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On Covering Smooth Manifolds with a Q-arrangement of Simplicies: An inductive Characterization of Q-matrices

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Abstract

This paper is concerned with a covering problem of smooth manifolds of dimension $n-1$ by stitching 2^n n-simplices formed with 2 n-lists of points along their common $(n-1)$-facets. The n-simplices are in bijective correspondence with the vertices of an n-dimensional hypercube; they could be degenerate and are allowed to overlap. We leverage the underlying inductive nature of the problem to give a (non-constructive) topological characterization. We show that for low dimensions such characterization reduces to studying the local geometry around the specific points serving to form the simplices, solving thereby the problem for $n \leq 3$. This covering problem provides a geometric equivalent reformulation of a relatively old, yet unsolved, problem that originated in the optimization community: under which conditions on the $n \times n$ matrix M, does the so called linear complementarity problem given by $w - Mz = q$, $w, z \geq 0$, and $w.z = 0$, have a solution (w, z) for all vectors $q \in \mathbb{R}^n$. If the latter property holds, the matrix M is said to be a Q-matrix.

Keywords. Simplicial covering, degeneracy and overlaps, spherical geometry, low-dimensional topology, linear complementarity problem.

1 Introduction

Given a vector $q \in \mathbb{R}^n$ and an $n \times n$ matrix M over the reals, the *linear complementarity problem* [4] asks whether there exist $w, z \in \mathbb{R}^n$ satisfying $w - Mz = q$, $w, z \geq 0$, and $w.z = 0$, where $x \geq 0$ means that x belongs to $\mathbb{R}^n_+$, the non-negative orthant (or cone) of $\mathbb{R}^n$, and $w.z$ is the scalar product of w and z. Let $\mathrm{LCP}(q, M)$ denote the set of solutions for q. When $\mathrm{LCP}(q, M)$ is non-empty for every vector q in $\mathbb{R}^n$, the matrix M is called a *Q-matrix*. When $\mathrm{LCP}(q, M)$ is reduced to a singleton, that is there exists a unique solution for every q, M is called a *P-matrix*. While a satisfying characterization for P-matrices has been known since the late fifties [16], the same cannot be said for Q-matrices despite several insights and serious attempts over the past sixty years.

To better appreciate the Q-matricity problem, and consequently its inherent difficulty, let us proceed by revisiting the related literature while applying Occam's razor to remove the unnecessary structure present in the original formulation, but superfluous for the existence of solutions.

The first step would be to interpret the existence of solutions as a covering problem of $\mathbb{R}^n$, thereby stepping away from thinking of Q-matricity as a property of the linear application represented by M [16, 12]. [1] To do so, let us start by inverting the original question: instead of fixing $q \in \mathbb{R}^n$ and asking what are its solutions, we rather fix (w, z) and ask what are the vectors q that could be solved by (w, z)? The constraint $w.z = 0$ implies that at least n components of w and z must vanish, whereas the constraints $w, z \geq 0$ force the n remaining components to be non-negative. Let J denote the subset of indices corresponding to those non-negative components and let M_J denote the $n \times n$ square matrix obtained by restricting the $n \times 2n$ matrix $\begin{pmatrix} I_n & -M \end{pmatrix}$ to J. Then, any q in the image set $M_J(\mathbb{R}^n_+)$ must have a solution by construction. Therefore, asking for a solution for each $q \in \mathbb{R}^n$ is equivalent to asking whether the 2^n cones $M_J(\mathbb{R}^n_+)$, called *complementary cones*, cover $\mathbb{R}^n$.

The second step would be to notice that one can drop the positive homogeneity of cones [10]. Indeed, as soon as $q \in \mathbb{R}^n$ has a solution (w, z), the vector λq, for any positive scalar λ has a solution as well, namely $(\lambda w, \lambda z)$. Moreover, the null vector has the trivial solution $(0, 0)$ regardless of M. Thus, instead of $\mathbb{R}^n$, one can rather focus on a sphere S^{n-1}, where each (proper) ray of $\mathbb{R}^n$ is represented uniquely by one point of the sphere. Thus complementary cones are represented by spherical $(n-1)$-simplices and each two neighbors share a common (spherical) $(n-2)$-simplex.

It now becomes clear that the choice of $\mathbb{R}$ as the base field as well as the entire vector space structure of $\mathbb{R}^n$ can be dropped all together for they do not play any particular role when interpreting Q-matricity as a covering problem. One therefore should not expect a decomposition exploiting the vector space structure of $\mathbb{R}^n$ to be insightful or useful to better understand the covering problem.

[1] From this perspective, the terminology Q-matricity itself might seem to be a misnomer. Nonetheless, thinking of Q-matricity as a property of the linear application M proved to be useful [3].

Contributions. We formally state the *Q-covering* problem (Definition 1) as an abstract, yet equivalent, formulation of Q-matricity. The key insight of such a reformulation resides in its inductive nature on which we rely to state a non-constructive topological characterization (Theorem 1). We show that in low dimensions such characterization reduces to studying the local geometry around the points serving to form the simplices, solving thereby the problem for $n \leq 3$. In particular, Section 3 lists exhaustively under which conditions the Q-arrangement is covering for the 2-sphere (Proposition 4). Conjecture 1 is a challenging quest that asks whether our current understanding of the problem still hold for $n \geq 4$. It pertains to understanding the vertex set of the holes left after laying all the simplices on the sphere.

2 Q-covering: definition and inductive characterization

We start by formally stating the original covering problem before stating the formulation we will be interested in. Let E^n be the n-dimensional Euclidean space and let $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ be two n-lists of non-vanishing vectors in E^n. A *complementary cone* is a cone spanned by n vectors $\{a_1, \ldots, a_n\}$ satisfying $a_i \in \{s_i, t_i\}$ for all i, that is the set $\{x \in E^n \mid \exists \lambda_1, \ldots, \lambda_n \geq 0.\ x = \sum_i \lambda_i a_i\}$. A complementary cone is *non-degenerate* if its generators are linearly independent. The two n-lists are said to be *non-degenerate* if all complementary cones are non-degenerate. The original covering problem asks whether E^n is included in the union of all complementary cones. [2]

The arrangement of complementary cones, known as the *Q-arrangement*, is not arbitrary: each cone has n *neighbors* where two cones are neighbors if and only if they have $n-1$ common generators. In fact, the cones are in correspondence with the vertices of an n-dimensional hypercube graph Q_n where the neighboring relationship is represented by the adjacency of the vertices of Q_n. Stitching together all the cones, one at a time, along their common $(n-1)$-facet amounts to following the longest Hamiltonian cycle of Q_n (of length 2^n). Notice that the collection of simplices need not form a simplicial set [7] as the intersections of simplices need not be reduced to lower dimensional simplices.

