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► **To cite this version:**

Nadhir Ben Rached, Abdul-Lateef Haji-Ali, Gerardo Rubino, Raúl Tempone. Efficient Importance Sampling for Large Sums of IID Random Variables. UQ 2022 - SIAM Conference on Uncertainty Quantification, Apr 2022, Atlanta, United States. pp.1-31. hal-03899073

HAL Id: hal-03899073

<https://inria.hal.science/hal-03899073>

Submitted on 15 Dec 2022

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SIAM UQ 2022 - Theory and simulation of failure probabilities and rare events mini-symposium

Efficient Importance Sampling for Large Sums of IID
Random Variables

Nadhir Ben Rached



joint work with Abdul-Lateef Haji-Ali (Heriot-Watt University), Gerardo Rubino (INRIA, Rennes - Bretagne Atlantique), and Raúl Tempone (KAUST, RWTH Aachen)

April 12, 2022

- Estimate using **importance sampling (IS)**

$$\mathbb{P}\left(\sum_{i=1}^N X_i \leq \gamma\right), \quad (1)$$

where X_1, X_2, \dots, X_N is a sequence of nonnegative independent and identically distributed (i.i.d) random variables (RVs).

- Particularly interested in the **rare event regime** when N is large and/or γ is small.
- **Application**: performance evaluation/prediction of wireless communication systems operating over fading channels ([Simon and Alouini 2005](#)).

- Need the use of **variance reduction techniques** as the **naive Monte Carlo (MC) estimator** is computationally expensive in **the regime of rare events**.
- Various **variance reduction techniques** have been proposed (Alouini et al. 2018, Beaulieu and Luan 2019, Ben Rached et al. 2016, Ben Rached et al. 2018, Botev et al. 2019, Asmussen et al. 2016, Gulisashvili and Tankov 2016):
 - **Logarithmically efficient** estimator in the case of i.i.d. sums Log-normal RVs using **exponential twisting** (Asmussen et al. 2016).
 - Under mild assumptions, **logarithmically efficient** in case of sums of correlated Log-normal RVs based on shifting the mean of the multivariate Gaussian distribution (Gulisashvili and Tankov 2016).
 - Under an assumption, **asymptotically vanishing relative error** for sums of correlated Log-normal RVs by combining IS with the control variate techniques (Alouini et al. 2018).
 - Two unified IS using **hazard rate twisting (HRT)** for sums of independent RVs. The first estimator is shown to be **logarithmically efficient**, whereas the second achieves **bounded relative error** under mild assumptions (Ben Rached et al. 2016).

- Efficiency of previous estimators was studied when N is fixed and $\gamma \rightarrow 0$.
- Efficiency is considerably affected when N increases.
- **Aim:** efficiently estimate (1) in the rare event regime when N is large and/or γ is small.

- Let $\mathbf{x} = (x_1, \dots, x_n)^t$ and $h_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^N f_X(x_i)$ be the joint PDF of the random vector $(X_1, \dots, X_N)^t$. We consider the estimation of

$$\alpha(\gamma, N) = \mathbb{P}_{h_{\mathbf{X}}} \left(\sum_{i=1}^N X_i \leq \gamma \right),$$

where $\mathbb{P}_{h_{\mathbf{X}}}(\cdot)$ is the probability under which the random vector $\mathbf{X} = (X_1, \dots, X_N)^t$ is distributed according to $h_{\mathbf{X}}(\cdot)$.

- Motivating example:** write the quantity of interest as

$$\mathbb{P}_{h_{\mathbf{X}}} \left(\sum_{i=1}^N X_i \leq \gamma \right) = \mathbb{E}_{h_{\mathbf{w}}} \left[(F_X(\gamma))^N \mathbf{1}_{(\sum_{i=1}^N w_i \leq 1)} \right] \quad (2)$$

where w_i is equal in distribution to $\frac{X_i}{\gamma}$ conditional on the event $\{X_i \leq \gamma\}$, $i = 1, 2, \dots, N$, and $h_{\mathbf{w}}(\mathbf{w}) = \prod_{i=1}^N f_w(w_i)$ with $f_w(\cdot)$ is the PDF of w_i .

