An Integro-differential-delay Operator Approach to Transformations of Linear Differential Time-delay Systems
Alban Quadrat

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1. INTRODUCTION

The goal of the paper is to further develop the study of rings of integro-differential-delay operators considered as noncommutative polynomial algebras satisfying standard calculus identities. Within the algebraic analysis approach, we show that transformations and reductions of linear differential time-delay systems can be interpreted as homomorphisms and isomorphisms of finitely presented left modules over an algebra of integro-differential-delay operators. In particular, we show how Fiagbedzi-Pearson’s transformation can be found again and generalized. This transformation maps the solutions of a first-order differential linear system with state and input delays to the solutions of a purely state-space linear system. Fiagbedzi-Pearson’s transformation reduces to the well-known Artstein’s reduction when the system has no state delay and yields an isomorphism of the solution spaces.

Abstract: In this paper, we further develop the study of rings of integro-differential-delay operators considered as noncommutative polynomial algebras satisfying standard calculus identities. Within the algebraic analysis approach, we show that transformations and reductions of linear differential time-delay systems can be interpreted as homomorphisms and isomorphisms of finitely presented left modules over an algebra of integro-differential-delay operators. In particular, we show how Fiagbedzi-Pearson’s transformation can be found again and generalized. This transformation maps the solutions of a first-order differential linear system with state and input delays to the solutions of a purely state-space linear system. Fiagbedzi-Pearson’s transformation reduces to the well-known Artstein’s reduction when the system has no state delay and yields an isomorphism of the solution spaces.

Keywords: Linear systems, systems with time-delays, polynomial methods, delay compensation

2. ALGEBRAIC ANALYSIS APPROACH

If we consider the following functional operators...
where
\[ y(t) = y(t - h), \quad \delta y(t) = y(t - r), \]
which act on functions \( y \), then we can easily check that:
\[ \partial \circ \partial = \partial \circ \partial, \quad \partial \circ \tau = \tau \circ \partial, \quad \partial \circ \tau = \tau \circ \partial. \]  

Hence, we can define the commutative polynomial ring \( D = \mathbb{R}[\partial, \delta, \tau] \) formed by all the differential time-delay operators in \( \partial, \delta, \) and \( \tau \) with coefficients in \( \mathbb{R} \). Every element \( P \in D \) can be uniquely written as \( P = \sum_{0 \leq |\nu| \leq \lambda} a_{\nu} \partial^{\nu} \), where \( \nu = (\nu_1 \nu_2, \nu_3) \in \mathbb{Z}_0^2 \) is a multi-index of length \( |\nu| = \nu_1 + \nu_2 + \nu_3 \), \( x^\nu = \partial^{\nu_1} \partial^{\nu_2} \partial^{\nu_3}, \) \( a_{\nu} \in \mathbb{R} \), and \( r \in \mathbb{Z} \geq 0 \). This ring \( D \) can be used to develop a polynomial approach to linear differential constant time-delay systems with real coefficients. Since “linear algebra over a ring” corresponds to module theory in mathematics, module theory was naturally introduced and used in this study.

This approach has largely been studied in the literature. See, e.g., Fliss and al. (1998) and the references therein.

Let us consider the following \( n \times (n + m) \)-matrices:
\[
\begin{align*}
R &= (\partial \cdot 1_n - A - B), \\
R' &= (\partial \cdot 1_n - A - 1_n \tau - B_0 - B_1 \delta),
\end{align*}
\]
with entries in \( D \), i.e., \( R, R' \in D^{n \times (n + m)} \), \( 1_n \) denotes the \( n \times n \) identity matrix. If we note \( \eta = (x^T \ y^T)^T \) and \( \eta' = (x^T \ y'^T)^T \), then the above linear functional systems can respectively be rewritten as follows:
\[ R \eta = 0, \quad R' \eta' = 0. \]

If \( \mathcal{F} \) is a \( D \)-module, i.e., \( d_1, d_2, d_2, d_2, d_2 \in \mathcal{F} \) for all \( d_1, d_2 \in D \) and for all \( f_1, f_2 \in \mathcal{F} \), then we can define the behaviours:
\[
\begin{align*}
\ker_f(R) &= \{ \eta \in \mathcal{F}^{(n + m) \times 1} | R \eta = 0 \}, \\
\ker_f(R') &= \{ \eta' \in \mathcal{F}^{(n + m) \times 1} | R' \eta' = 0 \}.
\end{align*}
\]

In this paper, the linear systems (1) and (2) will be studied which act on functions \( \lambda \). Hence, we can define the commutative polynomial ring \( D \) of length \( R \). This ring \( D \) can be used to develop a polynomial approach to linear differential constant time-delay systems with real coefficients. Since “linear algebra over a ring” corresponds to module theory in mathematics, module theory was naturally introduced and used in this study.

