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# On the construction of Shape Functions for Spacetime Trefftz-DG Formulations of Wave Problems with Perfectly Matched Layers

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WAVES 2022 – JULY 25-29, 2022

# On the construction of Shape Functions for Spacetime Trefftz-DG Formulations of Wave Problems with Perfectly Matched Layers

V. Vasanthan<sup>1</sup>, H. Barucq<sup>1</sup>, H. Calandra<sup>1</sup>, J. Diaz<sup>1</sup>

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firstname.lastname@inria.fr

- Aim of our work ?
  - ↔ to develop a Trefftz method in time domain with the Tent-Pitcher algorithm, for wave problems.
- Why ?
  - ↔ gain precision,
  - ↔ reduce computational time.
- What kind of problems ?
  - ↔ acoustic,
  - ↔ elastodynamic,
  - ↔ elasto-acoustic wave propagation systems.
- What kind of methods ?
  - ↔ Trefftz-Discontinuous Galerkin methods with Tent-Pitching.
- What kind of basis ?
  - ↔ polynomial basis functions,
  - ↔ Green's functions.

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# Acoustic Wave equation

First order acoustic wave equation :

$$\left\{ \begin{array}{ll} \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{v} = f & \text{in } \Omega, \\ \rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 & \text{in } \Omega, \\ \mathbf{v}(\cdot, 0) = \mathbf{v}_0, p(\cdot, 0) = p_0 & \text{in } \mathcal{D}, \\ \mathbf{v} \cdot \mathbf{n}_{\mathbf{x}} = 0 & \text{in } \partial \mathcal{D} \times \mathcal{I}. \end{array} \right. \quad \begin{array}{l} \Omega = \mathcal{D} \times \mathcal{I} \\ \mathcal{I} = [0, T] \end{array}$$

with

$$\mathbf{x} = (x_1, x_2),$$

$$p = p(\mathbf{x}, t), \text{ pressure,}$$

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t) = (u, v), \text{ velocity,}$$

$$f = f(\mathbf{x}, t), \text{ source term,}$$

$$c = c(\mathbf{x}), \text{ wave speed,}$$

$$\rho = \rho(\mathbf{x}), \text{ density.}$$



- Basis functions are solutions of the initial PDEs
- Physical characteristics of the solution are included in the discrete solution
- We work on the skeleton of the mesh only
- Requires lower number of degrees of freedom
- Basis functions are discontinuous → Discontinuous Galerkin discretization
- Flexibility in choice of basis functions (plane waves, Green's function...)
- Spacetime implicit scheme

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How to reduce computational costs ?

- solve element-by-element if mesh respects constraints

→ Tent-Pitcher algorithm

- introduced for hyperbolic problems in spacetime domain,
  - consists in building a causal mesh, which respects the wave propagation speed,
  - conducive to parallel computing.
- to our knowledge, this algorithm has not been applied to Trefftz-DG + Perfectly Matched Layers (PML) yet

I. Perugia et al [2020]

A. Üngör et al [2005]



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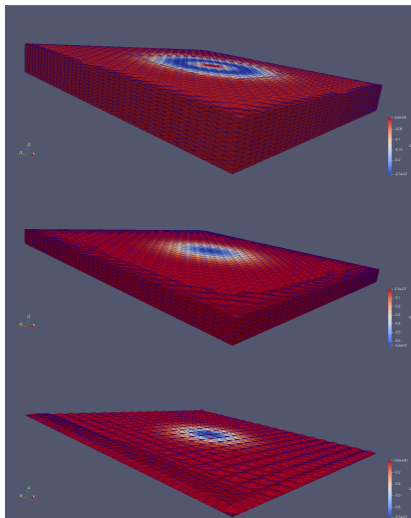


Figure: Visualisation of the evolution of the mesh in time

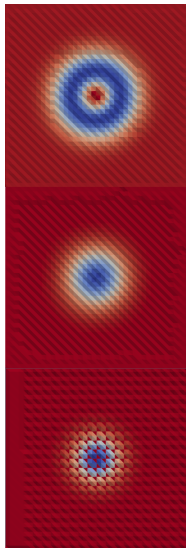
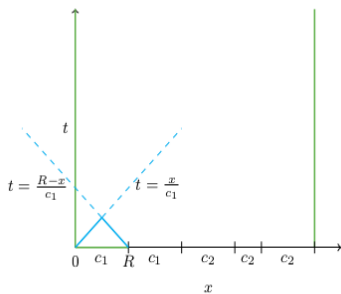


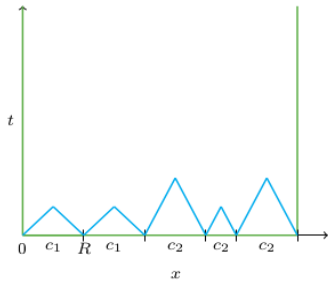
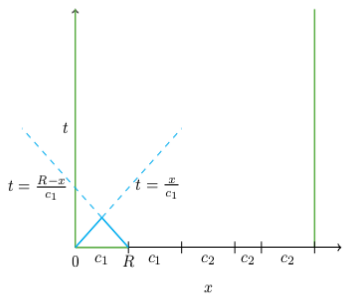
Figure: Pressure from above

