



**HAL**  
open science

# On the construction of Shape Functions for Spacetime Trefftz-DG Formulations of Wave Problems with Perfectly Matched Layers

Vinduja Vasanthan, H el ene Barucq, Henri Calandra, Julien Diaz

► **To cite this version:**

Vinduja Vasanthan, H el ene Barucq, Henri Calandra, Julien Diaz. On the construction of Shape Functions for Spacetime Trefftz-DG Formulations of Wave Problems with Perfectly Matched Layers. Waves 2022 - 15th International Conference on Mathematical and Numerical Aspects of Wave Propagation, Jul 2022, Palaiseau, France. hal-03883655

**HAL Id: hal-03883655**

<https://inria.hal.science/hal-03883655v1>

Submitted on 8 Jul 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destin ee au d ep ot et  a la diffusion de documents scientifiques de niveau recherche, publi es ou non,  emanant des  tablissements d'enseignement et de recherche franais ou  trangers, des laboratoires publics ou priv es.

WAVES 2022 – JULY 25-29, 2022

# On the construction of Shape Functions for Spacetime Trefftz-DG Formulations of Wave Problems with Perfectly Matched Layers

V. Vasanthan<sup>1</sup>, H. Barucq<sup>1</sup>, H. Calandra<sup>1</sup>, J. Diaz<sup>1</sup>

1. Project-Team Makutu, Inria, TotalEnergies, E2S UPPA, CNRS, France,  
firstname.lastname@inria.fr

- Aim of our work ?
  - ↔ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↔ gain precision,
  - ↔ reduce computational time.
- What kind of problems ?
  - ↔ acoustic,
  - ↔ elastodynamic,
  - ↔ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↔ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↔ polynomial basis functions,
  - ↔ Green's functions.

# Introduction

- Aim of our work ?
  - ↪ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↪ gain precision,
  - ↪ reduce computational time.
- What kind of problems ?
  - ↪ acoustic,
  - ↪ elastodynamic,
  - ↪ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↪ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↪ polynomial basis functions,
  - ↪ Green's functions.



# Introduction

- Aim of our work ?
  - ↪ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↪ gain precision,
  - ↪ reduce computational time.
- What kind of problems ?
  - ↪ acoustic,
  - ↪ elastodynamic,
  - ↪ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↪ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↪ polynomial basis functions,
  - ↪ Green's functions.



# Introduction

- Aim of our work ?
  - ↪ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↪ gain precision,
  - ↪ reduce computational time.
- What kind of problems ?
  - ↪ acoustic,
  - ↪ elastodynamic,
  - ↪ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↪ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↪ polynomial basis functions,
  - ↪ Green's functions.



# Introduction

- Aim of our work ?
  - ↪ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↪ gain precision,
  - ↪ reduce computational time.
- What kind of problems ?
  - ↪ acoustic,
  - ↪ elastodynamic,
  - ↪ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↪ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↪ polynomial basis functions,
  - ↪ Green's functions.



# Acoustic Wave equation

First order acoustic wave equation :

$$\left\{ \begin{array}{ll} \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, p(\cdot, 0) = p_0 & \text{in } \mathcal{D}, \\ \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}} = 0 & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad \begin{array}{l} \Omega = \mathcal{D} \times \mathcal{I} \\ \mathcal{I} = [0, T] \end{array}$$

with

$$\mathbf{x} = (x_1, x_2),$$

$$p = p(\mathbf{x}, t), \text{ pressure,}$$

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (u, v), \text{ velocity,}$$

$$f = f(\mathbf{x}, t), \text{ source term,}$$

$$c = c(\mathbf{x}), \text{ wave speed,}$$

$$\rho = \rho(\mathbf{x}), \text{ density.}$$

- Basis functions are solutions of the initial PDEs
- Physical characteristics of the solution are included in the discrete solution
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
- Basis functions are discontinuous → Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
- Spacetime implicit scheme

- Basis functions are solutions of the initial PDEs
- Physical characteristics of the solution are included in the discrete solution
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
  
- Basis functions are discontinuous → Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
  
- Spacetime implicit scheme

- Basis functions are solutions of the initial PDEs
- Physical characteristics of the solution are included in the discrete solution
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
  
- Basis functions are discontinuous → Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
  
- Spacetime implicit scheme

- Basis functions are solutions of the initial PDEs
- Physical characteristics of the solution are included in the discrete solution
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
  
- Basis functions are discontinuous → Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
  
- Spacetime implicit scheme

How to reduce computational costs ?

- solve element-by-element if mesh respects constraints
  - ↳ Tent-Pitcher algorithm
    - introduced for hyperbolic problems in spacetime domain,
    - consists in building a causal mesh, which respects the wave propagation speed,
    - conducive to parallel computing.
- to our knowledge, this algorithm has not been applied to Trefftz-DG + Perfectly Matched Layers (PML) yet

I. Perugia et al [2020]

A. Üngör et al [2005]



How to reduce computational costs ?

- solve element-by-element if mesh respects constraints
  - ↪ Tent-Pitcher algorithm
    - introduced for hyperbolic problems in spacetime domain,
    - consists in building a causal mesh, which respects the wave propagation speed,
    - conducive to parallel computing.
- to our knowledge, this algorithm has not been applied to Trefftz-DG + Perfectly Matched Layers (PML) yet

I. Perugia et al [2020]

A. Üngör et al [2005]



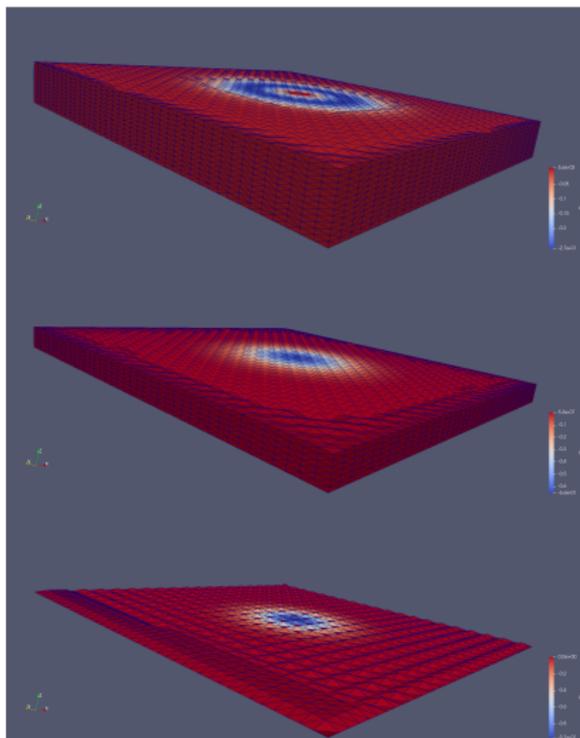


Figure: Visualisation of the evolution of the mesh in time

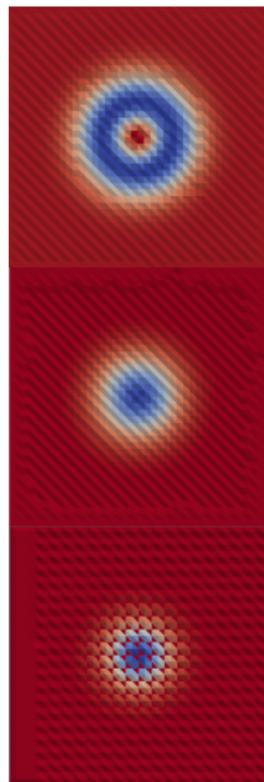
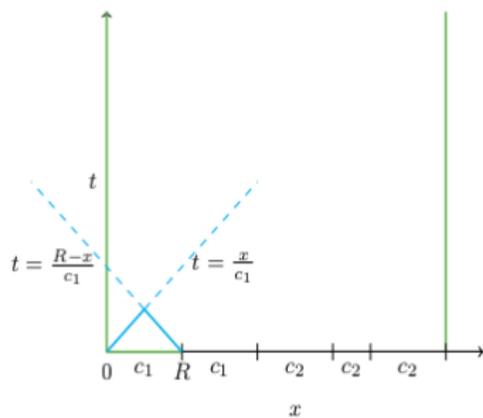
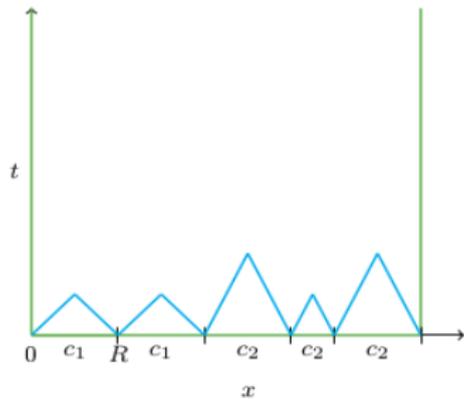
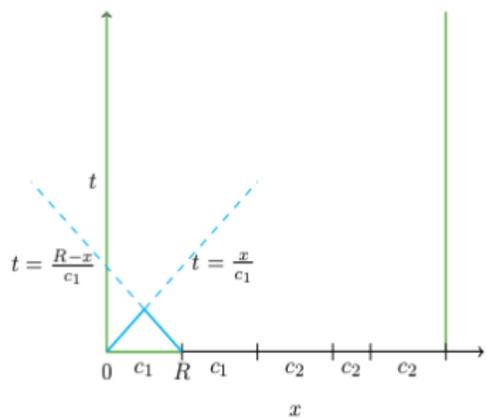