- Estimate the right-hand side term by the **naive MC** method.

- For fixed N , it achieves the **bounded relative error** property with respect to the rarity parameter γ for distributions that satisfy $f_w(x) \sim bx^p$ as x approaches zero and for $p > -1$ and $b > 0$ (Ben Rached et al. 2018).
- **But** using the Chernoff bound, we obtain for all $\eta > 0$

$$\mathbb{P}_{h_w} \left(\sum_{i=1}^N w_i \leq 1 \right) \leq \exp \left(\eta + N \log \left(\mathbb{E}_{f_w} [\exp(-\eta w)] \right) \right)$$

- The squared coefficient of variation of $\hat{\alpha}(\gamma, N)$ is given by

$$\frac{\text{var}_{h_w} [\hat{\alpha}(\gamma, N)]}{\alpha^2(\gamma, N)} = \frac{1 - \mathbb{P}_{h_w} \left(\sum_{i=1}^N w_i \leq 1 \right)}{\mathbb{P}_{h_w} \left(\sum_{i=1}^N w_i \leq 1 \right)}.$$

- When $\eta = 1$, the squared coefficient of variation is asymptotically equal to $1/\mathbb{P}_{h_w} \left(\sum_{i=1}^N w_i \leq 1 \right)$ and is lower bounded by

$$\exp \left(-1 - N \log \left(\mathbb{E}_{f_w} [\exp(-w)] \right) \right).$$

- The efficiency of the estimator deteriorates when N is large.

- When applicable, **exponential twisting** is expected to compare favorably, in most of the cases, to existing estimators.
- For **light tails** i.i.d. case and under regularity assumptions, **exponential twisting** is **logarithmically efficient** when the probability of interest is either $\mathbb{P}(\sum_{i=1}^N X_i > \gamma)$ and $\gamma \rightarrow +\infty$ or $\mathbb{P}(\sum_{i=1}^N X_i > \gamma N)$ and $N \rightarrow +\infty$ (Asmussen and Glynn 2007).
- In the left tail setting, **exponential twisting** achieves the **logarithmic efficiency** property in the case of i.i.d. Log-normal RVs when the probability of interest is $\mathbb{P}(\sum_{i=1}^N X_i < N\gamma)$ and either $N \rightarrow +\infty$ or $\gamma \rightarrow 0$ (Asmussen et al. 2016).

- **Exponential twisting PDF** minimizes the Kullback-Leibler (KL) divergence with respect to the underlying PDF under the constraint that the rare set $\{\mathbf{x} \in \mathbb{R}_+^N, \text{ such that } \sum_{i=1}^N x_i \leq \gamma\}$ is no longer rare (Ridder and Rubinstein 2007)

$$\inf_{h_{\mathbf{X}}^* \geq 0} \int h_{\mathbf{X}}^*(\mathbf{x}) \log \left(\frac{h_{\mathbf{X}}^*(\mathbf{x})}{h_{\mathbf{X}}(\mathbf{x})} \right) d\mathbf{x}$$

$$\text{s.t.} \quad \int h_{\mathbf{X}}^*(\mathbf{x}) d\mathbf{x} = 1 \tag{3}$$

$$\mathbb{E}_{h_{\mathbf{X}}^*} \left[\sum_{i=1}^N X_i \right] = \gamma$$

$$h_{\mathbf{X}}^*(\mathbf{x}) \geq 0, \quad x_i \geq 0 \text{ for all } i \in 1, 2, \dots, N.$$

