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► **To cite this version:**

Wenjie Mei, Rosane Ushirobira, Denis Efimov. On nonlinear robust state estimation for generalized Persidskii systems. *Automatica*, 2022, 10.1016/j.automatica.2022.110411 . hal-03903300

**HAL Id: hal-03903300**

**<https://inria.hal.science/hal-03903300>**

Submitted on 16 Dec 2022

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# On nonlinear robust state estimation for generalized Persidskii systems <sup>1</sup>

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## Abstract

A state observer is designed for a class of generalized Persidskii systems with nonlinear measurements, state disturbances, and output noise. We assume that all nonlinearities are diagonal and belong to a sector. The robust stability and convergence conditions for the estimation error are obtained by applying the theory of input-to-output stability. These conditions are established in the form of linear matrix inequalities. We provide numerical examples to demonstrate the efficacy and performances of the proposed observer.

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## 1 Introduction

The unmeasured state estimation for dynamical systems is a fundamental engineering problem whose solution is needed in numerous control or monitor applications [17,13]. The estimation theory is relatively well developed nowadays [4], and many different observers were proposed in the literature for linear plants (e.g., the most popular methods are the Kalman filter [14] or Luenberger observer [23]) or for nonlinear systems whose models can be approximated by linear ones under certain hypotheses [18,11,12]. For example, using the first-order approximations, the principal linear approaches can be extended to nonlinear plants (the extended/unscented Kalman filter [15,33], moving horizon estimation [31]). Another related popular direction is focused on linear parameter-varying (LPV) or quasi-LPV representation of nonlinear dynamics that permits the utilization of linear estimation tools [27,21]. Many differentiation algorithms are based on the state estimation techniques developed for a chain of integrators (a linear model) [32], as the seminal high order sliding mode differentiator [20]. Methods based on linear approximations may lose their validity if the state estimation is demanded at large (or in the presence of a severe signal and parametric uncertainty). Then some canonical forms of nonlinear dynamical

systems are considered, as homogeneous [22], Lur'e [2,7,6] or Persidskii models [25], for instance, in which Lur'e systems constitute a popular benchmark.

The main step in the design of an observer is the obligatory evaluation of convergence conditions of the state estimation error and its sturdiness to given classes of uncertainties, *i.e.*, analysis of the robust stability. The Lyapunov function method is the standard framework for the stability analysis of nonlinear dynamics [17], and the absence of constructive tools for the selection of a Lyapunov function for a general nonlinear system is its main drawback (thus, the canonical forms of nonlinear models, mentioned above, with known compositions of Lyapunov functions provide an answer to this weakness).

To illustrate another important issue in the nonlinear state estimation, consider a dynamical system:

$$\dot{x}(t) = f(x(t), d(t)), \quad y(t) = h(x(t)), \quad t \geq 0,$$

where  $x(t) \in \mathbb{R}^n$ ,  $d(t) \in \mathbb{R}^m$  and  $y(t) \in \mathbb{R}^p$  are the state, the unknown input and the measured output, respectively,  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$  are known nonlinear functions. An observer for this system is often chosen as another nonlinear dynamics in a general form:

$$\dot{z}(t) = r(z(t), y(t)), \quad \hat{x}(t) = g(z(t)),$$

where  $z(t) \in \mathbb{R}^q$  is the state of the observer and  $\hat{x}(t) \in \mathbb{R}^n$  is the estimate of  $x(t)$ ,  $r : \mathbb{R}^q \times \mathbb{R}^p \rightarrow \mathbb{R}^q$  and  $g : \mathbb{R}^q \rightarrow \mathbb{R}^n$  are functions to be selected so that the estimation error  $e := x - \hat{x}$  dynamics is asymptotically stable for  $d = 0$  (*i.e.*,  $\lim_{t \rightarrow +\infty} \|e(t)\| = 0$ ) and robustly stable in

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<sup>1</sup> This work is partially supported by the Chinese Scholarship Council. It is also partially supported by the Ministry of Science and Higher Education of the Russian Federation, passport of goszadanie no. 2019-0898.

the presence of disturbances  $d \neq 0$  (frequently, the input-to-state stability (ISS) framework [36,8] is applied). For example, a popular choice is  $r(z, y) = f(z, 0) + \varrho(y - h(z))$  and  $g(z) = z$ , where the output injection term  $\varrho : \mathbb{R}^p \rightarrow \mathbb{R}^n$  is properly adjusted. For this purpose, the estimation error dynamics is analyzed, and often it is assumed that it can be described by differential equations governed by exogenous disturbances/noises  $d$ , but independent of the system state  $x$ :

$$\dot{e}(t) = \ell(e(t), d(t)) \quad (1)$$

for some  $\ell : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ , which is always the case for linear models (i.e., if  $f(x, d) = Ax + d$  and  $h(x) = Cx$  for known matrices  $A$  and  $C$  of appropriate dimensions, then  $r(z, y) = Az + L(y - Cz)$ ,  $g(z) = z$  for an observer gain  $L$ , and  $\ell(e, d) = (A - LC)e + d$ ). Such a representation can also be obtained for plants close to linear ones. Nevertheless, the independence of the dynamics of  $e$  in  $x$  can be a restrictive hypothesis, and in a general scenario, this differential equation has to take the following form:

$$\dot{e}(t) = \tilde{\ell}(e(t), x(t), d(t)) \quad (2)$$

with  $\tilde{\ell} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Then the estimation error behavior has to be analyzed together with the observed system, and uniform stability or partial stability notions come to the attention (the same convergence and stability properties are required from  $e$ , but independently in  $x$  under imposed restrictions). One of the most popular concepts for robust partial stability analysis is given in the input-to-output stability (IOS) theory [34,35] (an extension of ISS).

In the present work, the robust state estimation problem is studied for a class of nonlinear dynamical systems that can be brought to the generalized Persidskii form [10,25] (the latter paper contains the results of our preliminary investigations formulated for a simpler class of models, under more restricted stability conditions and with academic illustrations). This class of systems is close to Lur'e ones [2,37], and there is a known canonical form of Lyapunov function for a Persidskii system [3,30,16], as well as for its generalized version with multiple nonlinearities [10,24], where ISS and IOS stability conditions have been proposed in the constructive form of linear matrix inequalities (LMIs). Using these results, the robust stability for both cases of representation of the estimation error dynamics, in autonomous (1) and general (2) settings, are investigated. Numerical experiments are given for a mechanical system and network epidemics dynamics to demonstrate the performance of the designed observer.

The organization of this paper is as follows. In Section 2 the preliminaries are presented. The system under consideration and the problem statement are described in Section 3. In sections 4 and 5, the LMI-based stability conditions for cases (1) and (2) are given.

## Notation

- $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{R}_+$  represent the sets of natural, real and nonnegative real numbers, respectively. The symbol  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^n$  (and the induced matrix norm  $\|A\|$  for a matrix  $A \in \mathbb{R}^{m \times n}$ ).
- For  $p, n \in \mathbb{N}$  with  $p \leq n$ , the notation  $\overline{p, n}$  is used to represent the set  $\{p, \dots, n\}$ .
- For a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ , its maximal eigenvalue is denoted by  $\lambda_{\max}(A)$ .
- $I_n$  and  $\mathbf{O}_{m \times n}$  stand for the  $n \times n$  identity matrix and the  $m \times n$  zero matrix, respectively.
- The  $m \times n$  block diagonal matrix with matrices  $v_i \in \mathbb{R}^{m_i \times n_i}$ ,  $i \in \overline{1, N}$  along the main diagonal, where  $m = \sum_{i=1}^N m_i$  and  $n = \sum_{i=1}^N n_i$ , is denoted by  $\text{diag}(v_1, \dots, v_N)$ , and the set of  $n \times n$  diagonal matrices with nonnegative elements on the diagonal is denoted by  $\mathbb{D}_+^n \subset \mathbb{R}_+^{n \times n}$ .
- For a Lebesgue measurable function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ , define the norm  $\|u\|_{[t_1, t_2]} = \text{ess sup}_{t \in [t_1, t_2]} \|u(t)\|$  for  $[t_1, t_2] \subset \mathbb{R}_+$ . We denote by  $\mathcal{L}_\infty^m$  the space of functions  $u$  with  $\|u\|_\infty := \|u\|_{[0, \infty)} < +\infty$ .
- A continuous function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_\infty$  if it is also unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is a decreasing function going to zero for any fixed  $r > 0$ .
- For a continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ , denote by  $\nabla V(\nu)f(\nu)$  the Lie derivative of  $V$  along the vector field  $f$  evaluated at point  $\nu \in \mathbb{R}^n$ .

## 2 Preliminaries

This section follows the results of [35,36]. Consider a nonlinear dynamical system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), d(t)), \quad t \in \mathbb{R}_+, \quad \text{with } x(0) = x_0, \\ y(t) &= h(x(t)), \end{aligned} \quad (3)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $d(t) \in \mathbb{R}^m$  is the external perturbation,  $d \in \mathcal{L}_\infty^m$  and  $y(t) \in \mathbb{R}^p$  is the output vector. Moreover  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $f(0, 0) = 0$  is a locally Lipschitz continuous function and  $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$  is a continuously differentiable function. For an initial state  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ , we denote the corresponding solution of the system (3) by  $x(t, x_0, d)$ , and the corresponding output is  $y(t, x_0, d) = h(x(t, x_0, d))$ . The system (3) is called *forward complete* if for all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ , the solution  $x(t, x_0, d)$  is uniquely defined for all  $t \in \mathbb{R}_+$ . In the rest of the paper, to lighten the notation the time-dependency of variables might remain implicitly understood, so we write  $x$  for the vector  $x(t)$ .

