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Stability criteria for singularly perturbed linear switching systems

Yacine Chitour[1], Ihab Haidar[2], Paolo Mason[1], and Mario Sigalotti[3‡]

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Abstract

We present some results on the uniform exponential stability of singularly perturbed linear systems undergoing switching. In absence of dwell-time constraints, the switching parameter can evolve on the same time scale as the fast variables, or even faster. We investigate the effect of switching laws evolving at a time scale comparable with the fast variables, describing the corresponding asymptotic effect on the slow variable. Based on this analysis, we propose stability criteria for the overall system, uniform for small values of the parameter of singular perturbation.

1 Introduction

Singularly perturbed linear switching systems of the type

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)y(t), \\ \varepsilon\dot{y}(t) &= C(t)x(t) + D(t)y(t), \end{aligned}$$

where ε denotes the small singular perturbation parameter, appear in several industrial and engineering applications (see, e.g., [3, 6, 7]). It has been noticed that, differently from the unswitching case [4], their stability properties cannot be deduced from those of their formal slow and fast reductions [5, 1].

We give in this paper a new necessary condition for the stability of the singularly perturbed linear switched system for small values of ε. This condition is obtained by considering switching laws which evolve on a time scale of order ε, and by accounting for their asymptotic effect, as ε goes to zero, on the slow variable x.

More precisely, let us consider a signal $t \mapsto (A(t), B(t), C(t), D(t)) =: \sigma(t)$ defined on an interval $[0, T]$ and all its reparameterizations $\sigma_\varepsilon : t \mapsto \sigma(t/\varepsilon)$ defined on $[0, \varepsilon T]$. The flow of the switching system, for the value of the singular perturbation parameter set to ε, corresponding to σ_ε and evaluated at time εT, turns out to have an effect of order $O(\varepsilon)$ on the coordinate x (since the velocities are of order 1 and the length of the time interval is of order ε) and of order $O(1)$ on the coordinate y, provided that the switching system $\dot{y}(t) = D(t)y(t)$ is exponentially stable. Suitable computations show that the evolution of x can actually be described as

$$x \mapsto x + \varepsilon T \Lambda(T, \sigma)x + O(\varepsilon^2),$$

where the term $\Lambda(T, \sigma)$ does not depend on the initial condition of the variable y. We then propose an auxiliary switching system, denoted $\check{\Sigma}$, having as possible modes all the matrices of the type $\Lambda(T, \sigma)$. We provide an explicit expression for such matrices and we prove that, indeed, if $\check{\Sigma}$ is exponentially unstable, then the same is true for the original singularly perturbed switching system, for all small enough values of the singular perturbation parameter ε.

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1.1 Notations

By $\mathbb{R}$ we denote the set of real numbers, by $|\cdot|$ the Euclidean norm of a real vector, and by $\|\cdot\|$ the induced matrix norm. We write $\lfloor x \rfloor$ to denote the evaluation of the floor function at a real number x. By $M_n(\mathbb{R})$ we denote the set of $n \times n$ real matrices. The $n \times n$ identity matrix is denoted by I_n. We use $\rho(M)$ to denote the spectral radius of a matrix $M \in M_n(\mathbb{R})$, defined as the largest modulus among the eigenvalues of M. Given a subset $\mathcal{N}$ of $M_n(\mathbb{R})$, we denote by $\mathcal{S}_\mathcal{N}$ the set of all measurable functions from $[0, +\infty)$ to $\mathcal{N}$.

2 Problem statement

Fix $n, m \in \mathbb{N}$ and a compact set of matrices $\mathcal{M} \subset M_{n+m}(\mathbb{R})$. Let $\Sigma = (\Sigma_\varepsilon)_{\varepsilon>0}$ where, for every $\varepsilon > 0$, Σ_ε denotes the linear switching system

$$\Sigma_\varepsilon : \quad \begin{array}{rcl} \dot{x}(t) &=& A(t)x(t) + B(t)y(t), \\ \varepsilon\dot{y}(t) &=& C(t)x(t) + D(t)y(t), \end{array}$$

where

$$t \mapsto \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}$$

is an arbitrary element of the set $\mathcal{S}_\mathcal{M}$ of measurable functions from $[0, +\infty)$ to $\mathcal{M}$. In particular, piecewise-constant functions (that is, with locally finite set of jumps) with values in $\mathcal{M}$ are admissible signals, but also continuous functions or functions with accumulating sequences of jumps.

For a given $\varepsilon > 0$, the usual stability notions, recalled in the following definition, apply to the linear switching system Σ_ε.

Definition 1. *Let $d \in \mathbb{N}$ and $\mathcal{N}$ be a bounded subset of $M_d(\mathbb{R})$. Consider the linear switching system*

$$\Sigma_\mathcal{N} : \quad \dot{x}(t) = N(t)x(t), \qquad N \in \mathcal{S}_\mathcal{N}, \qquad (1)$$

and denote by $\Phi_N(t, 0)$ the flow from time 0 to time t of $\Sigma_\mathcal{N}$ associated with the switching signal N. Then $\Sigma_\mathcal{N}$ is said to be

1. exponentially stable *if there exist $C > 0$ and $\delta > 0$ such that*

$$\|\Phi_N(t, 0)\| \le Ce^{-\delta t}, \qquad \forall\, t \ge 0, \forall\, N \in \mathcal{S}_\mathcal{N};$$