Figure: Pressure from above

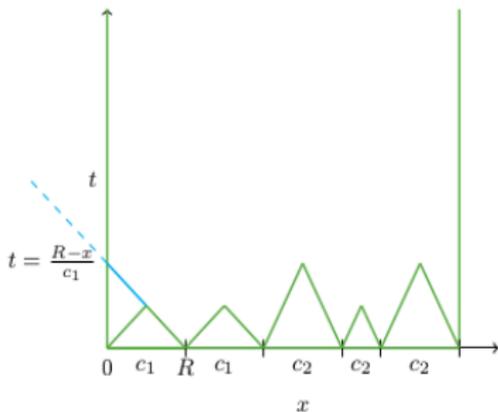
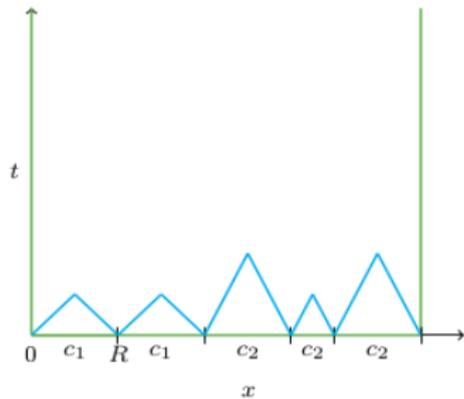
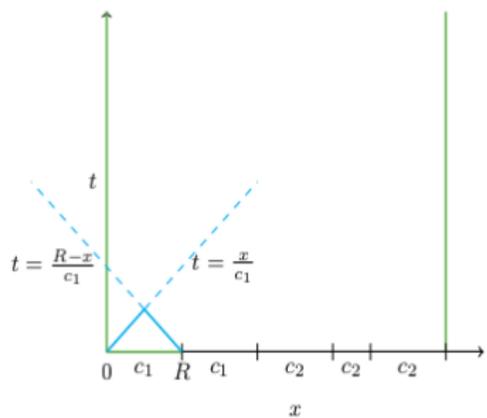
# Tent-pitching process 1d+t



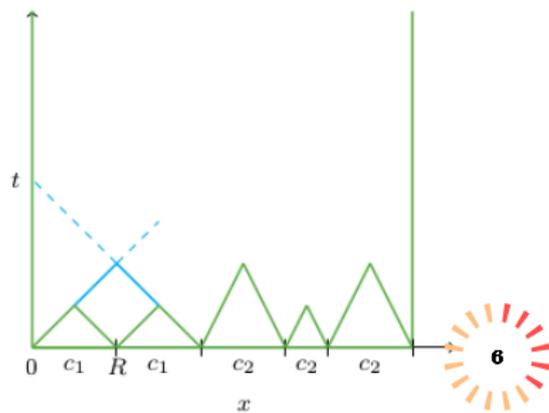
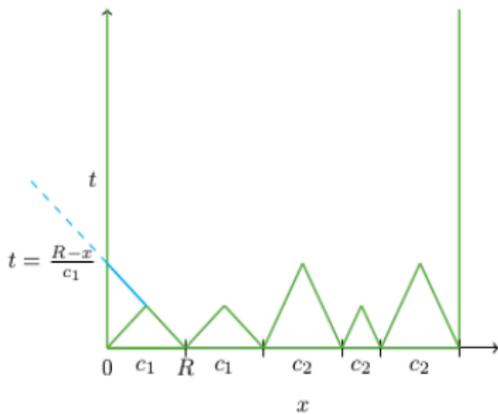
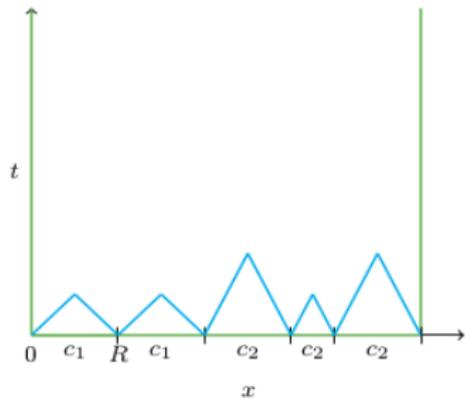
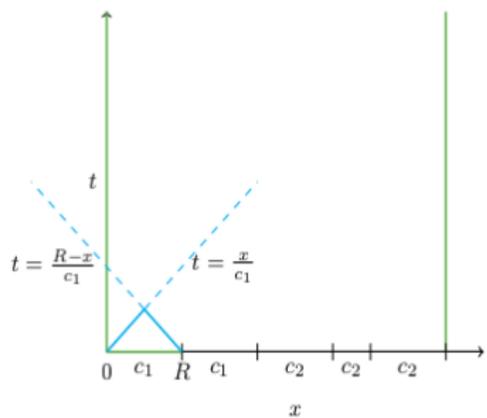
# Tent-pitching process 1d+t



