Computation of Koszul homology and application to involutivity of partial differential systems
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Abstract: The formal integrability of systems of partial differential equations plays a fundamental role in different analysis and synthesis problems for both linear and nonlinear differential control systems. Following Spencer’s theory, to test the formal integrability of a system of partial differential equations, we must study when the symbol of the system, namely, the top-order part of the linearization of the system, is 2-acyclic or involutive, i.e., when certain Spencer cohomology groups vanish. Combining the fact that Spencer cohomology is dual to Koszul homology and symbolic computation methods, we show how to effectively compute the homology modules defined by the Koszul complex of a finitely presented module over a commutative polynomial ring. These results are implemented using the ORE MORPHISMS package. We then use these results to effectively characterize 2-acyclicity and involutivity of the symbol of a linear system of partial differential equations. Finally, we show explicit computations on two standard examples.

Keywords: Systems of partial differential equations, multidimensional systems, behaviours, Cartan’s involutivity, formal integrability, Koszul homology, Spencer cohomology

1. INTRODUCTION

It is well-known that the study of structural properties of both linear and nonlinear systems, such as controllability, observability, differential flatness, etc., boils down to the study of the integrability or the involutivity of systems of vector fields. Moreover, many standard problems in nonlinear control theory, such as linearization by static feedback, (input-state, input-output) linearizations, model matching, minimal realization, etc., can be reduced to the study of the existence of a local solution of (nonlinear) systems of partial differential equations (PDEs) in the unknowns of the problem. It is thus not surprising that Frobenius theorem on integrability of geometric distributions plays an important role in control theory. The study of the existence of formal/locally convergent power series solutions of systems of PDEs has a long mathematical history and is nowadays called the formal integrability theory. Thus, control theorists have to face the problem of studying the formal integrability of linear or nonlinear systems of PDEs.

According to Spencer’s theory (Spencer (1969)), the formal integrability of linear PD systems can be reduced to the study of the Spencer cohomology, dual to the Koszul homology (Quillen (1964)), and integrability conditions.

In this paper, using computer algebra methods (Gröbner bases) and results on homomorphisms of finitely presented modules (Section 2) over the commutative polynomial ring $\mathcal{A} = \mathcal{B}[\chi_1, \ldots, \chi_n]$, where $\mathcal{B}$ is a commutative ring over a computable field $\mathcal{K}$ of characteristic 0, in Section 3, we first explain how to compute the Koszul homology $A$-modules $H_i(\chi, M)$, $i = 0, \ldots, n$, of a finitely presented $A$-module $M$. To simplify the presentation, we shall only consider the case $n = 3$, i.e., the most important situations in applications. But the general case exactly follows the same line. Koszul homology $A$-modules can be computed by standard computer algebra systems, e.g., using the ORE MORPHISMS package (Chuzeau and Quadrat (2009)), developed in Maple, as demonstrated in Section 6 with two standard examples of the literature (Pommaret (1994)). Then, following Quillen (1964); Malgrange (2005), using the natural total-order graduation of rings of PD operators and the fact that the study of Spencer cohomology of the symbol of a linear PD system reduces to the study of the graded Koszul homology $A$-modules $H_i(\chi, gr(M))$’s of the finitely presented graded $A$-module $gr(M)$, in Section 5, we give effective tests for $i$-acyclicity, $i = 0, \ldots, n$, and thus, for the involutivity of symbols of linear PD systems.

Notation. Let $\mathcal{K}$ be a field of characteristic 0 (e.g., $\mathcal{K} = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and $\mathbb{K}[x_1, \ldots, x_n]$ the commutative ring formed by all the polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $\mathcal{K}$. Let $\mathcal{D}$ be a noncommutative ring. Then, the set of all $q \times p$ matrices with entries in $\mathcal{D}$ is denoted by $\mathcal{D}^{q \times p}$. If $R \in \mathcal{D}^{q \times p}$, then we can consider the left $\mathcal{D}$-homomorphism (i.e., left $\mathcal{D}$-linear map) defined by:

$$\lambda = (\lambda_1 \ldots \lambda_q) \mapsto \lambda R.$$

(1)
We can also consider the following left $D$-modules:
\[
\ker_D(R) = \{ \lambda \in D^{1 \times q} \mid \lambda R = 0 \}, \quad \text{im}_D(R) = D^{1 \times q} R, \\
\text{coker}(R) = D^{1 \times p}/(D^{1 \times q} R).
\]

Note that $L := \text{coker}(R)$ is a quotient left $D$-module, i.e., $L$ is the left $D$-module defined by the residue classes $\pi(\mu) \in L$ for all $\mu \in D^{1 \times p}$, i.e., by definition, $\pi(\mu') = \pi(\mu)$ if $\mu' = \mu \in \text{im}_D(R)$, i.e., if there exists $\lambda \in D^{1 \times q}$ such that $\lambda R = -\mu$. For all $d_1, d_2 \in D$ and for all $\mu_1, \mu_2 \in D^{1 \times p}$, we then have $d_1 \pi(\mu_1) + d_2 \pi(\mu_2) \equiv (d_1 \mu_1 + d_2 \mu_2)$. Let $\pi : D^{1 \times p} \rightarrow L$ be the $D$-homomorphism defined by mapping $\mu \in D^{1 \times p}$ onto its residue class $\pi(\mu) \in L$. If $\mathcal{L}'$ is a left $D$-module, the $D$-module formed by all the homomorphisms from $L$ to $\mathcal{L}'$ is denoted by $\text{hom}_D(L, \mathcal{L}')$.