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\end{align*}
\]

Within the algebraic analysis approach, \( R' \in D^{n \times (n + m)} \), \( \mathcal{M}' = \text{coker}_D(R') \), and \( \text{ker}_D(R') \), then homomorphisms from \( \text{ker}_D(R') \) to \( \text{ker}_D(R') \) can be studied by homomorphisms from \( \mathcal{M} \) to \( \mathcal{M}' \). Let us briefly state again these results.

Let \( f \in \text{hom}_D(\mathcal{M}, \mathcal{M}') \). Then, it can be shown that \( f \) is defined by a pair of matrices \( P \in D^{p \times q} \) and \( Q \in D^{q \times q} \) satisfying \( RP = Q R' \). Then, we have \( f(\pi(\lambda)) = \pi(\lambda) \) for all \( \lambda \in D^{1 \times q} \), where \( \pi' : D^{1 \times q} \longrightarrow M' \) is the left \( D \)-homomorphism which sends \( \lambda' \in D^{1 \times q} \) onto its residue class \( \pi'(\lambda') \) in \( M' \). For more details, see Cluzeau et al. (2008). Using the above identity, we get the \( D \)-linear map \( f' : \text{ker}_D(R') \longrightarrow \text{ker}_D(R') \) defined by \( f'(\eta') = P \eta' \) for all \( \eta' \in \text{ker}_D(R') \). The left \( D \)-modules ker, im, and coker can be explicitly characterized (Cluzeau et al. (2008)). In particular, \( f \) is isomorphism if it is both injective and surjective, i.e., ker \( f \) = 0 and coker \( f \) = 0. For the algorithmic aspects of the computation of matrices \( P \) and \( Q \) for given matrices \( R \) and \( R' \), see Cluzeau et al. (2008) and the OreMorphisms package (Cluzeau et al. (2009)).

Using the above algebraic analysis approach to time-invariant linear differential constant time-delay systems, we can study the homomorphisms between the finitely presented \( D \) \( \mathcal{M} = \text{coker}_D(R) \) to \( \mathcal{M}' = \text{coker}_D(R') \), where the matrices \( R' \) are
defined by (4), and thus, the corresponding homomorphisms between the behaviours/linear systems \( \text{ker}_\tau(R) \) and \( \text{ker}_\tau(R') \). Since \( D \) is commutative, \( \text{hom}_D(M, M') \) inherits a \( D \)-module structure and it can be proved that \( \text{hom}_D(M, M') \) is a finitely presented \( D \)-module, and thus, it can be defined by a finite set of generators and a finite set of \( D \)-linear relations. Finally, using Gröbner basis methods, these generators and relations can be computed.

But, in the literature of delay systems, it is well-known that differential time-delay transformations are usually not enough to study interesting problems such as Artstein's reduction, finite spectrum assignment, stabilizability, etc., where integral transformations are required (e.g., distributed delays). For the study of the transformations from (1) to (2), we thus have to enlarge the ring \( D \) to include integral operators. We are naturally led to introduce rings of integro-differential constant time-delay operators as shown in Quadrat (2015); Cluzeau et al. (2018).

3. RINGS OF INTEGRO-DIFFERENTIAL-DELAY OPERATORS

As explained in Section 2, to study linear differential time-delay systems, it is natural to consider rings of integro-differential-delay operators. Let us first briefly introduce the rings of integro-differential operators. For more details, see Quadrat (2022b). To simplify, we shall consider here that differential time-delay transformations are usually centered around quadratic terms.