**Definition 1** A forward complete system (3) is said to be input-to-output stable (IOS) if there exist  $\beta \in \mathcal{KL}$

and  $\gamma \in \mathcal{K}$  such that

$$\|y(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_\infty), \quad \forall t \in \mathbb{R}_+ \quad (4)$$

for any  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ . When  $h(x) = x$ , the IOS property is called input-to-state stability (ISS).

An output stability property that is close to ISS is formulated next:

**Definition 2** A forward complete system (3) is said to be state-independent input-to-output stable (SIOS) if there exist  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$\|y(t, x_0, d)\| \leq \beta(\|h(x_0)\|, t) + \gamma(\|d\|_\infty), \quad \forall t \in \mathbb{R}_+$$

for any  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ .

**Definition 3** A forward complete system (3) is said to be uniformly bounded-input-bounded-state stable (UBIBS) if there exists  $\sigma \in \mathcal{K}$  such that

$$\|x(t, x_0, d)\| \leq \sigma \circ \max\{\|x_0\|, \|d\|_\infty\}, \quad \forall t \in \mathbb{R}_+$$

for all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ .

**Definition 4** For the system (3), a smooth function  $V: \mathbb{R}^n \rightarrow \mathbb{R}_+$  is

- (1) an IOS-Lyapunov function if there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\chi \in \mathcal{K}$  and  $\alpha_3 \in \mathcal{KL}$  such that

$$\begin{aligned} \alpha_1(\|h(x)\|) &\leq V(x) \leq \alpha_2(\|x\|), \\ V(x) \geq \chi(\|d\|) &\Rightarrow \nabla V(x)f(x, d) \leq -\alpha_3(V(x), \|x\|) \end{aligned} \quad (5)$$

for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{R}^m$ .

- (2) an SIOS-Lyapunov function if there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\chi \in \mathcal{K}$  and  $\alpha_3 \in \mathcal{K}$  such that

$$\begin{aligned} \alpha_1(\|y\|) &\leq V(x) \leq \alpha_2(\|y\|), \\ \|y\| \geq \chi(\|d\|) &\Rightarrow \nabla V(x)f(x, d) \leq -\alpha_3(\|y\|) \end{aligned} \quad (6)$$

for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{R}^m$ .

**Theorem 5 [35]** A UBIBS system (3) is IOS if and only if it admits an IOS-Lyapunov function.

**Theorem 6 [35]** A forward complete system (3) is SIOS if and only if it admits a SIOS-Lyapunov function.

### 3 Problem Statement

Consider a class of systems in the following (generalized Persidskii) form:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{j=1}^M A_j f^j(H_j x(t)) + Dw(t), \\ y(t) &= \begin{bmatrix} C_0 x(t) \\ C_1 f^1(H_1 x(t)) \\ \vdots \\ C_M f^M(H_M x(t)) \end{bmatrix} + v(t), \quad t \in \mathbb{R}_+, \end{aligned} \quad (7)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector;  $A_0 \in \mathbb{R}^{n \times n}$ ,  $A_j \in \mathbb{R}^{n \times k_j}$  and  $H_j \in \mathbb{R}^{k_j \times n}$  for  $j \in \overline{1, M}$ ;  $f^j(\ell) = [f_1^j(\ell_1) \dots f_{k_j}^j(\ell_{k_j})]^\top$  with  $\ell = [\ell_1 \dots \ell_{k_j}]^\top \in \mathbb{R}^{k_j}$  for  $j \in \overline{1, M}$  are the functions ensuring the existence of the solutions of the system (7) in the forward time at least locally (to shorten further writing we define  $k_0 = n$  and  $H_0 = I_n$ );  $y(t) \in \mathbb{R}^z$  is the output available for measurements,  $z = \sum_{s=0}^M z_s$  and  $C_s \in \mathbb{R}^{z_s \times k_s}$  for  $s \in \overline{0, M}$ ;  $D \in \mathbb{R}^{n \times p}$ ;  $w(t) \in \mathbb{R}^p$ ,  $v(t) \in \mathbb{R}^z$  are the external perturbations,  $w \in \mathcal{L}_\infty^p$ ,  $v \in \mathcal{L}_\infty^z$ .

In this work, if the upper bound is smaller than the lower one for an index, then the corresponding term (in sum or a sequence) must be omitted.

Following [30], the sector restrictions on  $f^j$ ,  $j \in \overline{1, M}$  are imposed:

**Assumption 7** Assume that for any  $i \in \overline{1, k_j}$  and  $j \in \overline{1, M}$ ,  $\nu f_i^j(\nu) > 0$ , for all  $\nu \in \mathbb{R} \setminus \{0\}$ .

Under Assumption 7, with a reordering of nonlinearities and their decomposition, there exists an index  $m \in \overline{0, M}$  such that for all  $s \in \overline{1, m}$  and  $i \in \overline{1, k_s}$ ,  $\lim_{\nu \rightarrow \pm\infty} f_i^s(\nu) = \pm\infty$ . Also, there exists  $\mu \in \overline{m, M}$  such that for all  $s \in \overline{1, \mu}$ ,  $i \in \overline{1, k_s}$ , we have  $\lim_{\nu \rightarrow \pm\infty} \int_0^\nu f_i^s(r) dr = +\infty$ . If  $m = 0$ , then all nonlinearities are bounded (at least for positive or negative argument). Clearly,  $\mu \geq m$ .

**Remark 8** The Lur'e models with the sector bounded nonlinearities [38, 19] represent an example of (7) under Assumption 7 with  $M = 1$ . The observer design for Lur'e models using quadratic Lyapunov functions was considered in [2, 7, 6].

In this study, an observer for (7) is proposed in the following conventional form:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_0 \hat{x}(t) + \sum_{j=1}^M A_j f^j(H_j \hat{x}(t)) + L(y(t) - \hat{y}(t)), \\ \hat{y}(t) &= \begin{bmatrix} C_0 \hat{x}(t) \\ C_1 f^1(H_1 \hat{x}(t)) \\ \vdots \\ C_M f^M(H_M \hat{x}(t)) \end{bmatrix}, \end{aligned} \quad (8)$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is the estimation of the state  $x(t)$ ;  $L = [L_0 \ L_1 \ \dots \ L_M] \in \mathbb{R}^{n \times z}$  is the matrix gain to be designed, with  $L_s \in \mathbb{R}^{n \times z_s}$  for  $s \in \overline{0, M}$ .

As introduced above, the goal is to ensure asymptotic convergence of  $\hat{x}$  to  $x$  in the case of no perturbations and boundedness of the estimates otherwise. To reach this objective, for this observer, we will analyze two cases of expression of dynamics of the estimation error  $e = x - \hat{x}$  given in the introduction: (1) and (2). For the latter case, we will investigate IOS conditions for the common system (7), (8) for the output  $e$ , while in the former scenario, SIOS conditions of the dynamics of  $e$  will be studied (in both cases, the inputs are represented by the disturbances  $w$  and  $v$ ). Observability or detectability issues of (7) are not considered in this work, and they will not be related to the conditions of stability of  $e$ .

#### 4 IOS analysis

Note that the output stability for the system (7), (8) is equivalent to a robust state synchronization of these two Persidskii systems under the influence of perturbations  $w$  and  $v$ . Therefore, the synchronization method developed in [24] can be adopted. To this end, let us write the common dynamics of (7), (8):

$$\dot{X} = \tilde{A}_0 X + \sum_{j=1}^M \tilde{A}_j F^j(\tilde{H}_j X) + \mathcal{D}, \quad \mathcal{D} = \begin{bmatrix} Dw \\ Lv \end{bmatrix}, \quad (9)$$

where  $X = [x^\top \ \hat{x}^\top]^\top \in \mathbb{R}^{2n}$  is the extended state,  $\mathcal{D} \in \mathbb{R}^{2n}$  is the augmented disturbance and

$$\tilde{A}_s = \begin{bmatrix} A_s & \mathbf{O}_{n \times k_s} \\ L_s C_s & A_s - L_s C_s \end{bmatrix}, \quad s \in \overline{0, M},$$

$$F^j(\tilde{H}_j X) = \begin{bmatrix} f^j(H_j x) \\ f^j(H_j \hat{x}) \end{bmatrix}, \quad \tilde{H}_j = \begin{bmatrix} H_j & \mathbf{O}_{k_j \times n} \\ \mathbf{O}_{k_j \times n} & H_j \end{bmatrix}, \quad j \in \overline{1, M}$$

with the output function given by the estimation error:

$$e = \Gamma X \quad \text{with} \quad \Gamma := \begin{bmatrix} I_n & -I_n \end{bmatrix}.$$

Clearly the system (9) also yields the generalized Persidskii form.