2. exponentially unstable *if there exist $C > 0$, $\delta > 0$, and a nonzero trajectory $t \mapsto x(t)$ of $\Sigma_\mathcal{N}$ such that*

$$|x(t)| \ge Ce^{\delta t}|x(0)|, \qquad \forall t \ge 0.$$

The maximal Lyapunov exponent *of $\Sigma_\mathcal{N}$ is defined as*

$$\lambda(\Sigma_\mathcal{N}) = \limsup_{t\to+\infty} \frac{1}{t} \sup_{N \in \mathcal{S}_\mathcal{N}} \log \|\Phi_N(t, 0)\|.$$

Remark 2. *Notice that, for every $t > 0$ and every $N \in \mathcal{S}_\mathcal{N}$, by Gelfand's formula,*

$$\rho(\Phi_N(t, 0)) = \lim_{k\to+\infty} \|\Phi_N(t, 0)^k\|^{\frac{1}{k}} = \lim_{k\to+\infty} \|\Phi_{\tilde{N}}(kt, 0)\|^{\frac{1}{k}}$$

where $\tilde{N}$ denotes the t-periodic signal obtained by periodization of $N|_{[0,t]}$. As a consequence,

$$\rho(\Phi_N(t, 0)) \le e^{t\lambda(\Sigma_\mathcal{N})}. \qquad (2)$$

Actually, it is well known (see, e.g., [8]) that an equivalent definition of the maximal Lyapunov exponent is

$$\lambda(\Sigma_\mathcal{N}) = \limsup_{t\to+\infty} \frac{1}{t} \sup_{N \in \mathcal{S}_\mathcal{N}} \log \rho(\Phi_N(t, 0)). \qquad (3)$$

Remark 3. *It follows from the definitions that $\Sigma_\mathcal{N}$ is exponentially stable if and only if $\lambda(\Sigma_\mathcal{N}) < 0$. Also, by (3), we have that $\Sigma_\mathcal{N}$ is exponentially unstable if and only if $\lambda(\Sigma_\mathcal{N}) > 0$. Moreover, the properties of exponential stability/instability and the value of $\lambda(\Sigma_\mathcal{N})$ keep unchanged if we replace $\mathcal{S}_\mathcal{N}$ by the class of piecewise-constant functions from $[0, +\infty)$ to $\mathcal{N}$.*

In the following we write $\Phi^\varepsilon_M(t, 0)$ to denote the flow from time 0 to time t of Σ_ε associated with a signal $M \in \mathcal{S}_\mathcal{M}$ and we introduce the following stability notions for the 1-parameter family Σ of switching systems.

Definition 4. *We say that* $\Sigma = (\Sigma_e)_{\varepsilon>0}$ *is*

1. ε-uniformly exponentially stable *if there exist* $\varepsilon^\star > 0$*,* $C > 0$*, and* $\delta > 0$ *such that*

$$\|\Phi^\varepsilon_M(t,0)\| \le Ce^{-\delta t},\ \forall t \ge 0,\ \forall M \in \mathcal{S}_\mathcal{M}, \forall \varepsilon \in (0,\varepsilon^\star); \tag{4}$$

2. ε-uniformly exponentially unstable *if there exist* $\delta > 0$*,* $C > 0$*, and* $\varepsilon^\star > 0$ *such that for every* $\varepsilon \in (0,\varepsilon^\star)$ *there exist* $M \in \mathcal{S}_\mathcal{M}$ *and* $z_0 \neq 0$ *for which*

$$|\Phi^\varepsilon_M(t,0)z_0| \ge Ce^{\delta t}|z_0|, \qquad \forall\, t \ge 0.$$

Remark 5. *The* ε*-uniform exponential stability of* Σ *is stronger than the property of global uniform asymptotic stability introduced in [1], since it not only guarantees that* $\lambda(\Sigma_\varepsilon) < 0$ *for every* ε *small enough, but also that the stability is uniform with respect to* ε*, i.e.,* $\lambda(\Sigma_\varepsilon)$ *is smaller than the negative number* $-\delta$*, independent of* ε *small, and the constant* C *appearing in* (4) *is independent of* ε *small. Similar considerations concern the* ε*-uniform exponential instability.*

3 Main results

In order to study the ε-uniform exponential stability of the two-time-scales system Σ, we are going to consider single-time-scale systems whose stability properties give information on the asymptotic behavior of Σ_ε for $\varepsilon > 0$ small.

Let us first discuss the system Σ_D corresponding to the evolution of the fast variable y with $x = 0$. More precisely, let

$$\mathcal{M}_D = \{D \mid \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{M}\}$$

and consider the linear switching system

$$\Sigma_D : \quad \dot y(t) = D(t)y(t), \qquad D \in \mathcal{S}_{\mathcal{M}_D}.$$

We first provide a useful result for the asymptotic behavior of Σ.

Lemma 6. *Assume that* Σ_D *is exponentially unstable. Then* Σ *is* ε*-uniformly exponentially unstable.*

Proof. Consider the family $(\Sigma_{\mathcal{N}_\varepsilon})_{\varepsilon\ge0}$ of linear switching systems, where, for every $\varepsilon \ge 0$, the compact set of matrices $\mathcal{N}_\varepsilon \subset M_{n+m}(\mathbb{R})$ is defined by

$$\mathcal{N}_\varepsilon = \left\{ \begin{pmatrix} \varepsilon A & \varepsilon B \\ C & D \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{M} \right\}.$$

Then one notices that, for every $\varepsilon > 0$, the time rescaling $t \mapsto \varepsilon t$ yields

$$\phi_{N_\varepsilon}(t,0) = \phi^\varepsilon_M(\varepsilon t, 0),$$

where M is an arbitrary signal in $\mathcal{M}$ and the signal $N_\varepsilon \in \mathcal{N}_\varepsilon$ is defined as

$$N_\varepsilon(t) = \begin{pmatrix} \varepsilon A(t) & \varepsilon B(t) \\ C(t) & D(t) \end{pmatrix}, \quad t \ge 0.$$

In particular,

$$\lim_{\varepsilon\to0} \phi^\varepsilon_M(\varepsilon t, 0) = \phi_{N_0}(t,0) = \begin{pmatrix} I_n & 0 \\ * & \Phi_D(t,0) \end{pmatrix}.$$

Since Σ_D is exponentially unstable, we can assume that the signal M is T-periodic for some $T > 0$ and such that $\rho(\phi_{N_0}(T,0)) = \rho(\Phi_D(T,0)) > 1$.