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The following identities in \( I \) can be derived from (6)

\[
e^2 = e, \quad eI = 0, \quad \partial e = 0,
\]

\[
\forall a \in A, \quad I a \partial = -I \partial (a) + a - e(a)e,
\]

\[
\forall a \in A, \quad I a I = [I(a), I],
\]

where \([a, b] = ab - ba\) denotes the commutator of \( a \) and \( b \). For instance, the fourth identity of (7) corresponds to the integration by parts. For more details, see Quadrat (2022b). Using (6) and (7), it is clear that \( J \) is a non-commutative polynomial ring in \( \partial, I \), and \( e \) with coefficients in \( A \), denoted by \( A(\partial, I, e) \). One can show that every element \( P \in J \) can be written uniquely as \( P = P_1 + P_2 + P_3 \), where \( P_1 = \sum_{i=0}^r a_i \partial^i \in \mathcal{A}(\partial) \), where \( A(\partial) \) is the ring of differential operators with coefficients in \( A \), namely, the non-commutative polynomials in \( \partial \) and \( a \in A \) satisfying the first identity of (6), \( P_2 \) belongs to the ring (without multiplicative identity) of integral operators

\[
J = A[I | I a I = [I(a), I]] = \left\{ \sum_{i=1}^r a_i b_i | a_i, b_i \in A \right\},
\]

and \( P_3 \) belongs to the only two-sided ideal \( \langle e \rangle = I e I \) of \( J \) defined by \( e \). Using the identities in \( I \), we can check again that \( P_3 \) is of the form \( P_3 = \sum_{i=0}^r a_i e \partial^i \), where \( a_i \in A \).

To define rings of integro-differential-delay operators, we add the new \( \mathbb{R} \)-endomorphism \( \delta \) of \( A \) defined by:

\[
\forall a \in A, \quad \delta(a)(t) = a(t - h).
\]

In this paper, we shall only consider the case where \( h \) is a fixed positive real number. Thus, \( \delta \) is a \( \mathbb{R} \)-endomorphism of \( A \) and \( \delta^{-1}(a)(t) = a(t + h) \) for all \( a \in A \). We also consider the \( \mathbb{R} \)-endomorphism \( e_h \) of \( A \) defined by:

\[
\forall a \in A, \quad e_h(a)(t) = a(t_0 + h).
\]

Note that we then have

\[
\forall a \in A, \quad (e_h \delta)(a)(t) = e_h(a(t - h)) = a(t_0) = e(a)(t),
\]

which shows the following identity in \( \text{end}_\mathbb{R}(A) \):

\[
e = e_h \delta.
\]

Thus, \( e \) is a consequence of \( \delta \) and \( e_h \). More generally, we have \( (e_h \delta)^{r+1}(a)(t) = a(t_0 - rh) \) for all \( r \in \mathbb{N} \), i.e., \( e_h \delta^{r+1} = e_h \delta^r \) is the evaluation at \( t_0 - rh \) for \( r \in \mathbb{N} \).

Similarly, we can easily check the following identities

\[
\delta^2 = \delta \delta, \quad \delta e_h = 0, \quad e I = e_h \delta I = 0, \quad \delta e_h = e_h,
\]

\[
e_h^2 = e_h, \quad I e_h = (t - t_0) e_h.
\]

Using the change of variables \( s = t + h \), we get

\[
(\delta I)(a)(t) = \int_{t_0}^{t+h} a(\tau) d\tau = \int_{t_0}^{t} a(s - h) ds = \int_{t_0}^{t} a(s - h) ds - \int_{t_0}^{t+h} a(s - h) ds,
\]

which yields the following identity:

\[
\delta I = I \delta - e_h I \delta = (I - e_h) I \delta.
\]

Using the change of variables \( s = t - h \), we also have

\[
(I \delta)(a)(t) = \int_{t_0}^{t-h} a(\tau - h) d\tau = \int_{t_0}^{t} a(s + h) ds = \int_{t_0}^{t} a(s + h) ds - \int_{t_0}^{t+h} a(s) ds,
\]

which shows that the identity holds in \( \text{end}_\mathbb{R}(A) \):

\[
I \delta = \delta I - e_h I \delta = I \delta - e_h \delta^2 I = (I - e_h \delta) I \delta = (I - e) \delta I.
\]
Let us check that (9) and (10) are equivalent. Multiply (9) by $1 - e_h \delta$ and using the identities $e_h = e_h^2 = e_h$, and $e I = e_h \delta I = 0$, we first have
\[
(1 - e_h \delta) \delta I = (1 - e_h \delta) (1 - e_h) I \delta = (1 - e_h \delta) e_h + e_h \delta e_h = (1 - e_h) I \delta = (1 - e_h) I = 1 - e_h \delta I = 1 - e_h \delta I,
\]
i.e., (10). Similarly, multiplying (10) by $1 - e_h$ and using $e_h^2 = e_h$ and $e I = e_h \delta I = 0$, we then have
\[
(1 - e_h) I \delta = (1 - e_h) (1 - e_h \delta) I \delta = (1 - e_h) - e_h \delta I = (1 - e_h) I \delta = \delta I - e_h \delta I = \delta I = 1 - e_h \delta I,
\]
i.e., (9), which proves that (9) and (10) are equivalent.