We say that (8) is an observer for (7) if the common dynamics (9) is IOS with the inputs  $w, v$  (or  $\mathcal{D}$ ) and the output  $e$  (this corresponds to the case (2) given in the introduction). The related conditions are as follows:

**Theorem 9** *Let Assumption 7 be satisfied. Let there exist  $0 \leq P_1 = P_1^\top \in \mathbb{R}^{n \times n}$ ;  $0 \leq P_2 = P_2^\top \in \mathbb{R}^{2n \times 2n}$ ;  $\Lambda^j = \text{diag}(\Lambda_1^j, \dots, \Lambda_{2k_j}^j) \in \mathbb{D}_+^{2k_j}$  ( $j \in \overline{1, M}$ );  $\Xi^s \in \mathbb{D}_+^{2k_s}$  ( $s \in$*

*$\overline{0, M}$ ),  $\Upsilon_{0,s} \in \mathbb{D}_+^{2k_s}$  ( $s \in \overline{1, M}$ );  $\{\Upsilon_{s,r}\}_{r=s+1}^M \subset \mathbb{D}_+^{2n}$  ( $s \in \overline{1, M-1}$ );  $\Theta \in \mathbb{D}_+^{2n}$ ;  $\Psi \in \mathbb{D}_+^{2n}$ ,  $\varrho \in \mathbb{R}$  and  $0 < \Phi = \Phi^\top \in \mathbb{R}^{2n \times 2n}$  such that*

$$P_1 > 0 \text{ or } P_2^{11} - 2P_2^{12} + P_2^{22} + \varrho P_1 > 0 \quad (10)$$

$$\text{or } \sum_{j=1}^{\mu} \tilde{\Lambda}^j + \varrho P_1 > 0;$$

$$P_2 \leq \Theta, \quad Q = Q^\top = (Q_{a,b})_{a,b=1}^{M+2} \leq 0, \quad (11)$$

where  $P_2^{11}, P_2^{12}, P_2^{22} \in \mathbb{R}^{n \times n}$ ,  $P_2 = \begin{bmatrix} P_2^{11} & P_2^{12} \\ P_2^{12} & P_2^{22} \end{bmatrix}$ ,  $\tilde{\Lambda}^j = H_j^\top \text{diag}(\Lambda_1^j, \dots, \Lambda_{k_j}^j) H_j + H_j^\top \text{diag}(\Lambda_{k_j+1}^j, \dots, \Lambda_{2k_j}^j) H_j$ ,

$$Q_{1,1} = \tilde{A}_0^\top P_\Gamma + P_\Gamma \tilde{A}_0 + \Xi^0 + \Gamma^\top \Psi \Gamma, \quad P_\Gamma = \Gamma^\top P_1 \Gamma + P_2;$$

$$Q_{j+1,j+1} = \tilde{A}_j^\top \tilde{H}_j^\top \Lambda^j + \Lambda^j \tilde{H}_j \tilde{A}_j + \Xi^j, \quad j \in \overline{1, M};$$

$$Q_{1,j+1} = P_\Gamma \tilde{A}_j + \tilde{A}_0^\top \tilde{H}_j^\top \Lambda^j + \tilde{H}_j^\top \Upsilon_{0,j}, \quad j \in \overline{1, M};$$

$$Q_{s+1,r+1} = \tilde{A}_s^\top \tilde{H}_r^\top \Lambda^r + \Lambda^s \tilde{H}_s \tilde{A}_r + \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r,$$

$$s \in \overline{1, M-1}, r \in \overline{s+1, M};$$

$$Q_{1,M+2} = P_\Gamma, \quad Q_{M+2,M+2} = -\Phi,$$

$$Q_{j+1,M+2} = \Lambda^j \tilde{H}_j, \quad j \in \overline{1, M},$$

and for some  $\xi > 0$ :

$$P_1 \leq \xi \Psi, \quad \Theta + \sum_{j=1}^M \tilde{H}_j^\top \Lambda^j \tilde{H}_j \leq \xi \left( \sum_{k=0}^m \tilde{H}_k^\top \Xi^k \tilde{H}_k \right. \\ \left. + 2 \sum_{r=1}^m \tilde{H}_r^\top \Upsilon_{0,r} \tilde{H}_r + 2 \sum_{s=1}^{m-1} \sum_{r=s+1}^m \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r \right). \quad (12)$$

Then a forward complete system (9) is IOS.

**PROOF.** Consider a candidate Lyapunov function

$$V(X) = X^\top P_\Gamma X + 2 \sum_{j=1}^M \sum_{i=1}^{2k_j} \Lambda_i^j \int_0^{\tilde{H}_j^i X} F_i^j(\tau) d\tau, \quad (13)$$

where  $\tilde{H}_j^i$  is the  $i^{\text{th}}$  row of the matrix  $\tilde{H}_j$ . Let us check the lower bound for  $V$  from the first condition in (5), which is valid if one of the following inequalities are satisfied:

$$X^\top P_2 X > 0, \quad X^\top \left( \sum_{j=1}^{\mu} \tilde{H}_j^\top \Lambda^j \tilde{H}_j \right) X > 0$$

under the constraints  $X^\top \Gamma^\top P_1 \Gamma X = 0$ ,  $\Gamma X \neq 0$ , *i.e.*, the matrix  $P_2$  or  $\sum_{j=1}^{\mu} \tilde{H}_j^\top \Lambda^j \tilde{H}_j$  should be positive definite on the subset of  $e \neq 0$  belonging to the kernel of  $P_1$  (the summation is for  $\mu$  terms since only unbounded nonlinearities are considered for radial unboundedness of  $V$ ). Hence, if  $P_1 > 0$ , then the constraints are self-excluding, and the case  $P_1 \geq 0$  is further considered. To simplify the formulation, define new coordinates

$Z = SX$ ,  $S = \begin{bmatrix} I_n & -I_n \\ I_n & I_n \end{bmatrix}$ , then  $e = \begin{bmatrix} I_n & \mathbf{0}_{n \times n} \end{bmatrix} Z$ ,  $S$  is not singular, and the above constraints take the form:

$$Z^\top \begin{bmatrix} I_n & \mathbf{0}_{n \times n} \end{bmatrix}^\top P_1 \begin{bmatrix} I_n & \mathbf{0}_{n \times n} \end{bmatrix} Z = 0, \begin{bmatrix} I_n & \mathbf{0}_{n \times n} \end{bmatrix} Z \neq 0,$$

which can be equivalently rewritten with respect to the first component of  $Z$  (the error  $e$ ),  $e^\top P_1 e = 0$ ,  $e \neq 0$ , together with the conditions to check:

$$e^\top (P_2^{11} - 2P_2^{12} + P_2^{22}) e > 0, e^\top \left( \sum_{j=1}^{\mu} \tilde{\Lambda}^j \right) e > 0$$

where  $P_2 = \begin{bmatrix} P_2^{11} & P_2^{12} \\ P_2^{12} & P_2^{22} \end{bmatrix}$  for  $P_2^{11}, P_2^{12}, P_2^{22} \in \mathbb{R}^{n \times n}$ , and

$$\tilde{\Lambda}^j = H_j^\top \text{diag}(\Lambda_1^j, \dots, \Lambda_{k_j}^j) H_j + H_j^\top \text{diag}(\Lambda_{k_j+1}^j, \dots, \Lambda_{2k_j}^j) H_j.$$

Using Finsler's Lemma [9], these conditions follow the first LMI given in the formulation of the theorem. So, in such a case there are  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that

$$\alpha_1(\|e\|) \leq V(X) \leq \alpha_2(\|X\|), \quad (14)$$

where

$$\alpha_2(\tau) \leq \lambda_{\max}(P_\Gamma) \tau^2 + \left( \sum_{j=1}^M 4k_j \right) \max_{\substack{j \in \{1, \dots, M\} \\ i \in \{1, 2k_j\}}} \left\{ \Lambda_i^j \int_0^{\|\tilde{H}_j\|^\tau} F_i^j(\gamma) d\gamma \right\},$$

then the first condition in (5) is verified. Next, consider the derivative of  $V$  (denote  $\dot{V} = \nabla V(X) \dot{X}$ ):

$$\begin{aligned} \dot{V} &= \dot{X}^\top P_\Gamma X + X^\top P_\Gamma \dot{X} + 2 \sum_{j=1}^M \dot{X}^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) \\ &= X^\top (\tilde{A}_0^\top P_\Gamma + P_\Gamma \tilde{A}_0) X + \left( \sum_{j=1}^M F^j(\tilde{H}_j X)^\top \tilde{A}_j^\top \right) P_\Gamma X \\ &\quad + X^\top P_\Gamma \sum_{j=1}^M \tilde{A}_j F^j(\tilde{H}_j X) + 2X^\top P_\Gamma \mathcal{D} + \\ &\quad 2 \sum_{j=1}^M \left( X^\top \tilde{A}_0^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) + \mathcal{D}^\top \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) + \right. \\ &\quad \left. \left( \sum_{s=1}^M F^s(\tilde{H}_s X)^\top \tilde{A}_s^\top \right) \tilde{H}_j^\top \Lambda^j F^j(\tilde{H}_j X) \right). \end{aligned}$$

Therefore, under (11) we obtain

$$\begin{aligned} \dot{V} &= \begin{bmatrix} X \\ F^1(\tilde{H}_1 X) \\ \vdots \\ F^M(\tilde{H}_M X) \\ \mathcal{D} \end{bmatrix}^\top Q \begin{bmatrix} X \\ F^1(\tilde{H}_1 X) \\ \vdots \\ F^M(\tilde{H}_M X) \\ \mathcal{D} \end{bmatrix} - X^\top (\Gamma^\top \Psi \Gamma + \Xi^0) X \\ &\quad - \sum_{j=1}^M F^j(\tilde{H}_j X)^\top \Xi^j F^j(\tilde{H}_j X) - 2 \sum_{j=1}^M X^\top \tilde{H}_j^\top \Upsilon_{0,j} F^j(\tilde{H}_j X) \\ &\quad - 2 \sum_{s=1}^{M-1} \sum_{r=s+1}^M F^s(\tilde{H}_s X)^\top \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r F^r(\tilde{H}_r X) + \mathcal{D}^\top \Phi \mathcal{D} \\ &\leq -X^\top (\Gamma^\top \Psi \Gamma + \Xi^0) X - \sum_{j=1}^M F^j(\tilde{H}_j X)^\top \Xi^j F^j(\tilde{H}_j X) \\ &\quad - 2 \sum_{j=1}^M X^\top \tilde{H}_j^\top \Upsilon_{0,j} F^j(\tilde{H}_j X) \\ &\quad - 2 \sum_{s=1}^{M-1} \sum_{r=s+1}^M F^s(\tilde{H}_s X)^\top \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r F^r(\tilde{H}_r X) + \mathcal{D}^\top \Phi \mathcal{D}. \end{aligned}$$

