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# The power-dual and the exponential-dual matrices – applications to the computation of matrix powers and matrix exponentials

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JMM 2022

virtual, April 7, 2022

# Talk's content

- 1 Introduction
  - Outline
  
- 2 Duality concepts
  - Power-duals
  - Exponential-duals
  
- 3 Back to general remarks

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# Original path

- The problem addressed here is to find the solution to a linear system of difference equations or of differential equations.
- The approach here described comes from the analysis of Markovian models, mainly queuing models, in discrete time or in continuous time.
- At the beginning, we developed a methodology based on the concept of duality in Markov processes proposed by Siegmund and developed in Anderson's book <sup>1</sup>, and we used it to derive closed-form expressions of the transient distributions of some basic queueing models ( $M/M/1$ ,  $M/M/1/H$ ,  $M/M/1$  and  $M/M/1/H$  with “catastrophes”, etc.).
- Later, we extended this duality idea to **arbitrary differential systems**, not necessarily coming from stochastic processes. This paper describes new results in this direction.

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<sup>1</sup> *Continuous-Time Markov Chains*, W. J. Anderson, published by Springer in 1991 (out of press).

# Using the dual to analyze Markovian queues

- From a given Markov process  $X$ , living on the nonnegative integers, (i) we build a new one, its Siegmund's dual  $X^*$ , **if it exists, which is not always the case**.
- **The transformation is linear (the generators are linearly related), and its main property is that the solutions to both systems are also linearly related, and in the two directions:  $X \rightarrow X^*$  and  $X^* \rightarrow X$ .**
- Next, (ii) we find the transient distribution of  $X^*$ , in general using **Uniformization**, to move to discrete time and thus analyzing the transient distribution of a Markov chain  $Y^*$  (typically counting paths by means of **lattice path combinatorics** tools).
- To finish, (iii) we come back at  $X$  using the inverse dual (linear) transformation.
- **In cases with some regularity in the structure, this can produce closed-forms solutions**, and it can also work when the dimension of the initial problem is infinite (e.g. for the  $M/M/1$  queue).

# Comments

- This procedure is useful when  $X^*$  is easier to analyze, in a way or another.
- As we see, there is an initial continuous time phase, then a discrete time part process, where a transient distribution is needed.
- In some cases, we have a discrete time problem for which we look for transients from the beginning (instead of ODEs we have then difference equations).
- In this talk we will briefly describe a general approach where transients in continuous time (algebraically speaking, matrix exponentials) or in discrete time (matrix powers) are needed. The former are built using the latter.
- To simplify the discussion, we only present here the finite case.

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# The power-dual of a matrix

- Let  $M$  be a square matrix (of reals, of complex numbers, ...) with dimension  $n$ , indexed on  $\{0, 1, \dots, n-1\}$ . The power-dual of  $M$  is a new matrix  $\tilde{M}$ , with dimension  $n+1$ , indexed on  $\{0, 1, \dots, n-1, n\}$ , defined as follows: for  $i = 0, 1, \dots, n-1$ ,

$$\tilde{M}_{i,j} = \begin{cases} \sum_{k=i}^{n-1} M_{j,k} - \sum_{k=i}^{n-1} M_{j-1,k} & \text{if } j \leq n-1, \\ 1 - \sum_{k=0}^{n-1} \tilde{M}_{i,k} & \text{when } j = n, \end{cases}$$

and the last row (when  $i = n$ ) is  $(0 \ 0 \ \dots \ 0 \ 1)$ .