We sum up the above identities in the following table, where $a r$ denotes the product of an element in the first column by an element in the first row.

<table>
<thead>
<tr>
<th>$a r$</th>
<th>$\partial$</th>
<th>$I$</th>
<th>$\delta$</th>
<th>$e_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial$</td>
<td>$\partial^*$</td>
<td>$1$</td>
<td>$\delta \partial$</td>
<td>$0$</td>
</tr>
<tr>
<td>$I$</td>
<td>$1 - e_h \delta$</td>
<td>$I I - I I$</td>
<td>$(1 - e_h \delta) I \delta$</td>
<td>$(1 - t_0) e_h$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$\delta \partial$</td>
<td>$(1 - e_h \delta) I \delta$</td>
<td>$e_h$</td>
<td>$e_h \delta$</td>
</tr>
<tr>
<td>$e_h$</td>
<td>$e_h \partial$</td>
<td>$e_h I$</td>
<td>$e_h \delta = e$</td>
<td>$e_h$</td>
</tr>
</tbody>
</table>

(11)

Moreover, we can check that the following identities hold:
\[
\forall a \in A, \quad \partial a = a \partial + \partial (a),
\]
\[
\delta a = \delta (a) \delta,
\]
\[
e_h a = e_h (a) e_h,
\]
\[
I a \partial = I \partial (a) + (1 - e_h \delta) a,
\]
\[
I a \delta = (1 - e_h \delta) I \delta^{-1} (a),
\]
\[
I a I = [I (a), I],
\]
\[
I a e_h = I (a) e_h.
\]

Definition 1. We call ring of integro-differential-delay operators over $A = C^\infty (R)$, denoted by $H = A(\partial, I, \delta, e_h)$, the $R$-subalgebra of $\text{end}_R (A)$ generated by $\partial, I, e_h, \delta$, and all the multiplication by elements of $A$. Equivalently, $H$ is the noncommutative polynomial ring $A(\partial, I, \delta, e_h)$ defined by $\partial, I, \delta, e_h$ which satisfy (11), (12), and $e_h \delta I = 0$.

Every element $h$ of $H$ can be written uniquely as $h = a(t, \delta, \partial) b(t, \partial) I c(t) + \partial (t) e_h j(t) \delta + \delta j(t) \delta \delta I e_h a(t, \delta)$, where $a(t, \delta, \partial)$ is a polynomial in $\partial$ and $\delta$ with coefficients in $A$, $b(t, \partial)$ is a polynomial in $\partial$ with coefficients in $A$, $c, d, g, i \in A$, $j$ a polynomial in $\partial$ with coefficients in $\mathbb{R}$ and $a$ a polynomial in $\partial$ and $\delta$ with coefficients in $\mathbb{R}$.

Consider the $R$-endomorphisms $\tau$ and $e_r$ of $A$ defined by
\[
\forall a \in A, \quad \tau (a)(t) = a(t - r), \quad e_r (a)(t) = a(t_0 + r),
\]
where $r$ is a fixed positive real number. Then, similar identities as (11) and (12) hold for $\tau$ and $e_r$ instead of $\partial$ and $e_h$, as well as $e_r, \tau I = 0$. Similarly as above, we can define the $\mathbb{R}$-subalgebra of $\text{end}_R (A)$ defined by $\partial, I, \delta, e_h, \tau, e_r$, satisfying the corresponding identities as well as:
\[
\delta \tau = \tau \delta, \quad e_r e_h = e_h, \quad e_h e_r = e_r.
\]

4. ALGEBRAIC ANALYSIS OVER RINGS OF INTEGRO-DIFFERENTIAL-DELAY OPERATORS

Following the algebraic analysis approach briefly introduced in Section 2, let $H = A(\partial, I, \delta, e_h, \tau, e_r)$ be the ring of integro-differential-delay operators defined in Section 3, $R, R' \in H^{(n \times n)}$ the matrices defined by (4), and $\mathcal{M} = \text{coker}_H (R)$ (resp., $\mathcal{M}' = \text{coker}_H (R')$) the left $H$ finitely presented by $R$ (resp., $R'$).