# Tent-pitching process 1d+t

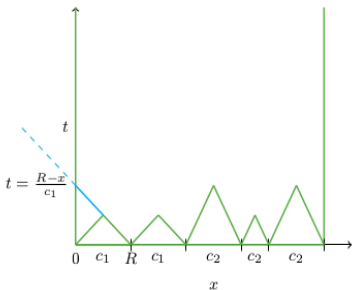
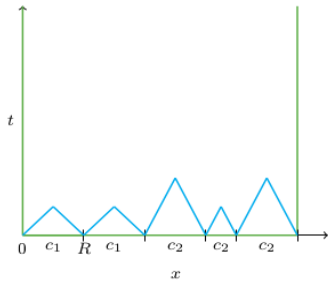
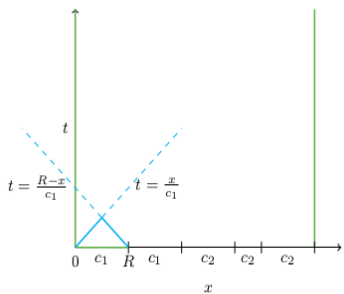




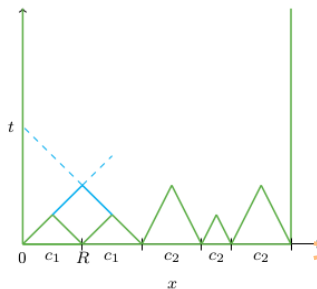
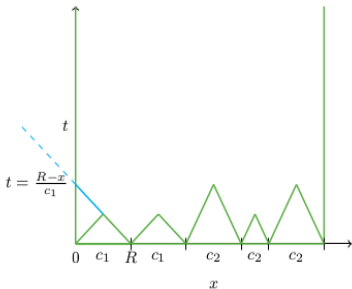
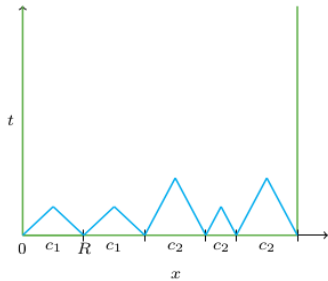
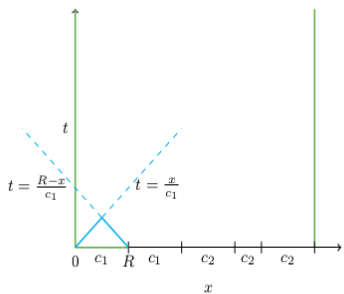
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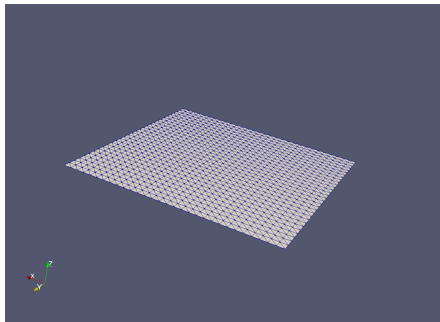


Figure: Initial spatial mesh

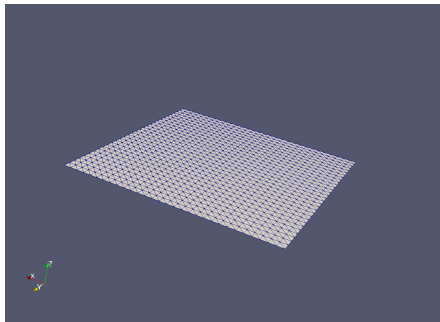


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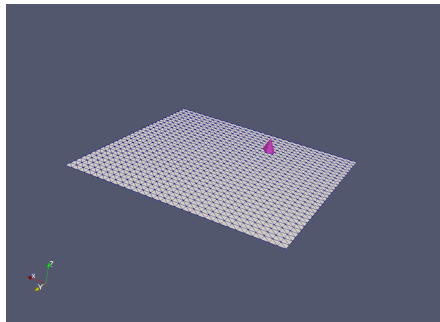


