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Event-Triggered Control for Linear Time-Varying Systems Using a Positive Systems Approach ⋆

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Abstract

We provide a new event-triggered output feedback control method for time-varying linear systems, using the notions of interval observers and positive systems. Our methods ensure uniform global exponential stability of the closed-loop system that is robust with respect to uncertainty in the coefficient matrices. Our illustrations include a curve tracking system from marine robotics.

Key words: Stability, event-triggered, observer, positive

1 Introduction

Event-triggered control plays a significant role in current controls research; see, e.g., the works [1], [5], [15], [17], [18], [19], [21], and [22]. An advantage of event-triggered control is its ability to reduce the computational burden associated with control implementations, by only changing control values when a significant event occurs. Such events are usually characterized as times when the state of the system enters a prescribed region. This differs from standard zero-order hold control strategies, which instead usually recompute the control values at times that are independent of the state. As shown in [7], event-triggered control is fundamentally a control problem for an interconnection consisting of the controlled plant and the event-triggering mechanism. Several previous event-triggered control design schemes can be unified from a small-gain perspective. Small-gain methods enjoy the desirable feature of ensuring robustness properties with respect to uncertainty, but can sometimes be conservative because they only base the design on mere knowledge of certain gain functions, and thus can lead to unnecessarily frequent control recomputations.

While emerging computing methods may facilitate recomputing control values, the increasing use of shared wired (or shared wireless) networked control systems strongly motivates designing controls that can take communication, computation, and energy constraints into account [5]. This has led to multiple event-triggered control designs, such as those in [6], [21], and [23]. A different body of current control theoretic research is based on positive systems, meaning, systems for which the nonnegative orthant is a positively invariant set, and this has led to new control analysis and designs that help overcome the challenges of applying traditional Lyapunov function approaches. Several works on positive systems are based on interval observers (as defined, e.g., in [3] and [16]), which yield intervals containing values of estimated states at each time, where the inequalities involving vector valued functions are viewed componentwise; see [13]. Interval observers and positive systems led to advances in aerospace engineering, mathematical biology, and other applications.

This (and the need to consider linear time-varying systems with uncertain coefficients and output feedback in applications) motivates the present work, where we develop a new event-triggered control technique based on the theory of positive systems. The positive system will be the dynamics for $(\overline{x}, -\underline{x})$, where $(\underline{x}, \overline{x})$ is the state of an interval observer (for an estimator $\hat{x}$ of unmeasured states) that will be essential for proving the desired exponential stability of our event-triggered closed-loop system; see (25). We believe that our work is the first to apply positive systems and interval observer methods to event-triggered output feedback control for systems with uncertain dynamics.

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Our global exponential stability proof uses interval observers as comparison systems, and the linear Lyapunov function approach from works such as [2], [4, Chapt. 3], and [13]. However, [13] does not cover event-triggering. Significant innovations in our work include (a) our covering time-varying systems, using new methods that are beyond the scope of traditional treatments of event-triggered linear time-invariant systems, such as our factoring of fundamental solutions as products of solutions of matrix differential equations, (b) our novel approaches to finding lower bounds on the times between event triggering times, (c) our new theorem that ensures uniform global exponential stability even when there is uncertainty in the dynamics, which uses a novel small-gain approach to output feedback control, without requiring a bound of the form $||\epsilon(t)|| \leq c||x(t)||$ relating the uncertainty ϵ and the state x at each time t for some constant c which was required in works like [22], and (d) our use of a nonstandard matrix 1-norm instead of the usual matrix 2-norm, which, in our examples, leads to larger lower bounds on the inter-sample times between the event-triggering times.

We review our definitions and notation in Section 2. We introduce our class of time-varying systems and our assumptions in Section 3. In Section 4, we present our event-triggered control design, and we state our theorem on the asymptotic convergence properties that our closed-loop systems enjoy when using this control design. In Section 5, we prove our theorem. In Section 6, we illustrate our method in a marine robotic dynamics and other cases. We end in Section 7 with our suggestions for future research.

In addition to addressing the essential difficulties of achieving event-triggered control for linear systems with outputs and uncertain coefficient matrices, this paper improves on our conference version [14] by covering time-varying systems (which call for using fundamental solutions instead of matrix exponentials), and by including complete proofs of our theorems and a new application to a marine robotic system. These three features were not present in [14], which was confined to time-invariant systems and which only provided sketches of proofs of theorems. Also, by including time-varying and uncertain coefficients and output feedback control, the theorem in this work includes the dynamics from all three theorems of [14], as special cases.

2 Definitions and Notation

We use these definitions and this notation, where the dimensions of our Euclidean spaces are arbitrary unless we otherwise indicate. The arguments of functions are omitted when no confusion can arise from the context. Set $\mathbb{Z}_0 = \{0, 1, 2, ...\}$ and $\mathbb{N} = \mathbb{Z}_0 \backslash \{0\}$. Given a matrix $G = [g_{ij}] \in \mathbb{R}^{r\times s}$, we set $|G| = [|g_{ij}|]$, i.e., the entries of $|G|$ are the absolute values of the corresponding entries of G. We also set $G^+ = [\max\{g_{ij}, 0\}]$ and $G^- = G^+ - G$, and $\sup_{\ell\in J} |G(\ell)| = [m_{ij}]$ where $m_{ij} = \sup_{\ell\in J} |g_{ij}(\ell)|$ when G is a time-varying and bounded matrix-valued function and J is a subset of the domain of G. A square matrix is called Metzler provided all of its off-diagonal entries are nonnegative. We use inequalities of matrices, in the following entry-wise sense. Given two matrices $D = [d_{ij}]$ an $E = [e_{ij}]$ of the same size, we write $D < E$ (resp., $D \leq E$) provided $d_{ij} < e_{ij}$ (resp., $d_{ij} \leq e_{ij}$) for all i and j. Also, we write $D \nleq E$ provided there is a pair (i, j) such that $d_{ij} > e_{ij}$. We use similar notation for vectors. A matrix M is called positive provided $M > 0$, where 0 is the zero matrix.

For a matrix $M = [m_{ij}]$ in $\mathbb{R}^{n\times n}$, we let D_M denote the diagonal matrix $\text{diag}\{m_{11}, m_{22}, \ldots, m_{nn}\}$ in $\mathbb{R}^{n\times n}$, so all the diagonal entries of $M - D_M$ are equal to zero. We let $R_M = D_M + (M - D_M)^+$ and $N_M = (M - D_M)^-$, so $M = R_M - N_M$. We let $|| \cdot ||$ denote the standard Euclidean norm of matrices and vectors, and we let $|| \cdot ||_\infty$ (resp., $|| \cdot ||_J$) denote the sup norm of matrix-valued functions in this norm over their domain (resp., an interval J in their domain). We let I be the identity matrix in any dimension, and we use the usual definition and properties of fundamental matrices, e.g., from [20, Appendix C]. We use $\Phi_{\mathcal{M}}$ to denote the fundmental matrix for a system of differential equations of the form $\dot{z} = \mathcal{M}(t)z$.

3 Class of Systems and Assumptions

We consider the system with an output y given by

$$\begin{cases} \dot{x}(t) = (A_0(t) + A_\delta(t))x(t) + (B_0(t) + B_\delta(t))u(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

with x valued in $\mathbb{R}^n$, y valued in $\mathbb{R}^q$, the control u valued in $\mathbb{R}^p$, known bounded continuous matrix-valued functions A_0 and B_0, piecewise continuous matrix-valued functions A_δ and B_δ that can represent uncertainties, and a known constant matrix $C \in \mathbb{R}^{q\times n}$. In some of what follows, we set $A(t) = A_0(t) + A_\delta(t)$ and $B(t) = B_0(t) + B_\delta(t)$. Our main assumptions are as follows:

Assumption 1 *There is a known bounded continuous matrix-valued function $L : \mathbb{R} \to \mathbb{R}^{n\times q}$ such that the system*

$$\dot{z}(t) = \big(A(t) + L(t)C\big)z(t) \tag{2}$$

is uniformly globally exponentially stable to the origin on $\mathbb{R}^n$. Also, there are known constants $a_* \geq 0$ and $b_* \geq 0$ such that $||A_\delta||_\infty \leq a_*$ and $||B_\delta||_\infty \leq b_*$. □

