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Modal computation for open waveguides

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Abstract

In this work, we are interested in mode computation in a three-dimensional open waveguide. We propose some absorbing boundary conditions to compute the modes in a bounded domain.

Keywords: open waveguides, Helmholtz equation, absorbing boundary conditions.

1 Introduction

The study of electromagnetic (EM) wave propagation is essential for considering the impact of Human’s technologies on the environment. For instance, the offshore wind energy is transported through twisted dynamic cables, which armours prevent a significant portion of the waves to irradiate outside the cable. Nevertheless, a remaining and possibly significant part might escape from the cable. Hence, our aim is to look at their scattering in the large stretch of sea water.

To consider this problem, we propose to modelize the cable and the surrounding water by an open 3D waveguide, which is an invariant domain according to the cable direction and which is unbounded in the two other directions. We will then take a modal approach for the resolution. Moreover, although Maxwell equations govern the propagation of EM waves, they will be here simplified to the Helmholtz equation, by considering Transverse Magnetic waves.

Before studying the EM field far away from the cable, we propose to focus on its behavior in its vicinity. Thus, we will use absorbing boundary conditions around the cable. Then, we will discuss about their relevance, especially in a low frequency system.

2 The modal problem

We consider the following Helmholtz problem :

$$-\Delta U - \rho\omega^2 U = 0 \quad \text{in } \mathbb{R}^3 \quad (1)$$

where ω is the pulsation and where $\rho = \rho(x, y) \geq \rho_- > 0$ (invariant in z) and there is a constant

ρ_∞ such that $\rho - \rho_\infty$ is compactly supported in the plane $\{(x, y) \in \mathbb{R}^2\}$, and \mathcal{C} denotes its support. We also consider $\rho_+ = \|\rho\|_\infty$. We seek for modal solutions of the form $U(x, y, z) = u(x, y)e^{i\beta z}$ where $u \in H^1(\mathbb{R}^2)$ and $\beta \in \mathbb{C}$ is the eigenvalue. Such a decomposition is possible because the domain is a waveguide whose geometry does not vary with z . It leads us to the following eigenvalue problem:

$$-\Delta_{x,y} u - (\rho\omega^2 - \beta^2)u = 0 \quad \text{in } \mathbb{R}^2 \quad (2)$$

where $\Delta_{x,y} = \partial_{x^2}^2 + \partial_{y^2}^2$ is the transverse Laplace operator.

Lemma 1 *If the eigenfunction u is in $H^1(\mathbb{R}^2)$, then the eigenvalue β^2 is in $\omega^2[\rho_-, \rho_+]$.*

This results is proven by contradiction arguments (using arguments of [1]) and will be detailed during the talk. Using radial decomposition of the solution, one can show that the solution u should satisfy the radiation condition, analogous to the Sommerfeld condition :

$$\lim_{r \rightarrow +\infty} \frac{\partial u}{\partial r} - i\sqrt{\rho_\infty\omega^2 - \beta^2} u = 0 \quad (3)$$

where $r = \sqrt{x^2 + y^2}$.

We can approximate this condition by imposing it at finite distance, for instance on a boundary $\partial\Omega$ (typically a circle of center 0 with radius R , such that the induced disk Ω includes the domain \mathcal{C} in the plane Oxy). It implies that we restrict (3) on Ω , and that we can write :

$$\frac{\partial u}{\partial r}|_{\partial\Omega} - i\sqrt{\rho_\infty\omega^2 - \beta^2} u|_{\partial\Omega} = 0 \quad (4)$$

Such a condition is called absorbing boundary condition (ABC). The new difficulty of the eigenproblem is in the non-linearity of this condition with the eigenvalue. The next section presents how to handle this.

3 Different sorts of ABC

The first idea to linearize the ABC is to consider that $\beta^2 \sim \rho^+ \omega^2$ (but still lesser). This consideration, with the fact that $\rho = \rho_\infty$ on $\partial\Omega$, leads us to approximate the square-root of the ABC as follows:

$$\frac{\partial u}{\partial r} |_{\partial\Omega} - i\omega \sqrt{\rho_\infty - \rho^+} u |_{\partial\Omega} = 0 \quad (5)$$

This expression is a common Robin boundary condition, independent of β , which will ease the resolution of the eigenproblem. However, we have to keep in mind that it results from a quite rough approximation. Using (5) might thus reduce the accuracy of the numerical results.