We claim that there exist $\delta, C > 0$ such that, for ε small enough there exists $w_\varepsilon \in \mathbb{R}^{n+m} \setminus \{0\}$ with

$$|\phi_{N_\varepsilon}(t,0)w_\varepsilon| \ge Ce^{\delta t}|w_\varepsilon|, \quad \forall t \ge 0. \tag{5}$$

Let $H_\varepsilon = \Phi_{N_\varepsilon}(T,0)$ for $\varepsilon \ge 0$. Denote by π the projector on a generalized eigenspace corresponding to an eigenvalue of maximal module of $\phi_{N_0}(T,0)$. Since $\lim_{\varepsilon\to0} H_\varepsilon = H_0$, by classical results (see [2, Theorem 5.1, Chapter II]) there exists a sum π_ε of projectors on generalized eigenspaces of H_ε, satisfying $\lim_{\varepsilon\to0} \pi_\varepsilon = \pi$, and the corresponding eigenvalues also converge. Let v_ε be a possibly complex eigenvector of the restriction of H_ε to the image of π_ε, associated with an eigenvalue α_ε. If α_ε is real then v_ε can be taken real as well, otherwise we assume without loss of generality that v_ε satisfies $|\mathrm{Re}(v_\varepsilon)| = \min_{\theta\in\mathbb{R}} |\mathrm{Re}(e^{i\theta}v_\varepsilon)|$. In particular

$$|\mathrm{Re}(\beta v_\varepsilon)| \ge |\beta||\mathrm{Re}(v_\varepsilon)|, \qquad \forall \beta \in \mathbb{C}.$$

Note that, for every positive integer k, one has that $H^k_\varepsilon v_\varepsilon = \alpha^k_\varepsilon v_\varepsilon$ and $H^k_\varepsilon \bar v_\varepsilon = \bar\alpha^k_\varepsilon \bar v_\varepsilon$ which implies that

$$H^k_\varepsilon \mathrm{Re}(v_\varepsilon) = \mathrm{Re}(\alpha^k_\varepsilon v_\varepsilon).$$

Set

$$w_\varepsilon = \mathrm{Re}(v_\varepsilon)$$

and consider the trajectory $t \mapsto z_\varepsilon(t) := \Phi_{N_\varepsilon}(t,0)w_\varepsilon$. For every nonnegative integer h and ε small enough, we have

$$|z_\varepsilon(hT)| \geq |\alpha_\varepsilon|^h |w_\varepsilon| \geq \bar{\alpha}^h |w_\varepsilon|,$$

where $\bar{\alpha}$ is some fixed real value in the open interval $(1, \rho(\phi_{N_0}(T,0)))$, independent of ε. Setting $\kappa = 2\max_{M\in\mathcal{M}} \|M\|$ it follows that

$$|z_\varepsilon(hT+\tau)| \geq |z_\varepsilon(hT)|e^{-\kappa\tau}$$

for every $\tau \in [0,T]$. This completes the proof of Equation (5). We then conclude by noticing that $|\phi^\varepsilon_M(t,0)w_\varepsilon| \geq Ce^{\delta t/\varepsilon}|w_\varepsilon| \geq Ce^{\delta t}|w_\varepsilon|$. □

The above result motivates the following assumption.

Assumption 7. *The switching system Σ_D is exponentially stable.*

In particular, all matrices in $\mathcal{M}_D$ are Hurwitz (and then invertible).

Lut us now introduce an auxiliary switching system, denoted by $\check{\Sigma}$, which could be thought of as the slow dynamics corresponding to signals whose switching occurs at rate exactly $1/\varepsilon$. In order to define $\check{\Sigma}$, let us associate with every $T > 0$ and every $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{S}_\mathcal{M}$ the matrices

$$\Lambda_0(T,\sigma) = \int_0^T \Phi_D(T,s)C(s)ds,$$

$$\Lambda_1(T,\sigma) = \int_0^T \left(A(s) + B(s)\Lambda_0(s,\sigma)\right)ds,$$

$$\Lambda_2(T,\sigma) = \int_0^T B(s)\Phi_D(s,0)ds,$$

and

$$\Lambda(T,\sigma) = \frac{\Lambda_1(T,\sigma) + \Lambda_2(T,\sigma)\left(I_m - \Phi_D(T,0)\right)^{-1}\Lambda_0(T,\sigma)}{T}.$$

The following property holds true.

Lemma 8. *Under Assumption 7 the set*

$$\check{\mathcal{M}} := \{\Lambda(T,\sigma) \mid T > 0,\ \sigma \in \mathcal{S}_\mathcal{M}\}$$

is bounded.