Due to the form of the function  $V$ , there exists  $\alpha \in \mathcal{K}_\infty$  such that

$$\begin{aligned} \alpha(V(X)) &\leq X^\top (\Gamma^\top \Psi \Gamma + \Xi^0) X + \sum_{j=1}^M F^j(\tilde{H}_j X)^\top \Xi^j F^j(\tilde{H}_j X) \\ &\quad + 2 \sum_{j=1}^M X^\top \tilde{H}_j^\top \Upsilon_{0,j} F^j(\tilde{H}_j X) \\ &\quad + 2 \sum_{s=1}^{M-1} \sum_{r=s+1}^M F^s(\tilde{H}_s X)^\top \tilde{H}_s^\top \tilde{H}_s \Upsilon_{s,r} \tilde{H}_r^\top \tilde{H}_r F^r(\tilde{H}_r X) \end{aligned}$$

under the conditions (12), which have to be verified for some  $\xi > 0$  (only the first  $m$  nonlinearities and the quadratic term are radially unbounded). Finally, under the conditions of the theorem:

$$\dot{V} \leq -\alpha(V) + \mathcal{D}^\top \Phi \mathcal{D}$$

for all  $X \in \mathbb{R}^{2n}$  and  $\mathcal{D} \in \mathbb{R}^{2n}$ . Hence, the second relation in (5) can be recovered:

$$V \geq \alpha^{-1}(2\mathcal{D}^\top \Phi \mathcal{D}) \Rightarrow \dot{V} \leq -\frac{1}{2}\alpha(V),$$

and the IOS property is guaranteed (if the right-hand side of the estimate for  $\dot{V}$  is in the form of a function of class  $\mathcal{X}$  (as above) and not of class  $\mathcal{KL}$  (as in (5)), then UBIBS property can be omitted, and forward completeness is enough).

We can require a more strict property for the nonlinearities of the system (7), which can be viewed as an incremental passivity condition [29,39]:

**Assumption 10** For any  $j \in \overline{1, M}$ :

$$X^\top \Gamma^\top \Gamma \tilde{H}_j^\top F^j(\tilde{H}_j X) > 0, \forall X \in \mathbb{R}^{2n} \setminus \{Z \in \mathbb{R}^{2n} : \Gamma Z = 0\}.$$

Under these additional restrictions imposed on the system (9), a relaxed stability result can be obtained:

**Corollary 11** Let assumptions 7 and 10 be satisfied. If there exist  $0 \leq P_1 = P_1^\top \in \mathbb{R}^{n \times n}$ ;  $0 \leq P_2 = P_2^\top \in \mathbb{R}^{2n \times 2n}$ ;  $\{\Xi^k\}_{k=0}^M$ ,  $\{\Upsilon_{s,r}\}_{r=s+1}^M \subset \mathbb{D}_+^n$  ( $s \in \overline{0, M-1}$ ),  $\{\Lambda^j\}_{j=1}^M \subset \mathbb{D}_+^{2k_j}$  and  $\varrho \in \mathbb{R}$  such that

$$P_2 + \varrho \sum_{q=1}^{\mu} \tilde{H}_q^\top \Lambda^q \tilde{H}_q^\top > 0; Q = Q^\top \leq 0,$$

$$\begin{aligned} \text{where } Q_{1,1} &= \tilde{A}_0^\top P_\Gamma + P_\Gamma \tilde{A}_0 + \Gamma^\top \Xi^0 \Gamma, P_\Gamma = \Gamma^\top P_1 \Gamma + P_2, \\ Q_{j+1,j+1} &= \tilde{A}_j^\top \tilde{H}_j^\top \Lambda^j + \Lambda^j \tilde{H}_j \tilde{A}_j + \tilde{H}_j \Gamma^\top \Xi^j \Gamma \tilde{H}_j^\top, j \in \overline{1, M}, \\ Q_{1,j+1} &= P_\Gamma \tilde{A}_j + \tilde{A}_0^\top \tilde{H}_j^\top \Lambda^j + \tilde{H}_j \Gamma^\top \Upsilon_{0,j} \Gamma, j \in \overline{1, M}, \\ Q_{s+1,r+1} &= \tilde{A}_s^\top \tilde{H}_r^\top \Lambda^r + \Lambda^s \tilde{H}_s \tilde{A}_r + \tilde{H}_s \Gamma^\top \Upsilon_{s,r} \tilde{H}_r^\top, \\ & s \in \overline{1, M-1}, r \in \overline{s+1, M}, \end{aligned}$$

and

$$\sum_{k=0}^M \Xi^k + 2 \sum_{s=0}^{M-1} \sum_{r=s+1}^M \Upsilon_{s,r} > 0.$$

Then the system (9) with  $\|\mathcal{D}\|_\infty = 0$  has globally bounded trajectories and  $\lim_{t \rightarrow +\infty} \|e(t)\| = 0$ .

**PROOF.** Consider the Lyapunov function (13). By Finsler's Lemma [5], the first LMI of the corollary implies positive definiteness of  $V$  with respect to  $X$ , then

$$\alpha_1(\|X\|) \leq V(X) \leq \alpha_2(\|X\|)$$

for some functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ . Repeating the steps of the proof of Theorem 9, the derivative of  $V$  with  $\mathcal{D} = 0$  can be upper bounded as follows:

$$\begin{aligned} \dot{V} &\leq -X^\top \Gamma^\top \Gamma \Xi^0 \Gamma X - \sum_{j=1}^M F^j(\tilde{H}_j X)^\top \tilde{H}_j \Gamma^\top \Xi^j \Gamma \tilde{H}_j^\top F^j(\tilde{H}_j X) \\ &\quad - 2 \sum_{j=1}^M X^\top \Gamma^\top \Gamma \Upsilon_{0,j} \tilde{H}_j^\top F^j(\tilde{H}_j X) \\ &\quad - 2 \sum_{s=1}^{M-1} \sum_{r=s+1}^M F^s(\tilde{H}_s X)^\top \tilde{H}_s \Gamma^\top \Upsilon_{s,r} \tilde{H}_r^\top F^r(\tilde{H}_r X) \end{aligned}$$

under the condition  $Q \leq 0$ . Since

$$\sum_{k=0}^M \Xi^k + 2 \sum_{s=0}^{M-1} \sum_{r=s+1}^M \Upsilon_{s,r} > 0$$

and due to assumptions 7 and 10, there exists  $\alpha \in \mathcal{K}$  such that  $\dot{V} \leq -\alpha(\|e\|)$ , which implies boundedness of all solutions in (9) with  $\|\mathcal{D}\|_\infty = 0$ . Applying LaSalle arguments [17], we obtain for all initial conditions,  $\lim_{t \rightarrow +\infty} \|e(t)\| = 0$ .

Previously, the observer gain  $L$  was assumed to be given. To find the gain as a solution of LMI, the next corollary considers an equivalent expression of (9):

$$\dot{X} = (\bar{A}_0 + W L_0 \bar{C}_0) X + K \bar{F}(\mathcal{H} X) + \begin{bmatrix} D w \\ L v \end{bmatrix}; \quad (15)$$

$$\begin{aligned} \text{with } \bar{A}_s &= \text{diag}(A_s, A_s), s \in \overline{0, M}; W = \begin{bmatrix} \mathbf{O}_{n \times n} & I_n \end{bmatrix}^\top; \\ \bar{F}(\mathcal{H} X) &= [(F^1(\tilde{H}_1 X))^\top \dots (F^M(\tilde{H}_M X))^\top]^\top, \\ \mathcal{H} &= [\tilde{H}_1^\top \dots \tilde{H}_M^\top]^\top; \bar{C}_s = [C_s \quad -C_s], s \in \overline{0, M}; \\ K &= [(\bar{A}_1 + W L_1 \bar{C}_1) \dots (\bar{A}_M + W L_M \bar{C}_M)]. \end{aligned}$$

The conditions of Theorem 9 can then be developed:

**Corollary 12** Let Assumption 7 be satisfied and  $H_j = I_n$  for  $j \in \overline{1, M}$ . If there exist  $0 < P_1 = P_1^\top \in \mathbb{R}^{n \times n}$ ;  $0 < P_2 = P_2^\top \in \mathbb{R}^{2n \times 2n}$ ;  $\{\Xi^j\}_{j=1}^M \subset \mathbb{D}_+^{2n}$ ,  $0 < \Phi_w = \Phi_w^\top \in \mathbb{R}^{p \times p}$ ,  $0 < \Phi_v = \Phi_v^\top \in \mathbb{R}^{z \times z}$  and  $L_j \in \mathbb{R}^{n \times z_j}$  ( $j \in \overline{1, M}$ ) such that  $G = G^\top = (G_{a,b})_{a,b=1}^{M+3} \leq 0$  where

$$\begin{aligned} P_\Gamma &= \Gamma^\top P_1 \Gamma + P_2; G_{1,1} = P_\Gamma^{-1} \bar{A}_0^\top + \bar{A}_0 P_\Gamma^{-1} + P_\Gamma^{-1}, \\ G_{j+1,j+1} &= \bar{A}_j^\top + \bar{A}_j + \bar{C}_j^\top L_j^\top W^\top + W L_j \bar{C}_j + \Xi^j, j \in \overline{1, M}, \\ G_{1,j+1} &= \bar{A}_j + W L_j \bar{C}_j + P_\Gamma^{-1} \bar{A}_0^\top, j \in \overline{1, M}, \\ G_{s+1,r+1} &= \bar{A}_s^\top + \bar{C}_s^\top L_s^\top W^\top + \bar{A}_r + W L_r \bar{C}_r, \\ & s \in \overline{1, M-1}, r \in \overline{s+1, M}, \end{aligned}$$

$$G_{s+1, M+2} = [D^\top \quad \mathbf{O}_{p \times n}]^\top, s \in \overline{0, M},$$

$$G_{s+1, M+3} = \begin{bmatrix} \mathbf{O}_{z \times n} & \mathbf{O}_{n \times z_0} & L_1 & \dots & L_M \end{bmatrix}^\top, s \in \overline{0, M},$$

$$G_{M+2, M+2} = -\Phi_w, G_{M+3, M+3} = -\Phi_v, G_{M+2, M+3} = \mathbf{O}_{p \times z},$$

then a forward complete system (9) is IOS for the observer gain  $L = \begin{bmatrix} \mathbf{O}_{n \times z_0} & L_1 & \dots & L_M \end{bmatrix}$ .