Figure: 1<sup>st</sup> iteration

# Tent-pitching process 2d+t

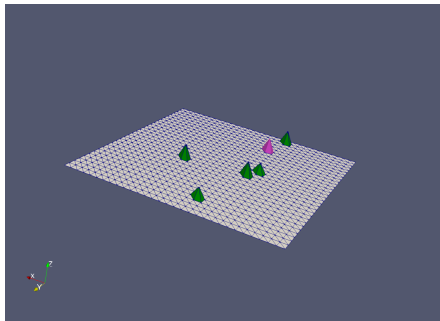


Figure: 5<sup>th</sup> iteration

# Tent-pitching process 2d+t

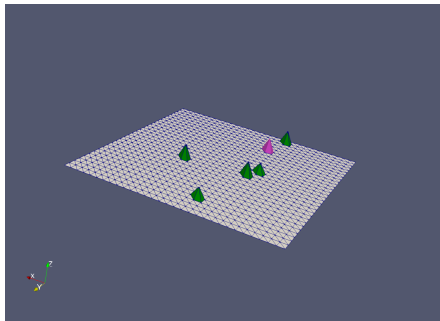


Figure: 5<sup>th</sup> iteration

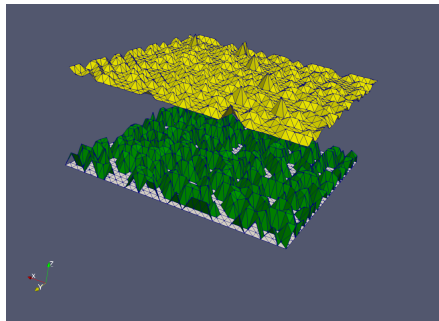
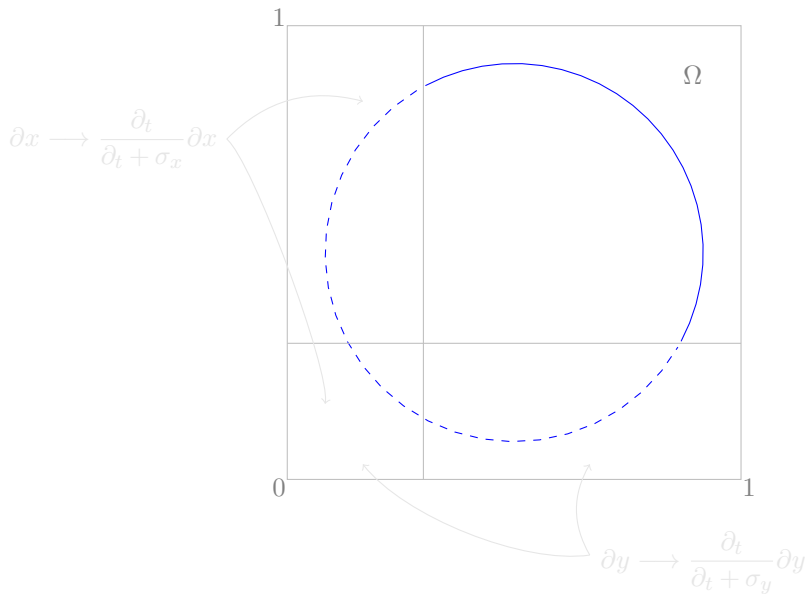