Proof. By Assumption 7 we have that

$$\|\Phi_D(t_2,t_1)\| \leq ce^{-\alpha(t_2-t_1)} \qquad (6)$$

for every $t_2 \geq t_1$, for some constants $\alpha > 0$ and $c \geq 1$ independent of the switching law $D \in \mathcal{S}_{\mathcal{M}_D}$. As a consequence of (6) and the boundedness of $\mathcal{M}$, we have $\|\Lambda_0(T,\sigma)\| \leq C_1 \min\{1,T\}$ and $\|\Lambda_2(T,\sigma)\| \leq C_1 \min\{1,T\}$ for some $C_1 > 0$, and $\|\Lambda_1(T,\sigma)\| \leq C_2 T$ for some $C_2 > 0$. Moreover since $\rho(\Phi_D(T,0)) < 1$ for every $T > 0$ and $D \in \mathcal{S}_{\mathcal{M}_D}$ according to (2), $(I_m - \Phi_D(T,0))^{-1}$ is well defined and expandable as an absolutely convergent power series in $\Phi_D(T,0)$. Letting $\bar{T} = \frac{\log(2c)}{\alpha}$ so that $\|\Phi_D(\tau,0)\| \leq \frac{1}{2}$ for every $\tau \geq \bar{T}$, we can write

$$\begin{aligned}
(I_m - \Phi_D(T,0))^{-1} &= \sum_{k\geq 0} \Phi_D(T,0)^k \\
&= \sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h + \Phi_D(T,0)^{\lfloor \frac{\bar{T}}{T} \rfloor + 1} \left(\sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h\right) \\
&\quad + \Phi_D(T,0)^{2(\lfloor \frac{\bar{T}}{T} \rfloor + 1)} \left(\sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h\right) + \dots \\
&= \left(\sum_{k\geq 0} \Phi_D(T,0)^{(\lfloor \frac{\bar{T}}{T} \rfloor + 1)k}\right) \left(\sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h\right).
\end{aligned}$$

As $\Phi_D(T,0)^h$ corresponds to the flow of Σ_D for a T-periodic signal at time hT we have $\|\Phi_D(T,0)^h\| \leq c$ by (6). Hence

$$\left\|\sum_{h=0}^{\lfloor \frac{\bar{T}}{T} \rfloor} \Phi_D(T,0)^h\right\| \leq \left(1 + \lfloor \frac{\bar{T}}{T} \rfloor\right) c \leq c + \frac{c\bar{T}}{T}.$$

Moreover, as $(\lfloor \frac{\bar{T}}{T} \rfloor + 1)T \geq \bar{T}$,

$$
\begin{aligned}
\left\| \sum_{k\geq 0} \Phi_D(T,0)^{(\lfloor \frac{\bar{T}}{T} \rfloor + 1)k} \right\| &\leq \sum_{k\geq 0} \|\Phi_D(T,0)^{(\lfloor \frac{\bar{T}}{T} \rfloor + 1)}\|^k \\
&\leq \sum_{k\geq 0} \frac{1}{2^k} \\
&= 2.
\end{aligned}
$$

Summing up

$$
\begin{aligned}
\|\Lambda(T,\sigma)\| &\leq C_2 + \frac{C_1^2 \min\{1,T\}^2 (2c + 2c\bar{T}/T)}{T} \\
&= C_2 + 2cC_1^2 \min\left\{ \frac{1}{T} + \frac{\bar{T}}{T^2}, T + \bar{T} \right\} \\
&\leq C_2 + 2cC_1^2 (1 + \bar{T}),
\end{aligned}
$$

concluding the proof of the lemma. □

The switching system $\check{\Sigma}$ is then defined as

$$
\check{\Sigma}: \quad \dot{\check{x}}(t) = M(t)\check{x}(t), \quad M \in \mathcal{S}_{\check{\mathcal{M}}}. \tag{7}
$$

We can now state the main result of this contribution.

Theorem 9. *Suppose that Assumption 7 holds and that system $\check{\Sigma}$ is exponentially unstable. Then system Σ is ε-exponentially unstable.*

Therefore, under Assumption 7, a necessary condition for Σ to be stable for small values of ε is that $\check{\Sigma}$ has no trajectory whose norm diverges exponentially.

4 Proof of Theorem 9

Before entering in the core of the proof of the theorem, let us prove a useful technical result.

Lemma 10. *Let $T > 0$, $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{S}_{\mathcal{M}}$, and $\mu \in \mathbb{R}$. For every $\varepsilon > 0$ denote by $\mathscr{M}(\varepsilon)$ the flow at time εT of Σ_ε corresponding to the signal $\sigma(\cdot/\varepsilon)$. Then there exists $P(\varepsilon)$ of the form*

$$
P(\varepsilon) = \begin{pmatrix} I_n & 0 \\ Q(\varepsilon) & I_m \end{pmatrix} \tag{8}
$$

such that

$$
P(\varepsilon)^{-1}\mathscr{M}(\varepsilon)P(\varepsilon) = \begin{pmatrix} I_n + \varepsilon T \Lambda(T,\sigma) + O(\varepsilon^2) & O(\varepsilon) \\ 0 & \Phi_D(T,0) + O(\varepsilon) \end{pmatrix}, \tag{9}
$$

where

$$
Q(\varepsilon) = \Big(I_m - \Phi_D(T,0)\Big)^{-1} \Lambda_0(T,\sigma) + O(\varepsilon) \tag{10}
$$

and the functions $O(\varepsilon^k)$ are such that $\|O(\varepsilon^k)\| \leq C\varepsilon^k$ for ε small, for some positive constant C independent of ε and σ.