**PROOF.** Consider Theorem 9 and its proof under substitutions  $H_j \rightarrow I_n$  ( $j \in \overline{1, M}$ ),  $\Lambda^j \rightarrow I_{2n}$  ( $j \in \overline{1, M}$ ),  $\Upsilon_{s,r} \rightarrow \mathbf{O}_{n \times n}$  ( $s \in \overline{0, M-1}, r \in \overline{s+1, M}$ ),  $\Theta \rightarrow P_2$ ,  $\Xi^0 \rightarrow P_2$ ,  $\Psi \rightarrow P_1$ ,  $\Phi \rightarrow \text{diag}(\Phi_w, \Phi_v)$  and  $L \rightarrow \begin{bmatrix} \mathbf{O}_{n \times z_0} & L_1 & \dots & L_M \end{bmatrix}$ . Since  $P_1 > 0$  and  $P_2 > 0$ , the relations from (5) about positive definiteness of the Lyapunov function  $V$  in (13) are satisfied. The time derivative of  $V$  with respect to (15) is:

$$\begin{aligned} \dot{V} &= \begin{bmatrix} X \\ F^1(X) \\ \vdots \\ F^M(X) \\ w \\ v \end{bmatrix}^\top Q_0 \begin{bmatrix} X \\ F^1(X) \\ \vdots \\ F^M(X) \\ w \\ v \end{bmatrix} - X^\top P_\Gamma X \\ &\quad - \sum_{j=1}^M F^j(X)^\top \Xi^j F^j(X) + w^\top \Phi_w w + v^\top \Phi_v v \\ &\leq -X^\top P_\Gamma X - \sum_{j=1}^M F^j(X)^\top \Xi^j F^j(X) + w^\top \Phi_w w + v^\top \Phi_v v, \end{aligned}$$

where  $Q_0 = E^\top GE$  and  $E = \text{diag}(P_\Gamma, I_{2n}, \dots, I_{2n}, I_p, I_z)$ . Since  $P_2 > 0$ , there exists  $\alpha \in \mathcal{K}_\infty$  such that

$$\alpha(V(X)) \leq X^\top P_\Gamma X + \sum_{j=1}^M F^j(X)^\top \Xi^j F^j(X).$$

Therefore,  $V \geq \alpha^{-1}(2(w^\top \Phi_w w + v^\top \Phi_v v))$  implying  $\dot{V} \leq -\frac{1}{2}\alpha(V)$ , so the IOS property is guaranteed.

Thus, to calculate  $L$  as a solution of LMIs, the conditions of Corollary 12 introduce several additional restrictions to the ones proposed in Theorem 9, where positive definiteness of  $P_1$ ,  $P_2$ , and substitution  $\Lambda^j = I_{2n}$  for  $j \in \overline{1, M}$  are the most constraining.

**Example 1** Consider a perturbed two-mass spring damper system on a horizontal plane in the form of (15) from [1]:

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -0.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 \end{bmatrix},$$

$$C_0 = C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad \phi(s) = \min\{1, |s|^{0.1}\} \text{sign}(s),$$

$$D = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad f^1(x) := \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \\ \phi(x_3) \\ \phi(x_4) \end{bmatrix}, \quad F^1(X) := \begin{bmatrix} f^1(x) \\ f^1(\hat{x}) \end{bmatrix},$$

where  $x \in \mathbb{R}^n$  for  $n = 4$  is composed of the relative position of the first mass with its velocity and the same information for the second mass. It is assumed that the position of the first mass  $x_1$  is measured, then we can also assume that  $\phi(x_1)$  is available;  $M = 1$  and the nonlinearity approximating the dry friction respects Assumption 7; for simulation  $w(t) = 0.2 \sin(t)$ ,  $v(t) = 0$  for all  $t \in \mathbb{R}_+$ ;  $X = [x^\top \hat{x}^\top]^\top \in \mathbb{R}^{2n}$  is the extended state. The selected observer gains are

$$L_0 = [1.1715 \ 1.4461 \ 0.4227 \ 0.1555]^\top, \quad L_1 = [10 \ 0 \ 0 \ 0]^\top.$$

Then the LMIs proposed in Theorem 9 are verified. A state trajectory of the system (7), and the logarithm of  $|e(t)|$  in (7), (8) for several initial conditions are presented in Fig. 1 and Fig. 2, respectively (the blue curve in Fig. 2 corresponds to the trajectory in Fig. 1). The simulation results illustrate that the behavior of the system (7) (it has nonlinearities in the state and in the output equations) is well estimated by the observer (8).

## 5 SIOS analysis

To represent the dynamics of estimation error  $e$  for (7), (8) in the form (1), let us introduce a short-hand notation

$$\delta f^j(x, \hat{x}) := f^j(H_j x) - f^j(H_j \hat{x})$$

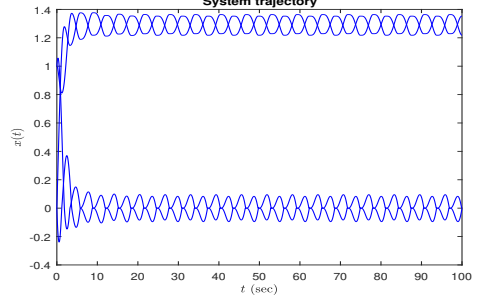


Fig. 1. The state trajectory

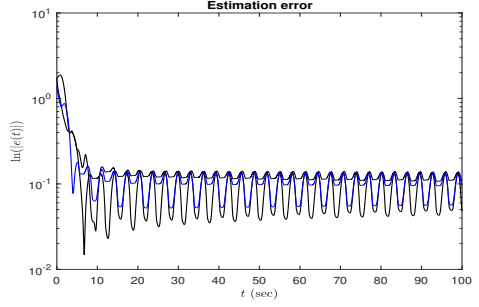


Fig. 2. The logarithm of estimation error

for all  $j \in \overline{1, M}$  (further we will often skip the arguments  $(x, \hat{x})$  of  $\delta f^j$  for brevity). Then the following properties are required in the sequel for nonlinear functions in (7) (a nonlinear version of quadratic constraints used in [2,7,6]):

**Assumption 13** Assume that there exist  $S_0^j, S_1^j, S_2^j, \Sigma_0^j, \Sigma_1^j, \Sigma_2^j \in \mathbb{D}_+^{k_j}$  and  $S_3^{j,q}, \Sigma_3^{j,q} \in \mathbb{D}_+^{n_j}$  with  $j, q \in \overline{1, M}$  such that

$$\begin{aligned} (\delta f^j)^\top \delta f^j &\leq e^\top H_j^\top S_0^j H_j e + 2e^\top H_j^\top S_1^j (\delta f^j) \\ &+ 2e^\top H_j^\top S_2^j f^j(H_j e) + 2 \sum_{q=1}^M (\delta f^j)^\top H_j S_3^{j,q} H_q^\top f^q(H_q e); \\ f^j(H_j e)^\top f^j(H_j e) &\leq e^\top H_j^\top \Sigma_0^j H_j e + 2e^\top H_j^\top \Sigma_1^j (\delta f^j) \\ &+ 2e^\top H_j^\top \Sigma_2^j f^j(H_j e) + 2 \sum_{q=1}^M (\delta f^j)^\top H_j \Sigma_3^{j,q} H_q^\top f^q(H_q e) \end{aligned}$$

for all  $x, \hat{x} \in \mathbb{R}^n$  with  $e = x - \hat{x}$ .

Since the right-hand side of the first inequality above is proportional to the estimation error  $H_j e$ , this hypothesis implies Assumption 10 (further, we recall it in the formulations of all results to avoid confusion). The dynamics of  $e$  can be written as:

$$\dot{e} = \mathcal{A}_0 e + \sum_{j=1}^M \mathcal{A}_j \delta f^j + \mathcal{D}, \quad (16)$$

with  $\mathcal{D} = Dw - Lv$  another auxiliary bounded input and



$\mathcal{A}_s = A_s - L_s C_s$ ,  $\forall s \in \overline{0, M}$ . The error dynamics (16) can be interpreted as the autonomous system (1).