Now, let $P = \partial I - A_0 \in H^{(n \times n)}$, $\Theta$ be the transition matrix of the linear differential system $\dot{x}(t) = A_0 x(t)$, and $Q = \Theta I \Theta^{-1} \in H^{(n \times n)}$. Then, using (12), we have
\[
PQ = (\partial I - A_0) (\Theta I \Theta^{-1}) = (\Theta \partial - \Theta) I \Theta^{-1} - A_0 \Theta I \Theta^{-1} = \Theta I \Theta^{-1} + (\Theta - A_0) \Theta I \Theta^{-1} = 1_n,
\]
which shows that $Q$ is a right inverse of $P$. Using the fact that $\Psi := \Theta^{-1}$ is a transition matrix of $\Psi + \Psi A_0 = 0$ and the integration by parts, we get the idempotent of $H^{(n \times n)}$:
\[
Q P = (\Theta I \Theta^{-1}) (\partial I - A_0) = \Theta I \Theta^{-1} - \partial (\Theta^{-1}) - \Theta I \Theta^{-1} A_0 = (1_n - \Theta e - \Theta e \Theta^{-1} - \Theta I \Theta^{-1} (A_0 + A_1) \tau) + \Theta I \Theta^{-1} (B_0 + B_1 + B_2),
\]
Note that $R'' = Q R'$ yields $\text{im}_H (R'') \subseteq \text{im}_H (R')$, $R'' = (P Q) R' = P R'R''$ yields $\text{im}_H (R') \subseteq \text{im}_H (R'')$, which proves $\text{im}_H (R'') \subseteq \text{im}_H (R')$. Therefore, we have $\mathcal{M}' = \text{coker}_H (R') \subseteq \text{coker}_H (R')$, which proves that $\mathcal{M}'$ is equivalently defined by the following representation:
\[
x = \Theta e x + \Theta I \Theta^{-1} (A_0 + A_1) x + \Theta I \Theta^{-1} (B_0 + B_1 + B_2) u.
\]
(13)

Using the properties of the transition matrix $\Theta$, namely, $\Theta^{-1} (t_0, t_0) = \Theta (t_0, t_0) \Theta (t_0, t_1) \Theta (t_1, t_2) = \Theta (t_0, t_2)$, we get:
\[
Q (y)(t) = \Theta (t, t_0) \int_{t_0}^{t} \Theta (t, \tau) y(\tau) d\tau = \int_{t_0}^{t} \Theta (t, \tau) y(\tau) d\tau.
\]
Thus, the linear system (1) is equivalent to:
\[
x(t) = \Theta e x + \Theta I \Theta^{-1} (A_0 + A_1) x(t) + \Theta I \Theta^{-1} (B_0 + B_1) u(t)
\]
\[
= \Theta (t, t_0) x(t_0) + \int_{t_0}^{t} \Theta (t, \tau) A_0 (\tau) x(\tau) d\tau
\]
\[
+ \int_{t_0}^{t} \Theta (t, \tau) A_1 (\tau) (\tau - r) d\tau + \int_{t_0}^{t} \Theta (t, \tau) B_0 (\tau) u(\tau) d\tau
\]
\[
+ \int_{t_0}^{t} \Theta (t, \tau) B_1 (\tau) (\tau - h) d\tau.
\]
Using (12) and (10) with $\tau$ instead of $\delta$, we have:
\[
I \Theta^{-1} A_1 \tau = I \tau \left( (\Theta^{-1} A_1) (\cdot + r) \right)
\]
\[
= (1 - e) \tau I \left( (\Theta^{-1} A_1) (\cdot + r) \right),
\]
\[
I \Theta^{-1} B_1 \delta = I \delta \left( (\Theta^{-1} B_1) (\cdot + h) \right)
\]
\[
= (1 - e) \delta I \left( (\Theta^{-1} B_1) (\cdot + h) \right).
\]
(13)

Therefore, (13) becomes:
\[
x = \Theta e x + \Theta I \Theta^{-1} A_0 x + \Theta (1 - e) \tau I \left( (\Theta^{-1} A_1) (\cdot + r) \right) x
\]
\[
+ \Theta I \Theta^{-1} B_0 (\cdot + r) u + \Theta (1 - e) \delta I \left( (\Theta^{-1} B_1) (\cdot + h) \right) u.
\]
The linear system (1) with the following initial condition
\[
\forall t \in [t_0 - h, t_0], \quad x(t) = \phi(t),
\]
can then equivalently be represented as follows:
\[
x(t) = \Theta(t, t_0) \phi(t_0) + \int_{t_0}^{t} \Theta(t, \tau) A_0(\tau) x(\tau) d\tau \\
+ \int_{t_0}^{t-r} \Theta(t, t + r) A_1(\tau + r) x(\tau) d\tau \\
+ \int_{t_0}^{t-h} \Theta(t, \tau) B_0(\tau) u(\tau) d\tau \\
+ \int_{t_0}^{t-b} \Theta(t, t + b) B_1(\tau + b) u(\tau) d\tau.
\]

The last representation of (1) is at the core to the well-known method of steps in time-delay system theory.