Proof. We first prove that $\varepsilon \mapsto \mathscr{M}(\varepsilon)$ admits a first order expansion

$$
\mathscr{M}(\varepsilon) = \mathscr{M}_0 + \varepsilon \mathscr{M}_1 + O(\varepsilon^2), \tag{11}
$$

for some matrices $\mathscr{M}_0, \mathscr{M}_1$ to be computed. To see that, we first apply the time rescaling $\tau = t/\varepsilon$ and we have that $\mathscr{M}(\varepsilon)$ is equal to $\Phi_{N_0+\varepsilon N_1}(T,0)$, where the signals N_0 and N_1 are defined as

$$
N_0(\tau) = \begin{pmatrix} 0 & 0 \\ C(\tau) & D(\tau) \end{pmatrix}, \qquad N_1(\tau) = \begin{pmatrix} A(\tau) & B(\tau) \\ 0 & 0 \end{pmatrix}.
$$

By the variation of constants formula, one has

$$
\Phi_{N_0+\varepsilon N_1}(T,0) = \Phi_{N_0}(T,0) + \varepsilon \int_0^T \Phi_{N_0}(T,\tau) N_1(\tau) \Phi_{N_0+\varepsilon N_1}(\tau,0)\, d\tau.
$$

One deduces that (11) holds true with

$$
\begin{aligned}
\mathscr{M}_0 &= \Phi_{N_0}(T,0), \\
\mathscr{M}_1 &= \int_0^T \Phi_{N_0}(T,\tau) N_1(\tau) \Phi_{N_0}(\tau,0)\, d\tau.
\end{aligned}
$$

It is easy to get that

$$
\mathscr{M}_0 = \begin{pmatrix} I_n & 0 \\ \Lambda_0(T,\sigma) & \Phi_D(T,0) \end{pmatrix}, \tag{12}
$$

$$
\mathscr{M}_1 = \begin{pmatrix} \Lambda_1(T,\sigma) & \Lambda_2(T,\sigma) \\ \Lambda_3(T,\sigma) & \Lambda_4(T,\sigma) \end{pmatrix}, \tag{13}
$$

where

$$\Lambda_3(T,\sigma) = \Lambda_0(T,\sigma)\Lambda_1(T,\sigma) - \int_0^T \Phi_D(T,\tau)\Lambda_0(\tau,\sigma)(A(\tau)+B(\tau)\Lambda_0(\tau,\sigma))d\tau$$

and

$$\Lambda_4(T,\sigma) = \Lambda_0(T,\sigma)\Lambda_2(T,\sigma) - \int_0^T \Phi_D(T,\tau)\Lambda_0(\tau,\sigma)B(\tau)\Phi_D(\tau,0)d\tau.$$

According to (11), (12), and (13), it follows that (9) holds true with $P(\varepsilon)$ and $Q(\varepsilon)$ as in (8) and (10). This concludes the proof of the lemma. $\square$

We can now provide a proof for Theorem 9.

Proof of Theorem 9. Since $\check{\Sigma}$ is exponentially unstable then, as recalled in Remark 3, $\lambda(\check{\Sigma}) > 0$. Then, according to (3), there exist $\ell \in \mathbb{N}$, ℓ matrices $\Lambda(T_1,\sigma_1),\dots,\Lambda(T_\ell,\sigma_\ell) \in \check{\mathcal{M}}$, and ℓ positive times $t_1,\dots,t_\ell$ so that

$$\rho\Big(e^{t_\ell\Lambda(T_\ell,\sigma_\ell)}\dots e^{t_1\Lambda(T_1,\sigma_1)}\Big) > 1. \tag{14}$$

Let ε be sufficiently small so that $\varepsilon T_k < t_k$ for every $k = 1,\dots,\ell$, and denote by $N_k = \lfloor \frac{t_k}{\varepsilon T_k} \rfloor$ the number of intervals of length εT_k contained in $[0,t_k]$.

Then, for every $k = 1,\dots,\ell$, consider the flow $\mathscr{M}_k(\varepsilon)$ of system Σ_ε corresponding to the signal $\sigma_k(\cdot/\varepsilon)$, evaluated at time εT_k. Thanks to Lemma 10, there exists $P_k(\varepsilon)$ given by $P_k(\varepsilon) = \begin{pmatrix} I_n & 0 \\ Q_k(\varepsilon) & I_m \end{pmatrix}$ such that

$$\mathcal{T}_k(\varepsilon) = P_k(\varepsilon)^{-1}\mathscr{M}_k(\varepsilon)P_k(\varepsilon)$$

satisfies

$$\mathcal{T}_k(\varepsilon) = \begin{pmatrix} I_n + \varepsilon T_k(\Lambda(T_k,\sigma_k)) + O(\varepsilon^2) & O(\varepsilon) \\ 0 & \Phi_D(T_k,0)+O(\varepsilon) \end{pmatrix}.$$

We repeat N_k times $\sigma_k(\cdot/\varepsilon)$ to get a signal on $[0,\varepsilon T_k N_k]$ and the corresponding flow of Σ_ε at time $\varepsilon T_k N_k$ is given by

$$\mathscr{M}_k(\varepsilon)^{N_k} = P_k(\varepsilon)\mathcal{T}_k(\varepsilon)^{N_k}P_k(\varepsilon)^{-1}. \tag{15}$$

We claim that

$$\mathcal{T}_k(\varepsilon)^{N_k} = \begin{pmatrix} e^{t_k\Lambda(T_k,\sigma_k)} + O(\varepsilon) & O(\varepsilon) \\ 0 & O(\varepsilon) \end{pmatrix}. \tag{16}$$