# Tent-pitching process 1d+t



# Tent-pitching process 1d+t



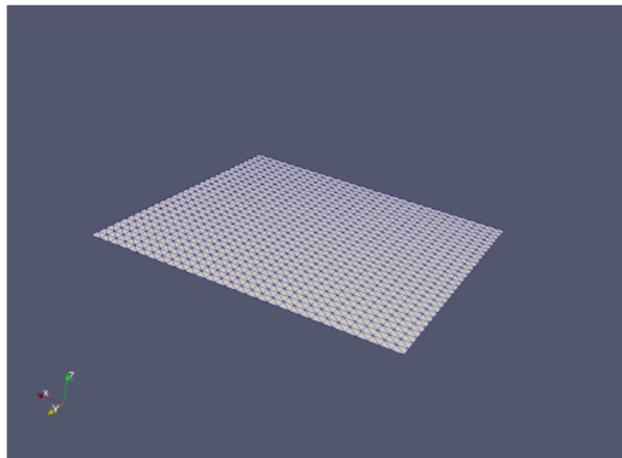


Figure: Initial spatial mesh

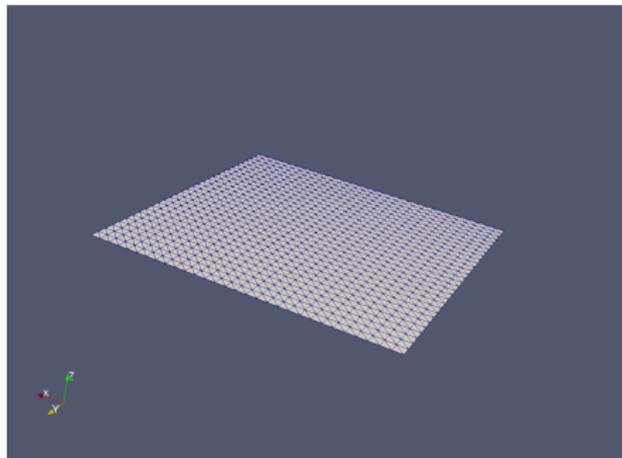


Figure: Initial spatial mesh

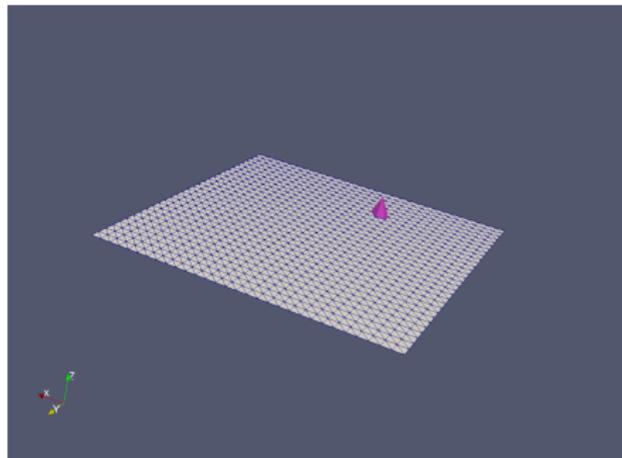


Figure: 1<sup>st</sup> iteration

# Tent-pitching process 2d+t

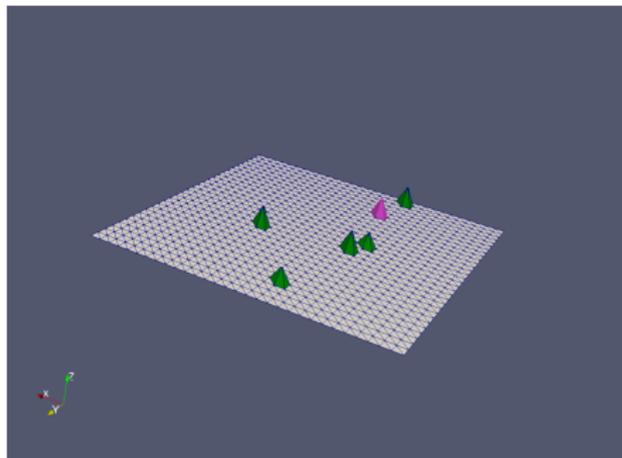


Figure: 5<sup>th</sup> iteration

# Tent-pitching process 2d+t

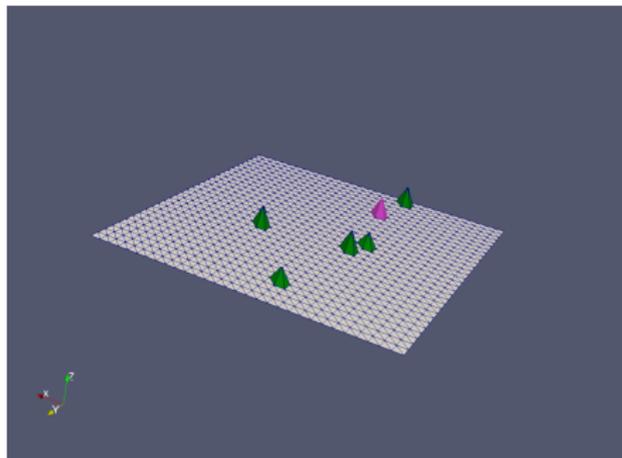


Figure: 5<sup>th</sup> iteration

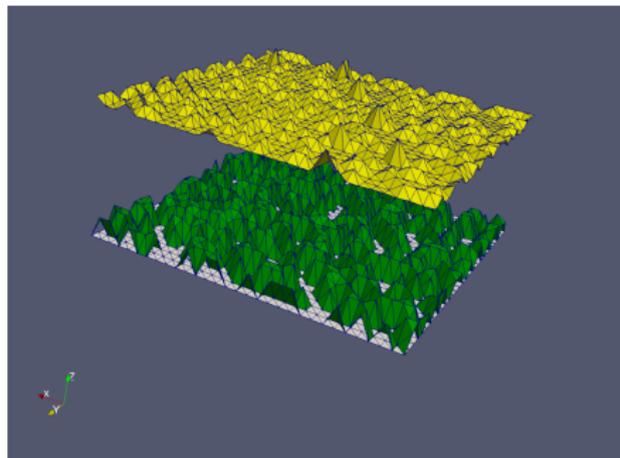
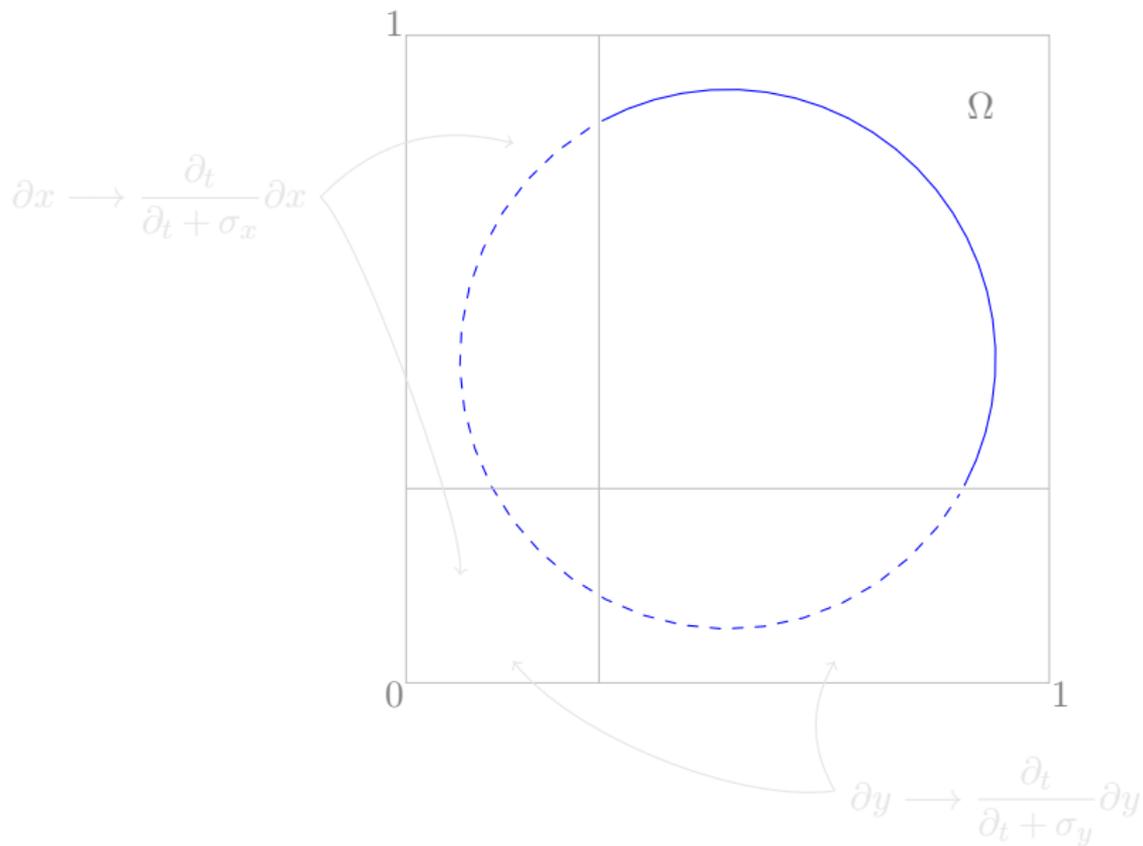
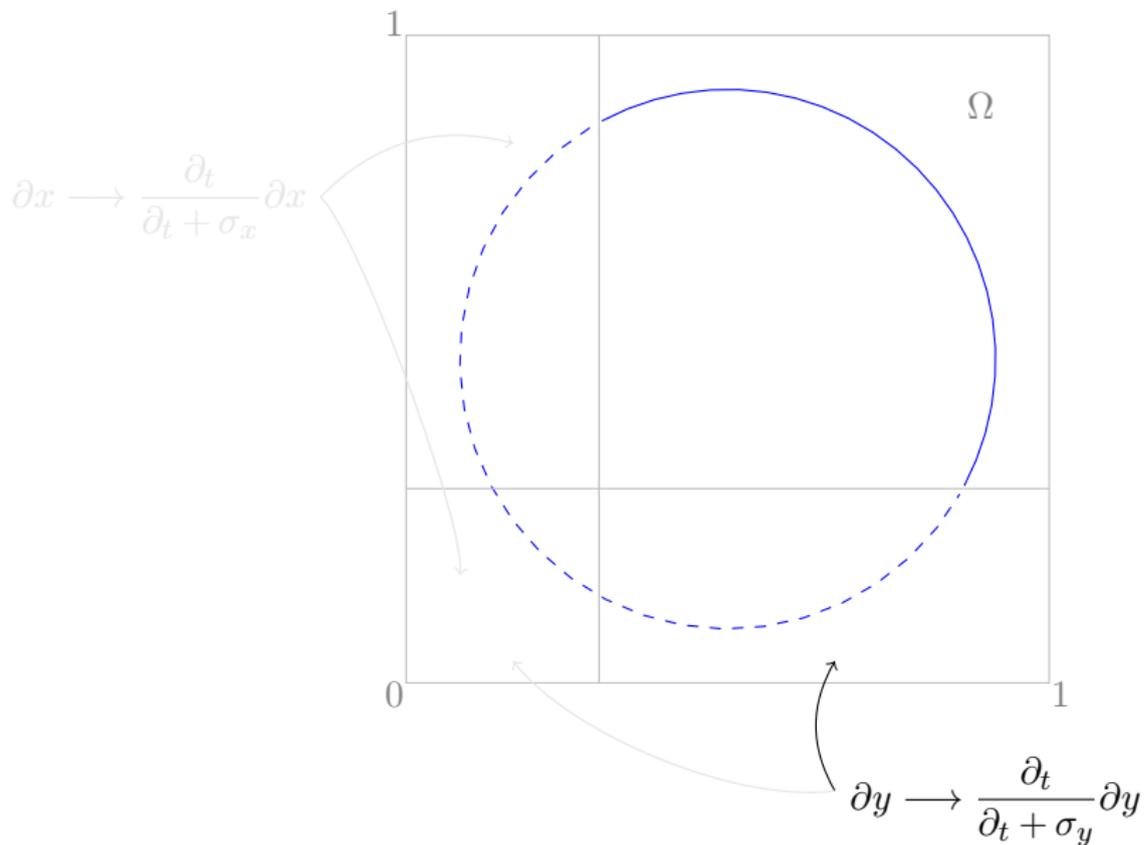


