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Combinatorial Proofs for Constructive Modal Logic

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Abstract

Combinatorial proofs form a syntax-independent presentation of proofs, originally proposed by Hughes for classical propositional logic. In this paper we present a notion of combinatorial proofs for the constructive modal logics $CK$ and $CD$, we show soundness and completeness of combinatorial proofs by translation from and to sequent calculus proofs, and we discuss the notion of proof equivalence enforced by these translations.

Keywords: combinatorial proofs, proof equivalence, arena nets, constructive modal logic.

1 Introduction

Combinatorial proofs have first been introduced by Hughes in order to give a “syntax-free” presentation of proof in classical propositional logic [19]. Their motivation is to capture the essence of a proof independently from any deductive proof system, such that we can speak about proof equivalence for proofs given in different formalisms [4]. Only recently it was possible to extend this idea to richer logics: (classical) modal logics [6], relevant logics [5,8], first order logic [20,21], and intuitionistic propositional logic [17]. In this paper we investigate combinatorial proofs for intuitionistic logic with modalities.

There are many different flavours of “intuitionistic modal logics” (see, e.g., [14,30,29,31,9,12]), depending on which additional variants of the classical k-axiom $\Box(A \supset B) \supset (\Box A \supset \Box B)$ are added. It is necessary to add more than just k, as k does not speak of the diamond modality $\Diamond$, which is in the intuitionistic case no longer the De Morgan dual of the box modality $\Box$.

We take here the minimal approach and only add $\Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$ in addition to the k-axiom, leading to what is now called constructive modal logics in the literature [30,9,18,28,13,23]. We chose this setting because (1) we would like to make as few assumptions as possible, (2) these logics have a sequent calculus presentation, which makes it easier to show soundness and completeness of combinatorial proofs, and (3) there is a close relation to game semantics for modalities [3], extending the work in [17], and lambda-calculus proof terms for constructive modal logics [9].
Combinatorial Proofs for Constructive Modal Logic

The main contribution of this paper is the definition of combinatorial proofs for the constructive modal logics CK and CD and to prove their soundness and completeness. We also show that they form a proof system in the sense of Cook and Reckhow [10], that is, checking the correctness of a combinatorial proof can be done in polynomial time in its size.

A combinatorial proof of a formula $F$ is a certain kind of homomorphism $f : G \rightarrow \llbracket F \rrbracket$ between two directed graphs. The directed graph $\llbracket F \rrbracket$ is a modal arena and encodes the formula $F$. Modal arenas are an extension of the arenas of [17], and are introduced in Section 4 of this paper. The directed graph $G$ is a modal arena net and encodes the “linear part” of the proof. Modal arena nets (introduced in Section 5) are modal arenas equipped with a partition on their vertices, carrying the information of axiom linkings in proof nets. Finally, the homomorphism $f$ is a skew fibration [19,17] and encodes the “resource management part” of the proof, i.e., it collects the information carried by the rule instances of contraction and weakening in the sequent calculus. We discuss skew fibrations in Section 6. Figure 1 shows an example of a combinatorial proof for the formula $F = (\Box(a \land a) \supset b, \Box c \supset (\Box a \supset b))$, where the solid and squiggly arrows are the edges of the arena $G$ and the dashed edges represent the partition of $G$ (encoding the $\text{ax}$ and $\text{K}^0$ rules). The dotted downwards directed arrows represent the skew fibration $f$ (encoding the $\text{W}$ and $\text{C}$ rules).

In order to establish a close correspondence between combinatorial proofs and syntactic proofs in a deductive system, we need to have a decomposition theorem which allows to factorize proofs into a linear part, capturing the logic interactions between the components of the proof, and a resource management part, capturing resources duplication or erasing.

The second contribution of this paper is such a decomposition theorem for the logics CK and CD. To obtain this result, we use a combination of the sequent calculus and deep inference. More precisely, we use the cut-free sequent systems given in [24], and we add deep rules for contraction and weakening, in a similar way as it has been done in [6].

However, in an intuitionistic setting, we have to distinguish between the left-hand side and the right-hand side of a sequent, where contraction and weakening apply only to one side. Consequently, the deep versions of these rules need to have access to the information on which side a subformula will eventually occur. For this reason, we use polarities, in a similar way as done in [25,32]. The polarized system and the
We say that a formula is modality-free if it contains no occurrences of $\Box$ and $\Diamond$. A formula is a $\supset$-formula (resp. a $\wedge$-formula, $\Box$-formula, or $\Diamond$-formula) if it is a formula of the form $A \supset B$ (resp. $A \wedge B$, $\Box A$, or $\Diamond A$).

The constructive modal logic $\mathcal{CK}$ is obtained by extending the propositional intuitionistic logic with the necessitation rule: if $F$ is provable then so is $\Box F$, and the two modal axiom schemes $k_1$ and $k_2$ shown below on the left:

\[
k_1: \Box(A \supset B) \supset ((\Box A) \supset \Box B) \quad k_2: \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \quad d: \Box A \supset \Diamond A
\]

The logic $\mathcal{CD}$ is obtained from $\mathcal{CK}$ by adding the axiom scheme $d$ on the right above.

We are now recalling the sequent system for these logics. For this, we denote by capital Greek letters $\Gamma$ or $\Delta$ a multiset of formulas, separated by comma. We write $\Omega\Gamma$ (resp. $\Omega\Delta$) for any such multiset made only of $\Omega$-formulas (resp. $\Omega$-formulas). A sequent $\Gamma \vdash A$ is a pair of a multiset of formulas and a formula.

In Figure 2, we show the sequent system $\mathcal{LI}$ (e.g., given in [33]) for disjunction-free intuitionistic logic, its linear fragment $\mathcal{IMLL}$, and the sequent systems $\mathcal{LCK}$ and $\mathcal{LCD}$ for the logics $\mathcal{CK}$ and $\mathcal{CD}$, respectively, as presented in [24]. The cut rule

\[
\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \text{cut}
\]

is admissible for all four systems, and we have the following:

**Theorem 2.1** Let $X$ for $X \in \{\mathcal{CK}, \mathcal{CD}\}$. The sequent system $\mathcal{LX}$ is a sound and complete proof system for the disjunction-free fragment of the logic $X$.

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*Intuitionistic multiplicative linear logic (see e.g. [26])

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Proof. Soundness follows from the observation that our systems are the disjunction-free versions of the calculi in [24]. Completeness follows from the cut elimination property.

If $X$ is a set of rules, we write $F' \vdash F$ if there is a derivation from $\vdash F'$ to $\vdash F$ using rules in $X$. Moreover, we write $\vdash F$ if there is a proof of $F$ in $X$, i.e., a derivation with empty premises of $\vdash F$ using rules in $X$.