Assumption 2 *There are a matrix $\Gamma > 0$, a bounded continuous matrix-valued function $K : \mathbb{R} \to \mathbb{R}^{p\times n}$, a constant $p > 0$, a function $V : [0, +\infty) \to \mathbb{R}^n$ of class C^1, and two vectors $\overline{V} > 0$ and $\underline{V} > 0$ such that, with the choices*

$$H(t) = A_0(t) + B_0(t)K(t) \text{ and } E(t) = R_H(t) + N_H(t), \tag{3}$$

we have

$$\underline{V} \leq V(t) \leq \overline{V} \tag{4}$$

and

$$\dot{V}(t)^\top + V(t)^\top\big(E(t) + |B_0(t)K(t)|\Gamma\big) \leq -pV(t)^\top \tag{5}$$

for all $t \geq 0$. □

The motivation for Assumptions 1-2 is as follows. Assumption 1 is a time-varying analog of the usual requirements from the constant coefficient unperturbed case that (A, C)

is observable, which is then robustified by including $A_\delta(t)$ in the coefficient matrix in (2) (so the required exponential stability condition for (2) will be satisfied if the unperturbed system $\dot{z}(t) = (A_0(t) + L(t)C)z(t)$ is uniformly globally exponentially stable to the origin on $\mathbb{R}^n$ when the sup norm of A_δ is small enough). Assumption 2 is a time-varying robustified analog of the usual requirement in the time-invariant case that H is Metzler and Hurwitz, in which case $H = R_H = E$; see [14] for our study of time-invariant cases, and [4, Lemma 2.3, p.41] for a proof that there are a positive constant vector V and a constant $p > 0$ such that $V^\top H \leq -pV^\top$ when H is a constant Metzler Hurwitz matrix.

Our rationale for calling Assumption 2 a robustified version of the preceding condition from [4] is that if there exist a constant vector $V > 0$ and a positive scalar p such that $V^\top H(t) \leq -pV^\top$ holds for all $t \geq 0$, and if $H(t)$ is Metzler for all $t \geq 0$, then Assumption 2 will be satisfied when Γ has small enough entries (because $H = E$). Therefore, taken together, Assumptions 1-2 are a time-varying robustified analog of the requirements from the emulation conditions in event-triggered control, and these assumptions will be used in the second part of our proof of our theorem, to establish the exponential convergence conclusion of our theorem. See Appendix A for ways to check whether Assumption 2 is satisfied.

We use the function

$$\Omega_0(t,s) = \Phi_{A_0}(t,s) + \int_s^t \Phi_{A_0}(t,m)B_0(m)K(m)\mathrm{d}m, \quad (6)$$

and we consider any constant $\nu > 0$ such that

$$\sup\{|I - \Omega_0^{-1}(t,s)| : 0 \leq s \leq t \leq s+\nu\} \leq \Gamma, \quad (7)$$

where Γ is from Assumption 2 and where the inverse values Ω_0^{-1} will exist for small enough $\nu > 0$ because $\Omega_0(s,s) = I$ for all $s \geq 0$, and because the boundedness of A_0, B_0, and K implies that $\sup\{||(\partial\Omega_0/\partial t)(t,s)|| : 0 \leq s \leq t \leq s+\nu\}$ is bounded for each $\nu > 0$. See Appendix B below for ways to find constants $\nu > 0$ to satisfy (7). In Section 4, we prove that our constant $\nu > 0$ that satisfies (7) is a lower bound on the intervals between the event triggering times t_i, so the Zeno phenomenon does not occur.

Fixing any constant $\nu > 0$ satisfying (7), and letting $\underline{v}$ denote the smallest entry of the constant vector $\underline{V} > 0$ from Assumption 2, and letting g denote the rate of uniform global exponential convergence in Assumption 1, and using the notation

$$\mathcal{B} = (I + \Gamma)\, E_\nu, \quad (8)$$

where

$$E_\nu = \sup\{|\Phi_{A_0}(t,s)L(s)| : 0 \leq s \leq t \leq s+\nu\} \quad (9)$$

where Γ, p, and L are from Assumptions 1-2, our last assumption is as follows, where C is the matrix used in (1):

Assumption 3 *The constants*

$$\begin{aligned} G_1 &= \tfrac{2}{g}\left(||A_\delta||_\infty + ||B_\delta K||_\infty\right) \text{ and} \\ G_2 &= \tfrac{2}{p\underline{v}}\left(||V^\top|B_0K|\mathcal{B}|C|||_\infty \nu + ||V^\top|LC|||_\infty\right) \end{aligned} \quad (10)$$

are such that the inequality

$$G_1 G_2 < 1 \quad (11)$$

is satisfied.

Condition (11) is a small-gain condition that we use in the second part of the proof of our theorem in Section 5; see Remark 2, and see [7] for background on small-gain methods.

4 Event-Triggered Control Design

In terms of the preceding matrices and constants and the matrices (8), and continuing our notation $A = A_0 + A_\delta$ and $B = B_0 + B_\delta$, we are now ready to define our event-triggered control. Our event-triggered feedback control and the corresponding event-triggering times t_i are defined by

$$u(t) = K(t)\hat{x}(t_i) \text{ for all } t \in [t_i, t_{i+1}) \text{ and } i \geq 0, \quad (12)$$

where $t_0 = 0$, and where for each integer $i \geq 0$, we have

(a) $\dot{x}(t) = A(t)x(t) + B(t)K(t)\hat{x}(t_i)$ if $t \in (t_i, t_{i+1})$,
(b) $\dot{\hat{x}}(t) = A_0(t)\hat{x}(t) + B_0(t)K(t)\hat{x}(t_i) + L(t)[C\hat{x}(t) - y(t)]$ if $t \in (t_i, t_{i+1})$,
(c) the inequality

$$\begin{aligned} &|\hat{x}(t) - \hat{x}(t_i)| \leq \\ &\Gamma|\hat{x}(t)| + \mathcal{B}\int_{\max\{0,t-\nu\}}^t |C\hat{x}(\ell) - y(\ell)|\mathrm{d}\ell \end{aligned} \quad (13)$$

is satisfied for all $t \in [t_i, t_{i+1})$, and
(d) For each $\epsilon > 0$ and each i such that $t_{i+1} < +\infty$, there is a $t_\ddagger \in (t_{i+1}, t_{i+1} + \epsilon)$ such that

$$\begin{aligned} &|\hat{x}(t_\ddagger) - \hat{x}(t_i)| \nleq \\ &\Gamma|\hat{x}(t_\ddagger)| + \mathcal{B}\int_{\max\{0,t_\ddagger-\nu\}}^{t_\ddagger} |C\hat{x}(\ell) - y(\ell)|\mathrm{d}\ell \end{aligned} \quad (14)$$

is satisfied,

where $\hat{x}(0) \neq 0$, $\mathcal{B}$ is from (8), and the constant $\nu > 0$ satisfies the requirements from Section 3. The system (a)-(d) is an event-triggered dynamics that resets the control values at the times t_i, using the following inductive argument, where the ϵ in (d) is used to express the fact that t_{i+1} is the infimum of all times $t \geq t_i$ when (13) fails to hold when i is such that $t_{i+1} < \infty$. Starting at time $t_0 = 0$, we choose $u(t) = K(t)x(0)$. Then, we continue to use the control formula $u(t) = K(t)x(0)$ as long as

$$\begin{aligned} &|\hat{x}(t) - \hat{x}(0)| \leq \\ &\Gamma|\hat{x}(t)| + \mathcal{B}\int_{\max\{0,t-\nu\}}^t |C\hat{x}(\ell) - y(\ell)|\mathrm{d}\ell \end{aligned} \quad (15)$$

continues to hold. This yields an interval $[0, t_*]$ along which the control takes the values $u(t) = K(t)x(0)$, and the proof of our theorem will show that $t_* \geq \nu$. If, at a later time, the inequality (15) is violated (as explained by condition (d) with $t_0 = 0$), then we change the control formula to $u(t) = K(t)x(t_1)$, where t_1 is the infimum of times when a violation of the inequality occurs. We repeat this process inductively, with t_0 replaced by t_1, then by t_2, and so on, to define the control values and triggering times t_i for all $t \geq 0$. Our main theorem is then as follows:

Theorem 1 *Let Assumptions 1-3 hold, and let K, L, Γ, and ν satisfy the requirements above. Then the closed-loop system given by (a)-(d) is globally exponentially stable to the origin on $\mathbb{R}^n$.* □

Remark 1 The preceding theorem is new, even in the special case where A_δ and B_δ are zero, because of our ability to achieve event-triggered output feedback stabilization with time-varying coefficients A_0 and B_0. □

5 Proof of Theorem 1

The proof has two parts. In the first part, we use our condition (7) on the constant $\nu > 0$ to show that the Zeno phenomenon does not occur. In the second part, we use Assumptions 1-3 to show that the uniform global exponential stability conclusion of the theorem holds.