**Theorem 14** *Let assumptions 7, 10 and 13 be satisfied. If there exist  $0 \leq P = P^\top \in \mathbb{R}^{n \times n}$ ;  $\Xi^0 \in \mathbb{D}_+^n$ ,  $\Lambda^j = \text{diag}(\Lambda_1^j, \dots, \Lambda_n^j)$ ,  $\Gamma_j, \Omega_j \in \mathbb{D}_+^{k_j}$  ( $j \in \overline{1, M}$ ),  $\{\Upsilon_{j,k}\}_{j,k=1}^M \subset \mathbb{D}_+^n$ ;  $0 < \Phi = \Phi^\top \in \mathbb{R}^{n \times n}$ ,  $\varrho \in \mathbb{R}$  and  $\gamma, \eta > 0$  such that*

$$\begin{aligned} P + \varrho \sum_{j=1}^{\mu} H_j^\top \Lambda^j H_j &> 0; \quad \Omega = \Omega^\top \leq 0, \quad (17) \\ \Xi^0 - \gamma \sum_{j=1}^M H_j^\top S_0^j H_j - \eta \sum_{j=1}^M H_j^\top \Sigma_0^j H_j &> 0, \\ \Gamma_j - \gamma S_1^j - \eta \Sigma_1^j &\geq 0, \\ \Omega_j - \gamma S_2^j - \eta \Sigma_2^j &\geq 0, \quad \Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k} \geq 0, \end{aligned}$$

where  $\Omega = (\Omega_{a,b})_{a,b=1}^4$ , with

$$\begin{aligned} \Omega_{1,1} &= \mathcal{A}_0^\top P + P \mathcal{A}_0 + \Xi^0, \quad \Omega_{2,2} = -\gamma I_{nM}, \quad \Omega_{1,2} = P \mathcal{A} + \bar{\Gamma}, \\ \Omega_{1,3} &= \mathcal{A}_0^\top \Lambda + \Omega, \quad \Omega_{2,3} = \mathcal{A}^\top \Lambda + \Upsilon, \quad \Omega_{3,3} = -\eta I_{nM}; \\ \Omega_{1,4} &= P, \quad \Omega_{2,4} = \mathbf{O}_{nM \times n}, \quad \Omega_{3,4} = \Lambda^\top, \quad \Omega_{4,4} = -\Phi, \\ \mathcal{A} &= [\mathcal{A}_1 \ \dots \ \mathcal{A}_M], \quad \bar{\Gamma} = [H_1^\top \Gamma_1 \ \dots \ H_M^\top \Gamma_M], \\ \Upsilon &= (H_j \Upsilon_{j,k} H_k^\top)_{j,k=1}^M, \\ \Lambda &= [H_1^\top \Lambda^1 \ \dots \ H_M^\top \Lambda^M], \quad \Omega = [H_1^\top \Omega_1 \ \dots \ H_M^\top \Omega_M], \end{aligned}$$

then the system (7), (8) is SIOS with respect to the estimation error  $e$ .

**PROOF.** Consider a candidate Lyapunov function:

$$V(e) = e^\top P e + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \Lambda_i^j \int_0^{H_j^i e_i} f_i^j(\tau) d\tau, \quad (18)$$

where  $H_j^i$  is the  $i^{\text{th}}$  row of the matrix  $H_j$ . Let us check the properties given in (6). Finsler's Lemma [5] and the first condition in (17) imply that the matrix  $P + \sum_{j=1}^{\mu} H_j^\top \Lambda^j H_j$  is positive definite, which ensures required definiteness of  $V$ . The time derivative of  $V$  for (16) admits the following representation:

$$\begin{aligned} \dot{V} &= \dot{e}^\top P e + e^\top P \dot{e} + 2 \dot{e}^\top \sum_{j=1}^M H_j^\top \Lambda^j f^j(H_j e) \\ &= \mathbf{v} \tilde{\mathcal{Q}} \mathbf{v}^\top + 2 \mathcal{D}^\top P e + 2 \mathcal{D}^\top \sum_{j=1}^M H_j^\top \Lambda^j f^j(H_j e), \end{aligned}$$

$$\begin{aligned} \text{where } \mathbf{v} &= \begin{bmatrix} e^\top & (\delta f^1)^\top & \dots & (\delta f^M)^\top \\ (f^1(H_1 e))^\top & \dots & (f^M(H_M e))^\top \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{Q}} &= \tilde{\mathcal{Q}}^\top = (\tilde{\mathcal{Q}}_{a,b})_{a,b=1}^3; \\ \tilde{\mathcal{Q}}_{1,1} &= \mathcal{A}_0^\top P + P \mathcal{A}_0, \quad \tilde{\mathcal{Q}}_{2,2} = \mathbf{O}_{\kappa \times \kappa}, \\ \tilde{\mathcal{Q}}_{1,2} &= P \mathcal{A}, \quad \tilde{\mathcal{Q}}_{1,3} = \mathcal{A}_0^\top \Lambda, \quad \tilde{\mathcal{Q}}_{2,3} = \mathcal{A}^\top \Lambda, \quad \tilde{\mathcal{Q}}_{3,3} = \mathbf{O}_{\kappa \times \kappa}, \end{aligned}$$

with  $\kappa = \sum_{j=1}^M k_j$ . Therefore, using the matrices introduced in Theorem 14, we have:

$$\begin{aligned} \dot{V} &= \mathbf{v}_{\mathcal{D}} \Omega \mathbf{v}_{\mathcal{D}}^\top + \gamma \sum_{j=1}^M (\delta f^j)^\top \delta f^j + \eta \sum_{j=1}^M f^j(H_j e)^\top f^j(H_j e) \\ &\quad - e^\top \Xi^0 e - 2 \sum_{j=1}^M e^\top H_j \Gamma_j (\delta f^j) - 2 \sum_{j=1}^M e^\top H_j \Omega_j f^j(H_j e) \\ &\quad - 2 \sum_{j=1}^M \sum_{k=1}^M (\delta f^j)^\top H_j \Upsilon_{j,k} H_k^\top f^k(H_k e) + \mathcal{D}^\top \Phi \mathcal{D}, \end{aligned}$$

where

$$\mathbf{v}_{\mathcal{D}} = [e^\top \ (\delta f^1)^\top \ \dots \ (\delta f^M)^\top \ (f^1(H_1 e))^\top \ \dots \ (f^M(H_M e))^\top \ \mathcal{D}^\top]^\top.$$

Since  $\Omega \leq 0$  due to (17) and applying Assumption 13,

$$\begin{aligned} \dot{V} &\leq -e^\top (\Xi^0 - \gamma \sum_{j=1}^M H_j^\top S_0^j H_j - \eta \sum_{j=1}^M H_j^\top \Sigma_0^j H_j) e \\ &\quad - 2 \sum_{j=1}^M e^\top H_j^\top (\Gamma_j - \gamma S_1^j - \eta \Sigma_1^j) (\delta f^j) \\ &\quad - 2 \sum_{j=1}^M e^\top H_j^\top (\Omega_j - \gamma S_2^j - \eta \Sigma_2^j) f^j(H_j e) \\ &\quad - 2 \sum_{j=1}^M \sum_{k=1}^M (\delta f^j)^\top H_j (\Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k}) H_k^\top f^k(H_k e) \\ &\quad + \mathcal{D}^\top \Phi \mathcal{D}. \end{aligned}$$

According to Theorem 6, to ensure the SIOS property of (7), (16) the right-hand side of the above estimate should be a positive definite and radially unbounded function of the error  $e$ . The corresponding term, which can guarantee these characteristics, is  $e^\top (\Xi^0 - \gamma \sum_{j=1}^M H_j^\top S_0^j H_j - \eta \sum_{j=1}^M H_j^\top \Sigma_0^j H_j) e$ . Therefore, if the conditions of Theorem 14 are satisfied, then we can substantiate that the system (7), (16) is SIOS with respect to estimation error  $e$  as desired.

**Remark 15** *Under an assumption that  $H_j = I_n$  for  $j \in \overline{1, M}$  and there exists  $\alpha \in \mathcal{K}_\infty$  such that  $\forall j \in \overline{1, M}$ :*

$$(\delta f^j)^\top \delta f^j \geq \alpha(\|e\|), \quad f^j(e)^\top f^j(e) \geq \alpha(\|e\|)$$

for all  $x, \hat{x} \in \mathbb{R}^n$  with  $e = x - \hat{x}$  (i.e., the functions  $\delta f^j$  and  $f^j$  are radially unbounded in terms of the estimation

error  $e$ ), the conditions (17) can be relaxed:

$$\begin{aligned}
& P + \varrho \sum_{j=1}^{\mu} \Lambda^j > 0, \quad \Omega \leq 0; \\
& \Xi^0 - \gamma \sum_{j=1}^M S_0^j - \eta \sum_{j=1}^M \Sigma_0^j \geq 0, \quad \Gamma_j - \gamma S_1^j - \eta \Sigma_1^j \geq 0, \\
& \Omega_j - \gamma S_2^j - \eta \Sigma_2^j \geq 0, \quad \Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k} \geq 0, \\
& \Xi^0 - \gamma \sum_{j=1}^M S_0^j - \eta \sum_{j=1}^M \Sigma_0^j + 2 \sum_{j=1}^M (\Gamma_j - \gamma S_1^j - \eta \Sigma_1^j) \\
& + 2 \sum_{j=1}^M (\Omega_j - \gamma S_2^j - \eta \Sigma_2^j) + 2 \sum_{j=1}^M \sum_{k=1}^M (\Upsilon_{j,k} - \gamma S_3^{j,k} - \eta \Sigma_3^{j,k}) > 0.
\end{aligned}$$

**Remark 16** If there exist  $\Delta_j = \text{diag}(\Delta_j^1, \dots, \Delta_j^{k_j}) \in \mathbb{D}_+^{k_j}$ ,  $\Delta_j^i \in \{0, 1\}$  for  $i \in \overline{1, k_j}$  and  $\Pi_j \in \mathbb{R}^{k_j \times z_0}$  such that  $\Delta_j H_j = \Pi_j C_0$  for  $j \in \overline{1, M}$  (i.e., a part of the argument of the nonlinearity is measured by the linear components of the output), then the observer (8) can be extended:

$$\begin{aligned}
\dot{\hat{x}}(t) &= A_0 \hat{x}(t) + \sum_{j=1}^M A_j f^j(H_j \hat{x}(t)) + L(y(t) - \hat{y}(t)) \\
&+ \sum_{j=1}^M \delta_j H_j^\top \Delta_j f^j(\Pi_j y_0(t) - \Pi_j C_0 \hat{x}(t)), \quad (19)
\end{aligned}$$

where  $y_0(t) \in \mathbb{R}^{z_0}$  is the first  $z_0$  elements of the output  $y(t)$ ,  $\delta_j > 0$  are tuning parameters. The error dynamics (16) is:

$$\dot{e} = A_0 e + \sum_{j=1}^M A_j \delta f^j - \sum_{j=1}^M \delta_j H_j^\top \Delta_j f^j(\Delta_j H_j e) + \mathcal{D}. \quad (20)$$

Clearly, the terms  $-\delta_j H_j^\top \Delta_j f^j(\Delta_j H_j e)$  are stabilizing and allow Assumption 13 to be relaxed (there is no need in the upper bound for the nonlinearities in these new items).