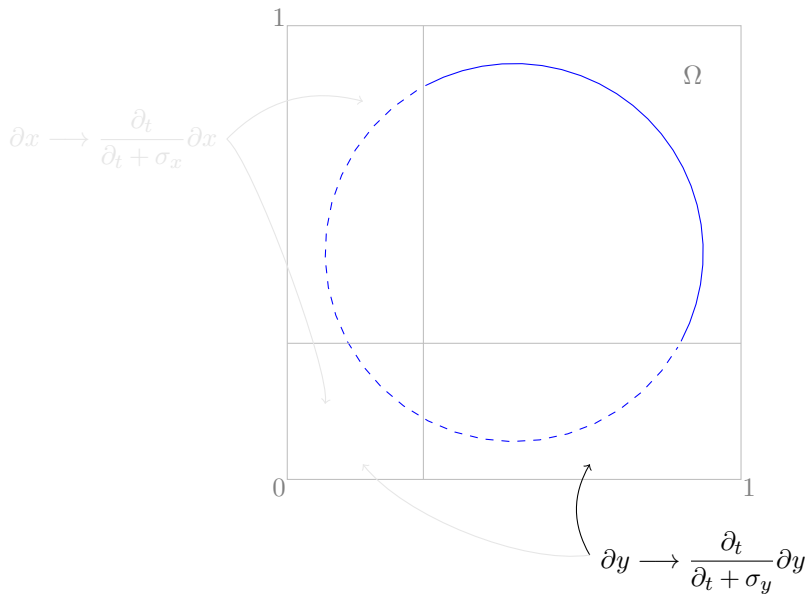
Figure: 6698<sup>th</sup> iteration

# Perfectly Matched Layers

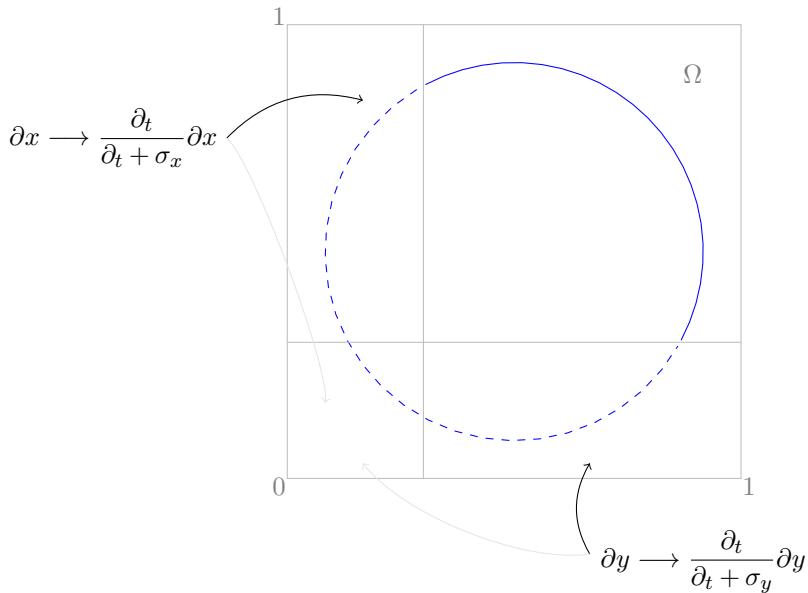




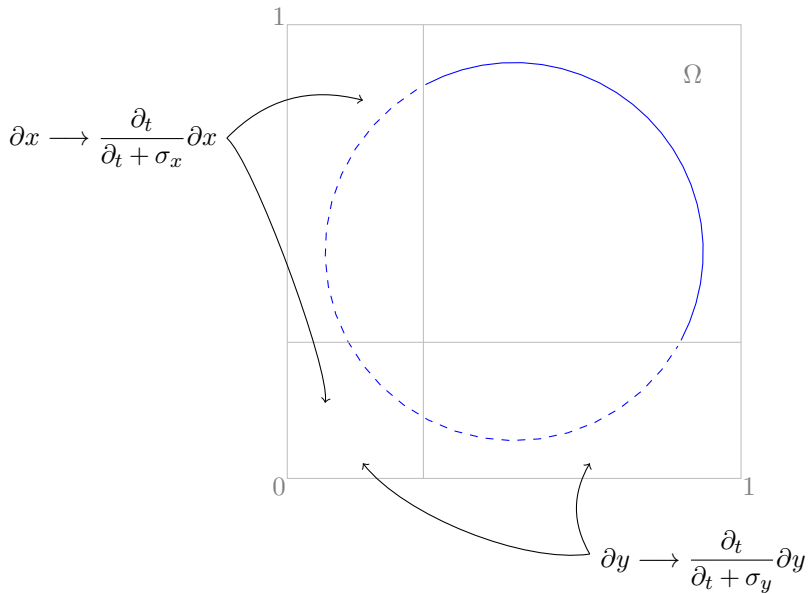
# Perfectly Matched Layers



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$$\begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_x} \frac{\partial p}{\partial x} = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_y} \frac{\partial p}{\partial y} = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial_t}{\partial_t + \sigma_x} \frac{\partial u}{\partial x} + \frac{\partial_t}{\partial_t + \sigma_y} \frac{\partial v}{\partial y} = 0. \end{cases}$$

in  $\Omega = \mathcal{D} \times \mathcal{I}$

with  $\sigma_x$  and  $\sigma_y$

absorbing coefficients

$$\begin{cases} \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} + p_1 = 0, \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} + p_2 = 0, \\ \frac{1}{c^2 \rho} \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u_1 + v_1 = 0. \end{cases}$$

$$(\partial_t + \sigma_x)p_1 = -\sigma_x \frac{\partial p}{\partial x}$$

$$(\partial_t + \sigma_y)p_2 = -\sigma_y \frac{\partial p}{\partial y}$$

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- $G_p^{pml}(\mathcal{X}, \mathcal{T}, \sigma_x, \sigma_y) = e^{-A} \cos(B) G_p(\mathcal{X}, \mathcal{T})$

- $G_u^{pml}(\mathcal{X}, \mathcal{T}, \sigma_x, \sigma_y) = \frac{e^{-A} G_p(\mathcal{X}, \mathcal{T})}{\rho r^2} \left( \mathcal{T} \mathcal{X}_1 \cos(B_1) + h \mathcal{X}_2 \sin(B_1) \right)$

- $G_v^{pml}(\mathcal{X}, \mathcal{T}, \sigma_x, \sigma_y) = \frac{e^{-A} G_p(\mathcal{X}, \mathcal{T})}{\rho r^2} \left( \mathcal{T} \mathcal{X}_2 \cos(B_2) + h \mathcal{X}_1 \sin(B_2) \right)$