Finally, if $D$ is commutative, then $\text{hom}_D(L, \mathcal{L}')$ inherits a natural $D$-module structure.

### 2. Algebraic Analysis Approach

#### 2.1 Finitely presented left $D$-modules & Behaviours

Let $R \in D^{1 \times p}$ and $L = \text{coker}(R)$. Moreover, let $\{e_i\}_{i=1}^{p}$ be the standard basis of $D^{1 \times q}$ and $\{f_j\}_{j=1}^{q}$ the standard basis of $D^{1 \times q}$. Finally, let $y_i = \pi(e_i)$ for $i = 1, \ldots, p$, and $y = (y_1 \ldots y_p)^T$. Using the fact that every $l \in L$ is of the form $l = \pi(\mu)$ for a certain $\mu = (\mu_1 \ldots \mu_p) \in D^{1 \times p}$, we then have $l = \pi(\mu) = \pi(\sum_{i=1}^{p} \mu_i e_i) = \sum_{i=1}^{p} \mu_i y_i = \mu y$, which shows that $\{y_1, \ldots, y_p\}$ is a finite set of generators of $L$. The left $D$-module $L$ is said to be finitely generated. Moreover, we have $f_j R = (R_1 \ldots R_p) \in \text{im}_D(R)$, and thus, $\sum_{k=1}^{p} R_{jk} y_k = y_j$ for $j = 1, \ldots, q$, which shows that the sets $\{y_i\}_{i=1}^{p}$ of $L$ satisfy the finite generating set of left $D$-linear relations $R y = 0$. The left $D$-module is said to be finitely presented. Equivalently, $L$ is defined by the following exact sequence of left $D$-modules
\[
D^{1 \times q} \xrightarrow{R} D^{1 \times p} \xrightarrow{\pi} L \rightarrow 0
\]
i.e., $\pi$ is surjective and $\ker(\pi) = \text{im}_D(R)$.

Consider a left $D$-module $F$ and the left $D$-homomorphism:
\[
R : F^{p \times 1} \rightarrow F^{q \times 1} \eta \mapsto R \eta.
\]

Then, $\text{ker}(\mathcal{R}) = \{ \eta \in F^{p \times 1} \mid R \eta = 0 \}$ is the $F$-solutions set of the linear system $R \eta = 0$, also called behaviour. A standard result in module theory asserts that:
\[
\text{ker}(\mathcal{R}) \cong \text{hom}_D(L, F).
\]

Indeed, for all $\phi \in \text{hom}_D(L, F)$, we have $\phi(0) = 0$, i.e., $\phi(\sum_{k=1}^{p} R_{jk} y_k) = \sum_{k=1}^{p} R_{jk} \phi(y_k) = 0$, which shows that $\phi(\pi(\lambda)) \equiv (\phi(e_1) \ldots \phi(e_p))^T \in \text{ker}(\mathcal{R})$. Conversely, if $\eta \in \text{ker}(\mathcal{R})$, then $\phi_\eta(\pi(\lambda)) := \lambda \eta \in F$ for all $\lambda \in D^{1 \times p}$ belongs to $\text{hom}_D(L, F)$, hence $\phi_\eta(\pi(\lambda)) = \eta$, and $\phi_\eta(\pi(\lambda)) = $.

#### 2.2 Homomorphisms of finitely presented left $D$-modules

In this section, we briefly state again useful results on homomorphisms of finitely presented left $D$-modules.

**Lemma 1.** (Cluzeau and Quadrat (2008)). Let us consider two finitely presented left $D$-modules $L = D^{1 \times p}/(D^{1 \times q} R)$ and $\mathcal{L}' = D^{1 \times p'}/(D^{1 \times q'} R')$. The existence of a homomorphism $f \in \text{hom}_D(L, \mathcal{L}')$ is equivalent to the existence of matrices $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying the relation:
\[
R P = Q R'.
\]

Then, the commutative exact diagram
\[
\begin{array}{ccc}
D^{1 \times q} & \xrightarrow{R} & D^{1 \times p} \\
\downarrow & & \downarrow \\
D^{1 \times q'} & \xrightarrow{R'} & D^{1 \times p'}
\end{array}
\]

holds, where $f : \text{ker}(\mathcal{R}) \rightarrow \text{ker}(\mathcal{R'})$ defined by:
\[
\forall \mu \in D^{1 \times p}, \quad f(\pi(\lambda)) = \pi'(\mu P).
\]