First part. Letting ν be the constant we defined above, we prove that the inter-event times $t_{i+1}-t_i$ are bounded below by ν, i.e., if there are any finite triggering times $t_i > 0$, then either (A) there are only a finite number of finite instants $t_0, ..., t_j$ and $\min_{l\in\{0,...,j\}}(t_{l+1} - t_l) \geq \nu$ or (B) there are infinitely many finite instants t_i and $\inf_{l\in\mathbb{Z}_0}(t_{l+1} - t_l) \geq \nu$. (The case where there are no trigger times gives $t_1 = +\infty$ and $u(t) = K(t)\hat{x}(0)$ for all $t \geq 0$.)

For any $i \in \mathbb{Z}_0$ such that $t_i < +\infty$, we apply the method of variation of parameters to the equation in (b) over $[t_i, t)$ with $t \in [t_i, t_i + \nu]$ with the initial state $\hat{x}(t_i)$ to get

$$\begin{aligned}&\hat{x}(t) = \mathcal{G}(t, t_i)\\&+\left[\Phi_{A_0}(t, t_i) + \textstyle\int_{t_i}^{t} \Phi_{A_0}(t, \ell)B_0(\ell)K(\ell)\mathrm{d}\ell\right]\hat{x}(t_i),\end{aligned} \tag{16}$$

where

$$\mathcal{G}(t, t_i) = \textstyle\int_{t_i}^{t} \Phi_{A_0}(t, \ell)L(\ell)[C\hat{x}(\ell) - y(\ell)]\mathrm{d}\ell. \tag{17}$$

Hence, $\hat{x}(t_i) = \Omega_0^{-1}(t, t_i)[\hat{x}(t)-\mathcal{G}(t, t_i)]$ for all $t \in [t_i, t_i+\nu]$, which is equivalent to

$$\hat{x}(t)-\hat{x}(t_i) = \left[I - \Omega_0^{-1}(t, t_i)\right]\hat{x}(t)+\Omega_0^{-1}(t, t_i)\mathcal{G}(t, t_i) \tag{18}$$

for all $t \in [t_i, t_i + \nu]$. Hence, (7) gives

$$|\hat{x}(t) - \hat{x}(t_i)| \leq \Gamma|\hat{x}(t)| + |\Omega_0^{-1}(t, t_i)\mathcal{G}(t, t_i)| \tag{19}$$

for all $t \in [t_i, t_i + \nu]$.

Since our formula (17) for $\mathcal{G}$ and our choice (9) of E_ν give

$$|\mathcal{G}(t, t_i)| \leq E_\nu \textstyle\int_{t_i}^{t} |C\hat{x}(\ell) - y(\ell)|\mathrm{d}\ell, \tag{20}$$

it follows from our choice (8) of $\mathcal{B}$ that we have

$$|\Omega_0^{-1}(t, t_i)\mathcal{G}(t, t_i)| \leq \mathcal{B} \textstyle\int_{\max\{0,t-\nu\}}^{t} |C\hat{x}(\ell) - y(\ell)|\mathrm{d}\ell \tag{21}$$

for all $t \in [t_i, t + \nu]$. Using (21) to upper bound the second right side term in (19), it follows that the inequality (13) holds for all $t \in [t_i, t_i + \nu]$, which gives $t_{i+1} \geq t_i + \nu$.

Second part. We prove the uniform global exponential stability of the 0 equilibrium of the closed-loop system (a)-(d). In what follows, all equalities and inequalities should be understood to hold for all $t \geq 0$, unless otherwise indicated.

Let $\tilde{x}(t) = \hat{x}(t) - x(t)$ and $G = A_0 + LC$. Then (a)-(b) give

$$\begin{aligned}\dot{\tilde{x}}(t) &= G(t)\tilde{x}(t) - A_\delta(t)x(t) - B_\delta(t)K(t)\hat{x}(t_i)\\&= [G(t) + A_\delta(t)]\tilde{x}(t)\\&\quad -A_\delta(t)\hat{x}(t) - B_\delta(t)K(t)\hat{x}(t_i)\end{aligned} \tag{22}$$

for all $t \in [t_i, t_{i+1})$ and all $i \geq 0$. Therefore, by our choice g of the uniform global exponential stability convergence rate from Assumption 1, and by applying the method of variation of parameters to the last equality in (22), it follows that the $\tilde{x}$ dynamics (22) satisfies an exponential ISS estimate of the form

$$||\tilde{x}(t)|| \leq \max\{\beta_1(||\tilde{x}(0)||, t), \gamma_1(||\hat{x}||_{[0,t]})\} \tag{23}$$

for all $t \geq 0$ and for a suitable function $\beta_1 \in \mathcal{KL}$ and the class $\mathcal{K}_\infty$ function $\gamma_1(s) = G_1 s$, where G_1 was defined in (10) in Assumption 3, and where the exponential property of the ISS means that there are positive constants c_1 and c_2 such that $\beta_1(s, t) = c_1 e^{-c_2 t}$ for all $s \geq 0$ and $t \geq 0$; see [7] for the standard definitions of $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions.

Also, setting $H = A_0 + B_0K$ and $\hat{\mu}(t) = \hat{x}(t_i) - \hat{x}(t)$ for all $t \in [t_i, t_{i+1})$ and $i \geq 0$, we have

$$\begin{aligned}\dot{\hat{x}}(t) &= H(t)\hat{x}(t) + B_0(t)K(t)\hat{\mu}(t) + L(t)C\tilde{x}(t)\\&= [R_H(t) - N_H(t)]\hat{x}(t)\\&\quad + B_0(t)K(t)\hat{\mu}(t) + L(t)C\tilde{x}(t).\end{aligned} \tag{24}$$

We next introduce the comparison system

$$\begin{cases}\dot{\overline{x}}(t) = R_H(t)\overline{x}(t) - N_H(t)\underline{x}(t)\\\qquad + (B_0(t)K(t)\hat{\mu}(t))^+ + (L(t)C\tilde{x}(t))^+\\\dot{\underline{x}}(t) = R_H(t)\underline{x}(t) - N_H(t)\overline{x}(t)\\\qquad - (B_0(t)K(t)\hat{\mu}(t))^- - (L(t)C\tilde{x}(t))^-\end{cases} \tag{25}$$

which we use as an interval observer for $\hat{x}$. We assume that

$$\underline{x}(0) < 0 < \overline{x}(0) \text{ and } \underline{x}(0) < \hat{x}(0) < \overline{x}(0). \tag{26}$$

Since $Y(t) = (\overline{x}(t), -\underline{x}(t))$ is a solution of a dynamics of the form $\dot{Y}(t) = \mathcal{H}(t)Y(t)+\mathcal{P}(t)$ for some piecewise continuous matrix-valued functions $\mathcal{H}$ and $\mathcal{P}$ such that $\mathcal{H}(t)$ is Metzler for all $t \geq 0$ and $\mathcal{P}(t) \geq 0$ for all $t \geq 0$, [12, Lemma 1] implies that

$$\underline{x}(t) \leq 0 \leq \overline{x}(t) \tag{27}$$

for all $t \geq 0$. Similar reasoning implies that $\overline{x}(t) - \hat{x}(t) \geq 0$ and $\hat{x}(t) - \underline{x}(t) \geq 0$ for all $t \geq 0$; this follows by applying [12, Lemma 1] to the $2n$ dimensional dynamics for $(\overline{x}(t) - \hat{x}(t), \hat{x}(t) - \underline{x}(t))$. Hence, for all $t \geq 0$, we obtain

$$\underline{x}(t) \leq 0 \leq \overline{x}(t) \text{ and } \underline{x}(t) \leq \hat{x}(t) \leq \overline{x}(t). \tag{28}$$

For the preceding initial conditions, set $x_\star(t) = \overline{x}(t)-\underline{x}(t)$. Then (28) gives $\underline{x}(t)-\overline{x}(t) \leq \hat{x}(t) \leq \overline{x}(t)-\underline{x}(t)$ and so also

$$|\hat{x}(t)| \leq x_\star(t). \tag{29}$$

Simple calculations give

$$\begin{aligned}\dot{x}_\star(t) &= (R_H(t) + N_H(t))x_\star(t)\\&\quad +|B_0(t)K(t)\hat{\mu}(t)| + |L(t)C\tilde{x}(t)|.\end{aligned} \tag{30}$$