This follows from the general formula

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}^N = \begin{pmatrix} A_{11}^N & W \\ 0 & A_{22}^N \end{pmatrix}$$

with

$$W = \sum_{j=0}^{N-1} A_{11}^j A_{12} A_{22}^{N-j-1},$$

applied to $N = N_k$,

$$A_{11} = I + B_{11}/N_k + O(\varepsilon^2) = e^{\frac{B_{11}}{N_k}} + O(\varepsilon^2),$$
$$A_{22} = \Phi_D(T_k,0) + O(\varepsilon),$$

where $B_{11} = t_k\Lambda(T_k,\sigma_k)$. Then, for ε small enough and $1 \le j \le N_k$ one gets that

$$\begin{aligned} \|A_{11}^j\| &\le \Big(1 + \frac{2\|B_{11}\|}{j}\Big)^j \le e^{2\|B_{11}\|}, \\ A_{11}^{N_k} &= e^{B_{11}} + O(\varepsilon). \end{aligned} \tag{17}$$

Moreover, since $\rho(A_{22}) < 1$ for ε small, it follows that

$$\|A_{22}^j\| \le K\lambda^j, \quad 1 \le j \le N_k, \tag{18}$$

for some $K > 0$ and $\lambda \in (0,1)$ independent of ε small enough and $k \in \{1,\dots\ell\}$. In particular,

$$\|A_{22}^{N_k}\| \le K\lambda^{\frac{t_k}{\varepsilon T_k}-1} = O(\varepsilon).$$

Using now (17), (18), and recalling that K can be taken so that $\|A_{12}\| \le K\varepsilon$, one deduces that

$$\begin{aligned} \|W\| &\le \sum_{j=0}^{N_k-1} \|A_{11}^j\|\|A_{12}\|\|A_{22}^{N_k-j-1}\| \\ &\le K^2 e^{2\|B_{11}\|}\varepsilon \sum_{j=0}^{N_k-1} \lambda^{N_k-1-j} \\ &= C\varepsilon, \end{aligned}$$

for some $C > 0$ independent of ε small enough and $k \in \{1,\dots,\ell\}$. This concludes the proof of (16).

We next use (10) and (16) in (15) to deduce that

$$\mathscr{M}_k(\varepsilon)^{N_k} = \begin{pmatrix} e^{t_k\Lambda(T_k,\sigma_k)} & 0 \\ r_k & 0 \end{pmatrix} + O(\varepsilon), \tag{19}$$

where

$$r_k = \Big(I_m - \Phi_{D_{\sigma_k}}(T_k, 0)\Big)^{-1}\Lambda_0(T_k,\sigma_k)e^{t_k\Lambda(T_k,\sigma_k)}.$$

Set $t_\varepsilon = \varepsilon\sum_{k=1}^{\ell} N_k T_k$ and notice that t_ε tends to $t_1 + \cdots + t_\ell$ as ε tends to zero. We concatenate the N_k times repetitions of $\sigma_k(\cdot/\varepsilon)$ for $k = 1,\ldots,\ell$ to get a signal on $[0, t_\varepsilon]$ and the corresponding flow of Σ_ε at time t_ε is given by the matrix product

$$\Upsilon_\varepsilon = \mathscr{M}_\ell(\varepsilon)^{N_\ell}\mathscr{M}_{\ell-1}(\varepsilon)^{N_{\ell-1}}\cdots\mathscr{M}_1(\varepsilon)^{N_1}.$$

Using (19), one deduces that

$$\Upsilon_\varepsilon = \begin{pmatrix} e^{t_\ell\Lambda(T_\ell,\sigma_\ell)}\cdots e^{t_1\Lambda(T_1,\sigma_1)} & 0 \\ r & 0 \end{pmatrix} + O(\varepsilon),$$

where the matrix r does not depend on ε.

Using (14), one gets that $\rho(\Upsilon_\varepsilon) > 1$ for ε small enough, yielding that $\lambda(\Sigma_\varepsilon) > 0$. In particular $\liminf_{\varepsilon\to 0^+}\lambda(\Sigma_\varepsilon) \geq 0$.

In order to show that Σ is ε-exponentially unstable, observe that the flow $\Upsilon_{\varepsilon,t}$ of Σ_ε at time $t \in [0, t_\varepsilon]$ corresponding to the signal defined above satisfies

$$\Upsilon_{\varepsilon,t} = \begin{pmatrix} e^{(t-\sum_{h=1}^{k-1}t_h)\Lambda(T_k,\sigma_k)}\prod_{\ell=1}^{k-1}e^{t_\ell\Lambda(T_\ell,\sigma_\ell)} + O(\varepsilon) & O(\varepsilon) \\ r(\varepsilon,t) & q(\varepsilon,t) \end{pmatrix}, \tag{20}$$

whenever $t \in [\sum_{h=1}^{k-1}t_h, \sum_{h=1}^{k}t_h]$, for some matrix functions r, q, where the terms $O(\varepsilon)$ are uniform with respect to $t \in [0, t_\varepsilon]$. The matrix $\Upsilon_{\varepsilon,t_\varepsilon}$ converges, as ε goes to zero, to

$$\bar\Upsilon = \begin{pmatrix} e^{t_\ell\Lambda(T_\ell,\sigma_\ell)}\cdots e^{t_1\Lambda(T_1,\sigma_1)} & 0 \\ \bar r & 0 \end{pmatrix},$$