- Example in dimension 2:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \tilde{M} = \begin{pmatrix} a+b & c+d-(a+b) & 1-(c+d) \\ b & d-b & 1-d \\ 0 & 0 & 1 \end{pmatrix}$$

- Example in dimension 3:

$$M = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -2 \\ 1 & -3 & 3 \end{pmatrix} \rightsquigarrow \tilde{M} = \begin{pmatrix} 2 & -2 & 1 & 0 \\ 1 & -3 & 2 & 1 \\ 2 & -4 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- A stochastic matrix:

$$P = \begin{pmatrix} 2/5 & 0 & 3/5 \\ 1/6 & 1/2 & 1/3 \\ 0 & 1/4 & 3/4 \end{pmatrix} \rightsquigarrow \tilde{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/5 & 7/30 & 1/6 & 0 \\ 3/5 & -4/15 & 5/12 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

as we see, in this case the power-dual is not a stochastic matrix;

- Another stochastic matrix:

$$P = \begin{pmatrix} 2/5 & 3/5 \\ 1/6 & 5/6 \end{pmatrix} \rightsquigarrow \tilde{P} = \begin{pmatrix} 1 & 0 & 0 \\ 3/5 & 1/15 & 1/3 \\ 0 & 0 & 1 \end{pmatrix};$$

here, the power-dual is also a stochastic matrix.

# Some properties

- **Result:** if  $\tilde{M}$  is the power-dual of matrix  $M$ , then the sum of the elements of any row of  $\tilde{M}$  is 1, and the last row of  $\tilde{M}$  is composed only of 0s except for the last element which is 1.
- **Inversion Lemma:** for  $0 \leq i, j \leq n - 1$ , we have

$$M_{i,j} = \sum_{k=0}^i \tilde{M}_{j,k} - \sum_{k=0}^i \tilde{M}_{j+1,k}.$$

- **Central result:** For any two square matrices  $A$  and  $B$  with the same dimension, we have

$$\widetilde{AB} = \widetilde{B}\widetilde{A}.$$

**Corollary:** for any square matrix  $M$  and any nonnegative integer  $k$ ,

$$\boxed{\widetilde{M^k} = \widetilde{M}^k.}$$

In words, the power-dual of the  $k$ th power of  $M$  is the  $k$ th power of  $M$ 's power-dual.



# The exponential-dual of a matrix

- Let  $M$  be a square matrix (of reals, of complex numbers, ...) with dimension  $n$ , indexed on  $\{0, 1, \dots, n-1\}$ . The exponential-dual of  $M$  is a new matrix  $\widehat{M}$ , with dimension  $n+1$  indexed on  $\{0, 1, \dots, n-1, n\}$ , defined as follows: for  $i < n$ ,

$$\widehat{M}_{i,j} = \sum_{k=i}^{n-1} M_{j,k} - \sum_{k=i}^{n-1} M_{j-1,k}$$

and the last row ( $i = n$ ) is  $(0 \ 0 \ \dots \ 0)$ .

- Example in dimension 2:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \widehat{M} = \begin{pmatrix} a+b & c+d-(a+b) & -(c+d) \\ b & d-b & -d \\ 0 & 0 & 0 \end{pmatrix}$$

- We can observe that if we denote by  $W$  a square matrix with the same dimension as  $M$ , composed of 0s except the last column composed of 1s, then we have  $\widehat{M} = \widetilde{M} - W$ .

- Example, dimension 3:

$$M = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & -3 \\ 0 & 2 & -2 \end{pmatrix} \rightsquigarrow \widehat{M} = \begin{pmatrix} 6 & -8 & 2 & 0 \\ 5 & -6 & 1 & 0 \\ 3 & -6 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- An infinitesimal generator:

$$M = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \rightsquigarrow \widehat{M} = \begin{pmatrix} 0 & 0 & 0 \\ \lambda & -(\lambda + \mu) & \mu \\ 0 & 0 & 0 \end{pmatrix};$$

as we see, the exponential-dual is also an infinitesimal generator;

- Another infinitesimal generator:

$$M = \begin{pmatrix} -3 & 1 & 2 \\ 6 & -7 & 1 \\ 1 & 2 & -3 \end{pmatrix} \rightsquigarrow \widehat{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & -9 & 5 & 1 \\ 2 & \boxed{-1} & -4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

here, the exponential-dual is not an infinitesimal generator.

