

An operator theoretical approach of some inverse problems

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Abstract

This work gives an overview of the solution to linear forward and inverse problems for a class of elliptic partial differential equations in two-dimensional domains Ω , in the framework of Banach spaces and operators. We focus on the equation $\nabla(\sigma\nabla u) = 0$, with σ in Sobolev spaces $W^{1,p}(\Omega)$, $1 < p < \infty$, with Dirichlet or Neumann or mixed boundary conditions. For domains Ω with Dini-smooth boundary $\partial\Omega$, the approach is based on the properties of generalized Hardy spaces.

1 Introduction

This paper aims at giving an overview of the solution of inverse problem with boundary data. The inverse problems we consider involve elliptic partial differential equations (PDE) of the form $\operatorname{div}(\sigma\nabla u) = 0$ where σ is a fixed function in a Sobolev space $W^{1,r}(\Omega)$ in a simply connected bounded domain Ω of the complex plane. The goal is to describe the existing results enhancing the connections between PDE, function spaces and operator theory for a deep understanding of inverse problems.

In [7], the authors established results in the Hardy spaces of the unit disk \mathbb{D} (for $\sigma = 1$, *i.e.* $\Delta u = 0$) for partial boundary data in $L^p(\mathbb{T})$. This work built the foundations of an approach to resolve a class of inverse problems which led to several other instances of these ideas applied in other, more complicated, settings. For example, the case where $\Omega = \mathbb{D}$ with $\sigma \in W^{1,\infty}(\mathbb{D})$ has been investigated in [8], and with $\sigma \in W^{1,r}(\mathbb{D})$, $r \geq 2$, in [2]. When the domain Ω is an annulus, the inverse problem for the Laplace equation ($\sigma = 1$) has been solved in [13, 26]; for σ in the space $W^{1,\infty}(\Omega)$, see [5, 18].

We consider the case where $\sigma \in W^{1,r}(\Omega)$, $r \geq 2$, on simply-connected Dini-smooth domains Ω of the complex plane, some remarks in conclusion will specify the validity of the results in less smooth domains and for multi-connected domains. The regularity of the boundary $\partial\Omega$ of the domain plays a fundamental role in the solution of inverse

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problem when the boundary data is in $L^p(\partial\Omega)$, for $1 < p < \infty$. Indeed, the conformal map from \mathbb{D} onto a simply-connected Dini-smooth bounded domain Ω has a derivative that can be extended continuously to the unit circle \mathbb{T} : it permits an extension of the results on the unit disk to the case of simply-connected Dini-smooth bounded domain using the conformal map.

The critical cases for such a study correspond to $r = 2$ (when the exponent r is equal to the dimension of the domain) for Dini-smooth domains Ω or $r \geq 2$ and Ω is only rectifiable (the above conformal map has a derivative that admits an $L^1(\mathbb{T})$ extension on \mathbb{T}) and the results cannot be obtained directly using conformal equivalence between \mathbb{D} and the rectifiable domain Ω . We will thus focus on $r > 2$ and a simply-connected Dini-smooth bounded domain Ω .

The paper is organized as follows. Section 2 installs the applied context of the PDE and the operators related to the considered inverse problems, together with notation and definitions. Section 3 is devoted to the treatment of the forward and the inverse problems; Section 3.1 describes the methods and the solutions for the Laplace equation (*i.e.* in the case $\sigma = 1$), while Section 3.2 is concerned with their generalizations to the conductivity equation. We finally provide some concluding remarks in Section 4.

2 Framework, notation, definitions

2.1 With PDE, examples

We consider linear elliptic PDEs from a family of conductivity equations:

$$\begin{cases} Au = \operatorname{div}(\sigma \nabla u) = f \text{ in } \Omega, \\ Bu = b \text{ on } I \subset \partial\Omega, & Bu = u|_I \text{ or } Bu = (\sigma \partial_n u)|_I, \end{cases} \quad (1)$$

for smooth enough domains $\Omega \subset \mathbb{R}^2$, conductivity coefficient σ with positive values, source term f and boundary data b . The particular case where $\sigma = 1$ corresponds to Poisson (Laplace) equation, in the inhomogeneous situation when $f \neq 0$.