Figure: 6698<sup>th</sup> iteration

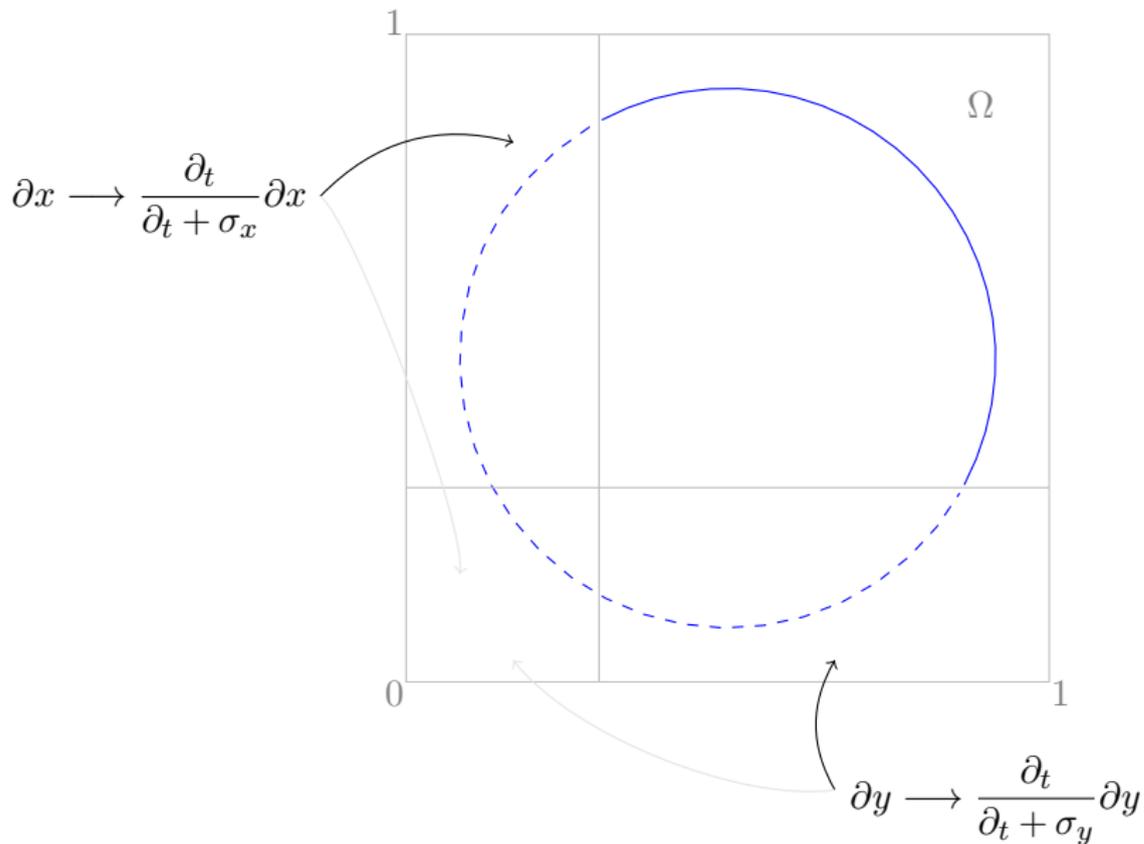
# Perfectly Matched Layers



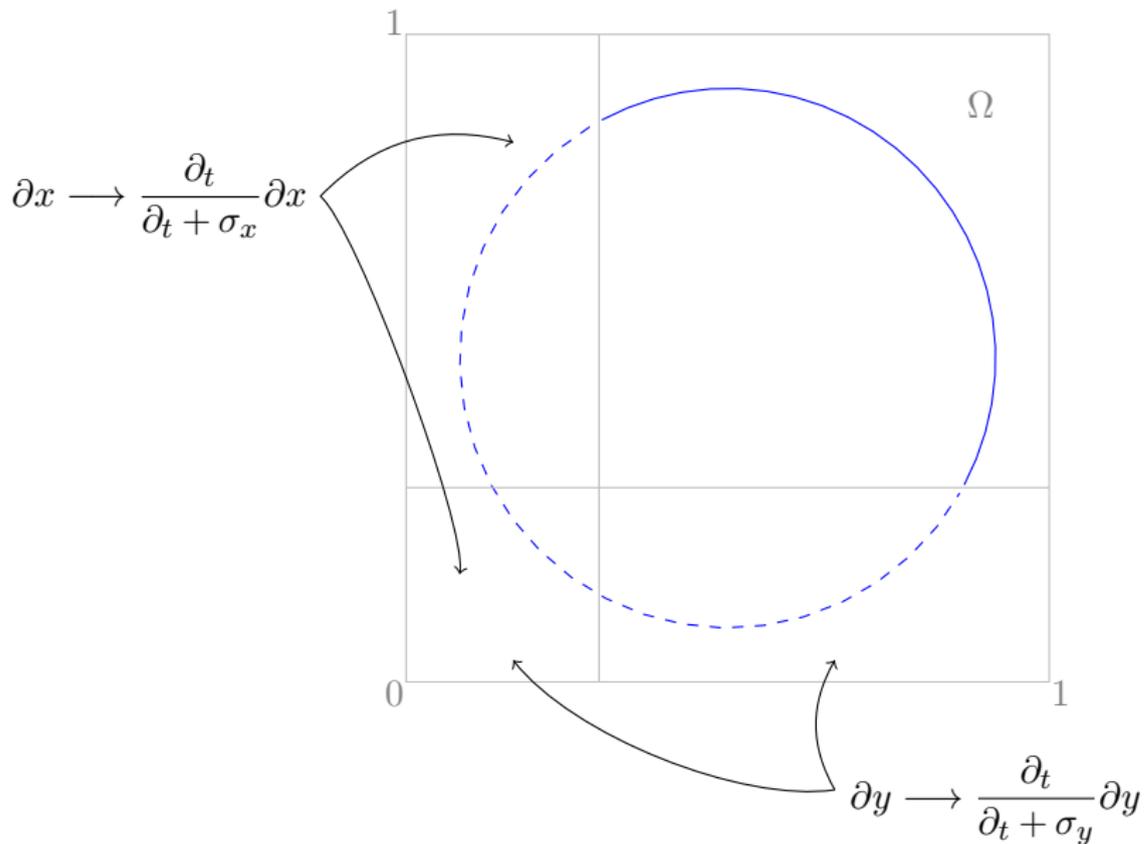
# Perfectly Matched Layers



# Perfectly Matched Layers



# Perfectly Matched Layers



# Perfectly Matched Layers

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_x} \frac{\partial p}{\partial x} = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_y} \frac{\partial p}{\partial y} = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_x} \frac{\partial u}{\partial x} + \frac{\partial_t}{\partial_t + \sigma_y} \frac{\partial v}{\partial y} = 0. \end{cases}$$

in  $\Omega = \mathcal{D} \times \mathcal{I}$

with  $\sigma_x$  and  $\sigma_y$

absorbing coefficients

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} + p_1 = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} + p_2 = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u_1 + v_1 = 0. \end{cases}$$

$$(\partial_t + \sigma_x)p_1 = -\sigma_x \frac{\partial p}{\partial x}$$

$$(\partial_t + \sigma_y)p_2 = -\sigma_y \frac{\partial p}{\partial y}$$

$$(\partial_t + \sigma_x)u_1 = -\sigma_x \frac{\partial u}{\partial x}$$

$$(\partial_t + \sigma_y)v_1 = -\sigma_y \frac{\partial v}{\partial y}$$

# Perfectly Matched Layers

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_x} \frac{\partial p}{\partial x} = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_y} \frac{\partial p}{\partial y} = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_x} \frac{\partial u}{\partial x} + \frac{\partial_t}{\partial_t + \sigma_y} \frac{\partial v}{\partial y} = 0. \end{cases}$$

in  $\Omega = \mathcal{D} \times \mathcal{I}$

with  $\sigma_x$  and  $\sigma_y$

absorbing coefficients

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} + p_1 = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} + p_2 = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u_1 + v_1 = 0. \end{cases}$$

$$(\partial_t + \sigma_x)p_1 = -\sigma_x \frac{\partial p}{\partial x}$$

$$(\partial_t + \sigma_y)p_2 = -\sigma_y \frac{\partial p}{\partial y}$$

$$(\partial_t + \sigma_x)u_1 = -\sigma_x \frac{\partial u}{\partial x}$$

$$(\partial_t + \sigma_y)v_1 = -\sigma_y \frac{\partial v}{\partial y}$$

- $G_p^{pml}(\mathcal{X}, \mathcal{T}, \sigma_x, \sigma_y) = e^{-A} \cos(B) G_p(\mathcal{X}, \mathcal{T})$