As alluded to in the introduction, it is relevant to observe that as soon as a non-zero vector q is covered, then so is λq for any positive scalar λ. Thus by introducing the equivalence relation $\simeq$ over E^n defined by $u \simeq v$ if and only if $u = \lambda v$ for some positive scalar λ, one can equivalently study the covering problem on the quotient space $E^n /\!\simeq$ instead of E^n. Since $q = 0$ is trivially covered as it belongs to all complementary cones, it suffices to study the covering of $(E^n \setminus \{0\}) /\!\simeq$ which can be represented by S^{n-1} (indeed, the space obtained by puncturing a single point from the Euclidean space E^n is homeomorphic to the sphere S^{n-1}). Although removed, the center of the sphere S^{n-1} plays a role in fixing the representatives of the original vectors s_i and t_i on S^{n-1}. It is therefore still needed to properly define the problem. In summary, in order to define the covering problem, the basic data that one needs consists of: a smooth manifold X of dimension n, a point $o \in X$, a neighborhood $U(o) \subseteq X$ of o and two n-lists of points on the boundary of $U(o)$. For convenience, we fix X to the sphere S^n in the sequel. The *Q-covering* problem is formally stated next and can be thought of as an abstract, yet equivalent, formulation of the original covering problem.

Definition 1 (Q-covering)**.** *Let o be a point in the unit sphere S^n, and let $B(o) \subseteq S^n$ be an open ball of radius $\epsilon \ll 1$ centered at o. Let $S^{n-1}(o) = \partial B(o)$, the boundary of $B(o)$ (which, as the notation suggests, is itself a sphere of dimension $n-1$). Given two n lists of points $\{s_1, \ldots, s_n\}$ and $\{t_1, \ldots, t_n\}$ on $S^{n-1}(o)$, consider the 2^n spherical $(n-1)$-simplices $C_j = \langle a_1, \ldots, a_n \rangle$ where $a_i \in \{s_i, t_i\}$. The* Q-covering *decision problem asks whether the Q-arrangement of the simplices C_j covers entirely $S^{n-1}(o)$? That is whether the inclusion $S^{n-1}(o) \subseteq \Sigma(o)$ holds, where*

$$\Sigma(o) = \bigcup_{a_i \in \{s_i, t_i\}} \langle a_1, \ldots, a_n \rangle \ .$$

One says that a point $q \in S^{n-1}(o)$ is covered *if it belongs to a simplex C_j.*

We will prefer the more concise notation $\bigoplus_{i=1}^n [s_i, t_i]$ to denote the union $\Sigma(o)$ of the simplices in Definition 1.

Remark 1. *While the use of the $\oplus$ operator is introduced and subsequently used as a notation convenience, it found its origin as (yet another) equivalent formulation of the original covering problem as the following Minkowski sum. Each pair $\{s_i, t_i\}$ encodes a* generalized line *L_i denoted $[s_i, t_i]$ and defined as $L_i = \{x \in E^n \mid \exists \lambda, \beta \geq 0;\ x = \lambda s_i + \beta t_i$ and $\lambda\beta = 0\}$. The union of all complementary cones is therefore equal to the Minkowski sum $\bigoplus_{i=1}^n [s_i, t_i]$.*

Once the points s_i and t_i are fixed on $S^{n-1}(o)$, drawing all the facets of all the simplices partitions the sphere $S^{n-1}(o)$ into *cells*.

[2] The requirement that all the vectors s_i and t_i are non-vanishing becomes justified since as soon as s_i (or t_i) vanishes for an index i, the covering problem asks whether finitely many degenerate cones can cover a full dimensional open set in E^n, which is impossible.

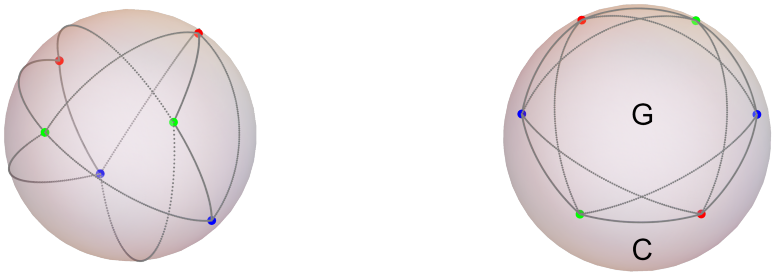


Figure 1: Cells ($n = 3$). The colored dots on the sphere represent the points s_i and t_i where each pair has a different color. The paths between them are the (spherical) 1-simplices. Complementary 2-simplices have three different colored dots as vertices (0-simplices). The cell G is a ghost cell. The cell C is not simplicial nor convex.

Definition 2 (Cell)**.** *A* cell *is a closed non-empty connected subset of $S^{n-1}(o)$ delimited by an $(n-2)$-facet in each direction and such that its interior does not intersect any other facet.*

A cell is not necessarily simplicial nor convex. In general, an $(n-1)$-simplex does not correspond to a unique cell, but could be partitioned into several cells. Figure 1 shows two possible configurations for $n = 3$. As shown below, however, a cell cannot be partially covered, it either belongs to $\Sigma(o)$ or it is entirely not covered.

Lemma 1. *Let $p, q \in \Sigma(o)$ such that there exists a continuous path from p to q that does not intersect any $(n-2)$-facet of all the $(n-1)$-simplices defined by the two n-lists of points s_i and t_i. Then any point in that path is also in $\Sigma(o)$.*

Proof. Without loss of generality, we can write the continuous path from p to q as $r(t) = \sum_{i=1}^{n} \alpha_i(t) v_i$ where t is a real parameter in the interval $[0, 1]$, $v_i \in \{s_1, \ldots, s_n, t_1, \ldots, t_n\}$, $\alpha_i(t) > 0$ for all t, $p = r(0)$, and $q = r(1)$. Since p is in $\Sigma(o)$, it must be in the interior of a $(n-1)$-simplex, C_p say, and likewise, q must belong to the interior of a (n-1)-simplex C_q. If there exists t such that $r(t) \notin \Sigma(o)$, then the path must exit $C_p \cup C_q$ crossing a $(n-2)$-facet and thereby contradicting the hypothesis. Thus no such t exists and the entire path must be in $\Sigma(o)$. □

Thus, it suffices to consider one point in the interior of a cell to check whether the cell is covered: if the point is covered, then the entire cell is in $\Sigma(o)$, otherwise, the entire cell is not covered. This means that the hole left uncovered once all the simplices are laid down on the sphere are a union of cells.