Consider the function $W(t,x_\star) = V^\top(t)x_\star$, where V is from Assumption 2. By (c) and (29), it satisfies

$$\begin{aligned}\dot{W}(t) &= \dot{V}^\top(t)x_*(t) + V^\top(t)(R_H(t)+N_H(t))x_\star(t)\\ &\quad +V^\top|B_0(t)K(t)\hat{\mu}(t)| + V^\top(t)|L(t)C\tilde{x}(t)|\\ &\le \dot{V}^\top(t)x_*(t) + V^\top(t)(R_H(t)+N_H(t))x_\star(t)\\ &\quad +V^\top(t)|B_0(t)K(t)|\big(\Gamma|\hat{x}(t)|\\ &\quad +\mathcal{B}\textstyle\int_{\max\{0,t-\nu\}}^t \hat{C}(\ell)\mathrm{d}\ell\big) + V^\top|L(t)C\tilde{x}(t)|\end{aligned} \tag{31}$$

for all $t \ge 0$ along solutions of (30), where $\hat{C}(\ell) = C\hat{x}(\ell) - y(\ell)$ and $\dot{W}(t)$ means $(d/dt)W(t,x_*(t))$. Using (5),(29), and (31), we get

$$\begin{aligned}\dot{W}(t) &\le -pW(t,x_\star(t)) + V^\top(t)|L(t)C\tilde{x}(t)|\\ &\quad +V^\top(t)|B_0(t)K(t)|\mathcal{B}\textstyle\int_{\max\{0,t-\nu\}}^t \hat{C}(\ell)\mathrm{d}\ell\\ &\le -pW(t,x_\star(t)) + \Xi(t), \text{ where}\\ \Xi(t) &= V^\top(t)|B_0(t)K(t)|\mathcal{B}|C|\textstyle\int_{\max\{0,t-\nu\}}^t |\tilde{x}(\ell)|\mathrm{d}\ell\\ &\quad +V^\top|L(t)C\tilde{x}(t)|,\end{aligned} \tag{32}$$

where the first inequality in (32) used the nonnegative valuedness of $x_*(t)$ (which follows from (29)) to maintain the inequality from (5) when (5) is right multiplied by $x_*(t)$.

By applying the method of variation of parameters to

$$\dot{W}(t) \le -pW(t,x_\star(t)) + \Xi(t) \tag{33}$$

from (32), it follows from the formula (32) for Ξ that the x_* dynamics satisfies an exponential ISS estimate of the form

$$||x_*(t)|| \le \max\{\beta_2(||x_*(0)||,t), \gamma_2(||\tilde{x}||_{[0,t]})\} \tag{34}$$

for all $t \ge 0$ and for a function $\beta_2 \in \mathcal{KL}$ and $\gamma_2(s) = G_2 s$, where G_2 was defined in (10). Using (29) to upper bound $|\hat{x}|$ in the ISS estimate (23), and then using the small-gain condition $G_1G_2 < 1$ from Assumption 3 to check that $\gamma_1(\gamma_2(s)) < s$ for all $s \ge 0$, it follows from the small-gain theorem (e.g., from [7, p.61]) that the $(x_*, \tilde{x})$ dynamics are uniform globally exponentially stable to 0 on $\mathbb{R}^{2n}$. The conclusion of Theorem 1 now follows because we can assume that $\bar{x}(0) \le \mathcal{M}|x(0)|$ and $\underline{x}(0) \ge -\mathcal{M}|x(0)|$ for a large enough constant $\mathcal{M} > 0$, which imply that our condition (29) gives $||x(t)|| = ||\hat{x}(t) - \tilde{x}(t)|| \le ||\hat{x}(t)|| + ||\tilde{x}(t)|| \le ||x_*(t)|| + ||\tilde{x}(t)|| \le 2||(x_*(t),\tilde{x}(t))||$ for all $t \ge 0$ and $||(x_*(0),\tilde{x}(0))|| \le (4\mathcal{M}+1)||x(0)||$, so we can convert the exponential stability estimate for the variable $(x_*, \tilde{x})$ into the required exponential stability estimate for $x(t)$.

Remark 2 Our small-gain argument from the proof differs significantly from standard small-gain approaches to event triggering, because we use a small-gain condition between $\tilde{x}$ and x_*, instead of between the original system and the event triggering rule. □

6 Illustrations

6.1 Output Feedback Event-Triggered Control

In this subsection, we illustrate Theorem 1, using a benchmark two-dimensional (i.e., planar) curve tracking dynamical system from [24]. While simpler than more complex ship models, or the analogous 3D curve tracking model from [9,24], it illustrates the value of Theorem 1 for using event-triggering to eliminate the need to continuously adjust the control values. We consider the system

$$\begin{cases}\dot{\rho}(t) = -\sin(\phi(t))\\ \dot{\phi}(t) = \frac{\kappa\cos(\phi(t))}{1+\kappa\rho(t)} + u(t)\\ y(t) = \rho(t)\end{cases} \tag{35}$$

which models curve tracking by a unit speed surface marine robot with a gyroscopic control u, where the available measurement ρ is the distance between the marine robot and the closest point (i.e., projection) on the curve being tracked (where we assume as in [24] that the closest point is unique at each time), ϕ is the heading angle (measuring the difference between the angles of the tangent lines of the path of the marine robot and of the path being traced out by the closest point on the curve being tracked), and κ is the curvature at each time; see [24] for a derivation of the model (35), and see Fig. 1 for the position and frame $(\mathbf{r_1}, \mathbf{x_1}, \mathbf{y_1})$ on the boundary curve being tracked at the closest point, and the position and frame $(\mathbf{r_2}, \mathbf{x_2}, \mathbf{y_2})$ for the vehicle, where $\rho = ||\mathbf{r_1} - \mathbf{r_2}||$, and ϕ is the angle between the tangent vector $\mathbf{x_1}$ at the projection point on the curve being tracked and the tangent vector $\mathbf{x_2}$ on the curve being traced out by the robot at each time. We assume for

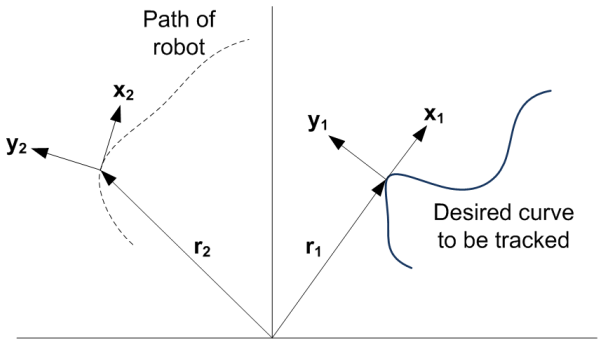

Fig. 1. Curve Tracking Frames and Positions from [24]

simplicity that κ is a positive constant.

The linear approximation of (35) around any choice of the reference trajectory $(\rho_r(t), \phi_r(t))$ is

$$\begin{cases}\dot{\rho}_a(t) = a_{12}(t)\phi_a(t)\\ \dot{\phi}_a(t) = a_{21}(t)\rho_a(t) + a_{22}(t)\phi_a(t) + u(t)\\ y(t) = \rho_a(t),\end{cases} \tag{36}$$

where

$$\begin{aligned}&a_{12}(t) = -\cos(\phi_r(t)),\ a_{21}(t) = -\tfrac{\kappa^2\cos(\phi_r(t))}{(1+\kappa\rho_r(t))^2},\\ &\text{and } a_{22}(t) = -\tfrac{\kappa\sin(\phi_r(t))}{1+\kappa\rho_r(t)}.\end{aligned} \tag{37}$$