Finally, we define the formula isomorphism as the equivalence relation $\sim$ over formulas generated by the following relations:

$A \land \top \sim A$  
$A \supset \top \sim \top \supset A$  
$A \land B \sim B \land A$  
$(A \land B) \land C \sim (A \land B) \lor C$  
$(A \land B) \supset C \sim A \supset (B \supset C)$  

(1)

3 Polarized System and Decomposition

We define the set of polarized formulas (or $P$-formulas) as the set generated by $A = \{a, b, \ldots\}$ using the following grammar

$$A, B ::= a^o \mid \top^o \mid A \land B^o \mid A^* \supset B^o \mid \Box A^o \mid \Diamond A$$  
$$A^*, B^* ::= a^* \mid \top^* \mid A^* \land B^* \mid A^* \supset B^* \mid \Box A^* \mid \Diamond A^*$$  

(2)

We say that formulas $A^o, B^o, \ldots$ are of even polarity and formulas $A^*, B^*, \ldots$ are of odd polarity. Note that the polarity of a formula determines the polarity of each subformula of that formula. For this reason we will omit the polarity markings for subformulas. A polarized formula is clean if it contains no subformulas of the shape $A \supset \top^* \cdot$

A polarized sequent is a sequent of $P$-formulas. We write $\Gamma \cdot$ for a sequent containing only formulas of odd polarity. Then $\Gamma^*, A^*$ is simply the polarized version of a sequent $\Gamma \vdash A$. A context is a (polarized) sequent $\Gamma\{\}$ in which an atom (or more generally a subformula) has been replaced by an hole $\{\}$. Then $\Gamma\{\Delta\}$ stands for the sequent obtained from $\Gamma\{\}$ by replacing $\{\}$ with $\Delta$.

We can now define the polarized sequent rules given in Figure 3. Observe that the upper part and the two rules $c^*$ and $w^*$ on the lower left are just the polarized version of the rules in Figure 2. The two rules $c^*_1$ and $w^*_1$ on the lower right are the deep version of $c^*$ and $w^*$, respectively. They can be applied deep inside any formula context. Note that they can only be applied to formulas of odd polarity. Figure 4 lists the various proof systems that are defined with these rules, and that we are using in this paper.

We can define the function $\lfloor\cdot\rfloor$ on polarized formulas that forgets the polarities. This function can be extended to polarized sequents with exactly one even formula via $[B^o_1, \ldots, B^o_n, A^o] = [B^*_1, \ldots, B^*_n] \vdash [A]$. Since all systems defined in Figure 4 can only prove such sequents, we have an immediate one-to-one correspondence between derivations in the polarized and the corresponding unpolarized systems.

However, the motivation for introducing the polarized systems is the following result.

\[\text{Note that the } \top^* \text{-rule is just a special case of the } w^* \text{-rule. It is introduced here to simplify the presentation of some of the results in this paper.}\]
We conclude by permuting the rules \( c^*_i \) and \( w^*_i \) below all other rules, while applying the transformation above whenever a non-clean formula is introduced. \( \square \)

### 4 Modal Arenas

A directed graph \( \mathcal{G} = (V_\mathcal{G}, \rightarrow_\mathcal{G}) \) is given by a set \( V_\mathcal{G} \) of vertices and a set \( \rightarrow_\mathcal{G} \subseteq V_\mathcal{G} \times V_\mathcal{G} \) of direct edges. A vertex \( v \) is a \( \rightarrow \)-root, denoted \( v \not\rightarrow \), if there is no vertex \( w \) such that \( v \rightarrow w \). We denote by \( \overline{\rightarrow_\mathcal{G}} \) the set of \( \rightarrow_\mathcal{G} \)-roots of \( \mathcal{G} \). A path from \( v \) to \( w \) of length \( n \) is a sequence of vertices \( v_0, \ldots, v_n \) such that \( v = v_0 \) and \( w = v_n \) and \( v_i \not\rightarrow v_{i+1} \) for \( i \in \{0, \ldots, n-1\} \). We write \( v \rightarrow^n w \) if there is a path from \( v \) to \( w \) of length \( n \). A directed acyclic graph (or dag for short) is a directed graph such that \( v \rightarrow w \) implies \( n = 0 \) for all \( v \in V \). A two-color directed acyclic graph (or 2-dag for short) \( \mathcal{G} = (V_\mathcal{G}, \rightarrow_\mathcal{G}, \not\rightarrow_\mathcal{G}) \) is given by a set of vertices \( V_\mathcal{G} \) and two disjoint sets of edges \( \rightarrow_\mathcal{G} \) and \( \not\rightarrow_\mathcal{G} \) such that the
Consider the sequent $\Gamma \vdash A$, we define the following operations on 2-dags of propositional variables occurring in formulas. We use the notation $\in V$ to denote the graphs consisting of a single vertex labeled by $a$, $\Box$ and $\Diamond$ to denote the graphs consisting of a single vertex labeled by $a$, $\Box$ and $\Diamond$, respectively.

**Definition 4.1** Let $\mathcal{G}, \mathcal{H}, \mathcal{F}$ be 2-dags with $\mathcal{F} \neq \emptyset$. We write $\mathcal{G} \rightarrow \mathcal{H}$ for the set of edges from the $\rightarrow$-roots of $\mathcal{G}$ to the $\rightarrow$-roots of $\mathcal{F}$, that is $\mathcal{G} \rightarrow \mathcal{H} = \{(u, v) \mid u \in R_G, v \in R_F\}$. We define the following operations on 2-dags:

- $\mathcal{G} \cup \mathcal{H} = (V_G \cup V_H, \rightarrow_G \cup \rightarrow_H, \sim_G \cup \sim_H)$
- $\mathcal{G} \rightarrow \mathcal{F} = (V_G \cup V_F, \rightarrow_G \cup \rightarrow_F, \sim_G \cup \sim_F)$ and $\mathcal{G} \rightarrow \emptyset = \emptyset$
- $\mathcal{G} \rightarrow \mathcal{H} = (V_G \cup V_H, \rightarrow_G \cup \rightarrow_H, \sim_G \cup \sim_H \cup R_{\mathcal{H}})$

We associate to each formula $F$ a $\mathcal{L}$-labeled 2-dag $\llbracket F \rrbracket$ as follows:

$$
\llbracket a \rrbracket = a \quad \llbracket A \supset B \rrbracket = \llbracket A \rrbracket \rightarrow \llbracket B \rrbracket \quad \llbracket \Box A \rrbracket = \Box \llbracket A \rrbracket \\
\llbracket \Box \rrbracket = \emptyset \quad \llbracket A \land B \rrbracket = \llbracket A \rrbracket \land \llbracket B \rrbracket \quad \llbracket \Diamond A \rrbracket = \Diamond \llbracket A \rrbracket
$$

(4)

For a sequent $B_1, \ldots, B_n \vdash A$, we define $\llbracket B_1, \ldots, B_n \vdash A \rrbracket$ as $\llbracket (B_1 \land \cdots \land B_n) \supset A \rrbracket$.

**Example 4.2** Consider the sequent $\Gamma \vdash A = \Box a \supset \Box (b \land c), d \supset \Diamond (e \supset f)$. We have

$$
\llbracket \Box a \supset \Box (b \land c), d \supset \Diamond (e \supset f) \rrbracket = \llbracket \Box a \rrbracket \rightarrow \llbracket \Box (b \land c) \rrbracket \rightarrow \llbracket d \rrbracket \rightarrow \llbracket \Diamond \Box (e \supset f) \rrbracket
$$

In the following, we give a characterization of those 2-dags that are encodings of formulas.

**Definition 4.3** A $\mathcal{L}$-labeled dag $\mathcal{G} = (V_G, \rightarrow_G)$ is an arena if $\mathcal{G}$ is

- $\Sigma$-free: if $a \rightarrow u$ and $a \rightarrow w \rightarrow v$ then $u \rightarrow v$;
- $\Sigma$-free: if $a \rightarrow v, a \rightarrow w, b \rightarrow w$ and $b \rightarrow u$ then $a \rightarrow u$ or $b \rightarrow v$.