Definition 3 (Surrounded Point)**.** *We say that a point $q \in S^{n-1}(o)$ is* surrounded *if and only if there exists an (open) neighborhood $U \subseteq S^{n-1}(o)$ of q such that U is covered, i.e. $U \subseteq \Sigma(o)$.*

Theorem 1 (Localization)**.** *If each cell has at least one surrounded vertex (0-simplex), then $S^{n-1}(o)$ is covered.*

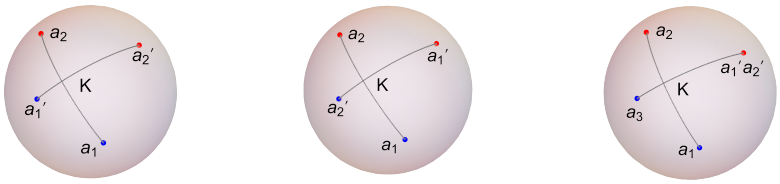
Proof. Fix q in $S^{n-1}(o)$. If q belongs to a (n-2)-facet of any cell, then it must belong to $\Sigma(o)$ since, by construction, those facets are subsets of facets of the involved complementary $(n-1)$-simplices.

If q belongs to the interior of a cell C, then by hypothesis, C has a surrounded vertex v, that is an (open) neighborhood $U \subseteq \Sigma(o)$ that intersects C°. Let $p \in U \cap C^\circ$ denote a point in this intersection. Since $p, q \in C^\circ$, which is a connected open set, one can construct a continuous path starting and ending at p, passing through q and entirely in C°. Thus, by lemma 1, q is also in $\Sigma(o)$. We thus proved, as required, that for any $q \in S^{n-1}(o)$, $q \in \Sigma(o)$. □

The characterization provided by Theorem 1 requires a test for at least one vertex per cell in order to check that the entire sphere is covered. Since a vertex is necessarily common to several cells, this already reduces the number of points to check at the cost of checking a stronger property (surrounding) than the simple covering. A natural question therefore arises: what would be the minimal set of such vertices? In order to answer this question, let $V(C)$ denote the set of vertices of a cell C. The set $V(C)$ may not contain any of the original points a_i. In this case, such cells are called *ghost cells* as they are not accessible from any a_i (in the sense that any path starting within a ghost cell must cross a boundary in order to reach one of the a_i's). In dimension 2, ghost cells do not exist. While it is possible to construct such cells in higher dimensions (for instance, the cell G in Figure 1), we remarked that ghost cells enjoy the following interesting property.

Proposition 1. *Let $n = 3$. Ghost cells are covered whenever they exist.*

Proof. Let K denote a ghost cell and let A be a 1-facet of K (edge). Assume without loss of generality that A is generated by a_1 and a_2. A divides the sphere into two hemispheres, one containing K, denoted by $H_A(K)$ and (the closure of) its complement. Let $a_3 \in \{s_3, t_3\}$. If $a_3 \in H_A(K)$ then K is covered by the 2-simplex given by $\langle a_1, a_2, a_3\rangle$ as otherwise K would be cut by a facet, contradicting the assumption of K being a cell. We are thus left with the case where $s_3, t_3 \in H_A(K)^c$. To form K, A must be crossed by another 1-facet B of K. For this crossing to hold, at least one generator g of B must be in $H_A(K)^c$. We distinguish three cases depicted below following the same order from left to right: (i) $g = a_1'$, (ii) $g = a_2'$ and (iii) $g = a_3$.



Case (i): If $g = a_1'$, at least one generator of B must be in $H_A(K)$ as otherwise B won't be crossing A and it must be a_2' (the only vector left) since $s_3, t_3 \in H_A(K)^c$. As discussed above, if s_3 or t_3 belong to $H_B(K)$ then K is covered. Thus, suppose next that $s_3, t_3 \in H_A(K)^c \cap H_B(K)^c$. Consider the 1-facet $C = \langle a_1, a_2'\rangle$ (which may support or not a 1-facet of K). None of A nor B are crossing C (they only touch in shared ends), and as A crosses B, both A and B must be in $H_C(K)$. So $K \subseteq H_A(K) \cap H_B(K) \cap H_C(K)$. If an $a_3 \in H_C(K)$ then $K \subseteq \langle a_1, a_2', a_3\rangle$ and is therefore covered (this relies on the specific position of a_3 regardless of whether C serves to form a 1-facet of K). Finally, if $s_3, t_3 \in H_C(K)^c$, then $\Sigma(o) \subseteq H_A(K)^c \cup H_B(K)^c \cup H_C(K)^c = (H_A(K) \cap H_B(K) \cap H_C(K))^c$ and therefore it is impossible to complete the construction of K (as a ghost cell) because $a_1, a_2' \in V(K)$, the set of vertices of K.

Case (ii): If $g = a_2'$, at least one generator of B must be in $H_A(K)$ as otherwise B won't be crossing A and it must be a_1' (the only vector left) since $s_3, t_3 \in H_A(K)^c$. If an $a_3 \in H_B(K)$ then K is covered. Thus, suppose next that $s_3, t_3 \in H_A(K)^c \cap H_B(K)^c$. Then $K \subseteq H_A(K) \cap H_B(K)$ and $\Sigma(o) \subseteq H_A(K)^c \cup H_B(K)^c = (H_A(K) \cap H_B(K))^c$ and therefore, like in the previous case, K cannot be a ghost cell. (So the only difference with the previous case is that in this case, one simply cannot construct a ghost cell, whereas in the previous case, we showed constructively that it is covered if it exists.)