Let us consider the reference trajectory (ρ_r, ϕ_r), where

$$\rho_r(t) = \tfrac{1}{\kappa}\left(1 - \tfrac{t\Delta_*}{1+t}\right) \text{ and } \phi_r(t) = \arcsin\left(\tfrac{\Delta_*}{\kappa(1+t)^2}\right), \tag{38}$$

where the physical meaning of the constant $\Delta_* \in [0, \min\{1,\kappa\})$ is that it determines how fast the robot is traveling towards the curve that is being tracked. Also,

for this reference trajectory, $(1-\Delta_*)/\kappa$ is the infimum of the distance between the position of the robot and the projection on the curve being tracked. Then (36)-(37) give

$$\begin{aligned} &a_{12}(t) = \mathcal{A}_1(t) - 1,\ a_{21}(t) = -\tfrac{\kappa^2}{4} + \mathcal{A}_2(t), \text{ and} \\ &a_{22}(t) = \mathcal{A}_3(t), \text{ where } \mathcal{A}_1(t) = 1 - \left(1 - \tfrac{\Delta_*^2}{\kappa^2(1+t)^4}\right)^{1/2}, \\ &\mathcal{A}_2(t) = \kappa^2 \left(\tfrac{1}{4} - \frac{\sqrt{1 - \frac{\Delta_*^2}{\kappa^2(1+t)^4}}}{\left(2 - \frac{t\Delta_*}{1+t}\right)^2} \right), \text{ and} \\ &\mathcal{A}_3(t) = -\tfrac{\Delta_*}{(1+t)^2} \tfrac{1}{2 - \frac{t\Delta_*}{1+t}}. \end{aligned} \tag{39}$$

Let us choose $\kappa = 2$. This produces the dynamics

$$\begin{cases} \dot{\rho}_a(t) &= (\mathcal{A}_1(t) - 1)\phi_a(t) \\ \dot{\phi}_a(t) &= -(1 - \mathcal{A}_2(t))\rho_a(t) + \mathcal{A}_3(t)\phi_a + u(t) \\ y(t) &= \rho_a(t). \end{cases} \tag{40}$$

The change of coordinates $(x_1(t), x_2(t)) = (\rho_a(t), \rho_a(t) - \phi_a(t))$ produces the dynamics $\dot{x}(t) = A(t)x(t) + Bu$ with output $y(t) = Cx(t)$, where

$$A(t) = \begin{bmatrix} \mathcal{A}_1(t) - 1 & 1 - \mathcal{A}_1(t) \\ \mathcal{A}_4(t) & 1 - \mathcal{A}_1(t) + \mathcal{A}_3(t) \end{bmatrix}, \tag{41}$$

$B = [0, -1]^\top$, $C = [1, 0]$, and $\mathcal{A}_4 = \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3$.

Then simple Mathematica calculations show that the requirements of Assumption 1-2 are satisfied with $A_\delta = 0$, $B_\delta = 0$, $V = [1, 2]^\top$, $K = [0, 1.75]$, each entry of Γ being 0.1, $L = [-2, -5]^\top$, $C = [1, 0]$, and $p = 0.01$ when $\Delta_* = 0.1$. The preceding choices of V, p, K, L, and Γ were obtained by first finding constant choices of V, p, K, L and Γ for which our assumptions are satisfied in the special case where $\Delta_* = 0$ (and with a constant choice of H that is both Metzler and Hurwitz), and then by choosing $\Delta_* > 0$ to be a small enough constant such that the requirements of our assumptions are still met with these choices of V, p, K, L, and Γ. For instance, to check that Assumption 1 is satisfied, we solved the Riccati equation $PM + M^\top P = -I$ for P with the choice $M = A + LC$ in the $\Delta_* = 0$ case, and then we checked that $\Delta_* = 0.1$ was small enough so that $V_0(z) = z^\top P z$ is a quadratic Lyapunov function for $\dot{z}(t) = (A(t) + LC)z(t)$. Then the requirement (B.1) from Lemma B.1 in Appendix B below is satisfied with $\nu = 0.026$. Hence, (7) is satisfied, so Theorem 1 applies in the special case where A_δ an B_δ are both the zero matrices.

If we replace (38) by the reference trajectory

$$\rho_r(t) = \tfrac{1}{\kappa}(1 - 0.1\sin(t)) \text{ and } \phi_r(t) = \arcsin\left(\tfrac{\cos(t)}{10\kappa}\right) \tag{42}$$

and replace V, p, Γ, and ν by $V = [1, 3.7]^\top$, $p = 0.001$,

$$\Gamma = \begin{bmatrix} 0.6075 & 0.648 \\ 0.081 & 0.243 \end{bmatrix} \tag{43}$$

and $\nu = 0.2125$, respectively, and keep the rest of the dynamics the same, then our assumptions and the requirements of Lemma B.1 are again satisfied. This can be checked by noting that

$$\sup\{|I - \Omega_0^{-1}(t, s)| : s \in [0, 2\pi], t \in [s, s + \nu]\} \leq \Gamma, \tag{44}$$

and then using the argument from Appendix B below for cases where $A(t)$ is periodic. The entries of Γ in (43) and ν were chosen using the following two step process. In the first step, the entries of Γ were chosen as the largest ones for which the requirements of Assumption 2 were satisfied with the preceding choices of A, B, V, K, and p. Then, in the second step, $\nu > 0$ was chosen to be the largest value such that (7) is satisfied with the preceding choices of A, B, K, and $\Gamma = [\Gamma_{ij}]$. However, with the preceding choices, our sufficient condition (B.1) from the proof of Lemma B.1 is not satisfied, because its left side with $\bar{b} = ||BK||$ and $\bar{a} = ||A||_\infty$ is $2.07515 > \min\{\Gamma_{11}, \Gamma_{12}, \Gamma_{21}, \Gamma_{22}, 1\} = 0.081$. This illustrates how numerical techniques (such as the alternative methods from Appendix B) can allow cases where (B.1) is violated but where our method still applies.

In Figs. 2-4 below, we plot MATLAB simulations that we obtained in the preceding periodic case with the preceding parameter values using the event-triggered control from our theorem (i.e., (a)-(d) above) applied to (40). In the top figure, we used $\rho_a(0) = \phi_a(0) = 0.1$ as the initial state. In Figs. 3-4, we plot the solutions that we obtained for different initial states $(\rho_a(0), \phi_a(0))$. Since the plots show convergence to the reference trajectory (42), they illustrate the value of our theorem.

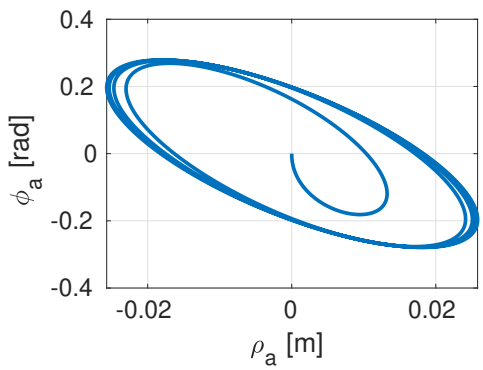

Fig. 2. Curve Tracking in Periodic Case: Phase Plane

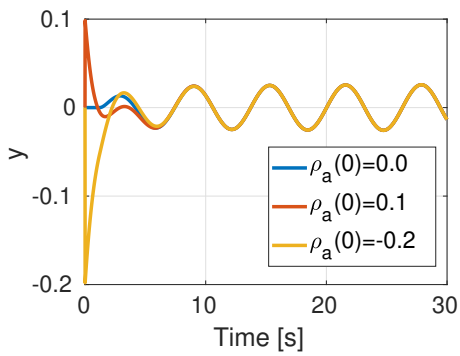

Fig. 3. Curve Tracking in Periodic Case: Output

The preceding cases are beyond the scope of works for time-invariant systems such as [21], because of our time-varying coefficient $A(t)$, unless $\Delta_* = 0$ in our reference trajectory

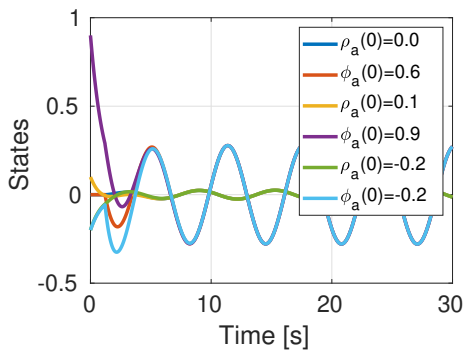

Fig. 4. Curve Tracking in Periodic Case: States

(38). In the special case where $\Delta_* = 0$, the matrix A is constant, and in that $\Delta_* = 0$ case, the example would be covered by [21] if we instead had $C = I$. Therefore, we next compare the lower bound ν that we would obtain from our method with the lower bound ν that could be obtained from [21, Corollary IV.1] in the $\Delta_* = 0$ and $C = I$ case. When $\Delta_* = 0$, our assumptions and our requirement (B.1) from Lemma B.1 below are satisfied with $\nu = 0.0925$ and the same choices of the other parameters that we used in our treatment of the $\Delta_* = 0.1$ case. On the other hand, if we apply [21, Corollary IV.1] with the same set of parameters (using $\sigma = \min\{\Gamma_{ij} : 1 \le i \le n, 1 \le j \le n\} = 0.1$ in [21, Corollary IV.1], which corresponds to our trigger condition $|e| \le \Gamma|x|$), then we would have gotten $\nu = 0.0592$, so our method provides a larger lower bound ν on the inter-sample times, and therefore ensures less frequent event triggers.