A modal arena $\mathcal{G} = (V_G, \rightarrow_G)$ is an $\mathcal{L}$-labeled 2-dag such that

- $\llbracket \Box a \rrbracket$ is an arena;
- $\mathcal{G}$ is properly labeled: if $v \sim_G w$, then $\ell(v) \in [\Box, \Diamond]$;
- $\sim_G$ is modal, that is:
Theorem 4.9

An we proceed by induction on the size of the formula F. The “if” direction has been shown in Lemma 4.7. For the “only if” direction, let a be a modal vertex of a modal arena G such that □a = 𝜓 in G. If V_0 = V_G then G = G_1 □ G_2 with G_1 a single vertex graph of the form □ or □. □

Lemma 4.4 ([17]) In an arena, if u → w and v → w then

either {x | v → x} ⊆ {x | w → x} or {x | w → x} ⊆ {x | v → x}.

Theorem 4.5 ([17]) An L-labeled 2-dag G is an arena iff there is a modality-free clean formula F such that G = [F].

We now extend this result to modal formulas.

Lemma 4.6 Let G be a modal arena, and let u, v, w ∈ V_G. If v → w then:

(i) v is a root iff w is a root; (ii) v → u iff w → u; (iii) if u → w then u → w.

Proof. The first statement follows from the fact that in a modal arena, if v → w, then v → u if w → u. The second statement is proven using the same argument, proceeding by induction on n making use of Lemma 4.4. The third statement is also proven using Lemma 4.4 and the fact that if v → w and u → v, then u → w. □

Lemma 4.7 If F is a formula, then the L-labeled 2-dag [F] is a modal arena.

Proof. By induction over the number of connectives and modalities of a formula. It suffices to remark that the graph operations + and → cannot introduce forbidden modal arena configurations. Similarly, the operation ∼ introduces no forbidden configurations whenever G = G_1 ∼ G_2 with G_1 a single vertex graph of the form □ or □. □

In order to prove the converse, we need the following definitions.

Definition 4.8 Let v be a modal vertex of a modal arena G. The scope of v is the set

Scope(v) = { w ∈ V_G | there is a u ∈ V_G such that v → u and w → u and w ∼ v }.

Intuitively, the scope of a modal vertex v in [F] is the set of vertices corresponding to modalities and atoms in the scope of the corresponding modality in F. To give an example, consider the arena in Example 4.2. There, e is in the scope of the □ while d is not. In fact, despite the existence of f such that □e → f and d → f and e → f, we have e ∈ Scope(□) since e → f and e → f, while d ∉ Scope(□) since d → f and d → □.

Theorem 4.9 An L-labeled 2-dag G is a modal arena iff G = [F] for some formula F.

Proof. The “if” direction has been shown in Lemma 4.7. For the “only if” direction, we proceed by induction on the size of G. If |V_G| = 1 then F = ⊤ if |V_G| = 1 then if v → w then F = a ∈ A, if v → w then F = a ∈ A, if v → ⊤ then F = ⊤ or F = ⊤, if v → □ then F = □ ⊤ or F = □ ⊤.
Definition 5.1 | modal arena nets and the linear proofs in LCK

We then show the correspondence between these with an equivalence relation over vertices, satisfying certain conditions capturing the idea of "axiom links" in proof nets. For any formulas $F$ and $G$ we have $F \sim G$ if and only if $F \equiv G$.

**Proposition 4.10** For any formulas $F$ and $G$ we have $F \vdash G$ iff $F = \llbracket G \rrbracket$.

**Proof.** This follows from the definition of the modal arena operations $\downarrow$, $\rightarrow$ and $\sim$. $\square$

5 Modal Arena Nets

We introduce the notion of CK- and CD-arena nets, which are modal arenas equipped with an equivalence relation over vertices, satisfying certain conditions capturing the idea of "axiom links" in proof nets. We then show the correspondence between these modal arena nets and the linear proofs in LCK$^*$ and LCD$^*$, respectively.

**Definition 5.1** A partitioned modal arena $\mathcal{G} = \langle V_\mathcal{G}, \rightarrow, \sim \rangle$ is given by a modal
arena \( \langle V_G, \rightarrow, \leadsto \rangle \) together with an equivalence relation \( \leadsto \) over vertices such that:

- if \( v \in V_G^\Delta \) and \( v \leadsto w \), then \( w \in V_G^\Delta \) and \( \ell(v) = \ell(w) \);

- if \( v \in V_G^\Delta \), then \( v \leadsto w \) for a unique \( w \in V_G^\Delta \).

In a partitioned modal arena we represent the equivalence relation \( \sim \) by drawing a (dashed non-oriented blue) edge \( v \sim w \) between two distinct vertices in the same \( \sim \)-class. For better readability, we only represent a minimal subset of these edges relying on the fact that \( \sim \) is an equivalence relation. By means of example, if \( \{u, v, w\} \) is an \( \sim \)-class, we may only draw \( u \sim v \sim w \) omitting the edge between \( u \) and \( w \).

We say that a formula (or \( P \)-formula) \( F \) is associated to \( G = \langle V_G, \rightarrow, \leadsto, \rangle \) if \( \llbracket F \rrbracket = \langle V_G, \rightarrow, \leadsto, \rangle \), and we denote by \( \emptyset \) the empty arena net.

**Remark 5.2** If \( v \) and \( w \) are vertices in a partitioned modal arena \( G \) such that \( v \leadsto w \), then \( v \in V_G^{\leq} \) iff \( w \in V_G^{\geq} \). If an \( \leadsto \) equivalence class contains more than two vertices then they are all labelled by \( \Diamond \) or \( \Box \).

If \( G \) is a modal arena and \( v \in V_G \), we define the depth of \( v \) (denoted \( d(v) \)) to be the length of the \( \rightarrow \)-paths from \( v \) to a \( \rightarrow \)-root \( w \in R_G \). This is well-defined as all such paths have the same length (see [17, Lemma 9]). The parity of a vertex \( v \) is the parity of \( d(v) \), which can be either even or odd. We write \( v^e \) or \( v^o \) if the parity of \( v \) is respectively even or odd. Note that if \( F^e \) is a \( P \)-formula, then the parity of the vertices in \( \llbracket F^e \rrbracket \) are the same as the polarity of the corresponding atoms (and modal subformulas) in \( F^e \).

The parity of an \( \rightarrow \)-edge \( v \rightarrow w \) is the parity of \( d(w) \). We say that an edge \( v \rightarrow w \) is a chord if there is a vertex \( u \) such that either \( v \rightarrow u \) and \( u \sim \rightarrow w \); or \( u \rightarrow w \) and \( u \sim \rightarrow v \). By means of example, in the following modal arenas the edges \( a \rightarrow b \) are chords.

![Chord Example]

We write by \( \leadsto_v \) the set of odd \( \rightarrow \)-edges in \( G \) that are not chords. In the following, we may depict \( \rightarrow \)-edges which are not \( \leadsto_v \)-edges using dotted edges.

If \( v \) is a vertex in a modal arena \( G = \llbracket F \rrbracket \), we denote by \( \hat{v} \) either \( v \) itself if there is no \( u \) such that \( v \in \text{Scope}(w) \), or the vertex \( w = \hat{v} \) such that \( v \in \text{Scope}(w) \) and \( w \in \text{Scope}(u) \) for all \( u \neq w \) with \( v \in \text{Scope}(u) \). That is, if \( v \neq \hat{v} \), then \( \hat{v} \) is the first modal vertex we encounter on in the path from \( v \) to the root of the formula tree of \( F \). By means of example, in Example 4.2 we have \( \hat{f} = \hat{e} = \hat{\Diamond} = \Diamond \) and \( \hat{d} = d \).