Case (iii): $g = a_3$ with $a_3 \in \{s_3, t_3\}$. The 1-facet B has $a_3 \in H_A(K)^c$ as a generator and must have another generator $a_i' \in H_A(K)$, $i \in \{1, 2\}$. Thus we distinguish two sub-cases. Sub-case(1): $a_i' = a_1'$. If $a_2' \in H_B(K)$, then $K \subset \langle a_1', a_2', a_3\rangle$. If $a_2' \in H_B(K)^c \cap H_A(K)$ then $K \subseteq \langle a_1, a_2', a_3\rangle$. If $a_3' \in H_A(K)^c \cap H_B(K)$ then $K \subseteq \langle a_1', a_2, a_3'\rangle$. We are thus left with $a_2', a_3' \in H_A(K)^c \cap H_B(K)^c$, but then $\Sigma(o) \subseteq H_A(K)^c \cup H_B(K)^c$ whereas $K \subseteq H_A(K) \cap H_B(K)$ and once more K cannot be a ghost cell. Sub-case (2): $a_i' = a_2'$ and $K \subseteq \langle a_1, a_2', a_3\rangle$. □

We conjecture that the property holds for any finite dimension n.

Conjecture 1. *Ghost cells are covered whenever they exist.*

Remark 2. *Among all the proofs we had for $n = 3$, the proof detailed above seems to be the best candidate for a possible generalization in higher dimensions, perhaps with the help of a proof assistant (like Coq [2]), to overcome the inherent combinatorial explosion.*

Assuming Conjecture 1, Theorem 1 tells us that Q-covering is equivalent to checking whether the points s_i and t_i are surrounded. A key observation to understand under which conditions a point s_i (or t_i) is surrounded is that, it suffices to "zoom in" on that point and analyze the (local) contributions of the different involved $(n-1)$-simplices. We formalize below this intuition. Since the index i is irrelevant for the upcoming discussion, we carry on with $i = n$ for simplicity.

Let a_n denote s_n or t_n and assume that $a_n \in S^{n-1}(o)$. Consider a ball $B(a_n) \subseteq S^{n-1}(o)$ of radius $\epsilon \ll 1$ centered at a_n. Let $S_\epsilon^{n-2}(a_n)$ denote $\partial B(a_n)$. and let $o(a_n)$ denote the opposite of a_n with respect to the point o. [3] Any point $v \in S^{n-1}(o)$ has a unique representative $\bar{v}$ on $S^{n-2}(a_n)$: it suffices to take the (unique) path of length π on $S^{n-1}(o)$ that starts at a_n, goes through v and ends at $o(a_n)$, which necessarily intersects $S^{n-2}(a_n)$ in a unique point $\bar{v}$. For completeness, we set $\overline{o(a_n)}$ to a_n.

Proposition 2. *The point a_n is surrounded if and only if $S^{n-2}(a_n) \subseteq \Sigma(a_n) \cup \Sigma'(a_n)$ where*

$$\Sigma(a_n) = \bigoplus_{i=1}^{n-1} [\bar{s}_i, \bar{t}_i] \quad \text{and} \quad \Sigma'(a_n) = \bigcup_{\substack{a_1,\dots,a_{n-1} \text{ s.t.} \\ a_n \in \langle a_1,\dots,a_{n-1},a_n'\rangle}} \bigoplus_{j=1}^{n-1} [\bar{a}_j, \bar{a}_n'] \ .$$

[3] The point $o(a_n)$ should not be confused with $-a_n$ in general since $B(a_n)$ is a ball on a sphere and not a ball on an Euclidean flat space.

Proof. The sum $\Sigma(a_n)$ encodes precisely the representatives of the $(n-1)$-simplices having a_n as a vertex. The sum $\Sigma'(a_n)$ encodes subsets of all the remaining $(n-1)$-simplices (precisely those having a'_n as a vertex) participating in surrounding a_n. Indeed, as soon as a_n belongs to an $(n-1)$-simplex $\langle a_1, \ldots, a_{n-1}, a'_n \rangle$, one has to account for exactly $n-1$ additional $(n-1)$-simplices contributing in surrounding a_n, namely those obtained from $\langle a_1, \ldots, a_{n-1}, a'_n \rangle$ by substituting each a_i by a_n regardless of whether a_n is in the interior or the boundary of the simplex (cf. to the gray regions shown in Figure 2 for an illustration in $n = 3$). □

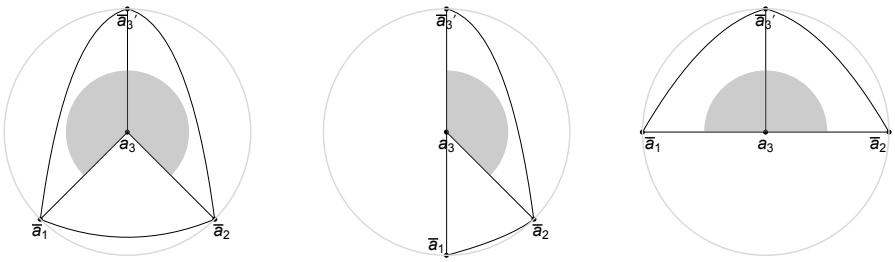


Figure 2: Three configurations where $a_3 \in \langle a_1, a_2, a_3 \rangle$ together with their (local) contribution depicted in gray. All can be encoded using the same sum $[\bar{a}_1, \bar{a}'_3] \oplus [\bar{a}_2, \bar{a}'_3]$ (as in $\Sigma'(a_3)$).

Remark 3. *Once the n-lists of points are fixed on $S^{n-1}(o)$, the Q-covering decision is independent from the point o in the following sense: only the relative positions of the involved simplices is relevant. This means that the Q-covering decision is* invariant *under any translation or uniform scaling of the sphere $S^{n-1}(o)$. This invariance is also valid for the inclusion $S^{n-2}(a_n) \subseteq \Sigma(a_n) \cup \Sigma'(a_n)$ of Proposition 2 and can be leveraged to symbolically compute the representative $\bar{v}$: it suffices to take the orthogonal projection of v on T_{a_n}, the tangent space to $S^{n-1}(o)$ at a_n, and consider in T_{a_n} the equivalence classes with respect to $\simeq$. Indeed, such a projection amounts to simply translate the entire $S^{n-2}(a_n)$ to T_{a_n}. This observation becomes relevant when it comes to checking Q-matricity algorithmically: it is easier to compute an orthogonal projection of a symbolic vector than to compute the intersection of a spherical path and a (hyper) circle.*

The main interest of proposition 2 resides in the inductive character it exhibits. Indeed, to solve the covering problem of $S^{n-1}(o)$, one is lead to solve finitely many surrounding problems which are themselves similar (but in general not identical) to covering problems in $S^{n-2}(a_i)$. Under some extra conditions, the inductive character becomes even more apparent. For instance, the following theorem states that when the considered collection of the 2^n simplices define a simplicial complex, the Q-covering is fully characterized inductively: whenever one zooms in around the points s_i and t_i, the same geometric structure is preserved.