When settled in dimension 3, such PDEs are involved in very many application areas in physics, among which those related to electromagnetism on Maxwell's equation under the quasi-static assumption (neglecting the time derivative of the electromagnetic fields), but also to gravimetry issues and Newton equation. Two-dimensional situations arise for instance whenever the 3D geometry admits symmetry properties while the solution is invariant in some direction (like for plasma in a toroidal *tokamak* where a 2D conductivity PDE arises from a cylindrical change of variable in Poisson-Laplace equation), but are also of interest by themselves since their solution can be handled with tools from complex analysis and analytic function spaces.

We have the following application in mind, related to electromagnetism, where u stands for a potential.

Coming from neurosciences and brain imaging, a model for electroencephalography (EEG) is given by (1) with Ω made of (disjoint union), say, 3 homogeneous nested layers

Ω_i , $i = 0, 1, 2$ (brain, skull, scalp), each of constant conductivity σ_i , and $f = \operatorname{div} \mu$ (current source) supported in the innermost layer Ω_0 , is to be estimated from the Cauchy data (electric potential, current flux) that are available on a part $I \subset \partial\Omega$ of the outer boundary. Due to continuity properties at interfaces, this can be separated into a pair of data completion (boundary value) Cauchy problems, and an inverse source estimation problem, for Poisson Laplace equations, see [1, 10] in \mathbb{R}^2 , and [15] in \mathbb{R}^3 . For $i = 2, 1$, we wish to find u such that

$$\begin{cases} \Delta u = 0 \text{ in } \Omega_i, \\ u|_I, (\partial_n u)|_I \text{ given on } I \subset \partial\Omega_i. \end{cases}$$

If the conductivity is allowed to vary, which is a reasonable assumption in particular within the intermediate layer Ω_1 , we are back to the general conductivity PDE:

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0 \text{ in } \Omega_i, \\ u|_I, (\sigma \partial_n u)|_I \text{ given on } I \subset \partial\Omega_i. \end{cases} \quad (2)$$

This solves the so-called cortical mapping step, and provides data on the the cortex $\partial\Omega_0$ from which we state the source estimation issue: find μ supported within Ω_0 (satisfying some assumptions, like being a linear combination of Dirac masses) such that

$$\begin{cases} \Delta u = \operatorname{div} \mu \text{ in } \Omega_0, \\ u|_{\partial\Omega_0}, (\partial_n u)|_{\partial\Omega_0} \text{ given on } \partial\Omega_0. \end{cases}$$

In this work, we will discuss the above boundary value problems (2), though not the EEG source estimation one.

2.2 With Banach spaces and their operators

Let V and W be two Banach spaces. Let $T : V \rightarrow W$ be a bounded linear map. We denote by V^* and W^* the respective dual spaces of V and W and we will write $\langle L, f \rangle_{V^*, V} = L(f)$ for $f \in V$, $L \in V^*$. The adjoint T^* of T is a bounded linear operator defined from W^* to V^* such that $\langle g, Tf \rangle_{W^*, W} = \langle T^*g, f \rangle_{V^*, V}$ for $f \in V$ and $g \in W^*$. In the case of Hilbert spaces, $V = \overline{\operatorname{Ran} T^*} \oplus \ker T$ and $W = \ker T^* \oplus \overline{\operatorname{Ran} T}$; the direct sums are orthogonal for the respective inner products on V and W , see [12, Rmk 17] or [9, Prop. 3].

Consider two elements $b \in W$ and $m \in V$, and T a linear and bounded operator from V to W such that $T(m) = b$. Given $b \in W$ and the linear bounded operator T , is it possible to find $m \in V$?

The problem is *well-posed in the Hadamard sense* [21] if the three following conditions are satisfied:

- a solution exists;
- it is unique;

- the solution m depends continuously on the initial data b .