We define the edge relation \( \leadsto_{v, a} \) as follows

\[ v \leadsto_{v, a} w \quad \text{if either} \quad w^e \quad \text{and} \quad w = \hat{v} \neq v, \quad \text{or} \quad v^* \quad \text{and} \quad v = \hat{w} \neq w \]

Intuitively, the \( \leadsto_{v, a} \)-edges connect a modality to the root of the formula in its scope “one step at the time” (see Figure 5). Note that \( v^* \leadsto_{v, a} w^e \) implies \( v^* \leadsto w^e \), while \( v^e \leadsto_{v, a} w^e \) implies \( w^e \leadsto_{v, a} v^e \).

**Definition 5.3** A partitioned modal arena \( G \) is linked if every \( \leadsto \)-class is of the form \( \{v^1, \ldots, v^r, w^e\} \). This induces the set directed edges \( \leadsto_{\sim} = \{(v, w) \mid v^* \leadsto_{\sim} w^e\} \). The
linking graph $\hat{G}$ of a modal arena is the directed graph with vertices $V_G$ and edges $\overset{\to}{\hat{G}} \cup \overset{\leftrightarrow}{\hat{G}} \cup \overset{\Rightarrow}{\hat{G}}$. We say that path in $\hat{G}$ is checked if it ends in a vertex in $R_G \cap \overline{R}_G$ and it contains no edge $v \to w$ with $w$ a modal vertex with $\text{Scope}(w) \neq \emptyset$.

A CK-arena net is a linked modal arena which satisfies Conditions (i)–(iv) below:

(i) $\hat{G}$ is acyclic: every checked path is acyclic;
(ii) $\hat{G}$ is functional: every checked path in $\hat{G}$ from a vertex $v^*$ to a root includes a vertex $w^*$ such that $v \to w$;
(iii) $\hat{G}$ is functorial: if $v \sim w$ and $w \sim w'$ then there is $v'$ such that $v \sim v'$ and $v' \sim w'$;
(iv) $\hat{G}$ is CK-correct: if $\{v_1^*, v_2^*, \ldots, v_n^*, w^*\}$ is a $\sim$-class of modal vertices, then either $v_1, v_2, \ldots, v_n, w \in V_G^0$ or there is a unique $i$ such that $v_i, w \in V_G^0$.

A linked modal arena is a CD-arena net if it satisfies Conditions (i)-(iii) above, plus the following:

(v) $\hat{G}$ is CD-correct: if $\{v_1^*, v_2^*, \ldots, v_n^*, w^*\}$ is a $\sim$-class of modal vertices, then either $v_1, v_2, \ldots, v_n, w \in V_G^D$ or $w \in V_G^\Rightarrow$ there is at most one $i \in \{1, \ldots, n\}$ such that $v_i \in V_G^\Rightarrow$.

A modal arena net is either a CK- or a CD-arena net. An arena net is a modal arena net with $V_G^\Rightarrow = \emptyset$. Note that in this case Conditions (iii)-(v) are vacuous.

The intuition for Conditions (iv) and (v) is that $\sim$-classes represent either atoms paired by an ax, or the set of modalities introduced by a same instance of a K\(^\circ\), K\(^\bullet\) or D-rule. Following this intuition, if $c = \{v_0, v_1, \ldots, v_n\} \subseteq V_G^\Rightarrow$ is a $\sim$-class, then the modal arena with vertices $\bigcup_{v \in c} \text{Scope}(v)$ corresponds to the sub-proof of the premise of any such rule.

**Lemma 5.4** Let $X \in \{\text{CK}, \text{CD}\}$ and $F$ be a clean $P$-formula. If $\frac{}{\vdash F}$, then there is a $X$-arena net $\hat{G} = \langle V_G, \overset{\to}{\hat{G}}, \overset{\leftrightarrow}{\hat{G}}, \overset{\Rightarrow}{\hat{G}} \rangle$ such that $\llbracket F \rrbracket = \langle V_G, \overset{\to}{\hat{G}}, \overset{\leftrightarrow}{\hat{G}}, \overset{\Rightarrow}{\hat{G}} \rangle$.

**Proof.** Let $\Sigma$ be a derivation of $F$ in $\text{LX}_f$. We proceed by induction on $\Sigma$ translating it into a derivation of the desired modal arena net $\hat{G}$ via the rules in Figure 6. By definition, each rule in $\text{LX}_f$ preserves $X$-arena net conditions, that is, if the premises of a rule are $X$-arena nets, then the conclusion is. Note that Condition (iv) fails for the rule $\Delta$, while Condition (v) holds. \(\square\)
Fig. 6. Translation of \( LCK^* \) and \( LCD^* \) sequent rules in modal arenas rules where \( \kappa \) is the equivalence class containing all vertices in the conclusion which are not in the premise. Note that \( G, H \) and \( K \) have to be non-empty.

**Lemma 5.5** Let \( X \in \{ CK, CD \} \) and \( F \) be a clean \( P \)-formula. If \( G \) is an \( X \)-arena net with associated \( P \)-formula \( F \), then \( \models^L X F \).

**Proof.** We prove the theorem for \( CK \)-arena nets since the proof for \( CD \)-arena nets is similar by considering also the rule \( D \).

If \( G = (0, 0, 0, 0) \), we conclude since \( \| T \| = 0 \) and \( \models^L T \). Otherwise to prove this theorem we define from the \( CK \)-arena net \( G \), with associated clean \( P \)-formula \( F \), an arena net \( \partial(G) \) with associated formula \( \partial(F) \). We then use use of the result in [17] on (non-modal) arena nets to produce an \( LCK^* \)-derivation of \( \partial(F) \). Then we conclude by showing how to define a \( LCK^* \)-derivation of \( F \) using the \( LCK^* \)-derivation of \( \partial(F) \).

**Step 1: definition of \( \partial(G) \).** Let \( G = (V_G, \rightarrow, \sim, \overrightarrow{\sim}) \) be a \( CK \)-arena net. We write \( v \overrightarrow{\sim} w \) if either \( v \sim w \) or if \( v = v' \) and \( w = \hat{w} \), that is, \( v \overrightarrow{\sim} w \) iff either both \( v \) and \( w \) are not in the scope of any modality, or both \( v \) and \( w \) belong to the scope of modalities in the same \( \sim \)-class.

We define the arena \( \partial(G) \) by removing all \( \sim \)-edges in \( G \) and keeping only the \( \rightarrow \) between vertices \( v, w \in V_G \) such that \( v \overrightarrow{\sim} w \). Then we replace each modal vertex \( v \) by a pair of \( \rightarrow \)-linked vertices \( v^i, v^o \) in such a way that the vertex \( v^i \) keeps track of the subformulas of the modality, while \( v^o \) is a placeholder to keep track of the interaction of the subformulas with the context.