6.2 Robustness

We next illustrate Theorem 1 using the special case

$$\begin{cases} \dot{x}_1(t) = (1+\delta_1(t))x_1(t) + \frac{1}{2}(1+\delta_2(t))x_2(t) + u \\ \dot{x}_2(t) = \frac{3}{2}(1+\delta_3(t))x_1(t) + u \end{cases} \tag{45}$$

of (1) with $B_0 = [1,1]^\top$, $B_\delta = 0$, $C = I$,

$$A_0 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & 0 \end{bmatrix}, \text{ and } A_\delta(t) = \begin{bmatrix} \delta_1 & \frac{1}{2}\delta_2 \\ \frac{3}{2}\delta_3 & 0 \end{bmatrix}, \tag{46}$$

where x_1 and x_2 are valued in $\mathbb{R}$, the piecewise continuous bounded unknown functions δ_i represent uncertainty, and u is the real valued input, which agrees with the example in [14] when each δ_i is the zero function. Then, using the notation $K = [K_1,\ K_2]$, our assumptions will be satisfied when the supremum of the δ_i's, $a > 0$, and the entries of $\Gamma > 0$ are small enough, provided

$$H = \begin{bmatrix} 1+K_1 & \frac{1}{2}+K_2 \\ \frac{3}{2}+K_1 & K_2 \end{bmatrix} \tag{47}$$

is Hurwitz and Metzler (again by [4, Lemma 2.3, p.41]). Since (47) is Hurwitz and Metzler if $K_1 + K_2 < -3/2$, $K_1 \ge -3/2$, and $K_2 \ge -1/2$, we will therefore choose $K_1 = -4/3$ and $K_2 = -1/3$. Then simple Mathematica calculations show that the requirements of Theorem 1 are satisfied when $\sup\{|\delta_i(t)| : t \ge 0\} \le 0.01$ for all i, by choosing $\nu = 0.02$ and each entry of Γ to be 0.045, because these choices ensure that (7) hold. Moreover, when the δ_i's are 0, we can satisfy the requirements with $\nu = 0.122$ with all other parameters as before. This gives the lower bound $\nu = 0.122$ on the intersampling times $t_{i+1} - t_i$.

Theorem 1 is outside the scope of notable works such as [22] (which required a bound of the form $||\epsilon(t)|| \le c||x(t)||$ relating the uncertainty ϵ and the state x at each time t for some constant c, which we do not require here). However, in the special case of the preceding example where each δ_i is the zero function, the system (45) is covered by the event-triggered results from [21]. If we had instead used [21, Corollary IV.1] to obtain a lower bound on the $t_{i+1} - t_i$'s (using $\sigma = \min\{\Gamma_{ij} : 1 \le i \le n, 1 \le j \le n\} = 0.045$ in [21, Corollary IV.1], which corresponds to our trigger condition $|e| \le \Gamma|x|$), then we would have gotten $\nu = 0.0811$ in the case where the δ_i's are the zero function, so our method again provides a larger ν and the potential advantage of ensuring less frequent control recomputations.

Fig. 5 shows MATLAB simulations of (45) that implement the event-triggered controller from Theorem 1, with a 20 second time horizon using the above parameter values with the δ_i's taken to be the constant values 0.01. For this time horizon, our controller from Theorem 1 produced 73 sample times when the control was recomputed.

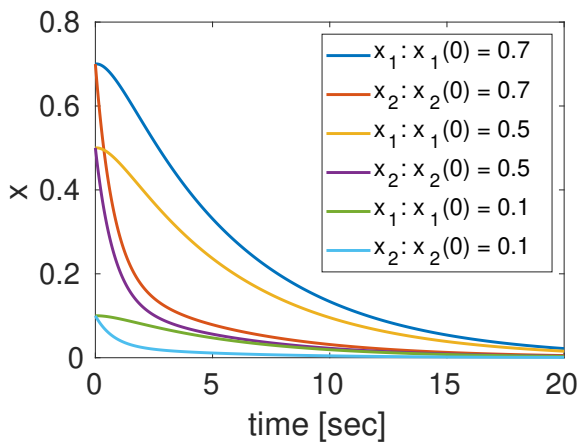

Fig. 5. Positive Systems Approach in Unperturbed Case

7 Conclusion

We proposed new event-triggered control designs, where instead of standard small-gain or other approaches, we used the theory of positive linear time-varying systems and interval observers. Our methods were motivated by the need to only infrequently recompute control values in marine robotic and other applications, instead of frequently changing control values. This allowed us to prove uniform global exponential stability properties for large classes of time-varying linear systems with output feedback and uncertainty in the coefficient matrices, by providing a robustness result with respect to uncertainty in the dynamics. Our examples illustrated how our method can lead to less conservative lower bounds on the inter-sampling times, com-

pared with other approaches, and so ensures less frequent sampling. In future work, we aim to apply our approach to generate event-triggered adaptive dynamic programming methods for solving data-driven optimization problems.

Appendix A: Checking Assumption 2

We provide two more remarks on how to check Assumption 2.

Remark A.1 Assumption 2 is an analog of [13, Assumption 1]. Assumption 2 can always be satisfied for a matrix $\Gamma > 0$ with small enough entries and a constant matrix K when (A, B) is a (constant) controllable pair, after a change of variables. This is done by choosing a constant matrix K such that all eigenvalues of $A + BK$ are negative real values, and then using a similarity transformation that transforms (A, B) into a new pair (PAP^{-1}, PB) where $H = P(A + BK)P^{-1}$ is the Hurwitz and Metzler Jordan canonical form of $A + BK$. Then we can satisfy the requirements of Assumption 1 using a constant vector $V > 0$ (by [4, Lemma 2.3, p.41]). □

Remark A.2 Assumption 2 can also be satisfied in time-varying cases where $(A_0(t), B_0(t))$ has the form $(A_c + \Delta_A(t), B_c + \Delta_B(t))$ when (A_c, B_c) is a controllable pair and the sup norms of the continuous time-varying parts (Δ_A, Δ_B) and the entries of $\Gamma > 0$ are small enough, again by a change of coordinates. On the other hand, for any constant $\alpha > 0$, Assumption 2 is also satisfied by

$$\begin{aligned} &A_0(t) = 1 - \tfrac{\alpha\cos(\alpha t)}{2+\sin(\alpha t)},\ B_0 = 1,\ K = -4,\ \Gamma = \tfrac{c}{4}, \\ &V = 2 + \sin(\alpha t),\ \overline{V} = 3,\ \underline{V} = 1,\ \text{and } p = 3 - c \end{aligned} \tag{A.1}$$

for any constant $c \in (0, 3)$ and $A_\delta = B_\delta = 0$, which illustrates how Assumption 2 also allows cases where the supremum of the time-varying part of A can be arbitrarily large (because $\alpha > 0$ can be arbitrarily large). Then we can use numerical methods to find values $\nu > 0$ such that the requirements from Lemma B.1 hold. For instance, for $\alpha = 50$, they hold with (A.1), $c = 2.99$, and $\nu = 0.1$. □

Appendix B: Methods for Choosing ν

A key ingredient in our theorem is the constant $\nu > 0$ satisfying our condition (7). The following lemma provides a sufficient condition for $\nu > 0$ to satisfy (7) (but see below for other ways to satisfy (7)):

Lemma B.1 *Let $\Gamma \in \mathbb{R}^{n\times n}$ be a positive matrix and Ω_0 be defined by (6). Fix constants $\overline{a} > 0$ and $\overline{b} > 0$ such that $||A_0||_\infty \le \overline{a}$ and $||B_0K||_\infty \le \overline{b}$. Let $\underline{\gamma} \in (0, \min\{1, r_0\})$ be a constant, where r_0 is the smallest of the entries of the matrix $\Gamma > 0$. Then for any constant $\nu > 0$ such that*

$$(1 + (\bar{b}/\bar{a}))(e^{\bar{a}\nu} - 1)e^{(1+(\bar{b}/\bar{a}))(e^{\bar{a}v}-1)} \le \underline{\gamma}, \tag{B.1}$$

our requirement (7) is satisfied. □

Proof: We show that $\Omega_0(t, s)$ is invertible and that

$$|I - \Omega_0^{-1}(t, s)| \le \Gamma \tag{B.2}$$

is satisfied when $t \in [s, s + \nu]$ and $s \ge 0$.