5. FIAGBEDZI-PEARSON’S TRANSFORMATION

The purpose of this section is to study the existence of \( f \in \text{hom}_H(M, M') \) defined by \( f(\pi(\lambda)) = \pi'(\lambda P) \), where
\[
P = \begin{pmatrix}
a_0 \tau I a_1 + a_2 I a_2 + a_4 & b_0 \delta I b_1 + b_2 I b_3 + b_4 \\
0 & c_0
\end{pmatrix},
\]
and \( a_1, a_2, b_1, b_2, b_3, b_4, c_0 \in A^{n \times n} \) are matrices to be determined. According to Section 2, \( P \) defines a left \( H \)-homomorphism from \( M \) to \( M' \) if and only if there exists \( Q \in H^{n \times n} \) satisfying \( RP = QR' \), i.e., such that:
\[
(\partial 1_n - A)(a_0 \tau I a_1 + a_2 I a_2 + a_4 + \delta + a_0 \tau(\tau) + a_4) \\
= Q(\partial 1_n - A - A\tau),
\]
(14)
\[
(\partial 1_n - A)(b_0 \delta I b_1 + b_2 I b_3 + b_4 - B c_0) \\
= -Q(\partial 1_n - A \delta). \quad (15)
\]

The normal form of the left-hand side of (14) is then:
\[
(\partial 1_n - A)(b_0 \delta I b_1 + b_2 I b_3 + b_4) - B c_0 = 0.
\]

Comparing this normal form with the right-hand side of (14) yields \( \text{deg}_Q Q = 0 \), \( \text{deg}_\delta Q = 0 \), \( \text{deg}_\tau Q = 0 \), \( \text{deg}_A Q \), \( Q \in A^{n \times n} \). Thus, (14) yields \( Q = a_4 \) and:
\[
(\partial 1_n - A)(a_0 \tau I a_1 + (a_2 - A a_2) I a_3 + a_4 \delta + a_0 \tau(\tau) + a_4) + a_4 = 0.
\]

Let us suppose that \( a_1 \neq 0 \) and \( a_3 \neq 0 \). Thus, we get:
\[
\begin{align*}
\hat{a}_0 & = -a_0 \tau a_1 = 0, \\
\hat{a}_2 & = -a_2, \\
\hat{a}_0 & = a_0 \tau, \\
\hat{a}_2 & = a_2 + a_3, \\
\hat{a}_3 & = a_3.
\end{align*}
\]

Let \( \Phi \) be the transition matrix of \( a_0 - A a_0 = 0 \). Integrating (16), we then obtain:
\[
\begin{align*}
\Phi(\tau, t_0) c_0, & \quad c_0 \in \mathbb{R}^{n \times n}, \\
\Phi(\tau, t_0) c_2, & \quad c_2 \in \mathbb{R}^{n \times n}, \\
\Phi(\tau, t_0) (\hat{a}_0(\tau) A(\tau) + a_4(\tau) A(\tau) A_0(\tau)), & \quad a_4(\tau) A(\tau) A_0(\tau).
\end{align*}
\]

Using that \( c_0 \) and \( c_2 \) are two constant matrices, we get:
\[
P_{11}(t) = \Phi(t, t_0) (\tau I A(t) + a_3(\tau) A(t) + a_4(\tau) A_0(t)) + I \Phi(t_0, t) (\hat{a}_4(t) + a_4(t) A_0(t)),
\]

Using the identities \( \partial \delta = \delta \partial = 1, \delta b_1 = b_1(1 - h) \delta \), we then get the following normal form of (15):
\[
(\hat{b}_0 - A b_0) \delta I b_1 + (b_0 b_1 \cdot a_4 B_1 + b_2 \cdot b_1 - A b_4 - B c_0 + a_4 B_0 = 0,
\]
Suppose that \( b_1 \neq 0 \) and \( b_3 \neq 0 \). The last equation yields:
\[
\begin{align*}
\hat{b}_0 - A b_0 & = 0, \\
b_0 b_1(1 - h) + a_4 B_1 & = 0, \\
b_2 - A b_2 & = 0, \\
b_3 & = 0, \\
b_2 b_3 + b_4 - A b_4 - B c_0 + a_4 B_0 & = 0, \\
b_0 - A b_0 & = 0, \\
b_2 - A b_2 & = 0, \\
b_4 & = 0, \\
b_0 b_1(1 - h) + a_4 B_1 & = 0, \\
b_2 b_3 + B c_0 + a_4 B_0 & = 0.
\end{align*}
\]