Formally we define \( \partial(G) = (\partial(V_G), \partial(\rightarrow \cup \overrightarrow{\sim}), \partial(\overleftarrow{\sim})) \) by:

- \( \partial(V_G) = V_G^i \cup (v^o, v^o) \mid v \in V_G^o \) ;

- \( \partial(\rightarrow \cup \overrightarrow{\sim}) \) is the union of the following five sets:

\[
\begin{align*}
\{ (r^o, r^i) \mid I \rightarrow r \} & \cup \left\{ (u, v) \mid u \overrightarrow{\sim} v \text{ and } u \rightarrow v \right\} \\
\{ (u, r^i) \mid u \overrightarrow{\sim} r \} & \cup \left\{ (\ell^o, u) \mid I \rightarrow \ell^o \right\} \\
\{ (u, m^o) \mid u \rightarrow m \text{ and } u \overrightarrow{\sim} m \} & \cup \left\{ (m^o, v) \mid m \rightarrow v \text{ and } m \overrightarrow{\sim} v \right\} \\
\{ (m^o, n^o) \mid m \rightarrow n \text{ and } m \overrightarrow{\sim} n \} & \cup \left\{ (m^o, n^o) \mid m \rightarrow n \text{ and } m \not\overrightarrow{\sim} n \right\} \\
\end{align*}
\]

where we assume \( u, v \in V_G^i \) and \( \ell^o, r^o, m, n, p \in V_G^o \) ;

- \( \partial(\overleftarrow{\sim}) \) is defined as: \( v^i \overleftarrow{\sim} w \) if \( v^i \overrightarrow{\sim} w \) and as \( \overleftarrow{\sim}(\partial(G)) v^o \) for each \( v \in V_G^o \).

See the first line of Figures 8 and 9 for a running example.

We observe that if \( \{ v_0^i, v_1^i, \ldots, v_s^i \} \) is a \( G \)-class of modal vertices, then a \( P \)-formula
Fig. 7. An example of the construction of the derivation of $F$ from the derivation of $\partial(F)$ assuming that in $G$ there is only one $\sim$-class of the form $[\Box_0, \ldots, \Box_n]$ 

associated to $G$ is of the form $H = H[\ell(v_0)A_0^\circ][\ell(v_1)A_1^\circ] \cdots [\ell(v_n)A_n^\circ]$ for an $(n + 1)$-ary context $H[\cdot \cdot \cdot \cdot]$. In this case, a $P$-formula associated to the arena $\partial(G)$ is of the form $\partial(H) = \partial(H)[v_0^{\text{out}}][v_1^{\text{out}}] \cdots [v_n^{\text{out}}][H^*_G]$ with $\partial(H)[\cdot \cdot \cdot \cdot]$ is an $(n + 2)$-ary context, fresh propositional variables $v_i^{\text{in}}, v_i^{\text{out}}$ for all $i \in \{0, \ldots, n\}$ and 

$$H^*_G = \left( (v_0^{\text{in}} \supset \partial(A_0^\circ) \land \cdots \land v_n^{\text{in}} \supset \partial(A_n^\circ)) \supset \partial(A_0^\circ) \right) \supset v_0^{\text{in}}$$

Step 2: proof that $\partial(G)$ is an arena net. We observe that, by definition of $\partial(G)$, every path $\partial(p)$ in $\widehat{\partial(G)}$ can be constructed from a checked path $p$ in $\widehat{G}$ by induction:

- the empty path is a path in both $\widehat{G}$ and $\widehat{\partial(G)}$;
- if $p = v \cdot p'$
  - if $v \in V_{\widehat{G}}^\text{in}$, then $\partial(p) = v \cdot \partial(p)'$;
  - if $v^\circ \in V_{\widehat{G}}^\text{out}$, then $\partial(p) = v^{\text{out}} \cdot v^\circ \cdot \partial(p)'$;
  - if $v^* \in V_{\widehat{G}}^\text{out}$, then $\partial(p) = v^{\text{in}} \cdot v^{\text{out}} \cdot \partial(p)'$.

We remark that the parity of atomic vertices is preserved by $\partial$, while the parity of a modal vertex $v \in V_{\widehat{G}}^\text{out}$ is the same as the corresponding vertex $v^{\text{out}} \in V_{\partial(G)}$. Since if $v^*$ then $v^{\text{out}} \rightarrow v^\circ$, and if $v^\circ$ then $v^{\text{out}} \rightarrow v^*$, then we have that in $\partial(G)$ an even (odd) vertex may occur only in an even (odd) position in a path in $G$. We conclude since from any path in $\widehat{\partial(G)}$ we obtain a path in $\widehat{G}$ by replacing every subpath $v^{\text{out}} \rightarrow v^\circ$ and $v^\circ \rightarrow v^{\text{out}}$ by the corresponding modal vertex $v$ in $G$.

By this correspondence between checked paths in $\widehat{G}$ and paths in $\widehat{\partial(G)}$ we conclude that $\widehat{\partial(G)}$ is acyclic and functional. That is, $\partial(G)$ is an arena net.

Step 3: construct the derivation associated to $\partial(G)$. Since $\partial(G)$ is an arena net, then we apply the result in [17] to produce a derivation in $\text{LK}^c_0$ of the formula $\partial(F)$. In such a derivation, by functionality and functoriality of $\widehat{G}$, whenever $v$ and $w$ are modal vertices such that $v \overset{G}{\rightarrow} w$, then if a path in $\partial(G)$ contains $v^{\text{out}}$, then it also contains $v^{\text{in}}, w^{\text{in}}, w^{\text{out}}$. This means that if $C = \{v_0^*, v_1^*, \ldots, v_n^*\}$ is an $\overset{G}{\sim}$-class of vertices in $\overset{G}{\sim}$, then any derivation of $\partial(F)$ in $\text{LK}^c_0$ contains a subderivation of the sequent
For a running example, refer to the lower line of Figures 8 and 9.

Let Theorem 5.6

In order to construct a derivation in the derivation, we replace every such subderivation in the derivation of an application of a induction over the number of Fig. 8. A deep rules
acterize the maps between modal arenas that characterize derivations built from the CK

After having characterized the linear part of a proof in CK or CD, we will now characterize the maps between modal arenas that characterize derivations built from the deep rules \( \ast \) for weakening and \( \ast \) for contraction (shown in Figure 3).

For a running example, refer to the lower line of Figures 8 and 9. □

By Lemma 5.4 and Lemma 5.5 we have the following theorem.

**Theorem 5.6** Let \( X \in \{ \text{CK}, \text{CD} \} \) and \( F \) be a clean P-formula. Then

\[
\nabla \Downarrow F \iff \text{there is a X-arena net } \mathcal{G} \text{ with } \mathcal{G} = \lfloor F \rfloor
\]

**6 Skew Fibrations**

After having characterized the linear part of a proof in CK or CD, we will now characterize the maps between modal arenas that characterize derivations built from the deep rules \( \ast \) for weakening and \( \ast \) for contraction (shown in Figure 3).

Let \( u, v, \) and \( w \) be vertices in a modal arena. We say that \( u \) is a meeting point of \( v \) and \( w \) whenever \( v \rightarrow u \) and \( w \rightarrow u \), and there is no vertex \( u' \neq u \) such that \( v \rightarrow u' \) and
By definition, the composition preserves \( \varphi \) if \( t \varphi \).

The composition of two skew fibrations is a skew fibration.

![Diagram](image)

**Fig. 9.** A K-arena \( G \) with associated formula \( (a \rightarrow b), \varnothing a, \varnothing b \) (the axiom \( \kappa_{1} \)), its corresponding arena \( \Phi(G) \), the derivation associated to \( G \) and the derivation associated to \( \Phi(G) \).

\( w \rightarrow w' \) and \( \varphi \rightarrow \varphi' \). The meeting depth of \( v \) and \( w \) is the depth of their meeting point, or \(-1\) if no meeting point exists. Note that this is well defined as all meeting points of \( v \) and \( w \) have the same depth (this follows from [17, Lemma 9]). Two distinct vertices \( v \) and \( w \) are *conject* denoted \( v \land w \) if their meeting depth is odd (or equal to \(-1\)).