Note that the identity (5) yields the homomorphism of behaviours $f^* : \text{ker}(\mathcal{R}) \rightarrow \text{ker}(\mathcal{R'})$ defined by:
\[
\forall \eta' \in \text{ker}(\mathcal{R'}), \quad f^*(\eta') = P \eta \in \text{ker}(\mathcal{R}).
\]

For the computation of the Koszul homology (see Section 3), we shall need to compute kernels and images of homomorphisms. Let us explicitly characterize the kernel and the image of $f \in \text{hom}_D(L, \mathcal{L}')$, where $L$ and $\mathcal{L}'$ are two presented left $D$-modules.

**Lemma 2.** (Cluzeau and Quadrat (2008)). Let us consider two finitely presented left $D$-modules $L = D^{1 \times p}/(D^{1 \times q} R)$ and $\mathcal{L}' = D^{1 \times p'}/(D^{1 \times q'} R')$. Let $f \in \text{hom}_D(L, \mathcal{L}')$ be a homomorphism defined by $P \in D^{p \times p'}$ and $Q \in D^{q \times q'}$ satisfying (5). Let $L = (P^T R')^{-1} \in D^{(p \times p') \times (p \times p')}$. Then:
\[
\ker(f) = (D^{1 \times p} S')/(D^{1 \times q} R), \quad \text{where } S' \in D^{r \times p'} \text{ is a matrix defined by:}
\]
\[
k\text{ker}(D, L) = D^{1 \times p}(S' - S''), \quad S'' \in D^{r \times q'}.
\]
(2) $\text{im}(f) = (D^{1 \times (p+q')})_{L}/(D^{1 \times q'} R')$.
(3) $\text{coker}(f) = D^{1 \times p'}/(D^{1 \times (p+q')} L)$.

3. THE KOSZUL COMPLEX

In this section, we shall briefly introduce the notions of Koszul complex and Koszul homology (Serre (1989)). In Section 4, we shall explain how to characterize the Koszul homology. In Section 5, we shall show how to use these results to effectively check the $i$-acyclicity and the involutivity of linear PD systems.

Let $A$ be a commutative ring and $T$ a free $A$-module of rank $n$, i.e., $T \cong A^n$. Let $u_1, \ldots, u_n$ be a basis of $T$, i.e., $T = A u_1 \oplus \cdots \oplus A u_n$. We can then consider the exterior algebra of $T$ defined by $\Lambda T = \bigoplus_{r=0}^{n} \Lambda^r T$, where $\Lambda^0 T = A$ and $\Lambda^r T$ is the homogeneous component of degree $r$ of $\Lambda T$. As an $A$-algebra, $(\Lambda T, +, \wedge)$ is generated by $u_1, \ldots, u_n$ satisfying the following relations:

$$u_i \wedge u_j = 0, \quad u_i \wedge u_j = -u_j \wedge u_i, \quad i \neq j.$$

Then, $\omega \in \Lambda^r T$ has the form $\omega = \sum_{i=1}^{n} a_i u_i$, where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, $a_i \in A$, $I$ is a finite set, and $\wedge$ is the wedge product. Hence, $\Lambda^r T$ is the free $A$-module generated by $u_1, \ldots, u_n$, where $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, i.e., we have $\Lambda^r T \cong A^C$.

Let $\mathcal{M}$ be a finitely generated $A$-module, $\chi_1, \ldots, \chi_n \in \mathcal{M}$. For $r = 1, \ldots, n$, let $d_r : \Lambda^r T \otimes \mathcal{M} \rightarrow \Lambda^{r-1} T \otimes \mathcal{M}$ be the $A$-homomorphism defined by

$$d_r (u_1, \ldots, u_r \cdot m) = \sum_{i=1}^{r} (-1)^{i+1} u_1 \wedge \cdots \wedge \hat{u_i} \wedge \cdots \wedge u_r \cdot \chi_i, m,$$

(6) for all $u_1, \ldots, u_r \cdot m \in \Lambda^r T \otimes \mathcal{M}$, where $\hat{u_i}$ means that this term is omitted. Set $\chi = (\chi_1, \ldots, \chi_n)$. The $d_r$'s induce the following complex $K(\chi, \mathcal{M})$ of $A$-modules:

$$0 \xrightarrow{d_{r+1}} \Lambda^{n-1} T \otimes \mathcal{M} \xrightarrow{d_r} \Lambda^n T \otimes \mathcal{M} \xrightarrow{d_{r-1}} \cdots \xrightarrow{d_2} \Lambda^2 T \otimes \mathcal{M} \xrightarrow{d_1} \Lambda^1 T \otimes \mathcal{M} \xrightarrow{d_0} 0,$$

i.e., $d_r \circ d_{r+1} = 0$ for $r = 0, \ldots, n$, called the Koszul complex of $\mathcal{M}$, and we can define the homology $A$-modules:

$$H_r(\chi, \mathcal{M}) = \text{ker}(d_r)/\text{im}(d_{r+1}), \quad r = 0, \ldots, n.$$