- $G_u^{pml}(\mathcal{X}, \mathcal{T}, \sigma_x, \sigma_y) = \frac{e^{-A} G_p(\mathcal{X}, \mathcal{T})}{\rho r^2} \left( \mathcal{T} \mathcal{X}_1 \cos(B_1) + h \mathcal{X}_2 \sin(B_1) \right)$

- $G_v^{pml}(\mathcal{X}, \mathcal{T}, \sigma_x, \sigma_y) = \frac{e^{-A} G_p(\mathcal{X}, \mathcal{T})}{\rho r^2} \left( \mathcal{T} \mathcal{X}_2 \cos(B_2) + h \mathcal{X}_1 \sin(B_2) \right)$

- $G_p(\mathcal{X}, \mathcal{T}) = \frac{1}{2\pi h}$ ,  $h = h(\mathcal{X}, \mathcal{T}) = \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}$ ,  $\Sigma_i(s) = \int_0^s \sigma_s(\tau) d\tau$ ,

- $A = A(\mathcal{X}, \mathcal{T}) = \left( \mathcal{X}_1 \Sigma_1(\mathcal{X}_1) + \mathcal{X}_2 \Sigma_2(\mathcal{X}_2) \right) \frac{\mathcal{T}}{r^2}$ ,

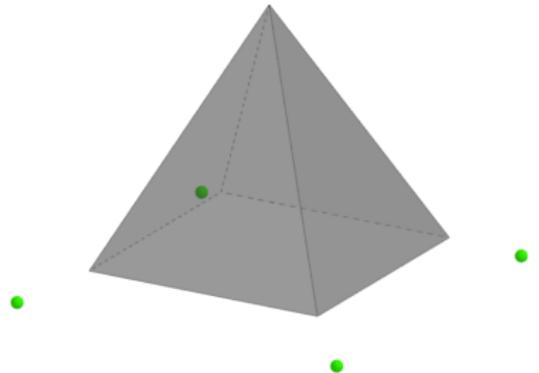
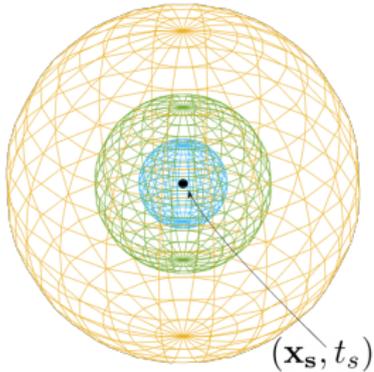
- $B = B(\mathcal{X}, \mathcal{T}) = \left( \mathcal{X}_1 \Sigma_2(\mathcal{X}_2) + \mathcal{X}_2 \Sigma_1(\mathcal{X}_1) \right) \frac{h}{r^2}$ ,

- $B_1 = B_1(\mathcal{X}, \mathcal{T}) = \left( \mathcal{X}_1 \Sigma_2(\mathcal{X}_2) - \mathcal{X}_2 \Sigma_1(\mathcal{X}_1) \right) \frac{h}{r^2}$ ,

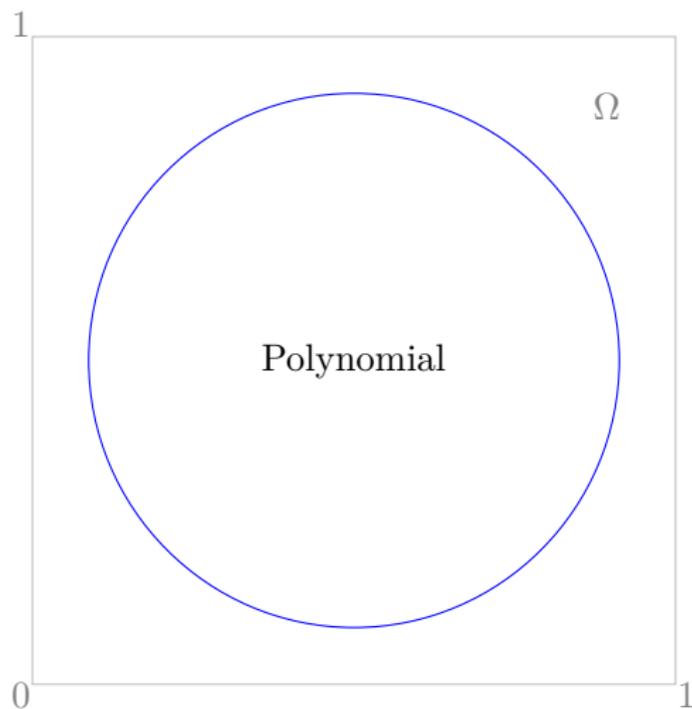
- $B_2 = B_2(\mathcal{X}, \mathcal{T}) = \left( \mathcal{X}_2 \Sigma_1(\mathcal{X}_1) - \mathcal{X}_1 \Sigma_2(\mathcal{X}_2) \right) \frac{h}{r^2}$ .

# Source Points

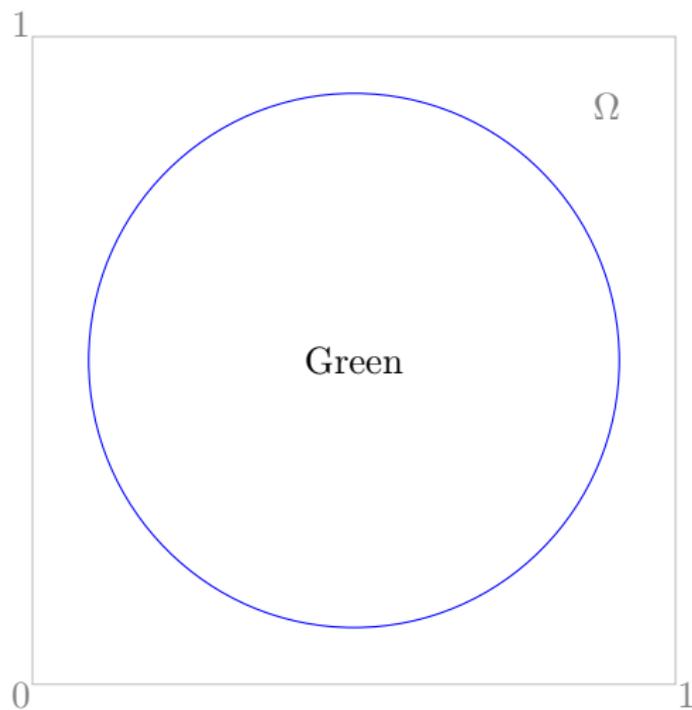
- $\mathcal{X} = \mathbf{x} - \mathbf{x}_s$
- $\mathcal{T} = t - t_s$
- $(\mathbf{x}_s, t_s)$  are source points at which we evaluate Green's functions.  
We can vary their number to achieve convergence.