**Definition 6.1** A modal arena homomorphism is either a map \( \theta_{G}: \emptyset \rightarrow G \) from the empty 2-diag to a modal arena \( G \), or a structure preserving map \( f: \mathcal{H} \rightarrow G \) between two modal arenas, i.e., its a function \( f: V_{\mathcal{H}} \rightarrow V_{G} \) that preserves:

\[
\begin{align*}
\rightarrow & : \text{ if } v \varphi \text{ then } f(v) \varphi \text{ } \& \text{ if } f(v) \varphi \text{ then } d(v) = d(f(v)) \\
\rightsquigarrow & : \text{ if } v \rightsquigarrow \varphi \text{ then } f(v) \rightsquigarrow f(\varphi) \text{ } \& \text{ if } \rightsquigarrow f(v) \rightsquigarrow f(\varphi) \text{ then } \ell(v) = \ell(f(\varphi))
\end{align*}
\]

A (modal) skew fibration is a modal arena homomorphism \( f: \mathcal{H} \rightarrow G \) which:

- preserves \( \land \) if \( v \land_{\mathcal{H}} w \) then \( f(v) \land_{G} f(w) \);  
- is a skew lifting: if \( f(v) \land_{\mathcal{H}} w \) then there exists \( u \) with \( u \land_{\mathcal{H}} v \) and \( f(u) \not\land_{G} w \).
- is a modal lifting: if \( f(v) \rightsquigarrow f(w) \), then there exists \( u \) with \( u \rightsquigarrow w \) and \( f(u) = f(v) \).

**Lemma 6.2** The composition of two skew fibrations is a skew fibration.

**Proof.** By definition, the composition preserves \( \rightarrow, \rightsquigarrow, \land \) and \( d \). Then the preservation of \( \land \) and the skew lifting condition of the composition are guaranteed as consequence of the preservation of \( d \) and \( \rightarrow \). Similarly, the modal lifting condition of the

\[3\] In [17] the definition of skew fibration only demands the weaker root preserving condition (that is, if \( v \in R_{\mathcal{H}} \) then \( f(v) \in R_{G} \)) instead of the depth preserving condition \( d(v) = d(f(v)) \) that we use here. However, in the same paper it is proven that root preservation is equivalent to depth preservation.
composition is guaranteed as consequence of the preservation of \( \rightsquigarrow \) and the fact that source and target of a skew fibration are modal arenas.

In order to prove the correspondence between \([c^*, w^*]\) derivations and skew fibrations, we provide the following definition.

**Definition 6.3** If \( f_1 : \mathcal{H}_1 \to \mathcal{G}_1 \) and \( f_2 : \mathcal{H}_2 \to \mathcal{G}_2 \) are modal arena homomorphisms such that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are disjoint modal arenas, we define the following modal arena homomorphisms:

\[
\begin{align*}
  f_1 + f_2 &= f_1 \cup f_2 : \mathcal{H}_1 + \mathcal{H}_2 \to \mathcal{G}_1 + \mathcal{G}_2 \\
  f_1 \rightsquigarrow f_2 &= f_1 \cup f_2 : \mathcal{H}_1 \rightsquigarrow \mathcal{H}_2 \to \mathcal{G}_1 \rightsquigarrow \mathcal{G}_2 \quad (\mathcal{H}_2, \mathcal{G}_2 \neq \emptyset) \\
  [f_1, f_2] &= f_1 \cup f_2 : \mathcal{H}_1 + \mathcal{H}_2 \to \mathcal{G}_1 \cup \mathcal{G}_2 \\
\end{align*}
\]

**Lemma 6.4** The operations from Definition 6.3 preserve skew fibration properties.

**Proof.** It suffices to check that if \( f_1 \) and \( f_2 \) are skew fibrations, then also \( f_1 + f_2 \), \( f_1 \rightsquigarrow f_2 \), \( f_1 \cup f_2 \), \( f_1 \rightsquigarrow f_2 \) and \([f_1, f_2]\) are.

**Lemma 6.5** Let \( H' \) and \( H \) be \( P \)-formulas. If \( H' \vdash H \), then there is a skew fibration \( f : [H'] \rightsquigarrow [H] \).

**Proof.** After Lemma 6.2, it suffices to prove that if \( H' \vdash H \) for a \( \rho \in [w^*, c^*] \), then there is a skew fibration \( f : [H'] \to [H] \). This is immediate after Lemma 6.4 after remarking that any map \( \emptyset : \emptyset \to \mathcal{G} \) is a skew fibration.

To prove the converse, we need some additional definitions and intermediate results.

**Definition 6.6** Two distinct vertices \( v \) and \( w \) in a modal arena they are *disjunct*, denoted \( v \uplus w \), if their meeting depth is even. An *odd skew fibration* is either a map \( \emptyset : \emptyset \to \mathcal{G} \), or a modal arena homomorphism \( f : \mathcal{H} \to \mathcal{G} \) which:

- preserves \( \vee \) if \( v \vee_{\mathcal{H}} w \) then \( f(v) \vee f(w) \);

- is an odd skew lifting: if \( f(v) \uplus f(w) \), then there exists \( u \) with \( v \geq_{\mathcal{H}} u \) and \( f(u) \notin f(w) \).

**Lemma 6.7** If \( f : \mathcal{H} \to \mathcal{G} \) be a modal arena homomorphism and \( \mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 \), then \( f = f_1 + f_2 \) with \( f_1 : \mathcal{H}_1 \to \mathcal{G}_1 \) and \( f_2 : \mathcal{H}_2 \to \mathcal{G}_2 \) modal arena homomorphisms for some \( \mathcal{H}_1, \mathcal{H}_2 \) such that \( \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 \).

**Proof.** Since \( f \) preserves \( \to \), then if \( v \rightsquigarrow w \) for a \( w \in \mathcal{G} \) then \( f(v) \rightsquigarrow f(w) \). Thus if \( \mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 \), then there is a partition \( \mathcal{G} = \mathcal{G}_1 \uplus \mathcal{G}_2 \). As remarked in the proof of Theorem 4.9, in such construction such partition, because of \( \rightsquigarrow \)-coherence, whenever \( v \rightsquigarrow w \) then \( v \) and \( w \) belong to the same subset. Then we can define \( V_{\mathcal{H}} \cap V_{\mathcal{H}_1} \) as the sets of vertices of \( \mathcal{H} \) which images by \( f \) admit a \( \rightsquigarrow \)-path to a vertex in \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), respectively. The modal arenas \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are defined from \( \mathcal{H} \) by the sets \( V_{\mathcal{H}_1} \) and \( V_{\mathcal{H}_2} \) respectively.

**Lemma 6.8** Let \( f : \mathcal{H} \to \mathcal{G} \) be a skew fibration or an odd skew fibration, with \( \mathcal{G} = \mathcal{G}_1 \rightsquigarrow \mathcal{G}_2 \) and \( \mathcal{G}_1 \) modal arenas. If there exist two modal arenas \( \mathcal{H}' \) and \( \mathcal{H}'' \) such that
Lemma 6.11

Every skew fibration is of the form $f(v) \in V_{G_1}$ for all $v \in V_{G_2}$.