Since $\Phi_{A_0}(t, s)$ is invertible, it follows from the semigroup property of fundamental solutions that the matrix $\Omega_0(t, s)$ is invertible if and only if $\mathcal{N}(s, t) = I + \int_s^t \Phi_{A_0}(s, m)B_0(m)K(m)\mathrm{d}m$ is invertible. When $t \ge s$, this matrix is invertible if

$$\mathcal{L}(t, s) < 1 \tag{B.3}$$

with

$$\mathcal{L}(t, s) = \int_s^t ||\Phi_{A_0}(s, m)||||B_0(m)K(m)||\mathrm{d}m \tag{B.4}$$

(by showing that the nullspace of $\mathcal{N}(s, t)$ is 0). Also,

$$\mathcal{L}(t, s) \le \int_s^t e^{\bar{a}(m-s)}\mathrm{d}m\bar{b} \le (\bar{b}/\bar{a})(e^{\bar{a}(t-s)} - 1) \tag{B.5}$$

when $t \ge s \ge 0$, e.g., by the Peano-Baker formula (e.g., from [20, Appendix C.4]). Since (B.1) implies that $(\bar{b}/\bar{a})(e^{\bar{a}\nu} - 1) < 1$, we deduce that the inequality (B.3) is satisfied for all $s \ge 0$ and $t \in [s, s + \nu]$, which implies that $\Omega_0(t, s)$ is invertible for all $s \ge 0$ and $t \in [s, s + \nu]$.

Next, we prove that (B.2) is satisfied for all $t \in [s, s + \nu]$, where $s \ge 0$ is arbitrary. Note that (B.2) is satisfied if

$$\left|\left|I - \Omega_0(t, s)^{-1}\right|\right| \le \underline{\gamma}. \tag{B.6}$$

This inequality is equivalent to

$$\left|\left|I - (I - \widetilde{\Omega}_0(t, s))^{-1}\right|\right| \le \underline{\gamma} \tag{B.7}$$

with $\widetilde{\Omega}_0(t, s) = I - \Omega_0(t, s)$. We prove that $||\tilde{\Omega}_0(t, s)|| < 1$ for all $t \in [s, s + \nu]$. Since

$$\begin{aligned} &\widetilde{\Omega}_0(t, s) = \\ &I - \Phi_{A_0}(t, s) - \int_s^t \Phi_{A_0}(t, m)B_0(m)K(m)\mathrm{d}m, \end{aligned} \tag{B.8}$$

the inequality

$$||\widetilde{\Omega}_0(t, s)|| \le ||I - \Phi_{A_0}(t, s)|| + (\bar{b}/\bar{a})(e^{\bar{a}(t-s)} - 1) \tag{B.9}$$

is satisfied when $t \ge s \ge 0$. The Peano-Baker formula gives

$$||\Phi_{A_0}(t, s) - I|| \le \sum_{k=1}^{+\infty} \tfrac{(t-s)^k||A_0||_\infty^k}{k!} \le e^{\bar{a}(t-s)} - 1 \tag{B.10}$$

when $t \ge s \ge 0$. Thus (B.9) implies that

$$\begin{aligned} ||\widetilde{\Omega}_0(t, s)|| &\le e^{\bar{a}(t-s)} - 1 + (\bar{b}/\bar{a})(e^{\bar{a}(t-s)} - 1) \\ &\le (1 + (\bar{b}/\bar{a}))(e^{\bar{a}(t-s)} - 1) \end{aligned} \tag{B.11}$$

when $t \ge s \ge 0$. From (B.1), we deduce that $||\widetilde{\Omega}_0(t, s)|| \le \underline{\gamma} < 1$ for all $t \in [s, s + \nu]$. Hence, by the matrix geometric series formula, the inequality (B.7) is equivalent to

$$\left|\left|\sum_{k=1}^{+\infty} \tfrac{\widetilde{\Omega}(t,s)^k}{k!}\right|\right| \le \underline{\gamma} \tag{B.12}$$

for all $t \in [s, s + \nu]$. It is satisfied if

$$||\widetilde{\Omega}_0(t, s)|| \sum_{k=1}^{+\infty} \tfrac{\left|\left|\widetilde{\Omega}_0(t,s)\right|\right|^{k-1}}{(k-1)!} \le \underline{\gamma} \tag{B.13}$$

for all $t \in [s, s + \nu]$. This inequality is equivalent to

$$||\widetilde{\Omega}_0(t, s)||e^{||\widetilde{\Omega}_0(t,s)||} \le \underline{\gamma}. \tag{B.14}$$

According to (B.11), (B.14) is satisfied if

$$(1 + (\bar{b}/\bar{a}))(e^{\bar{a}(t-s)} - 1)e^{(1+(\bar{b}/\bar{a}))(e^{\bar{a}(t-s)}-1)} \le \underline{\gamma}. \tag{B.15}$$

By (B.1), this is the case. This concludes the proof. □

The optimal (i.e., largest) ν satisfying the requirements of Lemma B.1 is the ν such that (B.1) holds with equality, which is the root ν of

$$(1+(\bar{b}/\bar{a}))(e^{\bar{a}\nu}-1)e^{(1+(\bar{b}/\bar{a}))(e^{\bar{a}v}-1)} = \underline{\gamma}, \tag{B.16}$$

because the left side of (B.1) is strictly increasing in ν. In practice, we can use numerical methods to solve for the root $\nu > 0$ of (B.16). The constant ν constructed in the proof of Lemma B.1 is in general not the largest possible ν such that (7) is satisfied for given choices of A_0, B_0, K, and Γ. In practice, direct or numerical techniques can be applied to find values for ν that satisfy (7) but violate (B.1). To see how, we next consider other ways to satisfy (7), including a result for periodic cases which we used to compute ν in Remark A.2; see Section 6.1 for an application to curve tracking.

First we consider the case where A_0, B_0, and K have the same period $p_* > 0$. Second, we consider the case where A_0 has the form $A_0(t) = A_c + \Delta_A(t)$ for a constant matrix A_c, where the known matrix-valued function Δ_A is bounded and continuous (but not necessarily periodic), under suitable bounds on the sup norm of Δ_A.

To cover the periodic case, first note that in this case, we have $\Phi_{A_0}(t,s) = \Phi_{A_0}(t-ip_*, s-ip_*)$ and so also $\Omega_0(t,s) = \Omega_0(t-ip_*, s-ip_*)$ for all integers $i \geq 0$, all $s \geq ip_*$ and all $t \geq s$, by our formula (6) for Ω_0. Hence, (7) will hold if $\nu > 0$ is such that (B.2) is satisfied for all $s \in [0, p_*]$ and $t \in [s, s+\nu]$. To see why, note that if (B.2) holds for all $s \in [0, p_*]$ and $t \in [s, s+\nu]$, and if $s_a \geq 0$ and $t_a \in [s_a, s_a+\nu]$ are given, then we can find an integer $i \geq 0$ such that $s_a \in [ip_*, (i+1)p_*]$, so

$$|I-\Omega_0^{-1}(t_a,s_a)| = |I-\Omega_0^{-1}(t_a-ip_*, s_a-ip_*)| \leq \Gamma, \tag{B.17}$$

because $s_a - ip_* \in [0, p_*]$ and $t_a - ip_* \in [s_a - ip_*, s_a - ip_* + \nu]$. For any $\nu > 0$, we can use numerical methods to check whether (B.2) holds for all $s \in [0, p_*]$ and $t \in [s, s+\nu]$, because we can factor $\Phi_{A_0}(t,m)$ for all $m \in [s,t]$, $s \in [0, p_*]$, and $t \in [s, s+\nu]$, e.g., by writing

$$\Phi_{A_0}(t,m) = \alpha_A(t)\beta_A(m) \tag{B.18}$$

for all $m \geq 0$ and $t \geq m$, where α_A and β_A are the unique solutions of the matrix differential equations

$$\dot{\alpha}_A(t) = A_0(t)\alpha_A(t) \text{ and } \dot{\beta}_A(m) = -\beta_A(m)A_0(m) \tag{B.19}$$

that satisfy $\alpha_A(0) = \beta_A(0) = I$, and by then checking if

$$\sup\{|I-\Omega_0^{-1}(t,s)| : s \in [0,p_*], t \in [s, s+\nu]\} \leq \Gamma; \tag{B.20}$$

see, e.g., [8], for the use of this decomposition of fundamental solutions in adaptive control. The factoring (B.18) makes it possible to compute the fundamental solution values that are needed to check Assumption 3 when $C \neq I$.