Assume that V and W are reflexive Banach spaces. This question is trivial when T is invertible: there is existence, uniqueness of a solution and the solution depends continuously on b since T^{-1} is invertible. The most interesting situation occurs when T is not invertible. Two situations happen: $R = \text{Ran } T$ is closed or not closed. If the range R of T is closed, one can write $T = P_R \tilde{T}$ where P_R denotes the (orthogonal?) projection from W onto R and \tilde{T} is an invertible linear bounded operator from $V \setminus \ker T$ onto $W \setminus \ker T^*$. Let $Q = \overline{\text{Ran } \tilde{T}^*} \subset V$. Then, the linear bounded operator $T^\dagger = P_Q \tilde{T}^{-1}$ is called the pseudo-inverse of T and whenever T is injective, $T^\dagger b$ is the unique element in V such that

$$\min_{m \in V} \|T(m) - b\|_W = \|T^\dagger b - m\|_V,$$

see [11, Thms 1, 5]. In the case of Hilbert spaces, the pseudo-inverse is given by $T^\dagger = T^* (T T^*)^{-1}$. The minimization problem is well-posed in the Hadamard sense.

If $\ker T \neq \{0\}$, the problem is said to have silent sources as elements m_S of V that are not detected by T . They lead to non-uniqueness properties, as $T(m + m_S) = T(m)$ for $m_S \in \ker T$.

Lemma 1 *When R is dense in W , for any $b \notin R$ there is a sequence $(m_k)_{k \geq 1}$ of elements in V such that $\|T(m_k) - b\|_W \xrightarrow[k \rightarrow +\infty]{} 0$. As a consequence, $\lim_{k \rightarrow +\infty} \|m_k\|_V = +\infty$.*

Proof: Assume that $\sup_{k \in \mathbb{Z}^+} \|m_k\|_V < \infty$ and there exists a subsequence $(m_{n_k})_{k \geq 1}$ converging to an element $m \in V$ and by continuity of T , $(T(m_k))_{k \geq 1}$ converges to $T(m) = b$ by uniqueness of a limit, which leads to a contradiction. ■

The limit of the norm of m_k has an impact on the error in the reconstruction: indeed, if for $k \geq 1$, $\eta_k = m_k - m$ then $\|\eta_k\|_V \geq \|m_k\|_V - \|m\|_V \xrightarrow[k \rightarrow +\infty]{} +\infty$. To avoid this phenomenon, we express a Tykhonov-like regularization process for classes of (ill-posed) inverse problems related to some elliptic partial differential equations in appropriate framework called bounded extremal problems denoted by BEP. The BEP is an approximation problem with a constraint in norm to avoid an arbitrary large norm of the element m in V such that $\|T(m) - b\|_W$ is small. It can be stated as follows:

$$\min_{\substack{g \in V \\ \|g\|_V \leq M}} \|T(g) - b\|_W. \quad (\text{BEP})$$

The BEP is formulated in normed functions spaces linked to the partial differential equation and using operators related the initial data b .

We will throughout use the terminology “solving an inverse problem” whenever a Tykhonov-like regularization method leads to its well-posedness, hence to existence and uniqueness of a solution. In the Hilbertian case, the solution is constructively given by an implicit equation.

In the sequel, we will write \tilde{T} instead of T to specify that the operator acts on a real Banach space V . For $I \subsetneq \partial\Omega$, we will write $J = \partial\Omega \setminus I$.

2.3 Notation, definitions

Let Ω be a domain of $\mathbb{C} \simeq \mathbb{R}^2$. We will write \mathcal{H}^1 for the Hausdorff measure on $\partial\Omega$.

A domain in \mathbb{C} is said to be Dini-smooth (see [31]) if its boundary $\partial\Omega$ has a parametrization ψ such that its derivative ψ' is Dini-continuous: its modulus of continuity $\omega_{\psi'}$ satisfies $\int_0^\varepsilon \frac{\omega_{\psi'}(t)}{t} dt < \infty$ for any $\varepsilon > 0$. If Ω is a simply connected Dini-smooth domain, a conformal map ψ from \mathbb{D} onto Ω is such that ψ' has a continuous extension to $\overline{\mathbb{D}}$ (see [31, Thm 3.5]). This result permits to extend most of the results from \mathbb{D} to a simply connected Dini-smooth domain and is still true for a conformal map ψ between two multi-connected bounded domains (see [5]). Throughout this paper, Ω is a Dini-smooth domain of \mathbb{C} .