If $A$ is a noetherian ring and $\mathcal{M}$ a finitely generated $A$-module, then the $H_r(\chi, \mathcal{M})$'s are finitely generated $A$-modules. They only depend on $T$ and not on the choice of the basis $u_1, \ldots, u_n$ of $T$. Moreover, we can check that:

$$H_0(\chi, \mathcal{M}) = \mathcal{M}/\text{im}(d_1) \cong \mathcal{M}/\left( \sum_{i=1}^{n} \chi_i \mathcal{M} \right),$$

$$H_n(\chi, \mathcal{M}) = \ker(d_n) = \{ m \in \mathcal{M} \mid \chi_i m = 0, \quad i = 1, \ldots, n \}.$$

Lemma 3. (Serre (1989)). The Koszul homology $A$-modules $H_r(\chi, \mathcal{M})$'s satisfy that $\chi_i H_r(\chi, \mathcal{M}) = 0$, for $i = 1, \ldots, n$.

If $A = \mathcal{B}[\chi_1, \ldots, \chi_n]$, where $\mathcal{B}$ is a commutative ring, then Lemma 3 shows that any element $t \in \mathcal{B}$ which has no constant term acts trivially over $H_r(\chi, \mathcal{M})$, i.e., for every homology class $\alpha \in H_r(\chi, \mathcal{M})$, the homology class $t \alpha$ vanishes. Thus, the $H_r(\chi, \mathcal{M})$'s are finitely generated $\mathcal{B}$-modules and if $\mathcal{B} = \mathcal{K}$ is a field, then the $H_r(\chi, \mathcal{M})$'s are finite-dimensional $\mathcal{K}$-vector spaces (Malgrange (2005)).

4. CHARACTERIZATION OF KOSZUL HOMOLOGY

Let $A$ be a commutative noetherian ring, $R \in A^{q \times p}$, and $M = \text{coker}(d_1) = A^{1 \times p}/(A^{1 \times q} R)$. In this section, we explain how to characterize the homology $A$-modules $H_r(\chi, \mathcal{M})$'s as $H_r(\chi, \mathcal{M}) = (A^{1 \times q'} S_r')/(A^{1 \times q} S_r)$, where $S_r$ and $S_r'$ represent elements of $A^{q \times q} A^{p \times p}$ whose classes modulo $A^{1 \times q} R$ are generators as $A$-modules of $\ker(d_r)$ and $\text{im}(d_{r+1})$, respectively. We shall only consider the case $n = 3$ that cover most of the examples studied in applications. The general case can be handled similarly.

The free $A$-modules $A^3 T$ and $A^2 T$ have respectively rank 1 and 3 with the following chosen bases:

- $\{e_1 \in A^3 T\}$ for $A^3 T$,
- $\{e_2 \in A^3 T, e_3 \in A^2 T\}$ for $A^2 T$.

The Koszul differential $d_3$ acts on $A^3 T \otimes \mathcal{M}$ as:

$$d_3 (e_1 \in A^3 T = e_2 \in A^3 T \otimes \mathcal{M} + e_3 \in A^3 T \otimes \mathcal{M} \xrightarrow{d_3} 0,$$

for all $m \in \mathcal{M}$, and $d_2$ acts on $A^2 T \otimes \mathcal{M}$ as:

$$d_2 (e_2 \in A^2 T = e_1 \in A^2 T \otimes \mathcal{M} \xrightarrow{d_2} 0,$$

where $d_2 (e_2 \in A^2 T) = e_1 \otimes u_1$ for all $m \in \mathcal{M}$ and $i = 1, 2, 3$. Let:

$$P_1 = \frac{A^{3 \times 1}}{\chi_1 \chi_2}, \quad P_2 = \left( \begin{array}{c} 0 \ -\chi_3 \chi_2 \\ \chi_3 \ 0 \ -\chi_1 \end{array} \right) \in A^{3 \times 3},$$

$$P_3 = \left( \begin{array}{c} 0 \ \chi_1 \ -\chi_2 \\ -\chi_1 \ 0 \ \chi_2 \\ -\chi_2 \ -\chi_1 \ 0 \end{array} \right) \in A^{3 \times 3}.$$