- $G_p(\mathcal{X}, \mathcal{T}) = \frac{1}{2\pi h}$ ,  $h = h(\mathcal{X}, \mathcal{T}) = \sqrt{\mathcal{T}^2 - \frac{r^2}{c^2}}$ ,  $\Sigma_i(s) = \int_0^s \sigma_s(\tau) d\tau$ ,

- $A = A(\mathcal{X}, \mathcal{T}) = \left( \mathcal{X}_1 \Sigma_1(\mathcal{X}_1) + \mathcal{X}_2 \Sigma_2(\mathcal{X}_2) \right) \frac{\mathcal{T}}{r^2}$ ,

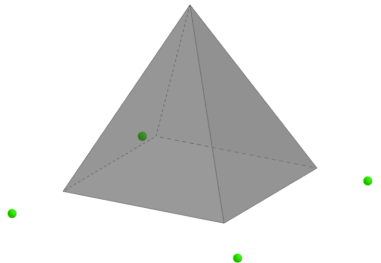
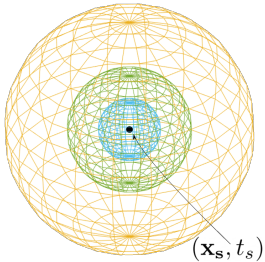
- $B = B(\mathcal{X}, \mathcal{T}) = \left( \mathcal{X}_1 \Sigma_2(\mathcal{X}_2) + \mathcal{X}_2 \Sigma_1(\mathcal{X}_1) \right) \frac{h}{r^2}$ ,

- $B_1 = B_1(\mathcal{X}, \mathcal{T}) = \left( \mathcal{X}_1 \Sigma_2(\mathcal{X}_2) - \mathcal{X}_2 \Sigma_1(\mathcal{X}_1) \right) \frac{h}{r^2}$ ,

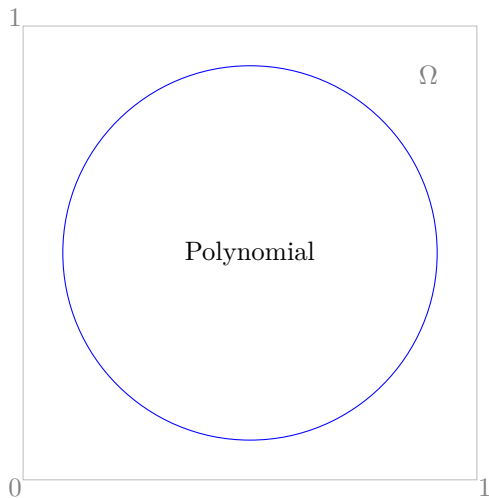
- $B_2 = B_2(\mathcal{X}, \mathcal{T}) = \left( \mathcal{X}_2 \Sigma_1(\mathcal{X}_1) - \mathcal{X}_1 \Sigma_2(\mathcal{X}_2) \right) \frac{h}{r^2}$ .

# Source Points

- $\mathcal{X} = \mathbf{x} - \mathbf{x}_s$
- $\mathcal{T} = t - t_s$
- $(\mathbf{x}_s, t_s)$  are source points at which we evaluate Green's functions.  
We can vary their number to achieve convergence.