Theorem 2 (Simplicial complex covering)**.** *The Q-covering defines a non-degenerate partition of $S^{n-1}(o)$ if and only if for all $i = 1, \ldots, n$, $S^{n-2}(s_i) \subseteq \Sigma(s_i)$, $S^{n-2}(t_i) \subseteq \Sigma(t_i)$, and $\Sigma'(s_i) = \Sigma'(t_i) = \emptyset$.*

Proof. Recall that in [16], it is shown that under the non-degeneracy assumption, the n-simplices partition the sphere if and only if, each pair $\{s_i, t_i\}$ is separated by all the great circles defined by dropping the index i in $a_1, \ldots, a_n$, where $a_j \in \{s_j, t_j\}$ (that is all the boundary simplices obtained by dropping the ith index).

Necessity. Under the stated separation condition, ghost cells do not exist as the $(n-2)$-simplices intersect only on their common vertices [16]. Therefore it suffices to focus on the s_i and t_i. By Theorem 1 and Proposition 2, $S^{n-2}(s_i) \subseteq \Sigma(s_i) \cup \Sigma'(s_i)$ (and similarly for t_i). If $\Sigma'(s_i) \neq \emptyset$, then either two simplices overlap or some simplices are degenerate. Each case, however, contradicts the separation hypothesis according to [16]. This leads to the stated condition.

Sufficiency. We have $S^{n-2}(s_i) \subseteq \Sigma(s_i)$. If there exists a great circle that doesn't separate t_i and s_i, then $\Sigma'(t_i)$ would not be empty as t_i belongs to a simplex having s_i as a vertex. This clearly contradicts the hypothesis. Thus, for all pairs, the separation condition must hold. □

We end this section by stating the following simple, yet useful, necessary condition on pairs $\{s_i, t_i\}$ for a Q-covering to hold. We will assume that this condition holds in the sequel.

Lemma 2. *If $s_i = t_i$ for an index i, then $S^{n-1}(o)$ cannot be covered.*

Proof. If $s_i = t_i$ then the point $o(s_i)(= o(t_i))$ cannot be covered by any simplex as the length of all 1-facets (spherical paths) is less than π. □

3 Characterizing Q-covering for $n = 3$

In this section we present a full characterization of Q-covering for $n = 3$. This means that we are able to provide a set of constraints on the involved vectors t_i, s_i that are satisfied if and only if all the sphere is covered by a Q-arrangement. Consequently, this also provides a set of constraints on the entries of a matrix M to be a Q-matrix. We start by characterizing the simpler planar case to showcase how the method works.

3.1 Planar Q-covering

For $n = 2$, ghost cells do not exist.

Theorem 3. *$S^1(o)$ is covered if and only if the four involved vectors s_1, s_2 and t_1, t_2 are surrounded.*

Proposition 2 states that to check if $a_2 \in \{s_2, t_2\}$ is surrounded, one has to check whether the following inclusion holds $S^0(a_2) \subseteq \Sigma(a_2) \cup \Sigma'(a_2)$ where

$$\Sigma(a_2) = [\bar{s}_1, \bar{t}_1] \quad \text{and} \quad \Sigma'(a_2) = \bigcup_{\substack{a_1 \text{ s.t.} \\ a_2 \in \langle a_1, a_2' \rangle}} [\bar{a}_1, \bar{a}_2'] \ .$$

The inclusion above can be interpreted as follows: either a_2' does not participate in covering a_2, thus $\Sigma'(a_2) = \emptyset$ and the inclusion $S^0(a_2) \subseteq [\bar{s}_1, \bar{t}_1]$ must hold, or a_2' does play a role in covering a_2 totally or partially. If a_2 is in the interior of a simplex $C = \langle a_1, a_2' \rangle$, then a_2 is clearly surrounded. If a_2 is on the boundary of C, then either $a_2 = a_2'$ or $a_2 = a_1$. If $a_2 = a_2'$, the global covering of $S^1(o)$ fails (see Lemma 2). If, however, $a_2 = a_1$, then the simplices with vertex a_1 can be also used to surround a_2. Out of the four simplices, one is redundant and degenerate (namely $\langle a_1, a_2 \rangle$), and surrounding with the two remaining simplices ($\langle a_1, a_2' \rangle$ and $\langle a_1', a_2' \rangle$) can be equivalently encoded as $S^0(a_2) \subseteq [\bar{a}_1', \bar{a}_2']$. In summary:

Proposition 3. *For $S^1(o)$ to be covered, the point a_2 must be surrounded in one of the following ways:*

1. *$S^0(a_2) \subseteq [\bar{s}_1, \bar{t}_1]$ ($\Sigma(a_2)$ suffices),*
2. *$a_2 \in \langle a_1, a_2' \rangle^\circ$ (a_2 is in the interior of $\langle a_1, a_2' \rangle$),*
3. *$a_2 = a_1 \wedge S^0(a_2) \subseteq [\bar{a}_1', \bar{a}_2']$ (a_2 coincides with a_1).*

In light of Remark 3, the inclusions $S^0(a_2) \subseteq [\bar{s}_1, \bar{t}_1]$ and $S^0(a_2) \subseteq [\bar{a}_1', \bar{a}_2']$ can be effectively checked using orthogonal projections on the tangent space T_{a_2} to $S^1(o)$ at a_2. If $\hat{v}$ denotes the orthogonal projection of v on T_{a_2}, this amounts to verify the respective inequalities $\hat{s}_1 \hat{t}_1 < 0$ and $\hat{a}_1' \hat{a}_2' < 0$. Moreover, checking $a_2 \in \langle a_1, a_2' \rangle^\circ$ amounts simply to check that the determinants of the three matrices $\begin{pmatrix} a_1 & a_2' \end{pmatrix}$, $\begin{pmatrix} a_2 & a_2' \end{pmatrix}$, and $\begin{pmatrix} a_1 & a_2 \end{pmatrix}$ have the same sign. As an immediate application, these conditions give a symbolic characterization of Q-matricity for $n = 2$. As custom in the literature, assuming $s_1 = e_1$, $s_2 = e_2$, and t_1, t_2 are the columns of the matrix $-\left(\begin{smallmatrix} m_1 & m_2 \\ m_3 & m_4 \end{smallmatrix}\right)$, with $m_1^2 + m_3^2 = m_2^2 + m_4^2 = 1$, the conditions of Proposition 3 translate, for $a_2 = e_2$, to the following set of constraints on the entries of M:

1. $-m_1 < 0$
2. $-m_4 > 0 \wedge m_2 > 0$ (if $a_1 = e_1$) or $-\det(M), m_2, -m_1$ have the same sign (if $a_1 = t_1$)
3. $-m_1 = 0 \wedge -m_3 = 1 \wedge -m_4 < 0$

A similar reasoning applies to t_2, e_1, and t_1 and combining together all the constraints (5 at most per point) leads to a full symbolic characterization for Q-matricity in the plane. For instance, we can readily use such a characterization to state that if two points a_1 and a_2 are opposite (with respect to o) then the Q-covering of $S^1(o)$ fails. As we shall see in the next section, it is possible to have a Q-covering of $S^2(o)$ with degenerate 3-simplices that are reduced to an arc of length π (equivalently non-pointed complementary cones with respect to the original covering problem).

3.2 Spacial Q-covering

For $n = 3$, by Proposition 1, ghost cells are covered whenever they exist. It follows that

Theorem 4. *$S^2(o) \subseteq \Sigma(o)$ if and only if the 6 points s_1, s_2, s_3 and t_1, t_2, t_3 are surrounded.*

According to Proposition 2, checking whether a_3 is surrounded amounts to check the inclusion $S^1(a_3) \subseteq \Sigma(a_3) \cup \Sigma'(a_3)$ where

$$\Sigma(a_3) = [\bar{s}_1, \bar{t}_1] \oplus [\bar{s}_2, \bar{t}_2] \quad \text{and} \quad \Sigma'(a_3) = \bigcup_{\substack{a_1, a_2 \text{ s.t.} \\ a_3 \in \langle a_1, a_2, a_3' \rangle}} [\bar{a}_1, \bar{a}_3'] \oplus [\bar{a}_2, \bar{a}_3'] \ .$$

Proposition 4. *If $S^2(o) \subseteq \Sigma(o)$ then a_3 is surrounded in one of the following ways (where $i \in \{1,2\}$ and $j \neq i$):*

1. $S^1(a_3) \subseteq [\bar{s}_1, \bar{t}_1] \oplus [\bar{s}_2, \bar{t}_2]$ *($\Sigma(a_3)$ suffices)*
2. $a_3 \in \langle a_1, a_2, a'_3 \rangle^\circ$ *(interior of a simplex)*
3. $a_3 \in \langle a_i, a'_3 \rangle^\circ \wedge S^1(a_3) \subseteq [\bar{a}_i, \bar{a}'_3] \oplus [\bar{a}_j, \bar{a}'_j]$ *(two simplices from $\Sigma(a'_3)$)*
4. $a_3 = a_i \wedge S^1(a_3) \subseteq [\bar{a}'_i, \bar{a}'_3] \oplus [\bar{a}_j, \bar{a}'_j]$ *(a_3 coincides with a_i)*
5. $a_3 \in \langle a_i, a'_3 \rangle^\circ \wedge \bar{a}'_j = \bar{a}'_3 \wedge S^1(a_3) \subseteq [\bar{a}'_i, \bar{a}_j] \oplus [\bar{a}'_3, \bar{a}_j]$ *(a mix from $\Sigma(a_3)$ and $\Sigma'(a_3)$, cf. Figure 3 left)*
6. $a_3 \in \langle a_1, a_2 \rangle^\circ \wedge a_3(\bar{a}'_i) \notin \langle \bar{a}_j, \bar{a}'_j \rangle \wedge S^1(a_3) \subseteq [\bar{a}_1, \bar{a}_2] \oplus [\bar{a}'_j, \bar{a}'_3]$ *(a mix from $\Sigma(a_3)$ and $\Sigma'(a_3)$, cf. Figure 3 right)*

Proof. In what follows, $a_i \in \{s_i, t_i\}$, and $a_{12} \in \{a_1, a_2\}$. If a_3 doesn't belong to any simplex $\langle a_1, a_2, a'_3 \rangle$ then a_3 is surrounded if and only if $S^1(a_3) \subseteq [\bar{s}_1, \bar{t}_1] \oplus [\bar{s}_2, \bar{t}_2]$. Assume a_3 belongs to one or several simplices having a'_3 as a vertex. Recall that $a_3 \neq a'_3$ by Lemma 2. If a_3 belongs to the interior of any simplex $\langle a_1, a_2, a'_3 \rangle$, then it is surrounded. Otherwise it belongs to one or several boundaries (1-facets) without being in the interior of any simplex. There are three possible cases (a simplex in $n = 3$ is a spherical triangle): (i) $a_3 = a_{12}$, (ii) a_3 belongs to the interior of $\langle a_{12}, a'_3 \rangle$, (iii) a_3 belongs to the interior of $\langle a_1, a_2 \rangle$.

Case (i). Fix a_{12} to a_1. (otherwise, it suffices to swap a_1 and a_2 in what follows). The four simplices having a_1 as a vertex can contribute to cover a_3. Out of the 8 available simplices having a_3 as a vertex, two are redundant and two are degenerate. Thus only four can be effectively used to surround a_3, namely $\langle a_1, a_2, a'_3 \rangle$, $\langle a_1, a'_2, a'_3 \rangle$, $\langle a'_1, a_2, a_3 \rangle$, and $\langle a'_1, a'_2, a_3 \rangle$. Since $a_3 = a_1$, the problem can be equivalently encoded as $S^1(a_3) \subseteq [\bar{a}_2, \bar{a}'_2] \oplus [\bar{a}'_1, \bar{a}'_3]$.