We will write $\mathcal{D}(\Omega)$ for the space of \mathcal{C}^∞ -smooth complex valued-functions with compact support in Ω , equipped with the usual topology. Its dual is the space of distributions $\mathcal{D}'(\Omega)$ on Ω .

For $1 \leq p < \infty$, $L^p(\Omega)$ denotes the space of complex-valued measurable functions f on Ω such that the norm $\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(z)|^p dz\right)^{1/p}$ is finite; $L^\infty(\Omega)$ is the space of essentially bounded functions on Ω , equipped with the norm defined by the essential *sup* of the modulus. The Sobolev space $W^{1,p}(\Omega)$ is the set of complex-valued functions $f \in L^p(\Omega)$ with distributional derivatives $\partial f := \partial_z f = \frac{1}{2}(\partial_x - i\partial_y)f$, $\bar{\partial} f := \partial_{\bar{z}} f = \frac{1}{2}(\partial_x + i\partial_y) \in L^p(\Omega)$ for $z = x + iy \in \mathbb{C}$. We equip $W^{1,p}(\Omega)$ with the norm

$$\|f\|_{W^{1,p}(\Omega)} := \left(\|f\|_{L^p(\Omega)}^p + \|\partial f\|_{L^p(\Omega)}^p + \|\bar{\partial} f\|_{L^p(\Omega)}^p \right)^{1/p}.$$

The space $W^{1,\infty}(\Omega)$ is defined similarly, with obvious modifications of the norm. For $1 \leq p \leq \infty$, the space $L^p_{\mathbb{R}}(\Omega)$ (respectively $W^{1,p}_{\mathbb{R}}(\Omega)$) denotes the space of real-valued functions in $L^p(\Omega)$ (in $W^{1,p}(\Omega)$ respectively). For p strictly greater than the dimension of Ω , $W^{1,p}(\Omega)$ embeds into $L^\infty(\Omega)$ [12, Cor. 9.14] and in the space of Hölder continuous functions. As a consequence, $W^{1,p}(\Omega)$ -functions extend continuously to $\partial\Omega$. For $p > 2$, a function in $W^{1,p}(\Omega)$ can be extended continuously to $\overline{\Omega}$: the extension to $\partial\Omega$ of a function $f \in W^{1,p}(\Omega)$ is denoted by $\text{tr}_{\partial\Omega}(f)$.

The space $W^{1-1/p,p}(\partial\Omega)$ is a real interpolation between the spaces $L^p(\partial\Omega)$ and $W^{1,p}(\partial\Omega)$; $W^{1-1/p,p}(\partial\Omega)$ is equipped with the norm:

$$\|f\|_{W^{1-1/p,p}(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega \times \partial\Omega} \frac{|f(t) - f(u)|^p}{(\mathcal{H}^1(t,u))^p} d\mathcal{H}^1(t) d\mathcal{H}^1(u) \right)^{1/p},$$

where $\mathcal{H}^1(t,u)$ denotes the length of the arc (t,u) on $\partial\Omega$. We will write $L^p_{loc}(\Omega)$ ($W^{1,p}_{loc}(\Omega)$ respectively) for the distributions on Ω such that their restriction to a compact subset Ω_0 of Ω belongs to $L^p(\Omega_0)$ ($W^{1,p}(\Omega_0)$ respectively).