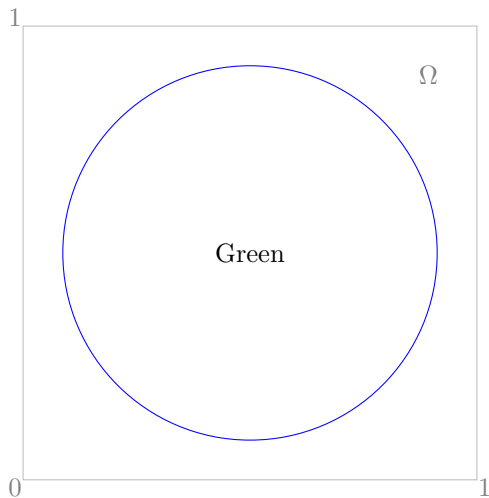


# What has been done

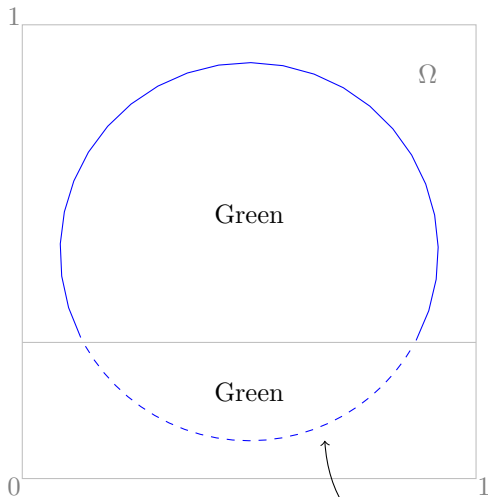




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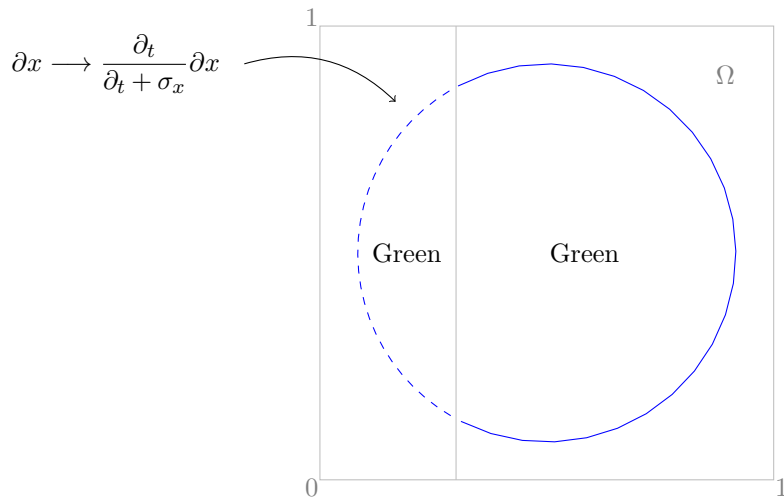


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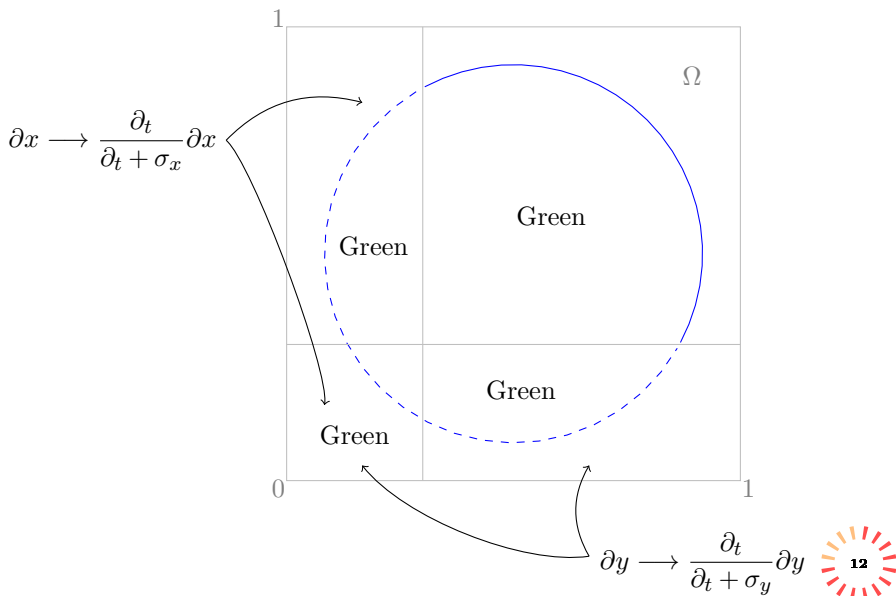


$$\partial y \rightarrow \frac{\partial_t}{\partial_t + \sigma_y} \partial y$$

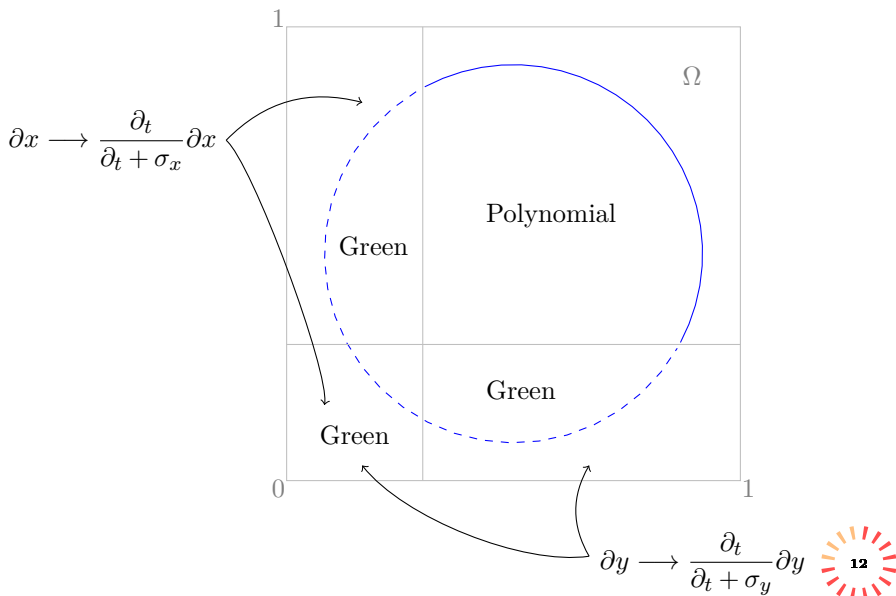
# What has been done



# What has been done



# What has been done



Pressure in domain with PML  
(left and bottom layers + bottom-left corner)