# Some properties

- Observe first that if  $\widehat{M}$  is the exponential-dual of matrix  $M$ , then the sum of the elements of any row of  $\widehat{M}$  is 0, and the last row of  $\widehat{M}$  is composed only of 0s.
- Inversion Lemma:**  $0 \leq i, j \leq n - 1$ , we have

$$M_{i,j} = \sum_{k=0}^i \widehat{M}_{j,k} - \sum_{k=0}^i \widehat{M}_{j+1,k}.$$

- Important result:** for any two square matrices  $A$  and  $B$  with the same dimension, we have

$$\widehat{AB} = \widehat{BA}.$$

**Corollary:** for any square matrix  $M$  and any nonnegative integer  $k$ ,

$$\widehat{M^k} = \widehat{M}^k.$$

In words, the exponential-dual of the  $k$ th power of  $M$  is the  $k$ th power of  $M$ 's exponential-dual.

## More results:

- Let  $A$  and  $B$  be two square matrices with the same dimension, and  $\alpha$  be a scalar. We have

$$\widehat{A+B} = \widehat{A} + \widehat{B}, \quad \widehat{\alpha M} = \alpha \widehat{M} \quad (\text{but } \widetilde{\alpha M} \neq \alpha \widetilde{M}).$$

- Let  $(M_k)_{k \geq 0}$  a matrix sequence, with  $M_k \rightarrow M$  as  $k \rightarrow \infty$ . Then,

$$\widehat{M}_k \rightarrow \widehat{M} \text{ as } k \rightarrow \infty.$$

If  $\sum_{k \geq 0} M_k = M$ , then

$$\sum_{k \geq 0} \widehat{M}_k = \sum_{k \geq 0} \widehat{M}_k = \widehat{M}.$$



# Main result

- Let  $M$  be an arbitrary square matrix. Using previous properties,

$$\widehat{e^M} = e^{\widetilde{M}}.$$

In words, the exponential-dual of the exponential of  $M$  is the exponential of the **power**-dual of  $M$ .

- This result explains how the evaluation of the power-dual appears when we work on matrix exponentials.
- In this short presentation with skipped some other technical results that are needed to prove previous theorem.
- Observe the difference:

$$\widetilde{M^k} = \widetilde{M}^k \quad \text{but} \quad \widehat{e^M} = e^{\widetilde{M}}.$$

# A path to compute $e^M$

Instead of

$$M \xrightarrow{\substack{\uparrow \\ \text{any algorithm}}} e^M,$$

we can try

$$M \xrightarrow[\substack{\uparrow \\ \text{(linear)}}]{\text{(linear)}} \widehat{M} \xrightarrow[\substack{\uparrow \\ \text{any algorithm}}]{\text{any algorithm}} e^{\widehat{M}} = e^{\widetilde{M}} \xrightarrow[\substack{\uparrow \\ \text{Inversion Lemma} \\ \text{of } \mathbf{power-dual} \\ \text{(linear)}}]{\text{Inversion Lemma of } \mathbf{power-dual} \text{ (linear)}} e^M.$$

For  $e^{Mt}$ ,

$$Mt \xrightarrow[\substack{\uparrow \\ \text{(linear)}}]{\text{(linear)}} \widehat{Mt} = \widehat{M}t \xrightarrow[\substack{\uparrow \\ \text{any algorithm}}]{\text{any algorithm}} e^{\widehat{Mt}} = e^{\widehat{M}t} = e^{\widetilde{Mt}} \xrightarrow[\substack{\uparrow \\ \text{Inversion Lemma} \\ \text{of } \mathbf{power-dual} \\ \text{(linear)}}]{\text{Inversion Lemma of } \mathbf{power-dual} \text{ (linear)}} e^{Mt}.$$

Other general properties:

- If  $P$  is a stochastic matrix, then the first row of  $\tilde{P}$  is  $(1 \ 0 \ \dots \ 0)$ .  
But recall that  $\tilde{P}$  needs not be stochastic.
- If  $A$  is an infinitesimal generator, then the first row of  $\hat{A}$  is  $(0 \ 0 \ \dots \ 0)$ . Recall that  $\hat{A}$  needs not be an infinitesimal generator.
- Connection with the dual of a Markov process as defined by Siegmund in 1976 and developed in Anderson's book: if  $A$  is the infinitesimal generator of Markov process  $X$  and if  $\hat{A}$  is also an infinitesimal generator, the associated Markov process  $X^*$  is the dual of  $X$ .

# Talk's content

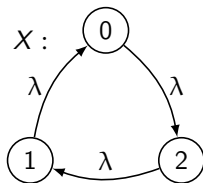
- 1 Introduction
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# Overview

- This work generalizes the use of duality in the theory of Markov processes as proposed by Siegmund (see Anderson's book). The dual doesn't always exist.
- In previous work we proposed to define what we call here exponential-dual, adopting **a purely algebraic approach**, when we realized that the stochastic framework was not necessary for the interest we had in the concept. Many proofs needed to be rewritten.
- Here, we decompose explicitly the cases of discrete time and continuous time, and exhibit their connections, making the construction more complete and consistent.

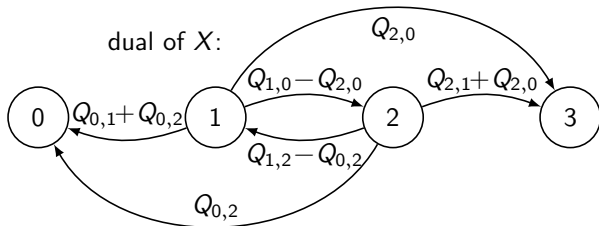
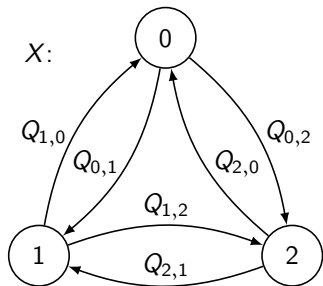
# On dual's existence (Markov processes)

- This concerns the historical origin of this work, in the stochastic processes area.
- For instance, consider an homogeneous “ring” illustrated here for 3 states:



- Whatever the size of the ring and the value of  $\lambda$ , the associated Markov process has no dual (for those familiar with duality in Markov processes, the problem is the monotonicity condition impossible to satisfy here).

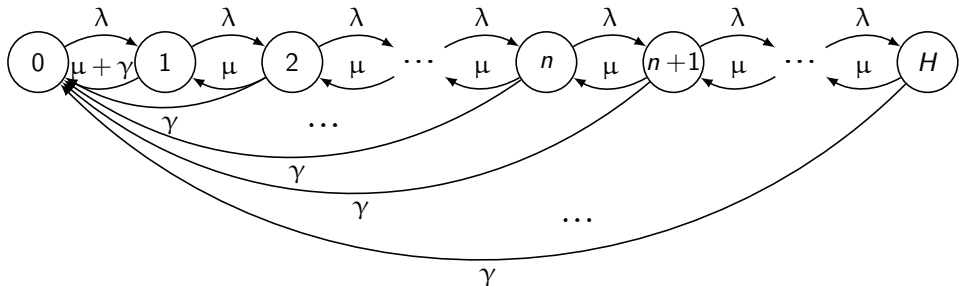
Consider now the general Markov process with 3 states:



Here, the dual exists iff  $Q_{1,0} > Q_{2,0}$  and  $Q_{1,2} > Q_{0,2}$ .