Case (ii). Fix a_{12} to a_1 (otherwise, it suffices to swap a_1 and a_2 in what follows). If the great circle $C = (a_1, a'_3)$ separates a_2 and a'_2, then a_3 is surrounded. This case can be equivalently encoded as $S^1(a_3) \subseteq [\bar{a}_1, \bar{a}'_3] \oplus [\bar{a}_2, \bar{a}'_2]$. Otherwise a_2 and a'_2 are on the same hemisphere. If both belong to C, then surrounding a_3 is impossible (all simplices belong to one hemisphere). So one of them, a_2 say, has to be in the interior of one of the hemispheres. For a_3 to be surrounded, C must separate a_2 and a'_1. If $\bar{a}_2 = a_3(\bar{a}'_1)$: then either $\bar{a}'_2 \in \langle \bar{a}_2, \bar{a}'_3 \rangle$ and thus a_3 cannot be surrounded, or $\bar{a}'_2 \in \langle \bar{a}_1, \bar{a}_2 \rangle^\circ$ and thus if a covering holds, it would mean that a_3 belongs to the interior of the simplex given by $\langle a'_1, a'_2, a'_3 \rangle$, a case we treated already; or $\bar{a}'_2 = \bar{a}'_1$ and $S^1(a_3) \subseteq [\bar{a}'_2, \bar{a}'_3] \oplus [\bar{a}'_1, \bar{a}_2]$. If $\bar{a}_2 \in \langle \bar{a}'_3, a_3(\bar{a}'_1) \rangle^\circ$: then necessarily $\bar{a}'_2 \notin \langle \bar{a}'_3, a_3(\bar{a}'_1) \rangle^\circ$, and since $\bar{a}_2$ and $\bar{a}'_2$ are not separated by C then either $\bar{a}'_2 \in \langle a_3(\bar{a}'_1), \bar{a}_1 \rangle^\circ$ or $\bar{a}'_2 = \bar{a}_1$. But in both cases $\Sigma(a_3)$ suffices to surround a_3, a case we also treated already. Finally, if $\bar{a}_2 \in \langle \bar{a}_1, a_3(\bar{a}'_1) \rangle^\circ$, then in order to cover $\langle \bar{a}'_1, \bar{a}'_3 \rangle$ (the only remaining simplex contributing to surrounding a_3), either $\bar{a}'_2 = \bar{a}'_3$ or $\bar{a}'_2 \in \langle a_3(\bar{a}'_1), \bar{a}'_3 \rangle^\circ$. In the latter case, $\Sigma(a_3)$ suffices to surround a_3 on its own. We are thus left with $\bar{a}'_2 = \bar{a}'_3$. This can be equivalently encoded as $S^1(a_3) \subseteq [\bar{a}'_1, \bar{a}_2] \oplus [\bar{a}'_3, \bar{a}_2]$ (cf. to Figure 3 (left) to help visualize the different contributions).

Case (iii). Let C denote the great circle (a_1, a_2). If a'_3 belongs to C, then we're back to case (ii). Suppose a'_3 belongs to the interior of one hemisphere. If Both a'_1 and a'_2 are both on the same hemisphere as a'_3, then a_3 cannot be surrounded. Thus, a'_1 say, must belong to the interior of the other hemisphere (otherwise, it suffices to swap the indices 1 and 2 in what follows). This separation can be equivalently encoded as $S^1(a_3) \subseteq [\bar{a}_1, \bar{a}_2] \oplus [\bar{a}'_1, \bar{a}'_3]$. It remains to cover $\langle \bar{a}_1, \bar{a}'_1 \rangle$ with the only two remaining simplices involving a'_2 which fails precisely when $a_3(\bar{a}'_2) \in \langle \bar{a}_1, \bar{a}'_1 \rangle$ (cf. to Figure 3 (right) to help visualize the different contributions). □

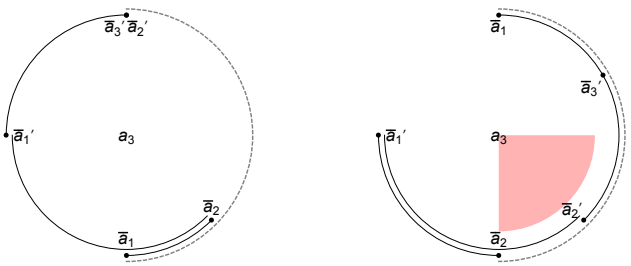


Figure 3: Cases 5 (left) and 6 (right) of Proposition 4. Black arcs are simplices from $\Sigma(a_3)$ whereas dashed arcs are from $\Sigma'(a_3)$. The red region in the left figure shows where $\bar{a}'_2$ should not be for a_3 to be surrounded.

Proposition 5. *Assuming non-degeneracy, if $S^2(o) \subseteq \Sigma(o)$ then a_3 is surrounded in one of the following ways (where $i \in \{1,2\}$ and $j \neq i$):*

1. $S^1(a_3) \subseteq [\bar{s}_1, \bar{t}_1] \oplus [\bar{s}_2, \bar{t}_2]$ *($\Sigma(a_3)$ suffices)*
2. $a_3 \in \langle a_1, a_2, a'_3 \rangle^\circ$ *(interior of a simplex)*
3. $a_3 \in \langle a_i, a'_3 \rangle^\circ \wedge S^1(a_3) \subseteq [\bar{a}_i, \bar{a}'_3] \oplus [\bar{a}_j, \bar{a}'_j]$ *(two simplices from $\Sigma(a'_3)$)*

As explained for $n = 2$, all these conditions could be easily translated to conditions on the entries of a 3×3 symbolic matrix M (25 at most per point in the general case) to get a symbolic characterization of Q-matricity for $n = 3$. (Notice that this requires to compute the orthogonal space of a symbolic vector, which can be done symbolically be selecting which components of the vector are non-vanishing.)

4 Related work

The current work is part of an extensive and rich literature, initiated almost seventy years ago, about characterizing and/or efficiently recognizing a Q-matrix. Below we restrict our attention to topological and geometric approaches that are not specific to particular sub-classes of the problem.

Spherical Geometry. In the late seventies, Kelly and Watson applied spherical geometry in their paper [10] providing a (non-constructive) characterization of non-degenerate Q-matrices. More precisely, the authors established that if $-s_i$ and $-t_i$ are in different components of the complement of the complex formed by the (complementary) $(n-1)$-simplices obtained when removing the index i, then one gets a Q-matrix. While the reasoning is simple, the characterization doesn't provide a uniform property that all the pairs (or points) need to satisfy. Moreover, Q-covering is specifically about understanding whether the complex formed by the complementary n-simplices is covering, and one is left with studying a different problem (namely two points being in different components) for (at most) n complexes involving each 2^{n-1} simplices.