Ongoing work :

- Parallelization
- Extension to unstructured meshes

Prospects :

- Coupling parallel with PML for unstructured meshes

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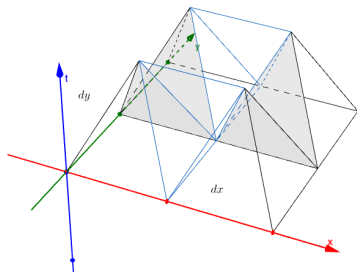
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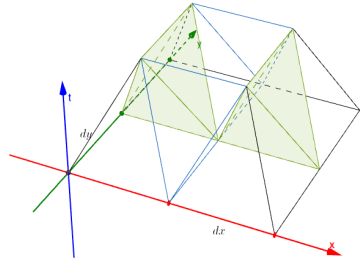
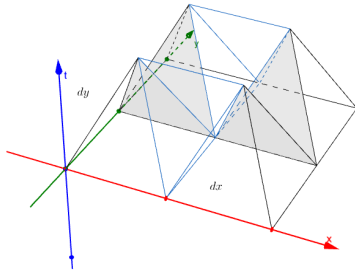


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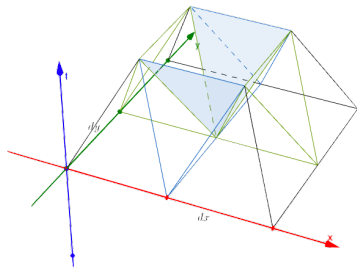
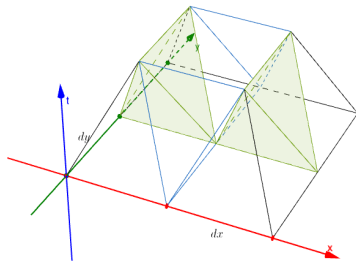
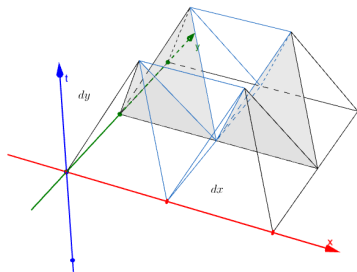
# Tent-pitching process $2d+t$



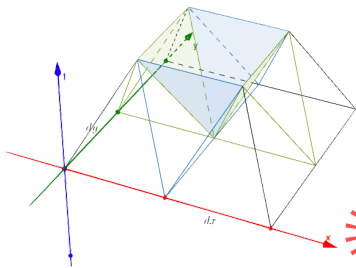
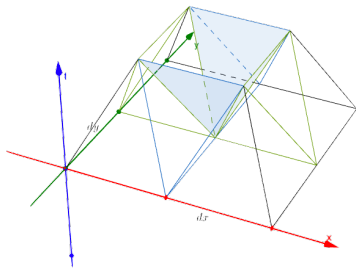
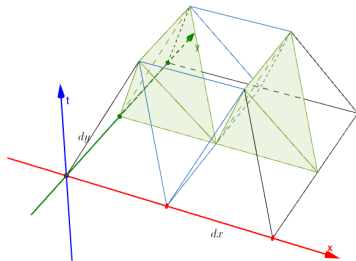
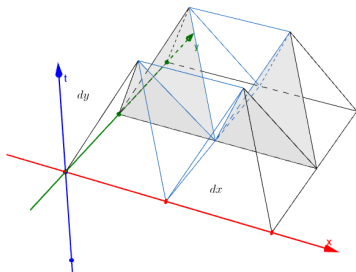
# Tent-pitching process $2d+t$



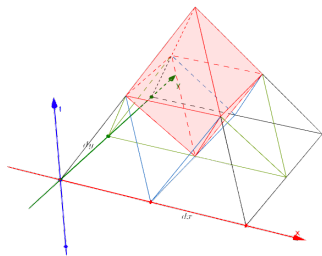
# Tent-pitching process 2d+t



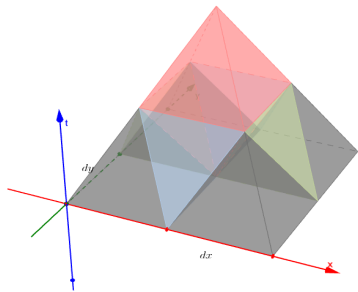
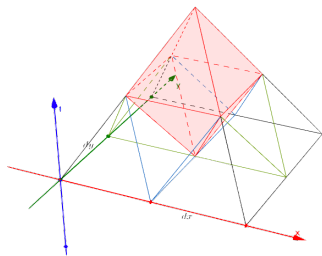
# Tent-pitching process 2d+t