In Theorem 14, the observer gains  $L_s$ ,  $s \in \overline{0, M}$  are assumed to be given. To find these gains as solutions of LMIs, the following equivalent representation of the error dynamics (16) will be used:

$$\dot{e} = (A_0 - L_0 C_0)e + (A - \bar{L}\bar{C})\delta f + Dw - Lv, \quad (21)$$

where we define the block matrices

$$\begin{aligned}
A &= [A_1 \dots A_M], \quad \bar{L} = [L_1 \dots L_M], \\
\bar{C} &= \text{diag}(C_1, \dots, C_M), \quad \delta f = [(\delta f^1)^\top \dots (\delta f^M)^\top]^\top.
\end{aligned}$$

We need (21) for compactness of notation in the next corollary:

**Corollary 17** Let assumptions 7, 10 and 13 with  $S_2^j = S_3^{j,k} = \Sigma_2^j = \Sigma_3^{j,k} = \mathbf{O}_{n \times n}$  and  $H_j = I_n$  for  $j, k \in \overline{1, M}$  be satisfied. If there exist  $0 < P = P^\top \in \mathbb{R}^{n \times n}$ ;  $\Xi^0 \in \mathbb{D}_+^n$ ;  $\{\Gamma_j\}_{j=1}^M \subset \mathbb{D}_+^n$ ;  $U_s \in \mathbb{R}^{n \times z_s}$  ( $s \in \overline{0, M}$ );  $0 < \Phi_w = \Phi_w^\top \in \mathbb{R}^{p \times p}$ ;  $0 < \Phi_v = \Phi_v^\top \in \mathbb{R}^{z \times z}$  and  $\gamma, \eta > 0$  such that

$$\begin{aligned}
P &\leq I_n, \quad \tilde{G} = \tilde{G}^\top = (\tilde{G}_{a,b})_{a,b=1}^5 \leq 0, \quad (22) \\
\Xi^0 - \gamma \sum_{j=1}^M S_0^j - \eta \sum_{j=1}^M \Sigma_0^j &> 0, \quad \Gamma_j - \gamma S_1^j - \eta \Sigma_1^j \geq 0,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{G}_{1,1} &= A_0^\top P + P A_0 - C_0^\top U_0^\top - U_0 C_0 + \Xi^0, \\
\tilde{G}_{1,2} &= P A - \bar{U} \bar{C} + \bar{\Gamma}; \quad \tilde{G}_{1,3} = A_0^\top P J - C_0^\top U_0^\top J, \\
\tilde{G}_{1,4} &= P D, \quad \tilde{G}_{1,5} = -U, \quad \tilde{G}_{2,2} = -\gamma I_{nM}, \quad \tilde{G}_{2,3} = A^\top P J \\
&- \bar{C}^\top \bar{U}^\top J, \quad \tilde{G}_{2,4} = \mathbf{O}_{nM \times p}, \quad \tilde{G}_{2,5} = \mathbf{O}_{nM \times z}, \quad \tilde{G}_{3,3} = -\eta I_{nM}, \\
\tilde{G}_{3,4} &= J^\top P D, \quad \tilde{G}_{3,5} = -J^\top U, \quad \tilde{G}_{4,4} = -\Phi_w, \quad \tilde{G}_{4,5} = \mathbf{O}_{p \times z}, \\
\tilde{G}_{5,5} &= -\Phi_v, \quad \bar{U} = [U_1 \dots U_M], \quad U = [U_0 \quad \bar{U}], \\
\bar{\Gamma} &= [\Gamma_1 \dots \Gamma_M], \quad J = [I_n \dots I_n] \in \mathbb{R}^{n \times nM},
\end{aligned}$$

then the system (7), (8) is SIOS with respect to the estimation error  $e$  with the observer gains  $L_s = P^{-1} U_s$  for all  $s \in \overline{0, M}$ .

**PROOF.** Consider a candidate Lyapunov function  $V$  from (18) with  $H_j = \Lambda^j = I_n$  for  $j \in \overline{1, M}$ , which is positive definite and radially unbounded since  $P > 0$ , and whose time derivative for (21) admits the representation:

$$\begin{aligned}
\dot{V} &= \dot{e}^\top P e + e^\top P \dot{e} + 2 \sum_{j=1}^M \sum_{i=1}^n f_i^j(e_i) \dot{e}_i \\
&= \mathbf{v} \hat{\mathbf{Q}} \mathbf{v}^\top + \gamma \sum_{j=1}^M (\delta f^j)^\top \delta f^j + \eta \sum_{j=1}^M f^j(e)^\top f^j(e) \\
&- e^\top \Xi^0 e - 2 \sum_{j=1}^M e^\top \Gamma_j (\delta f^j) + w^\top \Phi_w w + v^\top \Phi_v v, \quad \text{where} \\
\mathbf{v} &= [e^\top (\delta f^1)^\top \dots (\delta f^M)^\top (f^1(e))^\top \dots (f^M(e))^\top w^\top v^\top]^\top,
\end{aligned}$$

$\hat{\mathbf{Q}} \leq \tilde{E} \tilde{G} \tilde{E}$ ,  $\tilde{E} = \text{diag}(I_n, I_{nM}, P^{-1}, \dots, P^{-1}, I_p, I_z)$ , by setting  $U_s := P L_s$  for all  $s \in \overline{0, M}$  and substituting  $P^2 \leq I_n$ . Therefore, it holds that  $\hat{\mathbf{Q}} \leq 0$  if and only if  $\tilde{G} \leq 0$ . If  $\tilde{G} \leq 0$  and applying Assumption 13 with  $S_2^j = S_3^{j,k} = \Sigma_2^j = \Sigma_3^{j,k} = \mathbf{O}_{n \times n}$  for  $j, k \in \overline{1, M}$ , we have:

$$\begin{aligned}
\dot{V} &\leq -e^\top (\Xi^0 - \gamma \sum_{j=1}^M S_0^j - \eta \sum_{j=1}^M \Sigma_0^j) e \\
&- 2 \sum_{j=1}^M e^\top (\Gamma_j - \gamma S_1^j - \eta \Sigma_1^j) (\delta f^j) + w^\top \Phi_w w + v^\top \Phi_v v.
\end{aligned}$$

Following the proof of Theorem 14, the corresponding term guaranteeing SIOS property is  $e^\top (\Xi^0 -$

$\gamma \sum_{j=1}^M S_0^j - \eta \sum_{j=1}^M \Sigma_0^j) e$ . Therefore, if the conditions of (22) are satisfied, then the estimation error dynamics (7), (21) is SIOS and the observer gains  $L_s$ ,  $s \in \overline{0, M}$  can be obtained as desired.

Again the conditions of Corollary 17 are more restrictive than in Theorem 14 since it is assumed that  $0 < P \leq I_n$  and  $\Lambda^j = I_n$  for  $j \in \overline{1, M}$ .

**Example 2** Consider a multi-group susceptible-infected-susceptible (SIS) model [26,28]:

$$\dot{x}(t) = \text{diag}(1_n - x(t))(\beta Ax(t) + w(t)) - \gamma x(t), \quad (23)$$

where  $x(t) \in [0, 1]^n$  represents infected populations in  $n$  groups,  $1_n \in \mathbb{R}^n$  is the vector of ones;  $\beta > 0$  and  $\gamma > 0$  are the infection and the recovery rates, respectively;  $A \in [0, 1]^{n \times n}$  is the adjacency matrix of infection transmission between groups;  $w(t) \in [0, 1]^n$  corresponds to unmodelled cumulative infection receipt at each group. Following [28], assume that the infected population is measured in  $0 < p < n$  groups:

$$y(t) = Cx(t), \quad C = [I_p \quad \mathbf{0}_{p \times (n-p)}]. \quad (24)$$

To represent this system in the form (7), consider a change of variables  $z = \ln(1_n - x)$ ,  $x = 1_n - e^z$ , where application of an elementary function to a vector argument is understood elementwise, then

$$\dot{z}(t) = \beta Af_1(z(t)) - \gamma f_2(z(t)) - w(t), \quad y(t) = -Cf_1(z(t)), \\ f_1(z) = e^z - 1_n, \quad f_2(z) = 1_n - e^{-z}$$

and it is easy to check that assumptions 7 and 10 are satisfied. Assumption 13 holds locally with  $S_3^{j,j} = I_n$  and  $\Sigma_3^{j,j} = I_n$  and all other matrices equal zero for  $j \in \{1, 2\}$ . Then the state observation can be performed for a sufficiently small initial estimation error. Note that due to the form of the output, we obtain that  $\tilde{y} = Cz$  is an auxiliary measured signal. The observer is taken in the form (19):

$$\dot{\hat{z}}(t) = \beta Af_1(\hat{z}(t)) - \gamma f_2(\hat{z}(t)) + L(y(t) + Cf_1(\hat{z}(t))) \\ + \sum_{j=1}^2 m_j C^\top Cf_j(x(t) - \hat{z}(t)),$$

where  $\hat{z}(t) \in \mathbb{R}^n$  is the estimate of  $z(t)$ ,  $L \in \mathbb{R}^{n \times p}$  is the observer gain to be selected,  $Cf_1(x - \hat{z}) = e^{\tilde{y} - C\hat{z}} - 1_p$  and  $Cf_2(x - \hat{z}) = 1_p - e^{C\hat{z} - \tilde{y}}$  are dependent on the measured information only,  $m_j \geq 0$  are tuning parameters for  $j \in \{1, 2\}$ . In the original coordinates the observer can be rewritten as follows:

$$\dot{\hat{x}}(t) = \text{diag}(1_n - \hat{x}(t)) (L - m_2 C^\top \text{diag}(1_p - y(t))^{-1}) \times \\ (C\hat{x}(t) - y(t)) + \beta A\hat{x}(t) - \gamma \hat{x}(t) - m_1 C^\top (C\hat{x}(t) - y(t)),$$

and it is straightforward to check that  $\hat{x}(t) \in [0, 1]^n$  for all  $t > 0$  provided that  $\hat{x}(0) \in [0, 1]^n$  and  $(m_1 + m_2)C^\top - L$  is elementwise nonnegative.

