forks in that useful situation as follows. Let $a > 0$, and \bullet be “ ≤ 1 ” or “1”. Let X, d be a continuous complete quasi-metric space that is locally compact*

and coherent in its d -Scott topology, and also compact if \bullet is “1”. Let Z be the space of subnormalized forks on X if \bullet is “ ≤ 1 ”, of normalized forks if \bullet is “1”. Then:

- (Theorem 4.3) Z, d_{KR}^a is continuous complete, and arises as a 1-Lipschitz continuous retract of the space of quasi-lenses over $\mathbf{V}_\bullet X$ with the Vietoris topology, through r_{ADP} and s_{ADP}^\bullet . That retraction cuts down to an isometry between Z, d_{KR}^a and the space of convex quasi-lenses over $\mathbf{V}_\bullet X$, with the $(d_{KR}^a)_P$ quasi-metric. Directed suprema of formal balls are computed through naive suprema.
- (Theorem 4.4) If X, d is additionally algebraic, with a strong basis \mathcal{B} , then Z, d_{KR}^a is algebraic complete, with a strong basis of forks described in Theorem 4.4.
- (Proposition 4.8) The d_{KR}^a -Scott topology coincides with the weak topology on Z .

References

- [1] Abramsky, S., Jung, A., 1994. Domain theory. In: Abramsky, S., Gabbay, D. M., Maibaum, T. S. E. (Eds.), Handbook of Logic in Computer Science. Vol. 3. Oxford University Press, pp. 1–168.
- [2] Ali-Akbari, M., Honari, B., Pourmahdian, M., Rezaii, M. M., 2009. The space of formal balls and models of quasi-metric spaces. Mathematical Structures in Computer Science 19, 337–355.
- [3] de Brecht, M., Goubault-Larrecq, J., Jia, X., Lyu, Z., 2019. Domain-complete and LCS-complete spaces. Electronic Notes in Theoretical Computer Science 345, 3–35, proc. 8th International Symposium on Domain Theory (ISDT’19), also available as arXiv report 1902.11142 [math.GN].
- [4] Edalat, A., Heckmann, R., 1998. A computational model for metric spaces. Theoretical Computer Science 193, 53–73.
- [5] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M., Scott, D. S., 2003. Continuous Lattices and Domains. Vol. 93 of Encyclopedia of Mathematics and its Applications. Cambridge University Press.

- [6] Goubault-Larrecq, J., Sep. 2007. Continuous previsions. In: Duparc, J., Henzinger, T. A. (Eds.), Proceedings of the 16th Annual EACSL Conference on Computer Science Logic (CSL'07). Springer-Verlag LNCS 4646, Lausanne, Switzerland, pp. 542–557.
- [7] Goubault-Larrecq, J., Apr. 2010. De Groot duality and models of choice: Angels, demons, and nature. *Mathematical Structures in Computer Science* 20 (2), 169–237.
- [8] Goubault-Larrecq, J., 2013. *Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology*. Vol. 22 of *New Mathematical Monographs*. Cambridge University Press.
- [9] Goubault-Larrecq, J., 2017. Complete quasi-metrics for hyperspaces, continuous valuations, and previsions, arXiv 1707.03784, v2, <https://arxiv.org/abs/1707.03784>; v6, July 2020.
- [10] Goubault-Larrecq, J., 2017. Isomorphism theorems between models of mixed choice. *Mathematical Structures in Computer Science* 27 (6), 10321067.
- [11] Goubault-Larrecq, J., 2019. Formal ball monads. *Topology and its Applications* 263, 372–391.
URL <http://www.sciencedirect.com/science/article/pii/S0166864119302160>
- [12] Goubault-Larrecq, J., 2021. Kantorovich-Rubinstein quasi-metrics I: Spaces of measures and of continuous valuations. *Topology and its Applications* 295, 107673.
URL <https://www.sciencedirect.com/science/article/pii/S0166864121000870>
- [13] Goubault-Larrecq, J., 2022. Kantorovich-Rubinstein quasi-metrics II: Hyperspaces and powerdomains. *Topology and its Applications* 305, 107885.
URL <https://www.sciencedirect.com/science/article/abs/pii/S016686412100300X>
- [14] Goubault-Larrecq, J., 2022. Kantorovich-Rubinstein quasi-metrics III: Spaces of sublinear and superlinear previsions. *Topology and its Applications*

Applications 321, 108259.

URL <https://www.sciencedirect.com/science/article/pii/S0166864122002619>

- [15] Goubault-Larrecq, J., 2022. On weakly Hausdorff spaces and locally strongly sober spaces. *Topology Proceedings* Submitted, available from <https://arxiv.org/abs/2211.10400>.
- [16] Goubault-Larrecq, J., Ng, K. M., Oct. 2017. A few notes on formal balls. *Logical Methods in Computer Science* 13 (4:18), 1–34, special issue of *Domain Theory and Applications (the Domains XII Workshop)*, Cork, Ireland, August 2015.
- [17] Heckmann, R., 1993. Power domains and second-order predicates. *Theoretical Computer Science* 111, 59–88.
- [18] Heckmann, R., 1997. Abstract valuations: A novel representation of Plotkin power domain and Vietoris hyperspace. *Electronic Notes in Theoretical Computer Science* 6, in *Proc. 13th Intl. Symp. on Mathematical Foundations of Programming Semantics (MFPS'97)*.
URL <http://rw4.cs.uni-sb.de/~heckmann/papers/absval.ps.gz>
- [19] Keimel, K., 2008. Topological cones: Functional analysis in a T_0 -setting. *Semigroup Forum* 77 (1), 109–142.
- [20] Keimel, K., Plotkin, G., 2017. Mixed powerdomains for probability and nondeterminism. *Logical Methods in Computer Science* 13 (1).
URL <https://lmcs.episciences.org/2665>
- [21] Weihrauch, K., Schreiber, U., 1981. Embedding metric spaces into cpo's. *Theoretical Computer Science* 16 (1), 5–24.