Using $T \otimes A^{1 \times r} = A^{1 \times 3r}$, $A^3 T \otimes A^{1 \times r} = A^{3 \times 3r}$, and $\Lambda^3 T \otimes A^{1 \times r} = A^{1 \times r}$, for $r = q, p$, we then obtain the following commutative diagram of $A$-modules with exact rows and exact first two columns:

$$0 \xrightarrow{d_3} A^{1 \times q} R \xrightarrow{P_3} A^{1 \times p} \xrightarrow{P_2} A^3 T \otimes \mathcal{M} \xrightarrow{d_2} 0,$$

where $P_i \otimes I_p$ denotes the Kronecker product of the matrices $P_i$ and $I_p$ (and similarly for $P_i \otimes I_q$) for $i = 1, 2, 3$. Then, the Koszul homology $A$-modules are defined by:

$$H_0(\chi, \mathcal{M}) = \text{coker}(d_1), \quad H_1(\chi, \mathcal{M}) = \ker(d_2)/\text{im}(d_2), \quad H_2(\chi, \mathcal{M}) = \ker(d_3)/\text{im}(d_3).$$

Using Lemma 2, the third isomorphism theorem yields:

$$H_0(\chi, \mathcal{M}) = A^{1 \times p} \left( A^{1 \times q} \left( P_3 \otimes I_p \right) \right),$$

$$H_1(\chi, \mathcal{M}) = (A^{1 \times q} S_3')/(A^{1 \times q} S_3 \otimes I_3).$$
Remark 5. Alternatively, we can consider the restriction of the left $D$-homomorphism $\rho$ defined by (1) to $k[D_{\mathbb{Q}}] \subset D_{\mathbb{Q}}$ to get the left $B$-homomorphism $\rho : \mathcal{E} \to D_{\mathbb{Q}}$, and then its cokernel left $B$-module, i.e., $\mathcal{E}'_{\mathbb{Q}} = D_{\mathbb{Q}}/(D_{\mathbb{Q}} R)$ for all $s \in \mathbb{Z}$. Hence, we have $\mathcal{E}'_{\mathbb{Q}} = \mathcal{E}_{\mathbb{Q}}$ for all $s \in \mathbb{Z}$ if and only if (7) holds. If so, $R$ is said to be a strict morphism.

Note that (7) corresponds to an infinite number of integrability conditions. Fundamental results due to Quillen and Goldschmidt (Quillen (1964); Spencer (1969)) show that they can be reduced to a single integrability condition $(D_{\mathbb{Q}} R) \cap D_{\mathbb{Q}}' = D_{\mathbb{Q}}^0 R$, if the so-called symbol of the linear PD system is $2$-acyclic (see Section 5.2). More generally, a linear PD system is said to be involutive if it is formally integrable and if its symbol is $n$-acyclic, namely, involutive. In the next section, we shall introduce these concepts and show that they can be effectively tested using Koszul homology computations.

5.5 Acyclicity and involutivity of the symbol module

With the notations of Section 5.1, for all $r \in \mathbb{Z}$, let $\text{gr}(D_r) = D_r/D_{r-1}$ and $\sigma_r : D_r \to \text{gr}(D_r)$ be the canonical projection. Then, the graded ring of $D$ is defined by $\text{gr}(D) = \bigoplus_{r \in \mathbb{Z}} \text{gr}(D_r)$. If we set $\chi_i = \sigma_1(\delta_i)$ for $i = 1, \ldots, n$, then the identity $\delta_i b = b \delta_i + \delta_i(b)$ in $D_1$ yields $\chi_i b = b \chi_i$ in $\text{gr}(D_1)$ for all $b \in B$ and $i = 1, \ldots, n$. Moreover, we have $\chi_i \chi_j = \chi_j \chi_i$ in $\text{gr}(D_2)$. Thus, we have:

$$\forall s \in \mathbb{Z}, \quad (D^s_{\mathbb{Q}} R) \cap D^s_{\mathbb{Q}} = D^s_{\mathbb{Q}} R.$$  (7)

Conditions (7) are called integrability conditions.

The integrability conditions (7) mean that for all $s \in \mathbb{Z}$, the system equations of order $r$ and $s$, i.e., $(D^r_{\mathbb{Q}} R) \cap D^s_{\mathbb{Q}}$, are exactly the $s^\text{th}$-prolongation $\rho_{s/r}(R) := D^s_{\mathbb{Q}} R$ of the system equations. Note that we always have:

$$\forall s \in \mathbb{Z}, \quad \rho_{s/r}(R) \subseteq (D^s_{\mathbb{Q}} R) \cap D^s_{\mathbb{Q}}.$$  (Example 1)

Let $D = B(\delta_1, \delta_2)$ be the Weyl algebra, i.e., $B = K[x_1, x_2]$, $R = (\delta_1, \delta_2 + x_1)^T$ so that $r = 1$. We have $-\delta_1 x_2 + \delta_2 x_1 = \delta_2 + \delta_1 (\delta_2 + x_1)$ for all $d_1 \in D_1$. This implies that $(D^2_{\mathbb{Q}} R) \cap D_1 = D_1$. The linear PD system defined by $D$ is then not formally integrable since $\rho_0(R) = B(\delta_1, \delta_2 + x_1) \subseteq (D^2_{\mathbb{Q}} R) \cap D_1 = D_1$.