- The solution of this problem is given as

$$h_{\mathbf{X}}^*(\mathbf{x}) = \frac{h_{\mathbf{X}}(\mathbf{x}) \exp \left(\theta^* \sum_{i=1}^N x_i \right)}{\mathbb{E}_{h_{\mathbf{X}}} \left[\exp \left(\theta^* \sum_{i=1}^N X_i \right) \right]}, \quad \mathbf{x} \in \mathbb{R}_+^N \tag{4}$$

- The optimal twisting parameter θ^* solves

$$\frac{\mathbb{E}_{h_{\mathbf{X}}} \left[\sum_{i=1}^N X_i \exp \left(\theta^* \sum_{i=1}^N X_i \right) \right]}{\mathbb{E}_{h_{\mathbf{X}}} \left[\exp \left(\theta^* \sum_{i=1}^N X_i \right) \right]} = \gamma.$$

- By writing $h_{\mathbf{X}}^*(\mathbf{x}) = \prod_{i=1}^N f_{\mathbf{X}}^*(x_i)$, the optimal density is given by **exponentially twisting** each univariate PDF $f_X(\cdot)$

$$f_X^*(x) = \frac{f_X(x) \exp(\theta^* x)}{M(\theta^*)}, \quad x \geq 0,$$

with $M(\theta) = E_{f_X}[\exp(\theta X)]$ and the optimal twisting parameter θ^* satisfies

$$\frac{M'(\theta^*)}{M(\theta^*)} = \frac{\gamma}{N}.$$

- Since the left-tail is considered, $\theta^* \rightarrow -\infty$ as $\gamma \rightarrow 0$ and/or $N \rightarrow +\infty$ (Asmussen et al. 2016).

Observe the following drawbacks:

- Sampling according to $f_X^*(\cdot)$ is **not straightforward** \rightarrow needs the use of an acceptance-rejection technique, the complexity of which can be dramatic when the probability of acceptance is relatively small.
- Computations are much simpler if the moment generating function $M(\theta)$ is known in closed-form. **Such a requirement does not hold in general.**
- θ does not have, in general, a closed-form expression, and hence, it should be **approximated numerically**.

Goal

Propose an **alternative IS PDF** that approximately yields, for certain classes of distributions and in the rare event regime corresponding to **large N and/or small γ** , at least the same performance as the **exponential twisting** technique and at the same time does not introduce **serious limitations**.

$f_X(x) = x^p g(x)$ with $g(x) \sim b$ as x goes to 0, $d > -1$, and $b > 0$

- Distributions in this class: Exponential, Gamma, Nakagami-m, Weibull, Generalized Gamma, Rice, Gamma-Gamma, $\kappa - \mu$.
- The change of measure that we consider is

$$\tilde{f}_X(x) = \frac{x^p \exp(\theta x)}{\tilde{M}(\theta)}, \quad x \geq 0. \quad (5)$$

- The IS PDF corresponds to **the Gamma PDF** with shape parameter $p + 1$ and scale parameter $-1/\theta$.
- The value of θ is chosen to be equal to $\tilde{\theta}$ such that $\frac{\tilde{M}'(\tilde{\theta})}{\tilde{M}(\tilde{\theta})} = \frac{\gamma}{N}$

$$\tilde{\theta} = -\frac{N}{\gamma}(p + 1). \quad (6)$$

- The proposed IS estimator of $\alpha(\gamma, N)$ using M i.i.d. samples of \mathbf{X} from $\tilde{h}_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^N \tilde{f}_X(x_i)$ is

$$\hat{\alpha}_{is, M}(\gamma, N) = \frac{1}{M} \sum_{k=1}^M \mathbf{1}_{(\sum_{i=1}^N X_i^{(k)} \leq \gamma)} (\tilde{M}(\tilde{\theta}))^N \prod_{i=1}^N \frac{f_X(X_i^{(k)}) \exp(-\tilde{\theta} X_i^{(k)})}{(X_i^{(k)})^p}.$$

Main Result

For distributions satisfying $f_X(x) = x^p g(x)$ with $g(x) \sim b$ as x goes to 0, $p > -1$, and $b > 0$ is a constant, the proposed approach achieves approximately, as γ decreases to 0 and/or N increases, the same performance as the one given by the exponential twisting without introducing serious limitations.