Integrating the last system, we then obtain:
\[
\begin{align*}
P_{12}(t) & = \Phi(t, t_0) (b_0 b_1(1 - h) + a_4 B_1 + a_4(\tau) B_1(\tau) + a_4(\tau) B_0(\tau)), \\
P_{11}(t) & = a_4(\tau) A_0(\tau), \\
\Phi(t, t_0), & \quad \tau I A(t) + a_4(\tau) A(t) + a_4(\tau) A_0(\tau).
\end{align*}
\]

Theorem 1. A left \( H \)-homomorphism from \( M \) to \( M' \) is defined by \( f(\pi(\lambda)) = \pi'(\lambda P) \) for all \( \lambda \in H^{1 \times (n + m)} \), where \( P \) is a four block matrix defined by \( P_{21} = 0, P_{22} = c_0(t) \), \( P_{11}(t) = a_4(t) \\
-P(\tau, t_0) (\tau I A(t) + a_4(\tau) A(t) + a_4(\tau) A_0(\tau)), \\
+I \Phi(t_0, t) (\hat{a}_4(t) + a_4(t) A_0(t)) + I \Phi(t_0, t) (B(t) c_0(t) - a_4(t) B_0(t)),
\]
for all \( a_4 \in A^{n \times n} \) and \( c_0 \in A^{m \times m} \), where \( \Phi \) is transition matrix of \( a - A a = 0 \). Thus, the solutions of (1) are sent to solutions of (2) by the following integral transformation:
\[
\begin{align*}
z(t) & = a_4(\tau) x(t) \\
& - \int_{t_0}^{t-r} \Phi(\tau, \tau + r) A_4(\tau + r) A_1(\tau + r) x(\tau) d\tau \\
& + \int_{t_0}^{t} \Phi(\tau, \tau) A_4(\tau) A_0(\tau) - a_4(\tau) A_0(\tau) - \hat{a}_4(\tau) x(\tau) d\tau \\
& - \int_{t_0}^{t-h} \Phi(\tau, \tau + h) A_4(\tau + h) B_1(\tau + h) u(\tau) d\tau \\
& + \int_{t_0}^{t} \Phi(\tau, \tau) (B(t) c_0(\tau) - a_4(\tau) B_0(\tau)) u(\tau) d\tau,
\end{align*}
\]
\[
v(t) = c_0(t) u(t).
\]

If the following relations hold
The solutions of (1) are sent to solutions of (2) by:

\[
\begin{align*}
A(t,t) & = A(t) + \Phi(t,t) \Phi(t_0, t) A_1(t), \\
B(t) & = B(t) + \Phi(t,t) A_1(t) + \Phi(t,t) B_1(t),
\end{align*}
\]

Then the matrix \( P \) simplifies into:

\[
\begin{align*}
P_{11} & = A + \Phi(t,t_0) (I - \tau) I \Phi(t_0, t) A_1(t), \\
P_{12} & = \Phi(t,t_0) (I - \tau) I \Phi(t_0, t) + \Phi(t,t_0) B_1(t) + A_1(t + h).
\end{align*}
\]

If we suppose that \( a_4 \) and \( c_0 \) are invertible, then (18) yields:

\[
\begin{align*}
A & = a_4 a_4^{-1} + a_0 A_0 a_4^{-1} \\
+ \Phi(t,t_0) A_1(t) + \Phi(t,t_0) B_1(t) + A_1(t), \\
B & = a_4 B_0 c_0^{-1} + \Phi(t,t_0) B_1(t) + A_1(t + h) c_0^{-1}.
\end{align*}
\]

Finally, if we set \( a_4 = 1_n \) and \( c_0 = 1_m \) in (19), we then obtain the following corollary of Theorem 1.