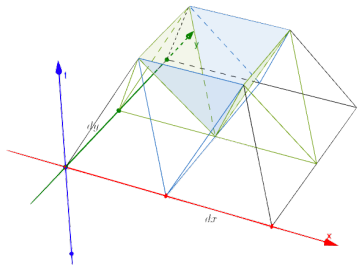
# Tent-pitching process $2d+t$



# Tent-pitching process $2d+t$



# Tent-pitching process



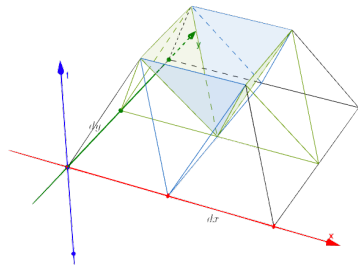
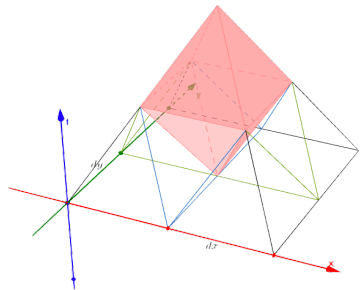
$$-\left(\frac{1}{2} + \beta_2\right) \int_{\mathcal{F}_h^{in}} \frac{1}{c^2 \rho} p_h^{in} q^{n_t}$$

$$-\left(\frac{1}{2} + \alpha_2\right) \int_{\mathcal{F}_h^{in}} \rho \mathbf{v}_h^{in} \cdot \mathbf{w}^{n_t}$$

$$-\frac{1}{2} \int_{\mathcal{F}_h^{in}} p_h^{in} \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h^{in} \cdot \mathbf{n}_x$$



# Tent-pitching process



$$\int_{\mathcal{F}_h^{out}} \frac{1}{c^2 \rho} p_h q n_t + \rho \mathbf{v}_h \cdot \mathbf{w} n_t + p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x$$

$$+ \left(\frac{1}{2} - \beta_2\right) \int_{\mathcal{F}_h^{in}} \frac{1}{c^2 \rho} p_h q n_t + \left(\frac{1}{2} - \alpha_2\right) \int_{\mathcal{F}_h^{in}} \rho \mathbf{v}_h \cdot \mathbf{w} n_t$$

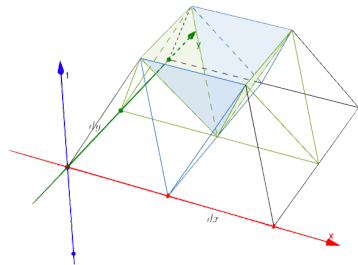
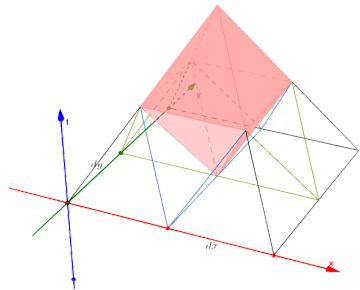
$$+ \frac{1}{2} \int_{\mathcal{F}_h^{in}} p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x$$

$$= -\left(\frac{1}{2} + \beta_2\right) \int_{\mathcal{F}_h^{in}} \frac{1}{c^2 \rho} p_h^{in} q n_t$$

$$- \left(\frac{1}{2} + \alpha_2\right) \int_{\mathcal{F}_h^{in}} \rho \mathbf{v}_h^{in} \cdot \mathbf{w} n_t$$

$$- \frac{1}{2} \int_{\mathcal{F}_h^{in}} p_h^{in} \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h^{in} \cdot \mathbf{n}_x$$

# Tent-pitching process



$$\int_{\mathcal{F}_h^{\text{out}}} \frac{1}{c^2 \rho} p_h q n_t + \rho \mathbf{v}_h \cdot \mathbf{w} n_t + p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x$$

$$+ \left(\frac{1}{2} - \beta_2\right) \int_{\mathcal{F}_h^{\text{in}}} \frac{1}{c^2 \rho} p_h q n_t + \left(\frac{1}{2} - \alpha_2\right) \int_{\mathcal{F}_h^{\text{in}}} \rho \mathbf{v}_h \cdot \mathbf{w} n_t$$

$$+ \frac{1}{2} \int_{\mathcal{F}_h^{\text{in}}} p_h \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h \cdot \mathbf{n}_x$$

$$- \left(\frac{1}{2} + \beta_2\right) \int_{\mathcal{F}_h^{\text{in}}} \frac{1}{c^2 \rho} p_h^{\text{in}} q n_t$$

$$- \left(\frac{1}{2} + \alpha_2\right) \int_{\mathcal{F}_h^{\text{in}}} \rho \mathbf{v}_h^{\text{in}} \cdot \mathbf{w} n_t$$

$$- \frac{1}{2} \int_{\mathcal{F}_h^{\text{in}}} p_h^{\text{in}} \mathbf{w} \cdot \mathbf{n}_x + q \mathbf{v}_h^{\text{in}} \cdot \mathbf{n}_x$$

$$Au = f$$