Standard Log-normal case

- Distributions that do not approach 0 polynomially are much more difficult to handle and need to be tackled on a **case-by-case basis**.
- We consider the case of the sum of i.i.d. standard **Log-normal** RVs whose PDF decreases to 0 at a faster rate than any polynomials.
- The Gamma distribution with fixed shape parameter will not recover the results given by the use of the **exponential twisting** technique.
- In [Asmussen et al. 2016](#), the **exponential twisting** was applied by:
 - 1 Providing an unbiased estimator of the moment generating function.
 - 2 Approximating the value of θ .
 - 3 Using acceptance-rejection to sample from the IS PDF.

Standard Log-normal case

We propose **two estimators**:

- The **first** is based on truncating the support $[0, +\infty]$ and only working on $[a, +\infty]$ with $a = \delta\gamma/N$
→ introduces a **bias** that needs to be controlled.
- In the regime of rare events, the **biased estimator** achieves approximately the same performances as the **Gamma IS PDF** with shape parameter equal to 2
→ propose a **second estimator** whose IS PDF is a **Gamma PDF** with optimized parameters.

Standard Log-normal case: biased estimator

- We rewrite the quantity of interest as

$$\mathbb{P}_{h_{\mathbf{X}}} \left(\sum_{i=1}^N X_i \leq \gamma \right) \approx \underbrace{\left(1 - F_X \left(\frac{\delta\gamma}{N} \right) \right)^N}_{\text{Known in closed form}} \times \mathbb{P}_{h_{\mathbf{X}}} \left(\sum_{i=1}^N X_i \leq \gamma \mid X_i > \frac{\delta\gamma}{N} \right), \quad (7)$$

where $\delta \in [0, 1)$.

- Let $\bar{f}_X(\cdot)$ be the PDF of $X_i \mid \{X_i > \frac{\delta\gamma}{N}\}$, $i = 1, 2, \dots, N$

$$\mathbb{P}_{h_{\mathbf{X}}} \left(\sum_{i=1}^N X_i \leq \gamma \mid X_i > \frac{\delta\gamma}{N} \right) = \mathbb{P}_{\bar{h}_{\mathbf{X}}} \left(\sum_{i=1}^N X_i \leq \gamma \right),$$

with $\bar{h}_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^N \bar{f}_X(x_i)$. The **exponential twisting** change of measure is then given by

$$\bar{f}_X^*(x) \propto \bar{f}_X(x) \exp(\theta x), \quad x \geq \frac{\delta\gamma}{N}.$$

Standard Log-normal case: biased estimator

- The approximate **exponential twisting** change of measure is given by

$$\tilde{f}_X(x) = \frac{\bar{f}_X \exp(\theta x) + (x - \frac{\delta\gamma}{N}) \bar{f}'_X \exp(\theta x)}{\tilde{M}(\theta)}, \quad x \geq \frac{\delta\gamma}{N}, \quad (8)$$

with $\bar{f}_X = \bar{f}_X(\frac{\delta\gamma}{N})$ and $\bar{f}'_X = \bar{f}'_X(\frac{\delta\gamma}{N})$. We assume that $\frac{\delta\gamma}{N} < \exp(-1)$ to ensure that $\tilde{f}'_X > 0$.

- Through a simple computation, we get

$$\tilde{M}(\theta) = -\frac{\exp(\theta\delta\gamma/N)}{\theta} \bar{f}_X + \frac{\exp(\theta\delta\gamma/N)}{\theta^2} \bar{f}'_X.$$

- The value of θ that solves $\frac{\tilde{M}'(\theta)}{\tilde{M}(\theta)} = \frac{\gamma}{N}$ is given by

$$\theta = -\frac{\bar{f}_X - c\bar{f}'_X + \sqrt{(\bar{f}_X - c\bar{f}'_X)^2 + 8\bar{f}_X\bar{f}'_X c}}{2c\bar{f}_X},$$

with $c = \frac{\gamma}{N}(1 - \delta)$.