**Proof.** Let $v \in H''$ such that $f(v) \in G_1$. Since $f$ preserves $d$, then $v \not\sim H$. Thus $H''$ cannot be a single-vertex modal arena. If $H''$ is a + of two modal arenas, then there is $z \in \overline{H'}$ such that $v \sim z \in H$ but $f(v) \not\sim f(z)$ in $G$. Therefore $f$ is not a skew fibration. Let $f(z) = w$. Then $f(v) \nleq w$ because $f(v) \in G_1$ and $w \in R_G$. If there is a $u$ with $v \nleq u$ in $H$ then there is $x \in V_{H'}$ such that $u \nleq x$ and $v \nleq x$. Since $x \nleq w$ we have $f(u) \nleq w$, which means that $f$ cannot be an odd skew fibration either. Then $H''$ has to be the shape $w \nleq H''$ and $f(w) \in G_2$ because $v \in R_H$. We can conclude as for the previous case that $f$ is not an even or odd skew fibration. Contradiction. □

**Lemma 6.9** Let $f : H \to G$ be an odd skew fibration, with $G = G_1 \to G_2$ for a modal arena $G_1$. If there is a modal arena $H'$ such that $H = H' \to H''$, then there are $H_1$ and $H_2$ such that $H = H_1 \to H_2$ and $f = f_1 \to f_2$ where $f_1 : H_1 \to G_1$ and $f_2 : H_2 \to G_2$ are modal arena homomorphisms.

**Proof.** By hypothesis, we can assume that $H$ is of the form $H = H' \to H''$ where $H''$ is not a $\to$ of two modal arenas. We conclude by Lemma 6.8 that $f(v) \in G_2$, for any $v \in V_{H'}$. If $V_{G_1} = f(V_{H'})$, then we conclude that $H_1 = H'$ and $H_2 = H''$. Otherwise, let $H' = H'_1 \to \cdots \to H'_n$ such that $H'_i$ is a $+$ of two modal arenas for no $i \in \{1, \ldots, n\}$. If $v, w \in V_{H'}$, there is a $(\to \cup \leftrightarrow)$-path from $v$ to $w$ in $V_{H'}$ if there is $i \in \{1, \ldots, n\}$ such that $v, w \in V_{H'_i}$. Since $\overline{R_G} \subset f(V_{H'})$, this implies that if there is $i \in \{1, \ldots, n\}$ such that $v, w \in V_{H'_i}$, then there is a $(\to \cup \leftrightarrow)$-path from $f(v)$ to $f(w)$ in $V_G \setminus \overline{R_G}$. That is, $f(V_{H'})$ is either a subset of $V_{G_1}$ or a subset of $V_{G_2}$ for all $i \in \{1, \ldots, n\}$.

Without loss of generality we assume there is $j$ such that $f(V_{H_j}) \subset V_{G_2}$ for all $i \leq j$. We conclude that $H_1 = H'_1 \to \cdots \to H'_j$ and $H_2 = (H'_{j+1} \to \cdots \to H'_n) \to H''$. □

**Lemma 6.10** Let $f : H \to G$ be a modal arena homomorphism and $G = v \nleq G'$. If $f$ is a skew fibration then $H = w \nleq H'$ and $f = 1_w \to f'$ with $f' : H' \to G'$ a skew fibration. If $f$ is odd skew fibration, then

- either $H = w \nleq H_2$ and $f = 1_w \to f_2 : H_2 \to G_2$ an odd skew fibration;
- or $H = (w \nleq H_1) \to H_2$ and $f = [f_1, f_2]$ with $f_1 : (w \nleq H_1) \to (v \nleq G_2)$ and $f_2 : H_2 \to (v \nleq G_2)$.

**Proof.** If $f$ is a skew fibration, then to conclude it suffices to remark there is a unique $w$ such that $f(w) = v$ since $v \in \overline{R_G}$. If $f$ is an odd skew fibration, let $w$ such that $f(w) = v$. If $V_{G_2} \setminus \{w\} = \text{Scope}(w)$, then we can conclude. Otherwise we conclude with $H_2$ be the modal arena with vertices in $V_{H} \setminus (\{w\} \cup \text{Scope}(w))$. □

**Lemma 6.11** Every skew fibration is of the form $1_G : \# + G^* : \# \to G^*$ or $1 \to G^*$. Every odd skew fibration is of the form $1_G : [\#', \#] : \# + \#' : \# \to G^*$, $1 \to G^*$, or $0_G$, where $\#'$ and $\#$ are skew fibrations, $\#$ and $\#'$ are odd skew fibrations, $v \in V_{G_1 \to G_2}$, and $G$ can be any modal arena.

**Proof.** By case analysis, let $f : H \to G$ be a modal arena homomorphism, remarking that for any modal arena $G$, the identity map $1_G$ is by definition an even and an odd skew fibration. If $f^* : H \to G$ is a skew fibration, then
- if $G$ is a single-vertex modal arena, then $H$ cannot be of the shape $H_1 + H_2$ or $H_1 \rightarrow H_2$ otherwise $f$ would not preserve $\lambda$, or of the shape $H_1 \rightarrow H_2$ otherwise it would not preserve $d$. Then $f = 1_v$, with $v$ the unique vertex in $V_H = V_G$;
- if $G = G_1 \cdot G_2$, then by Lemma 6.7 we have that $f' = f_2 \cdot f_1$ with $f_1$ and $f_2$ arena homomorphisms. Since $f'$ is an even skew fibration, it follows by definition of + that $f_1$ and $f_2$ are skew fibrations;
- if $G = G_1 \cdot G_2$, then we define $V_1 = \{ v \in V_H \mid f(v) \in G_1 \}$ and $V_2 = \{ v \in V_H \mid f(v) \in G_2 \}$. We have that $V_2 \neq \emptyset$ since $f$ preserve $d$. If $V_1 = \emptyset$, then $f = 0_G \cdot f_2$ with $f_2 : H \rightarrow G_2$. Otherwise, $V_1 \neq \emptyset$ and $H$ cannot be a single vertex. Similarly, $H$ cannot be of the shape $H_1 + H_2$ otherwise $f$ would not preserve $\lambda$, nor of the shape $\lambda \rightarrow H_2$ otherwise $f$ would not be modal. We conclude by Lemma 6.9 that $f = f_1 \cdot f_2$. Moreover, since $f$ is a skew fibration if follows that $f_2$ also preserves $\lambda$ and satisfies skew lifting while $f_1$ preserve $\nu$ and satisfies odd skew lifting;
- if $G = \nu \rightarrow G_2$, we conclude by Lemma 6.10.

If $f' : H \rightarrow G$ is an odd skew fibration, then we proceed similarly. If $G$ is a single-vertex modal arena, then $H$ cannot be of the shape $H_1 + H_2$ otherwise $f$ it would not be modal, or of the shape $H_1 \rightarrow H_2$ otherwise it would not preserve $d$. Let $H = H_1 + H_2$ such that $H_1 \neq H'_1 + H''_1$. Since $f'$ preserve $d$ and $\rightarrow$, then $H_1$ is a single-vertex modal arenas. Moreover, $f_2 : H_2 \rightarrow G_2$ is an odd skew fibration by definition. Then $f = [1_v, f_2]$ with $v$ the unique vertex in $V_H = V_G$;

If $G = G_1 \cdot G_2, G = G_1 \cdot G_2$ or $G = \nu \rightarrow G_2$ we apply a similar reasoning in the case of $f$ skew fibration.