Relaxations and Triangulation. By dropping the (only) non-linear condition $w.z = 0$, one gets a standard linear program which is well understood and easier to solve. Geometrically, such relaxation amounts to consider all n-simplices, complementary and non-complementary. If the non-complementary simplices are included in the complementary ones, the matrix M is known as a K-matrix. If moreover all the simplices are covering, then one immediately gets a characterization for Q-matrices. It is easy to see that if the opposite cone formed by the s_i intersects the interior of the cone formed by the t_i, then the entire space is covered by the union of all simplices (the feasible set is the entire space for each q). It thus remains to study under which conditions the non-complementary cones are included in the complementary ones. This was performed by Fredricksen, Watson and Murty in [6, Theorem 9] for $n = 3$. The main theorem states that if some specific set of finite points (precisely $3 \times 5! = 360$) are covered, then M is a Q-matrix. A similar generic approach consists in using all the points (s_i and t_i) at once to build a regular triangulation then check for inclusions in complementary simplices [5, Theorem 1].

Simplicial Homology. Given the problem at hand, it is very reasonable to investigate the usefulness and insights of standard homology groups to understand Q-covering. As the considered set of simplices need not define a simplicial complex (degeneracy and overlaps are allowed), in [13], Naiman and Stone exploited a related abstract simplicial complex called the *nerve* of the covering. Let $V = \{1, \ldots, m\}$ denote a set of vertices (0-simplices) where $m \leq 2^n$ denotes the number of the non-degenerate n-simplices involved in the covering. An (abstract) simplex of the nerve is a subset $W \subseteq V$ such that $\cap_{j \in W} C_j$ is non-empty. Unsurprisingly, the simplicial homology group H_{n-1} of the nerve is isomorphic to $\mathbb{Z}$ whenever the $S^{n-1}(o)$ is covered and is isomorphic to the trivial group 0 otherwise ([13, Corollary 13]). Apart from the fact that the effective computation of the simplicial homology group has a non-elementary complexity (given the exponential number of the n-simplices involved), such approach doesn't exploit the underlying structure of Q-covering. The nerve can be defined for any collection of subsets which makes it a powerful theoretical tool, totally non-insightful to capture the topological properties of the Q-covering. Something this paper strives to investigate from a pure geometrical view point.

Monotonicity. Several attempts tackled successfully either sub-classes or larger ones with computationally practical properties hoping it would lead to a better understanding of Q-matrices. For instance, following the very first characterization of P-matrices [16], the sign of principle minors of M were extensively studied. In [12], it has been shown that when $M \geq 0$, $M \in Q$ if and only if all the diagonal entries of M are positive. However in [11], Morris ended this quest by showing that Q-matrices cannot be characterized by signs of sub-determinants. As we have seen in Section 3, while checking the surrounding does require computing determinants, it involves non-trivial sets of vectors obtained by projecting on tangent spaces.

Solutions' Multiplicities. Another interesting direction tackled the number of solutions and/or their parity which shed some light on the inherent difficulty of the problem. For instance, R_0 matrices (also known as super-regular matrices) for which the LCP has a unique solution at $q = 0$ caught the attention of many. Karamardian [9] first proved that if in addition, for some strictly positive q the LCP has also a unique solution, then M is a Q-matrix. Shortly after, Saigal strengthened Karamardian's result in [15] by proving that if the LCP has a unique solution for $q = 0$ and an odd number of non-degenerate solutions for some other vector q, then M is necessarily a Q-matrix. Later in the same decade, in their paper [1], Aganagic and Cottle proved that among P_0 matrices (i.e. matrices for which all principal minors are non-negative), the subclass of Q-matrices is equal to the subclass of regular matrices. Not long after, Pang showed in [14] that a similar characterization to that of P_0 matrices, holds for L-matrices. It is interesting to notice that this direction was particularly fruitful and lead to several non-trivial conjectures generalizing known facts for larger well known classes like for semi-monotone matrices. Some turned out to be false like in [8] where Jeter and Pye provided a counter-example showcasing a semi-monotone 5×5-matrix that is Q but not R_0. Using our characterization, we were able to generate an example in $n = 3$ showcasing that a covering with (flat) non-pointed complementary cones is actually possible starting from dimension 3. For instance, consider s_1, s_2, s_3 as the vectors of the standard basis, and $t_1 = (-2, -4, -3)$, $t_2 = -e_1$, and $t_3 = (1, 1, 1)$. The cones $\langle s_1, t_2, s_3\rangle$ and $\langle s_1, t_2, t_3\rangle$ are two flat non-pointed cones and therefore the following 3×3 matrix is a Q-matrix but not an R_0-matrix.

$$\begin{pmatrix} 2 & 1 & -1 \\ 4 & 0 & -1 \\ 3 & 0 & -1 \end{pmatrix} .$$

Conclusion

To the best of our knowledge, no established theory in algebraic topology deals with the covering problem of Definition 1. On an abstract level, we regard the problem as a (strict) generalisation of the standard simplicial sets allowing for degeneracy and overlaps. Apart from using the nerve of the covering (which is possible but not insightful), our attempts to unravel a more specific algebraic structure were so far, unfortunately, unsuccessful. We thus tackled the problem from a purely geometric point of view hoping to pave the way towards an adequate "homology" where overlaps are built-in in the theory. We firmly believe that without such theory, no satisfactory answer could be stated for higher dimensions.

One advantage of the current line of work on the application side is its ability to provide a symbolic characterization for Q-matrices in full generality (regardless of degeneracy) in the three dimensional case which, up to our knowledge, did not exist before. We found such a characterization useful in order to generate automatically counter-examples for serious conjectures out of reach with our understanding of Q-covering. For instance, while we knew that non-pointed cones are not allowed in dimension 2 and possible in dimension 5, it wasn't clear whether they can occur in dimensions 3 or 4. More importantly perhaps, is that the point of view we presented sheds some light on the quest for a Murty-like test for Q-matrices (requiring to only test the covering of a finite set of vectors) regardless of the correctness of Conjecture 1: if it holds, the initial points are all one needs to check Q-matricity, otherwise, we know precisely what are the additional points one needs to consider and the inherent difficulty in checking whether they are surrounded or not. While we lean towards the correctness of the conjecture, we measure the difficulty (if not impossibility) for a human being to exhaustively enumerate all the possible cases to provide a formal proof for any finite n (like we did in Proposition 1 for $n = 3$). This is where a proof assistant (like Coq [2]) might be useful, with undoubtedly a mitigated satisfaction. This being said, we feel that bypassing this brute force approach requires a completely novel angle to attack the problem, which is yet to be found (algebraic insights are clearly missing).

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