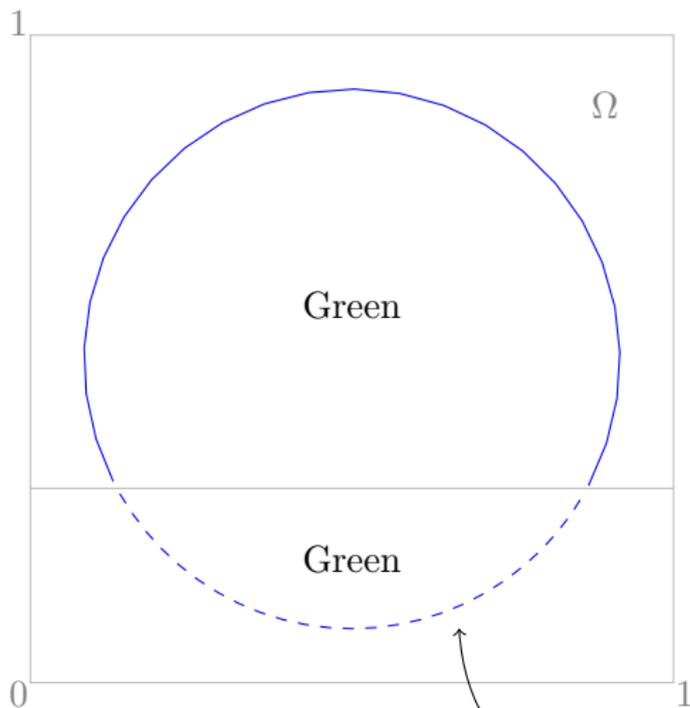
# What has been done



# What has been done

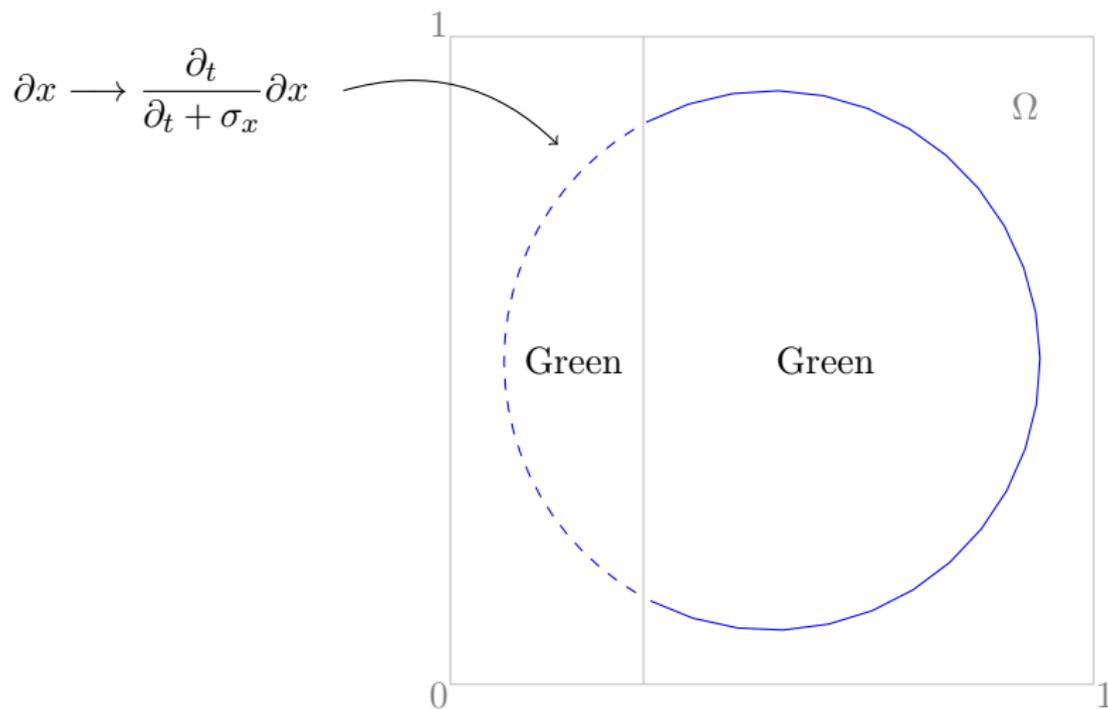


# What has been done

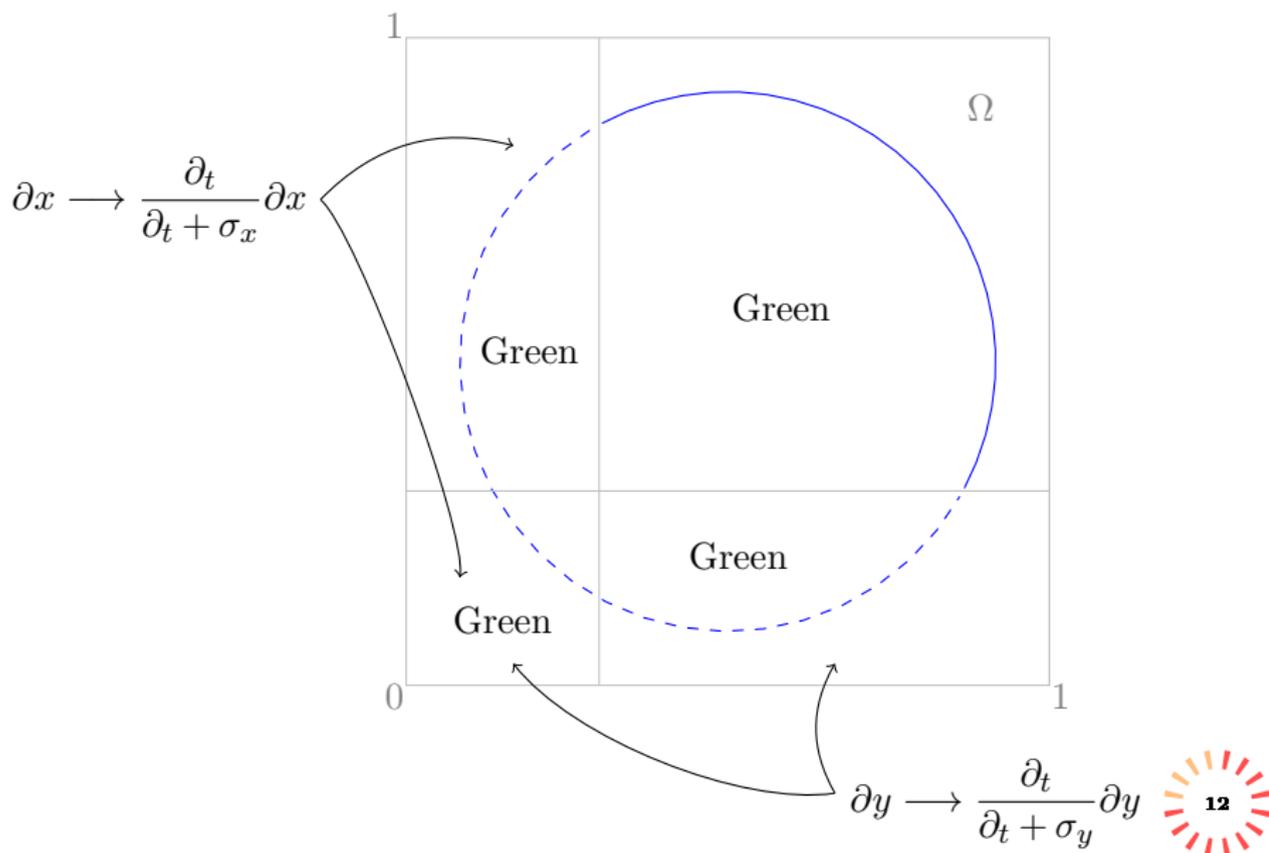


$$\partial y \rightarrow \frac{\partial_t}{\partial_t + \sigma_y} \partial y$$

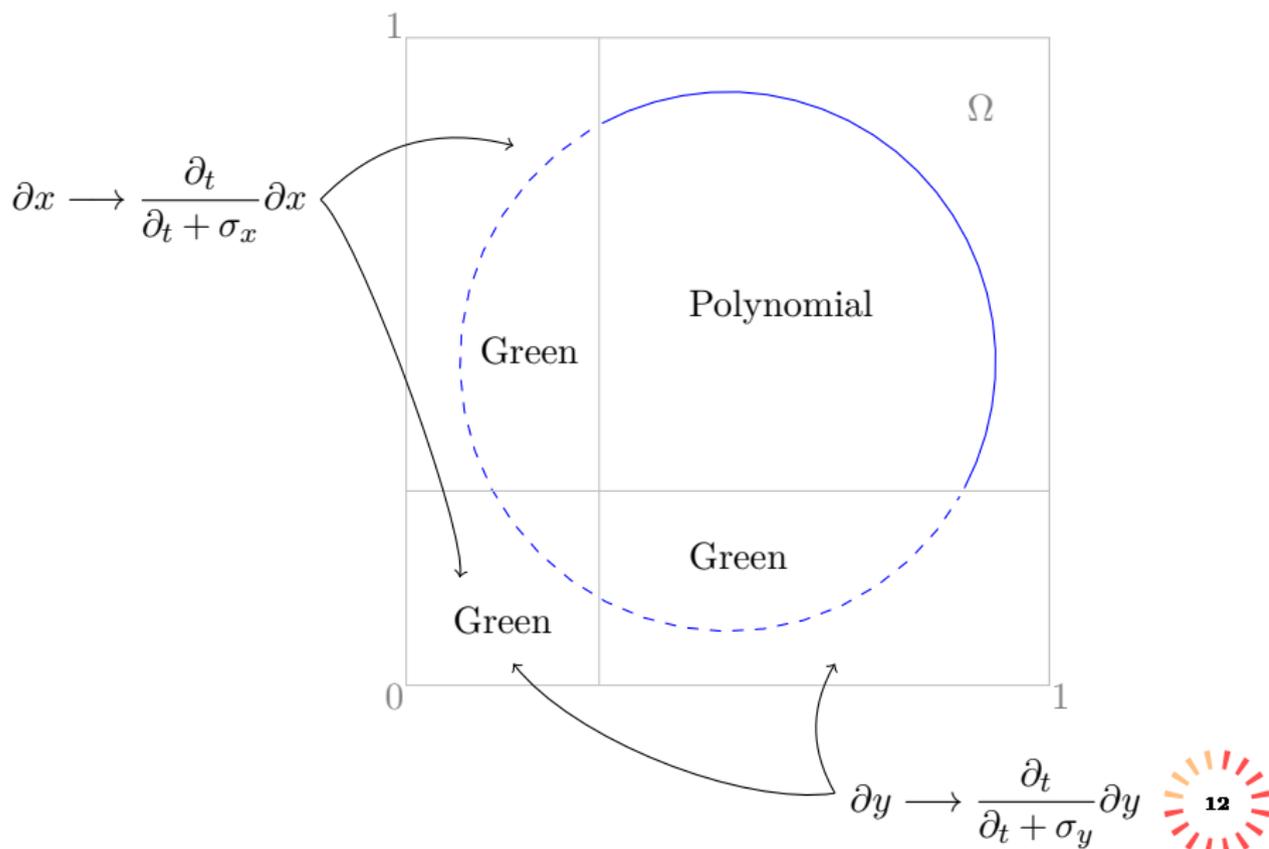
# What has been done



# What has been done



# What has been done



Pressure in domain with PML  
(left and bottom layers + bottom-left corner)