for some matrix $\bar r$. For ε small enough we construct a trajectory $z_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$ of Σ_ε satisfying $|z_\varepsilon(t)| \geq \hat C e^{\hat\lambda t}|z_\varepsilon(0)| > 0$ for every $t \geq 0$, with $\hat\lambda \in (0, \lambda(\check\Sigma))$ and $\hat C > 0$ independent of ε. Let π be the sum of the projectors on the generalized eigenspaces associated with the eigenvalues of $\bar\Upsilon$ of modulus $\rho(e^{t_\ell\Lambda(T_\ell,\sigma_\ell)}\cdots e^{t_1\Lambda(T_1,\sigma_1)}) > 1$. Since $\Upsilon_{\varepsilon,t_\varepsilon}$ converges to $\bar\Upsilon$ as ε goes to zero, by classical results (see [2, Theorem 5.1, Chapter II]) there exists π_ε, a sum of projectors on generalized eigenspaces of $\Upsilon_{\varepsilon,t_\varepsilon}$, satisfying $\lim_{\varepsilon\to 0}\pi_\varepsilon = \pi$, and the corresponding eigenvalues also converge. Let v_ε be a possibly complex eigenvector of the restriction of $\Upsilon_{\varepsilon,t_\varepsilon}$ to the image of π_ε, associated with an eigenvalue α_ε. If α_ε is real then v_ε can be taken real as well, otherwise we assume without loss of generality that v_ε satisfies $|\mathrm{Re}(v_\varepsilon)| = \min_{\theta\in\mathbb{R}}|\mathrm{Re}(e^{i\theta}v_\varepsilon)|$. In particular $|\mathrm{Re}(\beta v_\varepsilon)| \geq |\beta||\mathrm{Re}(v_\varepsilon)|$ for every $\beta \in \mathbb{C}$. Note that, for every positive integer k, one has that $(\Upsilon_{\varepsilon,t_\varepsilon})^k v_\varepsilon = \alpha_\varepsilon^k v_\varepsilon$ and $(\Upsilon_{\varepsilon,t_\varepsilon})^k\bar v_\varepsilon = \bar\alpha_\varepsilon^k\bar v_\varepsilon$ which implies $(\Upsilon_{\varepsilon,t_\varepsilon})^k\mathrm{Re}(v_\varepsilon) = \mathrm{Re}(\alpha_\varepsilon^k v_\varepsilon)$.

Consider the trajectory $z_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$ of Σ_ε obtained applying the flow $\Upsilon_{\varepsilon,t}$ to the initial condition $z_\varepsilon(0) = \mathrm{Re}(v_\varepsilon)$ and repeating periodically after time t_ε. Letting Π_x be the projection of a vector of $\mathbb{R}^{n+m}$ onto its first n components, it is easy to see that $|\Pi_x v| \geq C|v|$ for every v in the image of $\bar\Upsilon$, where

$$C = (1 + \|\bar r\|^2\|(e^{t_\ell\Lambda(T_\ell,\sigma_\ell)}\cdots e^{t_1\Lambda(T_1,\sigma_1)})^{-1}\|^2)^{-1/2}.$$

Hence, for every nonnegative integer h and ε small enough,

$$\begin{aligned} |x_\varepsilon(ht_\varepsilon)| &= |\Pi_x z_\varepsilon(ht_\varepsilon)| \\ &= |\Pi_x\pi_\varepsilon z_\varepsilon(ht_\varepsilon)| \\ &\geq |\Pi_x\pi z_\varepsilon(ht_\varepsilon)| - \|\Pi_x\|\|\pi - \pi_\varepsilon\|\,|z_\varepsilon(ht_\varepsilon)| \\ &\geq |\Pi_x\pi z_\varepsilon(ht_\varepsilon)| - \|\pi - \pi_\varepsilon\|\,|z_\varepsilon(ht_\varepsilon)| \\ &\geq C|\pi z_\varepsilon(ht_\varepsilon)| - \|\pi - \pi_\varepsilon\|\,|z_\varepsilon(ht_\varepsilon)| \\ &\geq C|z_\varepsilon(ht_\varepsilon)| - (1 + C)\|\pi - \pi_\varepsilon\|\,|z_\varepsilon(ht_\varepsilon)| \\ &\geq \frac{C}{2}|z_\varepsilon(ht_\varepsilon)|. \end{aligned} \tag{21}$$

As $z_\varepsilon(ht_\varepsilon) = \mathrm{Re}(\alpha_\varepsilon^h v_\varepsilon)$ we also have

$$|x_\varepsilon(ht_\varepsilon)| \geq \frac{C}{2}|\alpha_\varepsilon|^h|z_\varepsilon(0)|. \tag{22}$$

By (20), (21), and setting $\kappa = \max_{\Lambda\in\check{\mathcal{M}}}\|\Lambda\|$ it follows that

$$
\begin{aligned}
|x_\varepsilon(ht_\varepsilon+\tau)| &\geq |x_\varepsilon(ht_\varepsilon)|e^{-\kappa\tau} - |z_\varepsilon(ht_\varepsilon)|O(\varepsilon)\\
&\geq |x_\varepsilon(ht_\varepsilon)|e^{-\kappa\tau} - \frac{2}{C}|x_\varepsilon(ht_\varepsilon)|O(\varepsilon)\\
&\geq \frac{1}{2}|x_\varepsilon(ht_\varepsilon)|e^{-\kappa\tau} \qquad (23)
\end{aligned}
$$

for every $\tau \in [0, t_\varepsilon]$. Take ε small enough in such a way that $1 < \underline{\alpha} \leq |\alpha_\varepsilon| \leq \bar{\alpha}$ for some positive constants $\underline{\alpha}, \bar{\alpha}$, and $t_\varepsilon \leq \bar{t}$, where $\bar{t} = 2\sum_{h=1}^{\ell} t_h$. By (22) and (23) we thus get for every $t \geq 0$