Standard Log-normal case: biased estimator

- The question that remains is related to controlling the **bias** through a proper choice of the parameter δ .
- Let $\alpha_1(\gamma, N) = \left(1 - F_X\left(\frac{\delta\gamma}{N}\right)\right)^N \mathbb{P}_{h_{\mathbf{X}}} \left(\sum_{i=1}^N X_i \leq \gamma \mid X_i > \frac{\delta\gamma}{N}, \forall i\right)$

$$\left| \frac{\alpha(\gamma, N) - \hat{\alpha}_{1, is, M}}{\alpha(\gamma, N)} \right| \leq \frac{\alpha(\gamma, N) - \alpha_1(\gamma, N)}{\alpha(\gamma, N)} + \left| \frac{\alpha_1(\gamma, N) - \hat{\alpha}_{1, is, M}}{\alpha_1(\gamma, N)} \right|, \quad (9)$$

where $\hat{\alpha}_{1, is, M}$ is the IS estimator of $\alpha_1(\gamma, N)$ based on M i.i.d. realizations sampled according to $\tilde{h}_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^N \tilde{f}_X(x_i)$.

Lemma: On the choice of δ

The following expression of $\delta(\epsilon, N, \gamma)$

$$\delta(\epsilon, N, \gamma) = \frac{N}{\gamma} \exp \left(\Phi^{-1} \left(\frac{\epsilon}{2N} \frac{(\Phi(\log(\gamma/N)))^N}{(\Phi(\log(\gamma)))^{N-1}} \right) \right), \quad (10)$$

where $\Phi(\cdot)$ is the CDF of the standard Normal distribution, ensures that the **relative bias** is less than $\epsilon/2$.

Standard Log-normal case: The Gamma family as an IS PDF

- For a sufficiently small value of δ , the **biased estimator** achieves approximately the same performance as the **Gamma IS PDF** with shape parameter equal to 2 \rightarrow suggests using the Gamma family as IS PDF.
- Note that the advantage of using the Gamma family as IS PDFs is that the estimator is **unbiased**.
- Recall that the Gamma PDF is given by

$$\tilde{f}_X(x) = \frac{x^{k-1} \exp(-x/\theta)}{\Gamma(k)\theta^k}, \quad x > 0, \quad (11)$$

where $\theta > 0$ and $k > 0$ are the scale and shape parameters.

- The value of θ is chosen to be equal to $\theta = \frac{\gamma}{Nk}$ to ensure that the expected value of each of the X_i 's, $i = 1, 2, \dots, N$, under the PDF $\tilde{f}_X(\cdot)$ is equal to $\frac{\gamma}{N}$.
- The likelihood ratio is then given by

$$\mathcal{L}(x_1, x_2, \dots, x_N) = \frac{(\Gamma(k)\theta^k)^N \exp\left(\frac{\sum_{i=1}^N x_i}{\theta} - \frac{1}{2} \sum_{i=1}^N (\log(x_i))^2\right)}{\prod_{i=1}^N x_i^k (\sqrt{2\pi})^N}.$$

Standard Log-normal case: The Gamma family as an IS PDF

- The second moment of the **IS estimator** is bounded by

$$\begin{aligned} & \mathbb{E}_{\tilde{h}_{\mathbf{X}}} \left[\mathcal{L}^2(X_1, X_2, \dots, X_N) \mathbf{1}_{(\sum_{i=1}^N X_i \leq \gamma)} \right] \\ & \leq \left(\frac{\Gamma(k) \left(\frac{\gamma}{Nk}\right)^k}{\sqrt{2\pi}} \right)^{2N} \exp(2kN + k^2N). \end{aligned}$$