Ongoing work :

- Parallelization
- Extension to unstructured meshes

Prospects :

- Coupling parallel with PML for unstructured meshes

**Bérenger.**

A perfectly matched layer for the absorption of electromagnetic waves.

*Journal of Computational Physics*, 114(2):185 – 200, 1994.

**Björn Engquist and Andrew Majda.**

Absorbing boundary conditions for numerical simulation of waves.

*Proceedings of the National Academy of Sciences*, 74(5):1765–1766, 1977.

**I. Herrera.**

Trefftz method: A general theory.

*Eq.*, 16:561–580, 11 2000.

**Ralf Hiptmair, Andrea Moiola, and Ilaria Perugia.**

A survey of trefftz methods for the helmholtz equation.

06 2015.

**V.D. Kupradze and M.A. Aleksidze.**

An approximate method of solving certain boundary-value problems.

*Soobšč. Akad. Nauk Gruzin. SSR*, 30:529–536, 01 1963.

**Rudolf Mathon and R.L. Johnston.**

The approximate solution of elliptic boundary-value problems by fundamental solutions.

*SIAM Journal on Numerical Analysis*, 14, 09 1977.

**Ilaria Perugia, Joachim Schöberl, Paul Stocker, and Christoph Wintersteiger.**

Tent pitching and trefftz-dg method for the acoustic wave equation.

*Computers & Mathematics with Applications*, 2020.

**Elvira Shishenina.**

*Space-Time Discretization of Elasto-Acoustic Wave Equation in Polynomial Trefftz-DG Bases.*

PhD thesis, 2018.

**Erich Trefftz.**

Ein gegenstück zum ritzschen verfahren.

*In Proc. 2nd Int. Cong. Appl. Mech. Zurich*, pages 131–137, 1926.

**Alper Üngör and Alla Sheffer.**

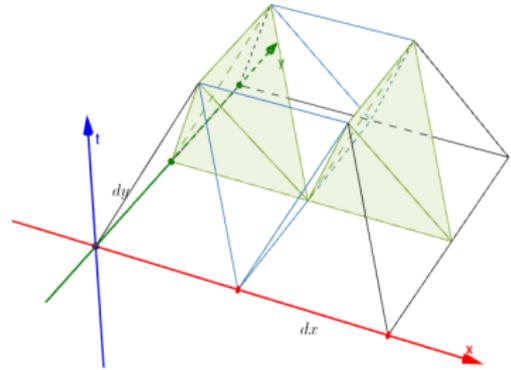
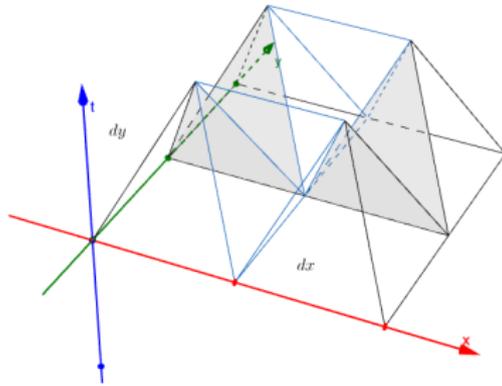
Tent-pitcher: A meshing algorithm for space-time discontinuous galerkin methods.

*In In proc. 9th int'l. meshing roundtable*, pages 111–122, 01 2000.

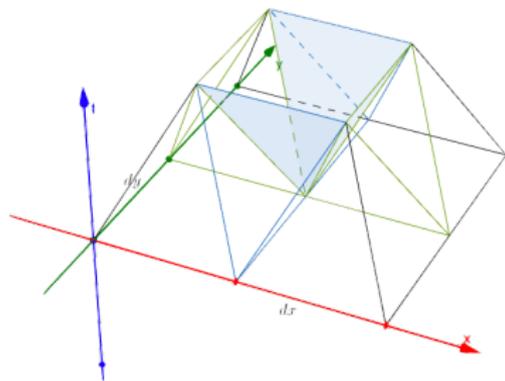
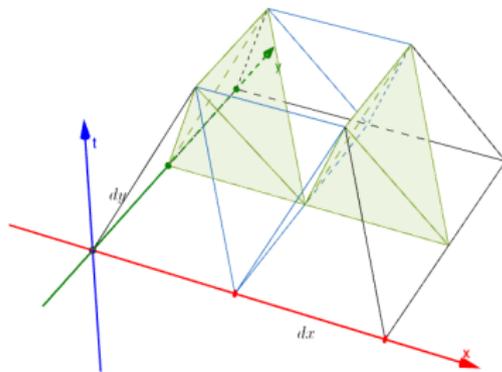
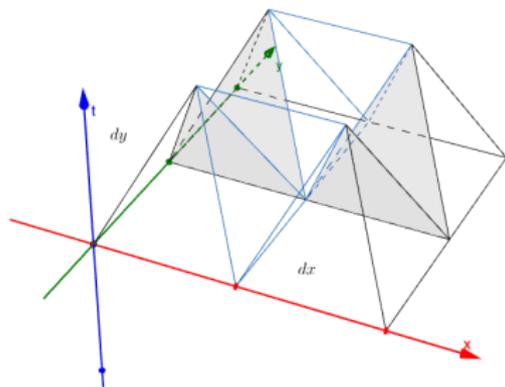
Thank you for your attention !



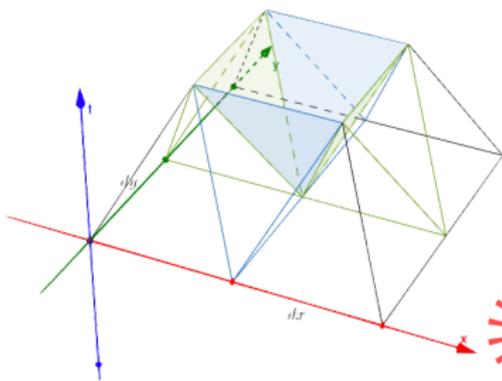
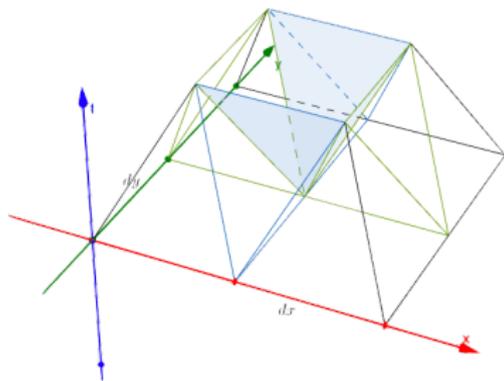
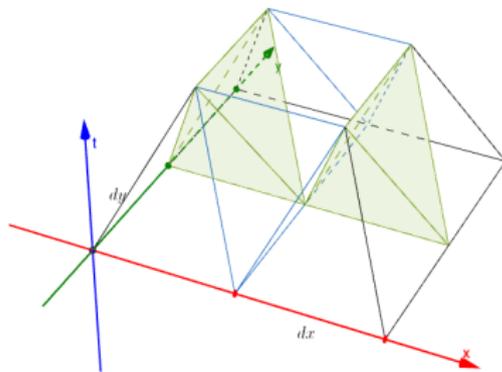
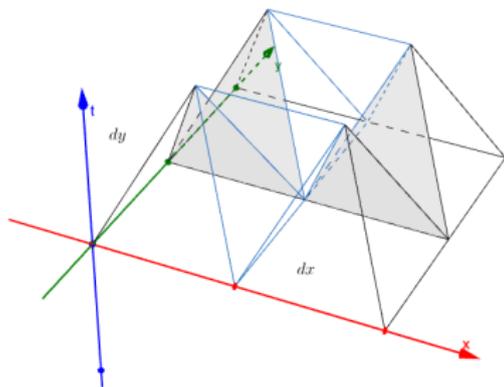
# Tent-pitching process $2d+t$