# Zooming on an example

Consider here the  $M/M/1/H$  queue plus “catastrophes” (or failures) as shown in the following Markovian graph:





- Take  $H = 4$ , to shorten the slides. Denote  $\Lambda = \lambda + \mu + \gamma$ . We have

$$A = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu + \gamma & -\Lambda & \lambda & 0 & 0 \\ \gamma & \mu & -\Lambda & \lambda & 0 \\ \gamma & 0 & \mu & -\Lambda & \lambda \\ \gamma & 0 & 0 & \mu & -(\mu + \gamma) \end{pmatrix}$$

- If we move to discrete time using Uniformization, the computation of  $e^{At}$  translates into that of evaluating  $U^k$  where using the notation  $p = \lambda/\Lambda$ ,  $q = \mu/\Lambda$ ,  $r = \gamma/\Lambda$ ,  $U = I - A/\Lambda$  is

$$U = \begin{pmatrix} q+r & p & 0 & 0 & 0 \\ q+r & 0 & p & 0 & 0 \\ r & q & 0 & p & 0 \\ r & 0 & q & 0 & p \\ r & 0 & 0 & q & p \end{pmatrix}$$

The powers of  $U$  are hard to obtain in closed-form here.

- If we use the exponential-dual (plus Uniformization), we must compute the powers of

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ p & 0 & q & 0 & 0 & r \\ 0 & p & 0 & q & 0 & r \\ 0 & 0 & p & 0 & q & r \\ 0 & 0 & 0 & p & q+r & r \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Due to those '1' at the corners, the use of lattice path combinatorics is much easier here than with matrix  $U$  of previous slide. That's an example of situation where we can develop the process until its end.

- The infinite case can also be solved in the same way.

The formal solution of the ODEs (Chapman-Kolmogorov equations) is given below, using the notation  $\rho = \lambda/\mu$ , for the case of the queue starting empty at time 0 (the general case is similar while more complex):

$$p_j(t) = \pi_j - \frac{2\mu\rho^{j/2}}{H+1} \sum_{1 \leq h \leq H} \left\{ \frac{e^{-(\lambda + \mu + \gamma - 2\sqrt{\lambda\mu} \cos(\frac{h\pi}{H+1}))t}}{\lambda + \mu + \gamma - 2\sqrt{\lambda\mu} \cos(\frac{h\pi}{H+1})} \times \right. \\ \left. \times \sqrt{\rho} \sin\left(\frac{h\pi}{H+1}\right) \left[ \sin\left(\frac{j h \pi}{H+1}\right) - \sqrt{\rho} \sin\left(\frac{(j+1)h\pi}{H+1}\right) \right] \right\}.$$

In the formula,  $\pi_j$  denotes  $p_j(\infty)$  (steady-state), easy to evaluate symbolically.

## Some references

- **Duality by Siegmund:**

*The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes*, D. Siegmund, Ann. Prob., Vol. 6, 914–924, 1976.

- **Anderson's book:**

*Continuous-time Markov chains: an applications-oriented approach*, W.J. Anderson, Springer, New York, 1991 (out of press).

- **Transient distributions of queues using duality:**

*Dual processes to solve single server systems*, A. Krinik, G. Rubino, D. Marcus, R.J. Swift, H. Kasfy, and H. Lam, Journal of Statistical Planning and Inference, Vol. 135, Nb. 1, 702–713, 2005.

## Some references

- **Recent work on the Exponential-Dual:**

*The Exponential-Dual Matrix Method: Applications to Markov Chain Analysis*, G. Rubino and A. Krinik, in *Stochastic Processes and Functional Analysis: New Perspectives*, R. Swift, A. Krinik, J. Switkes and J. Park ed., in Contemporary Mathematics, AMS, Vol. 774, 2021.

- **One of several references by P. Lorek:**

SIEGMUND DUALITY FOR MARKOV CHAINS ON PARTIALLY ORDERED STATE SPACES, P. Lorek, *Probability in the Engineering and Informational Sciences*, 32(4), 495–521, 2018.

**Thanks for your time.**