- Using **Stirling's formula**, we get for some constant C

$$\begin{aligned} & \mathbb{E}_{\tilde{h}_{\mathbf{X}}} \left[\mathcal{L}^2(X_1, X_2, \dots, X_N) \mathbf{1}_{(\sum_{i=1}^N X_i \leq \gamma)} \right] \\ & \lesssim C \exp(N(k^2 - 2k \log(N/\gamma) - \log(k))). \end{aligned}$$

- k is chosen such that it minimizes the above right-hand side term

$$k^* = \frac{1}{2} \left(\log\left(\frac{N}{\gamma}\right) + \sqrt{\left(\log\left(\frac{N}{\gamma}\right)\right)^2 + 2} \right). \quad (12)$$

- When N is large and/or γ is small, we obtain $k^* \sim \log\left(\frac{N}{\gamma}\right)$.

- We define the **squared coefficient of variation** of an **unbiased estimator** $\hat{\alpha}(\gamma, N)$ of $\alpha(\gamma, N)$ as

$$SCV(\hat{\alpha}(\gamma, N)) = \frac{\text{var}[\hat{\alpha}(\gamma, N)]}{\alpha^2(\gamma, N)}. \quad (13)$$

- From the **CLT**, the number of required samples to meet ϵ relative **statistical error** with 95% confidence is equal to $(1.96)^2 SCV(\hat{\alpha}(\gamma, N)) / \epsilon^2$.

Weibull Case

- $X_i, i = 1, 2, \dots, N$, are distributed according to the **Weibull distribution** whose PDF is

$$f_X(x) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} \exp(-(x/\lambda)^k), \quad x > 0.$$

- The comparison is made with respect to the **hazard rate twisting (HRT)** approach of **Ben Rached et al. 2016**.

Weibull Case

When $k = 1.5$, $\lambda = 1$, $\gamma = 0.5$, and $N = 12$, the proposed approach is approximately **270** times more efficient than the one based on HRT.

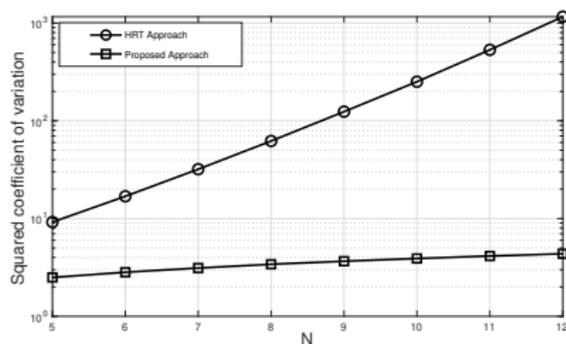


Figure 1: Squared coefficient of variation as a function of N where X_i are i.i.d. Weibull RVs with rate $\lambda = 1$, $k = 1.5$, and $\gamma = 0.5$.

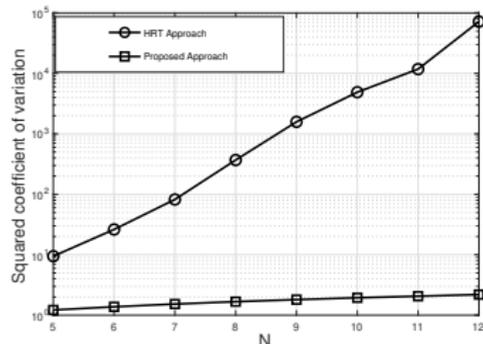


Figure 2: Squared coefficient of variation as a function of N where X_i are i.i.d. Weibull RVs with rate $\lambda = 1$, $k = 0.5$, and $\gamma = 0.01$.

The value of $\alpha(\gamma, N)$ ranges approximately from 10^{-20} to 10^{-6} .

The value of $\alpha(\gamma, N)$ ranges approximately from 10^{-16} to 10^{-6} .