Remark 5. From Section 2, we have $\mathcal{L} = \bigoplus_{s=1}^{\infty} D_y$. Then, we let $\mathcal{L}_t = \bigoplus_{s=1}^{t} D_y$, i.e., the quotient filtration of $\mathcal{L}$ induced by the filtration $\{D_t\}_{t \in \mathbb{Z}}$ of $D$, namely, $\mathcal{L} = \bigoplus_{t \in \mathbb{Z}} \mathcal{L}_t$, where we set $\mathcal{L}_t = 0$ for $t < 0$, and $D_y \mathcal{L}_t \subseteq \mathcal{L}_{t+1}$ for all $t \in \mathbb{Z}$. In other words, we have:

$$\forall s \in \mathbb{Z}, \quad \mathcal{L}_{r+s} = (D^s_{\mathbb{Q}} R + D^r_{\mathbb{Q}} R)/(D^r_{\mathbb{Q}} R) \cap \mathcal{L}_{r+s}.$$

Alternatively, we can consider the restriction of the left $D$-homomorphism $\rho$ defined by (1) to $k[D_{\mathbb{Q}}] \subset D_{\mathbb{Q}}$ to get the left $B$-homomorphism $\rho : \mathcal{E} \to D_{\mathbb{Q}}$, and then its symbol module $\rho_0(R)$.

5.5 Acyclicity and involutivity of the symbol module

Let $\mathcal{H} = \bigoplus_{s \in \mathbb{Z}} \mathcal{H}_s$ be the principal symbol of $\rho$, namely, the matrix obtained by the application of the $B$-homomorphism $\sigma_r$ to the entries of $R$. Let $\mathcal{M} = \text{coker}_A(\sigma_r(R))$ be the $A$-module finitely presented by $\sigma_r(R)$. All the entries of the rows $\sigma_r(R)$'s of $\sigma_r(R)$ are homogeneous polynomials of degree $r$. Thus, $\sigma_r(R) \in \text{hom}_A(A^{1 \times s}, A^{1 \times s})$ yields the graded homomorphism $A(-r)^{1 \times s} = A^0(-r)^{1 \times s}$ of degree 0. Its image $T := \bigoplus_{s \in \mathbb{Z}} T_{s \times s}$ is then a graded $A$-submodule of $A^{0 \times s}$ and the graded $A$-module $\text{gr}(M)$ is defined by:

$$\text{gr}(M) = A^{0 \times s}/\bigoplus_{s \in \mathbb{Z}} A(-r) \sigma_r(R)_s.$$

(8)
As explained above, we can now define the Koszul complex \( K(\chi, \text{gr}(M)) \) of the finitely generated \( A \)-module \( \text{gr}(M) \).

This Koszul complex has a natural grading defined by:

\[
\Lambda^r T \otimes B \text{gr}(M) = \bigoplus_{s \in \mathbb{Z}} \Lambda^{s+r} T \otimes B \text{gr}(M)_s, \ r = 0, \ldots, n,
\]

and using \( \chi_i \in A_1 \), the map \( d_r \) defined by (6) yields the following homogeneous \( B \)-homomorphism of degree 1:

\[
d_r : \Lambda^r T \otimes B \text{gr}(M)_s \longrightarrow \Lambda^{r-1} T \otimes B \text{gr}(M)_{s+1},
\]

\[
d_r (\chi_i \Lambda^r \otimes m_s) = \sum_{i=1}^r (-1)^{i+1} \chi_i \Lambda \otimes m_s,
\]

for all \( m_s \in \text{gr}(M)_s \). Then, \( K(\chi, \text{gr}(M)) \) is the direct sum of the \( B \)-complexes of equations (9):

\[
0 \longrightarrow \Lambda^n T \otimes B \text{gr}(M) \xrightarrow{d_n} \Lambda^{n-1} T \otimes B \text{gr}(M)_{s+1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \Lambda^2 T \otimes B \text{gr}(M)_{s+2} \xrightarrow{d_1} \text{gr}(M)_{s+n} \xrightarrow{d_0} 0, \ s \in \mathbb{Z}.
\]

Thus, we have

\[
H_r(\chi, \text{gr}(M)) = \bigoplus_{k \in \mathbb{Z}} H_r(\chi, \text{gr}(M))_k, \ r = 0, \ldots, n, \ (10)
\]

where \( H_r(\chi, \text{gr}(M))_k \) is the homology at degree

\[
H_2(\chi, \text{gr}(M)) = (\Lambda^1 \otimes B)^{(r)}/(\Lambda^1 \otimes B) S_2 \times T \cong B^1 \times (\text{gr}(M)_s \cong \Lambda 1 \times 2).
\]

Definition 6. Given integers \( t \geq 0 \) and \( i \in \{0, \ldots, n\} \), the symbol \( B \)-module \( \text{gr}(M) \) is said to be \( i \)-acyclic at degree \( t \) if \( B_0(\text{gr}(M)) = 0 \) for all \( d \geq t \) and \( k = 0, \ldots, i \).