The two other ABCs that we propose come from the first two steps of the Newton-Raphson algorithm, applied to $i\sqrt{\rho_\infty \omega^2 - \beta^2}$. We take a small parameter ε_0 , which will be our initial guess for this root. The first step of the algorithm gives us the following expression for the ABC :

$$\frac{\partial u}{\partial r} |_{\partial\Omega} + \left(-\frac{\varepsilon_0}{2} + \frac{\rho_\infty \omega^2 - \beta^2}{2\varepsilon_0} \right) u |_{\partial\Omega} = 0 \quad (6)$$

This one shall be a more accurate condition, and is linear in β^2 , which does not make the implementation more difficult.

The last ABC is obtained by applying the second step of the Newton-Raphson algorithm:

$$\frac{\partial u}{\partial r} |_{\partial\Omega} + \left(-\frac{\varepsilon_0}{4} + \frac{\rho_\infty \omega^2 - \beta^2}{4\varepsilon_0} + \frac{\rho_\infty \omega^2 - \beta^2}{\varepsilon_0 + \frac{\beta^2 - \rho_\infty \omega^2}{\varepsilon_0}} \right) u |_{\partial\Omega} = 0 \quad (7)$$

This condition is clearly non-linear with β , but we can linearize it pretty simply. Indeed, let v be a function defined on $\partial\Omega$ such that

$$\left(\varepsilon_0 + \frac{\beta^2 - \rho_\infty \omega^2}{\varepsilon_0} \right) v = u |_{\partial\Omega} \quad (8)$$

This expression is linear with β^2 , and leads to the following condition :

$$\frac{\partial u}{\partial r} |_{\partial\Omega} + \left(-\frac{\varepsilon_0}{4} + \frac{\rho_\infty \omega^2 - \beta^2}{4\varepsilon_0} \right) u |_{\partial\Omega} + (\rho_\infty \omega^2 - \beta^2) v = 0 \quad (9)$$

This condition is more difficult to implement, because of the new variable v which must be taken into account. However, we can reasonably expect better results.

4 Numerical results

To show our first results, we consider $\mathcal{C} = \{r \leq 0.2\}$, $\rho = 10 \cdot \mathbf{1}_{\mathcal{C}} + \mathbf{1}_{\Omega \setminus \mathcal{C}}$ and $\omega = 10$. We will compare the results with different size of Ω , that is to say with $R = 0.5$ and $R = 1$.

The modal computation of this example, with those two sizes of domain, gives us ten modes, whose eigenvalues are all in the interval $[100, 1000]$, satisfying the lemma 1. Figure 1 represents the first mode $u(x, y)$ (i.e. with largest β) for the two radii and with the ABC (5) on $\partial\Omega$.

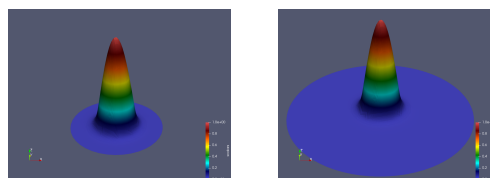


Figure 1: On the left, the first mode u with $R = 0.5$, and on the right, with $R = 1$, both with $\omega = 10$.

We can see that both radii give similar solutions. The same conclusion holds for the other ABCs. Actually, the frequency ω is high enough to enable the modes concentration near \mathcal{C} , giving few importance to the boundary condition.

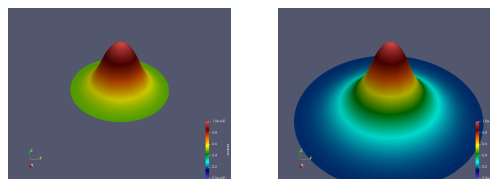


Figure 2: On the left, the first mode u with $R = 0.5$, and on the right, with $R = 1$, both with $\omega = 2$.

Let us see what the first ABC gives when the frequency is lower, for instance $\omega = 2$. On Figure 2, we can see that the solutions are no more neglectable on the boundary of the domain, so clearly in this case, the ABC must be well chosen to keep accurate results. This will be discussed in detail during the talk.

References

- [1] A.-S. Bonnet-BenDhia, S. Fliss, C. Hazard, A. Tonnoir, A Rellich type theorem for the Helmholtz equation in a conical domain, C. R. A. S. Paris Ser. I 354 (2016), 27– 32.