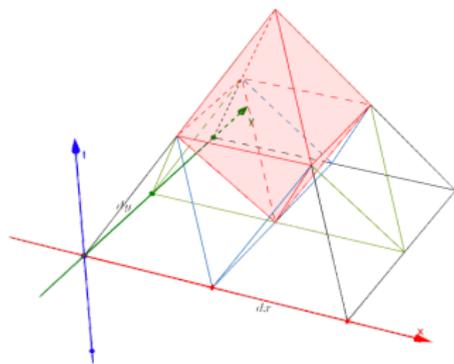
# Tent-pitching process 2d+t



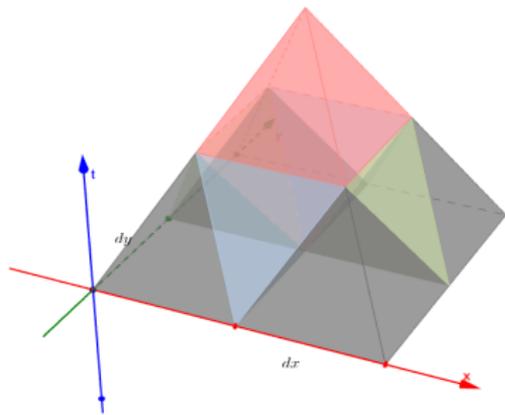
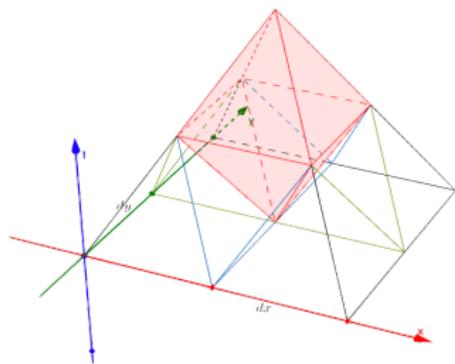
# Tent-pitching process 2d+t



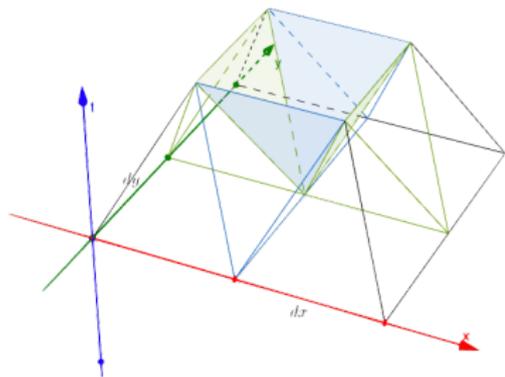
# Tent-pitching process $2d+t$



# Tent-pitching process $2d+t$



# Tent-pitching process

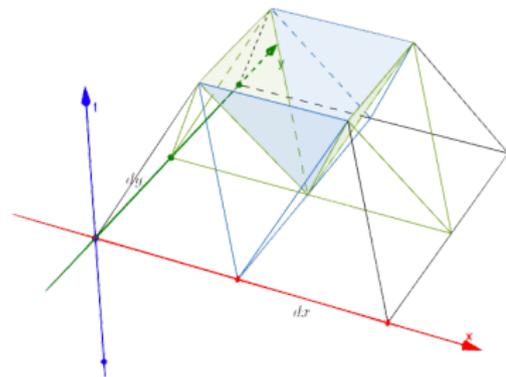
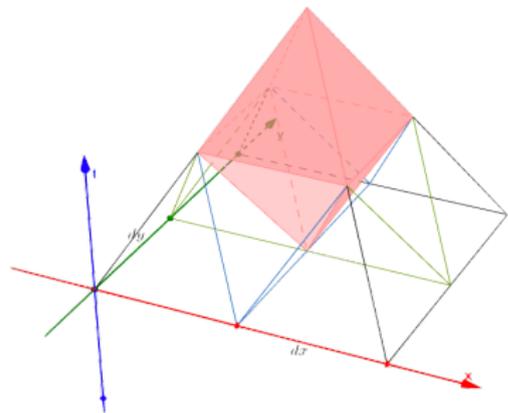


$$-\left(\frac{1}{2} + \beta_2\right) \int_{\mathcal{F}_h^{in}} \frac{1}{c^2 \rho} p_h^{in} q^{n_t}$$

$$-\left(\frac{1}{2} + \alpha_2\right) \int_{\mathcal{F}_h^{in}} \rho \mathbf{v}_h^{in} \cdot \mathbf{w}^{n_t}$$

$$-\frac{1}{2} \int_{\mathcal{F}_h^{in}} p_h^{in} \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h^{in} \cdot \mathbf{n}_x$$

# Tent-pitching process



$$\int_{\mathcal{F}_h^{\text{out}}} \frac{1}{c^2 \rho} p_h q n_t + \rho \mathbf{v}_h \cdot \mathbf{w} n_t + p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x$$

$$+ \left(\frac{1}{2} - \beta_2\right) \int_{\mathcal{F}_h^{\text{in}}} \frac{1}{c^2 \rho} p_h q n_t + \left(\frac{1}{2} - \alpha_2\right) \int_{\mathcal{F}_h^{\text{in}}} \rho \mathbf{v}_h \cdot \mathbf{w} n_t$$

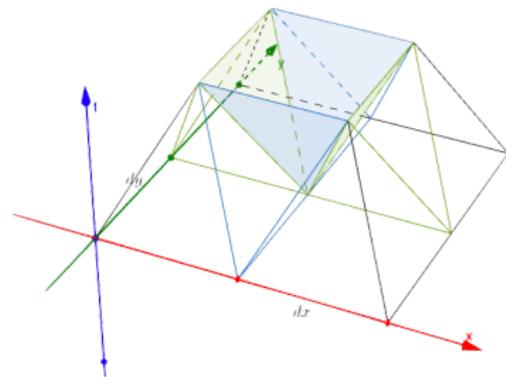
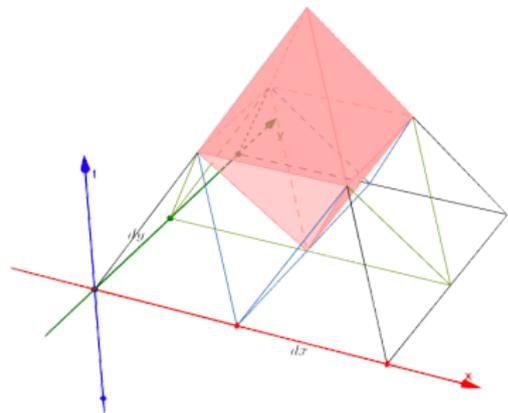
$$+ \frac{1}{2} \int_{\mathcal{F}_h^{\text{in}}} p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x$$

$$= -\left(\frac{1}{2} + \beta_2\right) \int_{\mathcal{F}_h^{\text{in}}} \frac{1}{c^2 \rho} p_h^{\text{in}} q n_t$$

$$- \left(\frac{1}{2} + \alpha_2\right) \int_{\mathcal{F}_h^{\text{in}}} \rho \mathbf{v}_h^{\text{in}} \cdot \mathbf{w} n_t$$

$$- \frac{1}{2} \int_{\mathcal{F}_h^{\text{in}}} p_h^{\text{in}} \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h^{\text{in}} \cdot \mathbf{n}_x$$

# Tent-pitching process



$$\int_{\mathcal{F}_h^{out}} \frac{1}{c^2 \rho} p_h q n_t + \rho \mathbf{v}_h \cdot \mathbf{w} n_t + p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x$$

$$+ \left(\frac{1}{2} - \beta_2\right) \int_{\mathcal{F}_h^{in}} \frac{1}{c^2 \rho} p_h q n_t + \left(\frac{1}{2} - \alpha_2\right) \int_{\mathcal{F}_h^{in}} \rho \mathbf{v}_h \cdot \mathbf{w} n_t$$

$$+ \frac{1}{2} \int_{\mathcal{F}_h^{in}} p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x$$

$$- \left(\frac{1}{2} + \beta_2\right) \int_{\mathcal{F}_h^{in}} \frac{1}{c^2 \rho} p_h^{in} q n_t$$

$$- \left(\frac{1}{2} + \alpha_2\right) \int_{\mathcal{F}_h^{in}} \rho \mathbf{v}_h^{in} \cdot \mathbf{w} n_t$$

$$- \frac{1}{2} \int_{\mathcal{F}_h^{in}} p_h^{in} \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h^{in} \cdot \mathbf{n}_x$$

$$Au = f$$