$$
\begin{aligned}
|z_\varepsilon(t)| \geq |x_\varepsilon(t)| &\geq \frac{C}{4}e^{-\kappa t_\varepsilon}|\alpha_\varepsilon|^{\lfloor \frac{t}{t_\varepsilon}\rfloor}|z_\varepsilon(0)|\\
&\geq \frac{Ce^{-\kappa t_\varepsilon}}{4|\alpha_\varepsilon|}e^{\frac{\log|\alpha_\varepsilon|}{t_\varepsilon}t}|z_\varepsilon(0)| \geq \hat{C}e^{\hat{\lambda}t}|z_\varepsilon(0)|,
\end{aligned}
$$

where $\hat{C} = \frac{Ce^{-\kappa\bar{t}}}{4\bar{\alpha}}$ and $\hat{\lambda} = \frac{\log\underline{\alpha}}{\bar{t}}$, which shows that Σ is ε-exponentially unstable. ■

Remark 11. *The standard approach to singular perturbations in the non-switching case consists in identifying separated slow and fast variables where different controllers for different time-scale variables can be designed in order to lead the overall system to its desired performance (see, e.g., [4]). We could apply formally such an approach in our case by saying that y tends instantaneously to $-D(t)^{-1}C(t)x(t)$ (the equilibrium of the equation for y when $x \equiv x(t)$) and obtaining the slow-variable switching system*

$$\dot{x}(t) = \big(A(t) - B(t)D(t)^{-1}C(t)\big)\,x(t). \qquad (24)$$

However, as it has already been observed in the literature, the switching system (24) may be exponentially stable even when, for every $\varepsilon > 0$, Σ_ε is unstable [5]. The modes of (24) are given by matrices in the set

$$\bar{\mathcal{M}} := \{A - BD^{-1}C \mid \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) \in \mathcal{M}\} \subset M_n(\mathbb{R}).$$

It is not difficult to check that $\bar{\mathcal{M}} \subset \check{\mathcal{M}}$. Indeed, letting $T > 0$ and $\bar{\sigma} \in \mathcal{S}_{\mathcal{M}}$ constantly equal to M on $[0, T]$, it holds

$$\Lambda(T, \bar{\sigma}) = A - BD^{-1}C.$$

Example 12. *Let us show by an example that $\check{\mathcal{M}}$ gives sharper necessary conditions for stability than the set $\bar{\mathcal{M}}$ introduced in the previous remark. Consider system Σ with*

$$\mathcal{M} = \left\{M_1 = \begin{pmatrix} -1 & 1\\ 0 & -0.1\end{pmatrix}, M_2 = \begin{pmatrix} -3 & 0\\ 2 & -0.1\end{pmatrix}\right\}.$$

The stability of singularly perturbed planar switching systems is completely characterized in [1, Theorem 2] through some necessary and sufficient conditions. Based on this characterization (cf., in particular, Item (SP5) in [1, Theorem 2]) the condition

$$\frac{1}{2}\left(\mathrm{tr}(M_1)\mathrm{tr}(M_2) - \mathrm{tr}(M_1M_2)\right) < -\sqrt{\det(M_1)\det(M_2)} \qquad (25)$$

implies that Σ_ε is exponentially unstable for all $\varepsilon > 0$. Condition (25) is satisfied in the case of this example since

$$\frac{1}{2}\left(\mathrm{tr}(M_1)\mathrm{tr}(M_2) - \mathrm{tr}(M_1M_2)\right) = -0.8$$

and

$$\det(M_1M_2) = 0.03.$$

Look now at systems $\bar{\Sigma}$ and $\check{\Sigma}$. We have $\bar{\mathcal{M}} = \{-1, -3\}$ and then system (24) is exponentially stable. Concerning system $\check{\Sigma}$, let us consider the switching signal

$$\sigma(t) = \alpha(t)M_1 + (1 - \alpha(t))M_2$$

associated with the 2-periodic function

$$\alpha(t) = \begin{cases} 1 & t \in [0,1],\\ 0 & t \in [1,2],\end{cases}$$

and take $T = 2$. For this choice of σ and T one can easily verify that

$$\Lambda(T, \sigma) = -2 + 100(1 - e^{-0.2})^{-1}(1 - e^{-0.1})^2 > 0.$$

Then $\check{\Sigma}$ is exponentially unstable, as illustrated in Figure 1.

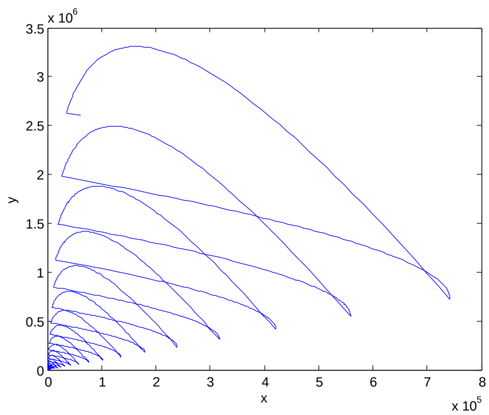

Figure 1: Fast and slow variables evolution of system Σ_ε with signal σ and $\varepsilon = 0.1$, starting from $(x_0, y_0) = (1, 1)$.

5 Conclusions

The main goal of this contribution is to identify linear modes (the elements of $\check{\mathcal{M}}$) that can be approximated arbitrarily well, in the slow-varying variables x, by a singularly perturbed switching linear system for sufficiently small values of the singular perturbation parameter. The corresponding signals are periodic with a period of order ε. In this way, we are able to single out a switched system with a single time-scale (the system $\check{\Sigma}$), whose stability is a necessary condition for the stability of the original singularly perturbed switching system for small values of ε.

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