Lemma 3. For any \( B \)-module \( F \), the so-called Spencer sequence. The Spencer cohomology of (11) at \( A \)-modules

\[
0 \longrightarrow \Lambda^d T \otimes B \text{gr}(M) \xrightarrow{d_2} \Lambda^{d+1} T \otimes B \text{gr}(M)_{s+1} \xrightarrow{d_1} \cdots \xrightarrow{d_0} 0,
\]

i.e., the so-called Spencer sequence. The Spencer cohomology of (11) at \( A \)-modules is denoted by \( H^{d+1}(B) \).

If \( B = C \), then the \( H^k(\chi, \text{gr}(M))_k \)'s are finite dimensional \( K \)-vector spaces. If \( K = \mathbb{R} \), Quillen (1964) proves that the \( K \)-vector space \( \text{gr}(M)_k \) is locally the dual of the symbol \( g_s \) of the sheafification version of the differential operator \( R : F^p \times \mathbb{R} \rightarrow F^q \times \mathbb{R} \), and \( H^i(\chi, \text{gr}(M))_k \) is the dual \( K \)-vector space of the Spencer cohomology \( H^{i-1}(B) \).

One can prove that \( H_0(\chi, \text{gr}(M))_k+1 = 0 \) for \( k \geq r \) and \( \text{gr}(M) \) is 1-acyclic at degree \( r \) (Malgrange (2005)).

Using Section 4, matrices \( S_2 \) and \( S_2 \) with homogeneous rows exist such that we have the following representation:

\[
H_2(\chi, \text{gr}(M)) = (\Lambda^1 \otimes B)^{(r)}/(\Lambda^1 \otimes B) S_2 \times T.
\]

From Theorem 7, we can effectively test the 2-acyclicity.

(1) Use Section 4 with \( M = \text{ker}(s_2, \text{gr}(M)) \), \( r = \text{ord}(R) \), to compute \( S_2 \) and \( S_2 \) such that (12).

(2) Let \( S_2 \) be the matrix obtained by deleting rows of \( S \) with degree strictly less than \( r \).

(3) Test if a factorisation of the form \( S_2 = T_2 \times T_2 \) exists, where \( T_2 \) is a matrix with entries in \( A \).

Hence, \( \text{gr}(M) \) is 2-acyclic at degree \( r \) if and only if a factorization such as in Step (3) exists.
6. EXAMPLES


Let \( B = \mathbb{Q}[x_1, x_2, x_3] \), \( D = B[\partial_1, \partial_2, \partial_3] \), \( A = B[x_1, x_2, x_3] \), \( \mathcal{L} = D/((D^{3 \times 3})^R) \), \( r = 2 \), \( M = A/(A^{1 \times 5}) \sigma_3(R) \), where \( R \in D^{3 \times 3} \) and \( \sigma_3(R) \in A^{1 \times 5} \) are defined by

\[
R = \begin{pmatrix}
\partial_3^2 & \partial_2 \partial_3 - \alpha \partial_1^2 \\
\partial_2 \partial_3 & \partial_1 \partial_3 & \partial_1 \partial_2
\end{pmatrix}, \quad \sigma_2(R) = \begin{pmatrix}
\chi_2^3 & \chi_2 \chi_3 & \chi_3^2 \\
\chi_1 \chi_3 & \chi_1 \chi_2 & \chi_1^2 \\
0 & 0 & 0
\end{pmatrix}
\]

and \( \text{gr}(M) = A(0)/(A(-1)^{5 \times 5}) \sigma_2(R) \). We study the 2-acyclicity of \( \text{gr}(M) \) at degree 2. Let \( p = \chi_2 \chi_3 - \alpha \chi_1^2 \) and:

\[
S'_2 = \begin{pmatrix}
\begin{pmatrix}
\chi_2 & 0 & \alpha \chi_1 \\
\chi_1 & 0 & 0 \\
\chi_2 & 0 & 0
\end{pmatrix} & \begin{pmatrix}
\chi_2 \chi_3 & \chi_1 \chi_3 \\
\chi_1 \chi_2 & \chi_1^2 \\
0 & \chi_1 \chi_3
\end{pmatrix} & \begin{pmatrix}
0 & \chi_2 \chi_3 \\
0 & \chi_1 \chi_2 & \chi_1 \chi_3 \\
0 & 0 & \chi_1 \chi_3
\end{pmatrix}
\end{pmatrix}, \quad S_2 = \begin{pmatrix}
\begin{pmatrix}
\chi_1 & \chi_2 & \chi_3 \\
\chi_2 & 0 & 0 \\
\chi_3 & \chi_1 \chi_2 & \chi_1 \chi_3 \\
0 & \chi_1 \chi_3 & 0 \\
0 & 0 & \chi_1 \chi_3
\end{pmatrix}
\end{pmatrix}
\]

We have \( H_2(\chi, M) = (A^{1 \times 11} S'_2)/(A^{1 \times 16} S_2) \). Then, we can check that the matrix \( S'_{2,2} \) formed by the last four rows of \( S_2 \) satisfies \( S'_{2,2} = T_{2,2} S_2 \) for a certain matrix \( T_{2,2} \). Thus, \( \text{gr}(M) \) is 2-acyclic at degree 2 for all \( \alpha \).