Weibull Case

The value of $\alpha(\gamma, N)$ ranges approximately from 10^{-16} to 10^{-6} for $N = 8$ and from 10^{-22} to 10^{-8} for $N = 10$.

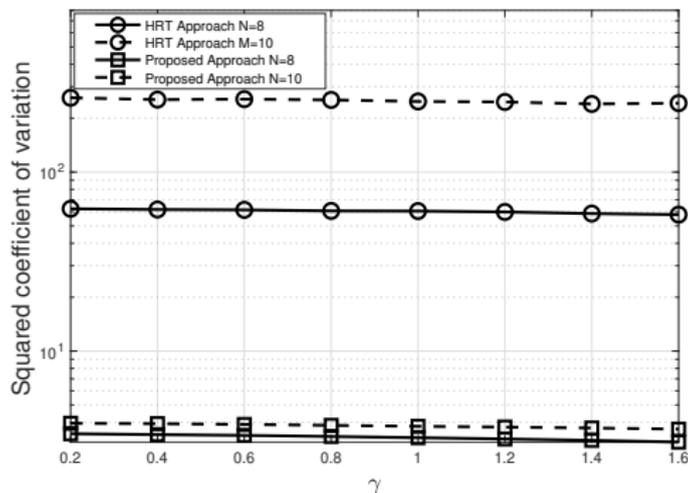


Figure 3: Squared coefficient of variation as a function of γ where X_i are i.i.d. Weibull RVs with rate $\lambda = 1$, $k = 1.5$.

Approximately **18** (respectively **64**) times more efficient than the HRT one when $N = 8$ (respectively $N = 10$) and $\gamma = 0.2$.

Log-normal Case

- The standard **Log-normal** PDF is given by

$$f_X(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\log(x))^2}{2}\right), \quad x > 0.$$

- Comparison with **exponential twisting** proposed in [Asmussen et al. 2016](#).

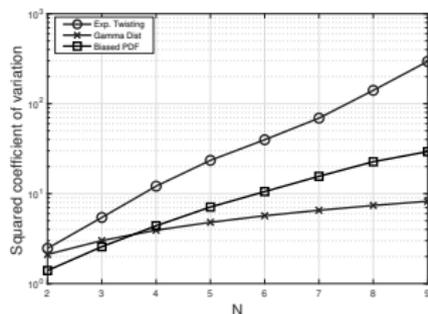


Figure 4: Squared coefficient of variation as a function of N where X_i are i.i.d. standard Log-normal RVs with $\gamma = 0.5$, and $\epsilon = 0.05$. When $N = 9$, approximately **30** times more efficient than the one based on **exponential twisting**.

The value of $\alpha(\gamma, N)$ ranges approximately from 10^{-20} to 10^{-2} .

Log-normal Case

The quantity of interest $\alpha(\gamma, N)$ ranges approximately from 10^{-15} to 10^{-6} for $N = 8$ and from 10^{-21} to 10^{-9} for $N = 10$.

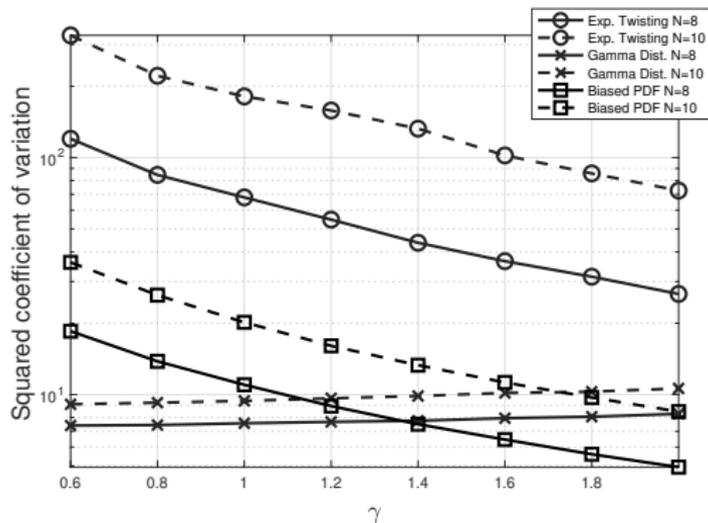


Figure 5: Squared coefficient of variation as a function of γ where X_i are i.i.d. standard Log-normal random variables with $\epsilon = 0.05$. Approximately 15 times (respectively 35) more efficient than the exponential twisting one when $N = 8$ (respectively $N = 10$) and $\gamma = 0.6$.