Lemma 6.11 is now enough to complete the proof of Theorem 6.12: Given a skew fibration $f$, we can decompose $f$ as an expression with the operations in Lemma 6.11, which can then be immediately transformed into a deep inference derivation using only $w^*_{\mu}$ and $c^*_{\rho}$. (This is a standard operation in deep inference, see e.g. [15].)

**Theorem 6.12** Let $H$ and $H'$ be P-formulas.

$$H' \overset{\mu, \rho}{\rightarrow} H$$

there is a skew fibration $f : \llbracket H' \rrbracket \rightarrow \llbracket H \rrbracket$.

**Proof.** To prove the “if” direction, it suffices to prove that if $H' \overset{\mu, \rho}{\rightarrow} H$ for a $\rho \in \{w^*_{\mu}, c^*_{\rho}\}$, then there is a skew fibration $f : \llbracket H' \rrbracket \rightarrow \llbracket H \rrbracket$. Then we conclude by Lemma 6.2.

By ?? 6.11, we have that any skew fibration can be written as composition from $[1_v] : \nu \rightarrow \nu$ and $0_G : \emptyset \rightarrow G$ via the operations in (5) above. In particular, each $0_G$ occurring in the decomposition corresponds to an application of a $w^*_{\mu}$, while each occurrence of $[\nu, \nu]$ corresponds to an application of a $c^*_{\rho}$. We conclude by reconstructing a derivation in $\{c^*_{\rho}, w^*_{\mu}\}$ using this decomposition and the correspondence between P-formulas and modal arenas (Theorem 4.9).

7 **Combinatorial Proofs**

We can now combine the results of the previous sections, to define combinatorial proofs for the logics CK and CD, and to prove their soundness and completeness.
then

Corollary 7.4 arenas is a skew fibration.

\[ X \]

Proof. All the following checks can be done in polynomial time: that a 2-arena modal arena; that a modal arena is an intuitionistic combinatorial proof, or CK-ICP, (resp. CD-intuitionistic combinatorial proof; or CD-ICP) if \( G \) is a CK-arena net (resp. CD-arena net).

The intuitionistic combinatorial proofs (or ICPs) from [17] are the special cases where no modalities occur, that is, an ICP is a skew fibration \( f: G \rightarrow [F] \) from an arena net \( G \) to the arena of a modality-free formula \( F \).

Theorem 7.2 (Soundness and Completeness) If \( F \) is a formula and \( X \in \{ \text{CK, CD} \} \), then \( \vdash X \) iff there is an X-ICP \( f: G \rightarrow [F] \).

Proof. By Theorem 3.1 there are P-formulas \( H \) and \( H' \) such that \( F = \{ H \} \) and \( H' \) is clean and \( \vdash X F \iff \vdash X H' \). In Theorem 5.6 we have shown that \( \vdash X H' \) iff there is an X-arena net \( G \) with \( H' \) the formula associated to \( G \). In Section 6 we have shown that \( H' \vdash H \) iff there is a skew fibration \( f: [H'] \rightarrow [H] \). This is equivalent to having an X-ICP \( f: G \rightarrow [F] \), since \( [H] = [F] \). \( \square \)

Lemma 7.3 Let \( X \in \{ \text{CK, CD} \} \). If \( \mathcal{H} \) and \( G \) are 2-dags and \( f: V_H \rightarrow V_G \), then it can be checked in polynomial time (in the size of \( \mathcal{H} \) and \( G \)) whether \( f \) is an X-ICP.

Proof. All the following checks can be done in polynomial time: that a 2-dag \( G \) is a modal arena; that a modal arena is an X-arena net; and that a map between two modal arenas is a skew fibration. \( \square \)

Corollary 7.4 Let \( X \in \{ \text{CK, CD} \} \). Then the X-ICPs form a sound and complete proof system in the sense of Cook and Reckhow [10].

8 On Proof Equivalence for Constructive Modal Logics

Let us now compare various notions of proof equivalence in constructive modal logics, building from the previous results in [17], where the authors show that intuitionistic combinatorial proofs capture a finer notion of proof equivalence than the one induced by the simply typed lambda calculus or by winning innocent strategies from games semantics [1,22,27].

In the following, we use \( \equiv \) to denote the proof equivalence over derivations generated by independent rule permutations, that is, permutations of inference rules whose active formulas and principal formulas are disjoint, as in the examples shown in Figure 10. Then, in Figure 11, we show examples of rule permutations which are non-independent. Based on these, we define the following proof equivalences:

\[ \equiv_{\text{ICP}} := (\equiv \cup \equiv_e \cup \equiv_c) \quad \equiv_i := (\equiv_{\text{ICP}} \cup \equiv_a) \quad \equiv_{\text{WHS}} := (\equiv_i \cup \equiv_{\text{DE}}) \]

Note that \( \equiv_{\text{ICP}} \) and \( \equiv_i \) can be defined in the same way for the non-modal case. Indeed, it has been shown in [17] that \( \equiv_{\text{ICP}} \) is the proof equivalence induced by intuitionis-
by the simply-typed $\lambda$-calculus below. In the modality-free case, tic combinatorial proofs. We extend this result to the modal case in Theorem 8.1 and let $\mathcal{D}$ and $\mathcal{D}'$ be derivations in $\mathcal{LX}$. Then we have $\mathcal{D} \equiv_{\text{ICP}} \mathcal{D}'$ iff $\mathcal{D}$ and $\mathcal{D}'$ are represented by the same $X$-ICP.

**Proof.** This is a direct consequence of the result on intuitionistic combinatorial proofs in [17]. It suffices to observe that if weakening and contraction rules could be per-
muted below/above K-rules, this would change the number of modalities handled by these rules. However, in CK-ICPs there is a one-to-one correspondence between ∼-classes and applications of K- and D-rules and at the same time a one-to-one correspondence between the number of modalities handled by each of these rules and the vertices in such ∼-equivalence class. Therefore the equivalence relation ≡ICP does not allow to permute weakening and contraction rules over K- and D-rules. □

The rule permutations in ≡ic are well-known in linear logic, as they correspond to the possibility of moving both weakening and contraction gates outside a !?-box in multiplicative exponential proof nets (see the notion of generalized ?-nodes introduced in [11] allowing to capture both the rule permutations in ≡c and ≡ic). It has been observed before (see, e.g., [2]), that including the ≡ic rule permutations in the proof equivalence of classical linear logic makes proof equivalence PSPACE-complete.

This immediately follows from the result in [16] about PSPACE-completeness of proof equivalence for multiplicative linear logic with units. More precisely, multiplicative linear logic proof nets require each ⊥-gate to be attached to an axiom by a so-called “jump” in order to guarantee a polynomial proof equivalence.

A similar phenomenon occurs in the constructive modal logics studied in our paper since in combinatorial proof each ⊤•-rule instance is linked to a K- or a D-rule instance occurring below it. Since rule permutations in ≡ic or ≡oc trigger a “jump-rewiring” mechanism similar to the one observed in [16], we conjecture that proof equivalence including these permutations is PSPACE-hard.

9 Conclusions and Future Works

We have presented the syntax of combinatorial proofs for the disjunction-free fragment of the constructive modal logics CK and CD. We have proved that (1) this syntax is a sound and complete proof system in the sense of Cook and Reckhow [10], and that (2) it enforces a notion of proof equivalence which is finer than the one provided by natural deduction proofs, but still coarser than plain sequent calculus.

In future work we want to further investigate the various notions of proof equivalence for constructive modal logics, and in a next step study combinatorial proofs for other variants of intuitionistic modal logics.

References