For  $n = 15$ , we selected the pair of matrices  $(A, C)$  in (23), (24) to be observable ( $A$  is not symmetric), then the LMIs of Theorem 14 induced by the error dynamics (20) are verified. A state trajectory of (23) is shown in Fig. 3, corresponding to the red curve in Fig. 4, representing the estimation error decay in logarithmic scale for different initial conditions. The error  $e$  converges to a vicinity of the origin proportional to the amplitude of disturbances (for simulation,  $w$  was chosen as a harmonic perturbation, and  $v \equiv 0$ ). This application confirms the efficacy and the generality of the developed estimation framework.

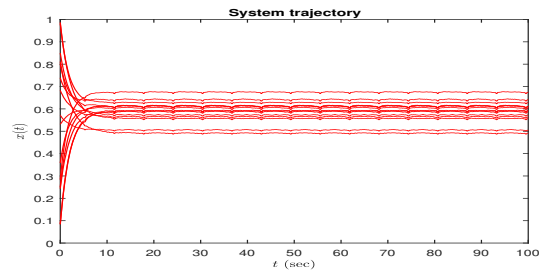


Fig. 3. The state trajectory

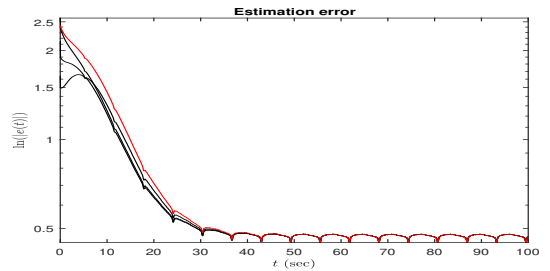


Fig. 4. The logarithm of estimation error

## 6 Conclusion

This work studied the problem of robust state estimation for a class of generalized Persidskii systems. A simple observer was proposed containing a copy of the system dynamics with a nonlinear output injection term. Two sets of stability conditions were developed, establishing IOS and SIOS properties of the common dynamics of the system and the observer with respect to the estimation error. These conditions were obtained in the form of LMIs. Two examples were shown (two-mass and SIS models) to verify the effectiveness of our framework. The design of adaptive or reduced-order observers, extension to the input estimation as in [7,6], or introduction of additional output injection in nonlinearities as in [2,6] can be considered directions for further investigations.

## References

- [1] V. Andrieu, D. Astolfi, and P. Bernard. Observer design via interconnections of second-order mixed sliding-mode/linear differentiators. *International Journal of Robust and Nonlinear Control*, 31(9):3631–3657, 2021.
- [2] M. Arcak and P. Kokotovic. Nonlinear observers: a circle criterion design and robustness analysis. *Automatica*, 37(12):1923–1930, 2001.
- [3] E.A. Barbashin. On construction of Lyapunov functions for nonlinear systems. In *Proc. 1st IFAC World Congress*, pages 742–751, Moscow, 1961.
- [4] G. Besançon. *Nonlinear observers and applications*, volume 363 of *Lecture notes in control and information*. Springer, 2007.
- [5] S. Boyd, L.E. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Studies in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, 1994.
- [6] A. Chakrabarty and M. J. Corless. Estimating unbounded unknown inputs in nonlinear systems. *Automatica*, 104:57–66, 2019.
- [7] A. Chakrabarty, M. J. Corless, G. T. Buzzard, S. H. Zak, and A. E. Rundell. State and Unknown Input Observers for Nonlinear Systems With Bounded Exogenous Inputs. *IEEE Transactions on Automatic Control*, 62(11):5497–5510, 2017.
- [8] S.N. Dashkovskiy, D.V. Efimov, and E.D. Sontag. Input to state stability and allied system properties. *Automation and Remote Control*, 72(8):1579–1614, 2011.
- [9] Y. Ebihara, D. Peaucelle, and D. Arzelier. *S-Variable Approach to LMI-Based Robust Control*. Communications and Control Engineering. Springer London, London, 2015.
- [10] D. Efimov and A. Aleksandrov. Robust stability analysis and implementation of Persidskii systems. *58th IEEE Conference on Decision and Control (CDC)*, 2019.
- [11] J. P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems applications to bioreactors. *IEEE Transactions on Automatic Control*, 37(6):875–880, 1992.
- [12] H. Hammouri and M. Kinnaert. A new procedure for time-varying linearization up to output injection. *Systems & Control Letters*, 28(3):151–157, 1996.
- [13] R. Isermann. *Fault-Diagnosis Systems: An Introduction from Fault Detection to Fault Tolerance*. Springer, Berlin, Heidelberg, 2006.
- [14] R.E. Kalman. A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82:35–45, 1960.
- [15] R. Kandepe, B. Foss, and L. Imsland. Applying the unscented Kalman filter for nonlinear state estimation. *Journal of Process Control*, 18:753–768, 2008.
- [16] E. Kaszkurewicz and H. Liu. Stability of nonlinear systems. *Automatica*, 15:609–614, 1979.
- [17] H. Khalil. *Nonlinear systems*. Prentice-Hall, Upper Saddle River, NJ, 2002.
- [18] A. J. Krener and W. Respondek. Nonlinear observers with linearizable error dynamics. *SIAM Journal on Control and Optimization*, 23(2):197–216, 1985.
- [19] M. R. Liberzon. Essays on the absolute stability theory. *Automation and Remote Control*, 67(10):1610–1644, 2006.
- [20] M. Livne and A. Levant. Proper discretization of homogeneous differentiators. *Automatica*, 50(8):2007–2014, 2014.
- [21] F. López-Estrada, D. Rotondo, and G. Valencia-Palomo. A review of convex approaches for control, observation and safety of linear parameter varying and Takagi-Sugeno systems. *Processes*, 7(11):814, 2019.
- [22] F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time observer design: Implicit Lyapunov function approach. *Automatica*, 87:52–60, 2018.
- [23] D. Luenberger. An introduction to observers. *IEEE Transactions on Automatic Control*, 16(6):596–602, 1971.
- [24] W. Mei, D. Efimov, and R. Ushirobira. Feedback synchronization in Persidskii systems. *Proc. 21st IFAC World Congress*, 2020.
- [25] W. Mei, D. Efimov, and R. Ushirobira. Towards state estimation of Persidskii systems. In *59th IEEE Conference on Decision and Control (CDC)*, pages 2881–2886, 2020.
- [26] W. Mei, S. Mohagheghi, S. Zampieri, and F. Bullo. On the dynamics of deterministic epidemic propagation over networks. *Annual Reviews in Control*, 44:116–128, 2017.
- [27] J. Mohammadpour and C.W. Scherer, editors. *Control of Linear Parameter Varying Systems with Applications*. Springer, Boston, 2012.
- [28] M. U. B. Niazi. *Aggregated Monitoring of Large-scale Network Systems and Control of Epidemics*. PhD thesis, GRENOBLE ALPES University, 2021.
- [29] A. Pavlov and L. Marconi. Incremental passivity and output regulation. *Systems & Control Letters*, 57(5):400–409, 2008.
- [30] S. Persidskii. Concerning problem of absolute stability. *Automation and Remote Control*, pages 5–11, 1969.
- [31] C. Rao, J. Rawlings, and D. Mayne. Constrained state estimation for nonlinear discrete-time systems: stability and moving horizon approximations. *IEEE Transactions on Automatic Control*, 48:246–258, 2003.
- [32] M. Reichhartinger, D. Efimov, and L. Fridman. Special issue on differentiators. *International Journal of Control*, 91(9):1980–1982, 2018.
- [33] S. Särkkä. On unscented Kalman filtering for state estimation of continuous-time nonlinear systems. *IEEE Transactions on Automatic Control*, 52:1631–1641, 2007.
- [34] E. Sontag and Y. Wang. Notions of input to output stability. *Systems & Control Letters*, 38(4-5):235–248, 1999.
- [35] E. Sontag and Y. Wang. Lyapunov characterizations of input to output stability. *SIAM Journal on Control and Optimization*, 39(1):226–249, 2000.
- [36] E. D. Sontag. The ISS philosophy as a unifying framework for stability-like behavior. In *Nonlinear control in the year 2000, Vol. 2 (Paris)*, volume 259 of *Lecture Notes in Control and Inform. Sci.*, pages 443–467. Springer, London, 2001.
- [37] V. A. Yakubovich, G. A. Leonov, and A. Kh. Gelig. *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*. WORLD SCIENTIFIC, 2004.
- [38] V.A. Yakubovich. Popov’s method and its subsequent development. *European Journal of Control*, 8(3):200–208, 2002.
- [39] F. Zhang, H.L. Trentelman, and J.M. A. Scherpen. Dynamic feedback synchronization of Lur’e networks via incremental sector boundedness. *IEEE Transactions on Automatic Control*, 61(9):2579–2584, 2016.