Log-normal Case

- In order to include the **computing time** in our comparison, we define the **Work Normalized Relative Variance (WNRV)** metric of an unbiased estimator $\hat{\alpha}(\gamma, N)$ of $\alpha(\gamma, N)$ (Ben Rached et al. 2018)

$$\text{WNRV}(\hat{\alpha}(\gamma, N)) = \frac{\text{SCV}(\hat{\alpha}(\gamma, N))}{M} \times \text{computing time in seconds.}$$

- The **computing time** is the time in seconds needed to get an estimator of $\alpha(\gamma, N)$ using M i.i.d. samples of $\hat{\alpha}(\gamma, N)$.
- When comparing two estimators, the one that exhibits less **WNRV** is more efficient than the other estimator: it achieves less **relative error** for a given **computational budget**, or equivalently it needs less **computing time** to achieve a fixed **relative error**.

Log-normal Case

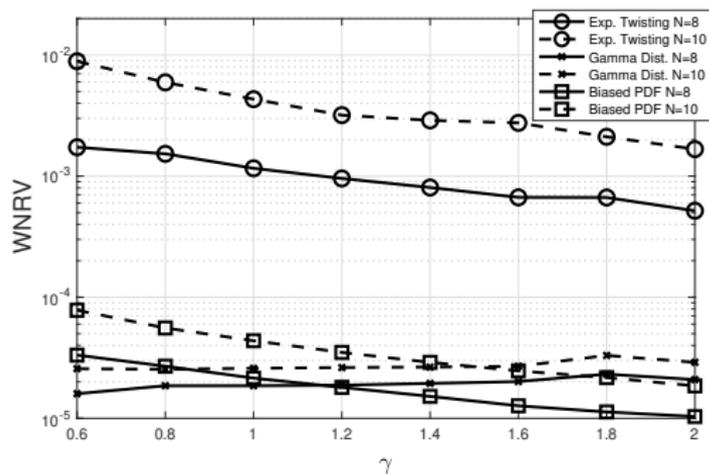


Figure 6: WNRV as a function of γ where X_i are i.i.d. standard Log-normal RVs with $\epsilon = 0.05$.

For $N = 10$ and $\gamma = 0.6$, the approach based on using the Gamma IS PDF is approximately **35** times (respectively **340** times) more efficient than the one based on exponential twisting when using the squared coefficient of variation metric (respectively the WNRV metric).

- Developed efficient importance sampling estimators to estimate rare event probabilities corresponding to the left-tail of the CDF of **large sums** of nonnegative i.i.d. RVs.
- Achieved at least the same performance as the **exponential twisting** technique.
- The **Gamma PDF** with suitably chosen parameters achieves for most of the well-practical distributions substantial amount of variance reduction, and at the same time avoids the restrictive limitations when using the **exponential twisting** technique.
- Validated **numerically** the efficiency in the rare event regime corresponding to **large N and/or small γ** .

A possible future direction: study the asymptotic properties of the proposed estimators as **$N \rightarrow \infty$ and/or $\gamma \rightarrow 0$** .

Further details in: N. Ben Rached, A.-L. Haji-Ali, G. Rubino, and R. Tempone “Efficient Importance Sampling for Large Sums of Independent and Identically Distributed Random Variables”, Statistics and Computing, vol. 31, number 6, 2021.

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Thank you for you attention