To cover cases where $A_0(t) = A_c + \Delta_A(t)$ for a constant matrix A_c and continuous and bounded Δ_A, assume that the coefficient matrix $B = B_0$ and K are known constant matrices, and that the requirements of Assumption 2 are satisfied for some $\Gamma > 0$ with a constant K, but similar reasoning applies if B and K are time varying.

Choose any positive matrices Γ_a and $\Gamma_b = [\gamma_{ij}]$ such that $\Gamma = \Gamma_a + \Gamma_b$. Then (7) is satisfied if $\nu > 0$ and Δ_A satisfy

$$\begin{aligned} &\sup_{\ell \in [0,\nu]} |I - \Omega_c^{-1}(\ell)| \leq \Gamma_a \text{ and} \\ &\left\{ b_* e^{\nu||A_c||} \left(e^{\nu||\Delta_A||_\infty} - 1\right) \right\} ||\Omega_c^{-1}||_{[0,\nu]} < c_0, \end{aligned} \tag{B.21}$$

where $b_* = 1 + \nu||BK||$,

$$\Omega_c(s) = e^{sA_c} + \textstyle\int_0^s e^{A_c(s-\ell)} \mathrm{d}\ell BK, \tag{B.22}$$

and the constant $c_0 \in (0,1)$ is such that

$$\tfrac{c_0}{1-c_0} ||\Omega_c^{-1}||_{[0,\nu]} \leq \min\{\gamma_{ij} : 1 \leq i \leq n, 1 \leq j \leq n\}. \tag{B.23}$$

To see why (7) is satisfied in this case, we first write

$$\begin{aligned} \Omega_0(t,s) &= \Omega_c(t-s) + \Phi_{A_0}(t,s) - e^{(t-s)A_c} \\ &\quad + \textstyle\int_s^t \left[\Phi_{A_0}(t,\ell) - e^{A_C(t-\ell)}\right] BK \mathrm{d}\ell \\ &= \Omega_c(t-s)\left[I + \eta(t,s)\right] \end{aligned} \tag{B.24}$$

for all $s \geq 0$ and $t \geq s$, where

$$\begin{aligned} \eta(t,s) &= \Omega_c(t-s)^{-1}\big[\Phi_A(t,s) - e^{A_c(t-s)} \\ &+ \textstyle\int_s^t \left(\Phi_{A_0}(t,\ell) - e^{A_c(t-\ell)}\right) BK \mathrm{d}\ell\big]. \end{aligned} \tag{B.25}$$

Using the bound

$$\begin{aligned} &\left|\left|\Phi_{A_0}(t,s) - e^{A_c(t-s)}\right|\right| \leq \\ &e^{\nu||A_c||}\left(e^{\nu||\Delta_A||_\infty} - 1\right) \end{aligned} \tag{B.26}$$

which holds for all $s \geq 0$ and $t \in [s, s+\nu$, e.g., by [10, Lemma 2], and recalling (B.21), it follows that $||\eta(t,s)|| \leq c_0 < 1$, and therefore that $I + \eta(t,s)$ is invertible (by checking that its nullspace is trivial). This implies that (B.24) is invertible when $s \geq 0$ and $t \in [s, s+\nu]$

Also, for $\bar{c} = \min_{ij} \gamma_{ij}$, and for any $S \in \mathbb{R}^{n\times n}$ that satisfies $||S|| \leq \bar{c}$, we have $|S| \leq \Gamma_b$. Specializing the preceding observation to the case where $S = \Omega_c(t-s)^{-1} - \Omega_0^{-1}(t,s)$, it follows from Lemma C.1 in Appendix C below (with $M_0 = \Omega_c^{-1}(t-s)$, N_0 being the quantity in squared brackets in (B.25), $\bar{m} = ||\Omega_c^{-1}||_{[0,\nu]}$, and $\bar{n}$ being the quantity in curly braces in (B.21)) that (B.23) gives

$$\begin{aligned} &\left|\left|M_0^{-1} - (M_0+N_0)^{-1}\right|\right| \\ &= \left|\left|\Omega_c(t-s)^{-1} - \Omega_0^{-1}(t,s)\right|\right| \leq \tfrac{||\Omega_0^{-1}||_{[0,\nu]} c_0}{1-c_0} \\ &\leq \min\{\gamma_{ij} : 1 \leq i \leq n, 1 \leq j \leq n\}, \end{aligned} \tag{B.27}$$

and so also

$$\left|\Omega_c(t-s)^{-1} - \Omega_0^{-1}(t,s)\right| \leq \Gamma_b, \tag{B.28}$$

by writing $\Omega_0(t,s) = M_0 + N_0$ and then noting that $||M_0^{-1}||||N_0|| \leq c_0$. Hence, the bound $|I - \Omega_0^{-1}(t,s)| \leq \Gamma$ for all $s \geq 0$ and $t \in [s, s+\nu]$ follows from the first inequality in (B.21) and (B.28) and our decomposition $\Gamma = \Gamma_a + \Gamma_b$. Conditions (B.21) can be checked numerically by computing Ω_0 in (B.22), and they hold for a small enough $\nu > 0$ because $\Omega_0(0) = I$, by the continuity of matrix inversion.

Using the preceding alternative conditions (B.21) instead of (B.1) from Lemma B.1, we can allow cases with bigger

values of ν, and therefore ensure less frequent recomputations of control values. For instance, consider the scalar case where $A_c = B_0 = 1$, $K = -2.2$, Δ_A is bounded by 0.1, $\Gamma = 0.35$, and $\nu = 0.16$. Then, simple Mathematica calculations show that condition (B.1) is violated, because with the choices $\bar{a} = 1.1$ and $\bar{b} = 2.2$, the left side of (B.1) is $1.02834 > \min\{1, \Gamma\} = 0.35$. On the other hand, the criteria (B.21) are satisfied with the preceding choices of A_c, B_0, K, Γ, and ν and any Δ_A that is bounded by 0.1, if we choose $c_0 = 0.0526$, $\Gamma_a = 0.79\Gamma$, and $\Gamma_b = 0.21\Gamma$.

Appendix C: Lemmas from [10]

We used the following lemmas (which are [10, Lemma 1], and a variant of [11, Lemma 1] with a similar proof, respectively) in Appendix B above:

Lemma C.1 *Let $M_0 \in \mathbb{R}^{n\times n}$ be an invertible matrix. Let $N_0 \in \mathbb{R}^{n\times n}$ be a matrix. Let $\bar{n}$ and $\bar{m}$ be two constants such that $||M_0^{-1}|| \leq \bar{m}$ and $||N_0|| \leq \bar{n}$. Assume that $\bar{m}\bar{n} < 1$. Then the matrix $M_0 + N_0$ is invertible and*

$$\left|\left|M_0^{-1} - (M_0 + N_0)^{-1}\right|\right| \leq \frac{\bar{m}^2\bar{n}}{1 - \bar{m}\bar{n}} \tag{C.1}$$

is satisfied. □

Lemma C.2 *Consider the system*

$$\dot{\zeta}(t) = [\mathcal{A}(t) + \mathcal{E}(t)]\,\zeta(t) \tag{C.2}$$

where $\mathcal{A} : [0, +\infty) \to \mathbb{R}^{n\times n}$ is a continuous matrix-valued function, ζ is valued in $\mathbb{R}^n$, and $\mathcal{E} : [0, +\infty) \to \mathbb{R}^{n\times n}$ is a bounded piecewise continuous function. Then for all $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$ such that $t_1 \geq t_2$, the inequality

$$\begin{aligned} &||\Phi_{\mathcal{A}+\mathcal{E}}(t_1, t_2) - \Phi_{\mathcal{A}}(t_1, t_2)|| \\ &\leq e^{||\mathcal{A}||_\infty (t_1 - t_2)} \left(e^{||\mathcal{E}||_\infty (t_1 - t_2)} - 1\right) \end{aligned} \tag{C.3}$$

is satisfied. □

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