To test the involutivity at degree 2, we have to consider \( H_3(\chi, M) \). Computing a representation of \( H_3(\chi, M) \), we obtain \( H_3(\chi, M) = (A^{1 \times 8} S'_3)/(A^{1 \times 5} S_3) \), where:

\[
S'_3 = \begin{pmatrix}
\begin{pmatrix}
\chi_2 & \chi_2 \chi_3 & \chi_1 \chi_3 & \chi_2^3 & \chi_1 \chi_2 & \alpha \chi_1^2 \\
0 & \chi_1 \chi_3 & 0 & 0 & 0 & 0
\end{pmatrix} & \begin{pmatrix}
\chi_2 \chi_3 & \chi_1 \chi_3 \\
\chi_1 \chi_2 & \chi_1 \chi_3 \\
0 & \chi_1 \chi_3 \\
0 & 0 & \chi_1 \chi_3 \\
0 & 0 & 0 & \chi_1 \chi_3
\end{pmatrix} & \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{pmatrix}, \quad S_3 = \begin{pmatrix}
\begin{pmatrix}
\chi_1 & \chi_2 & \chi_3 \\
\chi_2 & 0 & 0 \\
\chi_3 & \chi_1 \chi_2 \\
0 & \chi_1 \chi_3 & 0 \\
0 & 0 & \chi_1 \chi_3 \\
0 & 0 & 0
\end{pmatrix}
\end{pmatrix}
\]

We then have \( S'_{3,2} = S_3' \) and we can check that there is no factorization of the form \( S'_{3,2} = T_{3,2} S_3 \), which yields \( H_3(\chi, \text{gr}(M)) \) does not vanish at degree 2 for almost all \( \alpha \), i.e., \( \text{gr}(M) \) is not generally involutive at degree 2.

Considering the last entry of \( S'_3 \), the only particular value to check is \( \alpha = 0 \). We substitute \( \alpha = 0 \) into \( R \). In the same manner, we can compute \( S'_3 \) and \( S_3 \) such that \( H_3(\chi, M) = (A^{1 \times 2} S'_3)/(A^{1 \times 5} S_3) \), where \( S'_3 = (\chi_2 \chi_3 - \alpha \chi_1^2) \chi_1 \chi_3 \chi_1 \chi_2 \chi_1 \chi_3) \). Then, we have \( S'_{3,2} = S_3' \) and we can check that there is no factorization of the form \( S'_{3,2} = T_{3,2} S_3 \), which yields \( \text{deg}(S'_3) \) is strictly less than \( r = 2 \). Thus, \( \text{gr}(M) \) is 3-acyclic, i.e., involutive at degree 2. We find again the results obtained in Pommaret (1994).

Let us now study if \( \text{gr}(M) \) is 2-acyclic/involutive at degree 3. We consider the first prolongation \( \rho_1(R) \) of \( R \), its principal symbol \( \sigma_3(\rho_1(R)) \), i.e.,

\[
\sigma_3 = \begin{pmatrix}
\begin{pmatrix}
\partial_2 \partial_3 - \alpha \partial_1^2 \\
\partial_1 \partial_3 & \partial_1 \partial_2
\end{pmatrix} & \begin{pmatrix}
\partial_1 \partial_2^2 - \partial_2 \partial_3^2 \\
\partial_1^2 & \partial_2 \partial_3
\end{pmatrix} & \begin{pmatrix}
\partial_2 \partial_3^2 - \partial_2^2 \partial_3 \\
\partial_1 \partial_3 & \partial_1 \partial_2
\end{pmatrix}
\end{pmatrix}
\]

which corresponds to \( \text{gr}(M)_3 \), and the finitely presented \( A \)-module \( M_3 = \text{coker}_A(\rho_1(R)) \). Then, we get \( H_2(\chi, M_3) = (A^{1 \times 8} U'_2)/(A^{1 \times 19} V'_2) \) and \( H_3(\chi, M_3) = A^{1 \times 2} V'_2)/(A^{1 \times 5} V'_2) \), where the entries of \( U'_2 \) and \( V'_2 \) have degree strictly less than 3, which yields \( H_2(\chi, M_3) = 0 \), \( i = 2, 3 \), i.e., \( \text{gr}(M) \) is involutive at degree 3 (Pommaret (1994)). For the explicit computations (and more examples), see the webpage of Cluzeau and Quadrat (2008).

REFERENCES