**Corollary 2.** With the notations of Theorem 1, if the matrices \( A \) and \( B \) are chosen so that

\[
\begin{align*}
A(t) & = a_0(t) + \Phi(t,t) A_1(t), \\
B(t) & = B_0(t) + \Phi(t,t) B_1(t),
\end{align*}
\]

then the matrices \( P_{11} \) and \( P_{12} \) of Theorem 1 become:

\[
\begin{align*}
P_{11} & = a_4 + \Phi(t,t_0) (I - \tau) I \Phi(t_0, t) A_1(t), \\
P_{12} & = \Phi(t,t_0) (I - \tau) I \Phi(t_0, t) + \Phi(t,t_0) B_1(t) + A_1(t + h).
\end{align*}
\]

Thus, the solutions of (1) are sent to solutions of (2) by:

\[
\begin{align*}
z(t) & = x(t) + \int_{t-h}^{t} \Phi(t,t_0) A_1(t) x(t) \, dt \\
& + \int_{t-h}^{t} \Phi(t,t_0) B_1(t) u(t) \, dt,
\end{align*}
\]

If \( A_1 = 0 \), then (20) and (21) yield \( A = A_0 \) and:

\[
P = \begin{pmatrix} 1_n & \Phi(t,t_0) (1 - \tau) I \Phi(t_0, t) A_1(t) + \Phi(t,t_0) B_1(t) + A_1(t) \end{pmatrix}.
\]

**Example 1.** Let us consider the time-invariant case, i.e., \( A, A_0, A_1 \in \mathbb{R}^{n \times n} \) and \( B, B_0, B_1 \in \mathbb{R}^{m \times m} \). Then, we first get \( \Phi(t,t_0) = e^{A(t-t_0)} \). Setting \( t_0 = 0, a_4 \in \mathbb{R}^{n \times n} \) and \( c_0 \in \mathbb{R}^{m \times m} \) to simplify the expressions, Theorem 1 yields:

\[
\begin{align*}
z(t) & = a_4 x(t) \\
& - \int_{t-h}^{t} e^{A(t-t_0)} A_1(t) x(t) \, dt \\
& + \int_{t-h}^{t} e^{A(t-t_0)} A_4 x(t) \, dt \\
& - \int_{t-h}^{t} e^{A(t-t_0)} B_1(t) u(t) \, dt \\
& + \int_{t-h}^{t} e^{A(t-t_0)} (B c_0 - B_0) u(t) \, dt,
\end{align*}
\]

If \( a_4 \) and \( c_0 \) are invertible, then (19) yields

\[
\begin{align*}
A & = a_4 A_0 a_4^{-1} + e^{-A_1} A_1 a_4^{-1}, \\
B & = a_4 B_0 c_0^{-1} + e^{-A_1} B_1 c_0^{-1},
\end{align*}
\]

and the solutions of (1) are sent to solutions of (2) by:

\[
\begin{align*}
z(t) & = a_4 x(t) + \int_{t-h}^{t} e^{A(t-t_0)} A_1(t) x(t) \, dt \\
& + \int_{t-h}^{t} e^{A(t-t_0)} A_4 x(t) \, dt \\
& + \int_{t-h}^{t} e^{A(t-t_0)} B_1(t) u(t) \, dt,
\end{align*}
\]

Now, if we set \( a_4 = 1_n \) and \( c_0 = 1_m \), then we obtain:

\[
P = \begin{pmatrix} 1_n & e^{A(t-t_0)} A_1 + e^{-A_1} B_1 \end{pmatrix}.
\]

Thus, the solutions of (1) are sent to solutions of (2) by:

\[
\begin{align*}
z(t) & = x(t) + \int_{t-h}^{t} e^{A(t-t_0)} A_1(t) x(t) \, dt \\
& + \int_{t-h}^{t} e^{A(t-t_0)} B_1(t) u(t) \, dt,
\end{align*}
\]

We have just found again *Fiagbedzi-Pearson’s transformation* introduced in Fiagbedzi et al. (1986).

Finally, if \( A_1 = 0 \), i.e., if (1) has no state delay, then \( A = A_0, B = B_0 + e^{-A_1} B_1 \), and the matrix \( P \) becomes:

\[
P = \begin{pmatrix} 1_n & e^{A(t-t_0)} A_1 \end{pmatrix}.
\]

We then find again *Artstein’s reduction* (Artstein (1982))

\[
\begin{align*}
z(t) & = x(t) + \int_{t-h}^{t} e^{A(t-h)} B_1(t) u(t) \, dt,
\end{align*}
\]

which bijectively maps the solutions of (1) with \( A_1 = 0 \) to the solutions of (2). See Artstein (1982); Quadrat (2015).

**REFERENCES**