The distribution $\nu \overline{\partial} f$ with $\nu \in W_{\mathbb{R}}^{1,r}(\Omega)$ is defined using the Leibniz rule:

$$\langle \nu \overline{\partial} f, \phi \rangle = - \int_{\Omega} (\nu \overline{f} \overline{\partial} \phi + \overline{\partial} \nu \overline{f} \phi), \quad \text{for } \phi \in \mathcal{D}(\Omega),$$

where \langle, \rangle denotes the duality product between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. For $r > 2$, if $\sigma \in W^{1,r}(\Omega)$ with $0 < c < \sigma < C$ and $\operatorname{div}(\sigma \nabla u) = 0$ for $u \in W_{\mathbb{R}}^{1,p}(\Omega)$, then the normal derivative $\partial_n u$ is the unique element of $W^{-1/p,p}(\partial\Omega) = (W^{1-1/q,q}(\partial\Omega))^*$ with $1/p + 1/q = 1$

$$\langle \sigma \partial_n u, \phi \rangle = \int_{\Omega} \sigma \nabla u \nabla g, \quad \operatorname{tr}_{\partial\Omega} \phi = g.$$

3 Inverse boundary value problems

Let Ω be a Dini-smooth simply connected domain of \mathbb{C} and $I \subset \partial\Omega$, $J \subset \partial\Omega \setminus I$, with $\mathcal{H}^1(I) > 0$ and $\mathcal{H}^1(J) > 0$. We will denote by ϕ a conformal map from \mathbb{D} onto Ω . Let $1 < p < \infty$ and $b \in L_{\mathbb{R}}^p(I)$. Let $\sigma \in W_{\mathbb{R}}^{1,r}(\Omega)$ for $r > 2$ satisfying $0 < c_{\sigma} < \sigma < C_{\sigma} < 1$ with $c_{\sigma}, C_{\sigma} > 0$.

The inverse data extension problem can be stated as follows, with $I \subset \partial\Omega$: given $b = (b_D, b_N) \in (L_{\mathbb{R}}^p(I))^2$, is there m such that

$$\begin{cases} \operatorname{div}(\sigma \nabla m) = 0 & \text{on } \Omega, \\ m = b_D \text{ and/or } \sigma \partial_n m = b_N & \text{on } I? \end{cases} \quad (3)$$

Note that for $I = \partial\Omega$, only one of the two boundary conditions can be imposed. The operator $A = \operatorname{div}(\sigma \nabla \cdot)$ is an elliptic operator of the form $A = \operatorname{div}(M \nabla \cdot)$ in the isotropic case where $M = \sigma I_2$ where I_2 is the identity matrix of size 2.

Remark 1 *The existence of a solution m to Equation (3) can be written in terms of operators as follows, see (BEP): for which space V of functions $m : \Omega \rightarrow \mathbb{R}$ satisfying $\operatorname{div}(\sigma \nabla m) = 0$ on Ω such that m can be extended to $\partial\Omega$ as a function in $L^p(\partial\Omega)$ is the operator $\tilde{T} : V \rightarrow L_{\mathbb{R}}^p(I)$, $\tilde{T}(m) = b_D$ for $I = \partial\Omega$, $\tilde{T} : V \rightarrow (L_{\mathbb{R}}^p(I))^2$, $\tilde{T}(m) = (b_D, b_N)$ for $I \subset \partial\Omega$, continuous and surjective? See also Remark 2 below.*

3.1 Harmonic solutions, Hardy spaces

In the homogeneous situation where $\sigma = 1$, (3) rewrites as a Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{on } \Omega, \\ u = b_D \text{ and/or } \partial_n u = b_N & \text{on } I. \end{cases} \quad (4)$$

The condition $u = b_D$ on I refers to a Dirichlet boundary condition (D) and the condition $\partial_n u = b_N$ on I is a Neumann boundary condition (N). Coupling the two as in (4) leads to a Cauchy-type problem, with overdetermined partial data. In what follows, we assume that $b_D \in L_{\mathbb{R}}^p(I)$, $b_N \in W_{\mathbb{R}}^{1,p}(I)$ for $1 < p < \infty$.

3.1.1 Forward problems

The forward problem consists from a given harmonic function u on Ω in finding $\text{tr}_{\partial\Omega}u = b_D$ ((D) condition) and $\partial_n u = b_N$ ((N) condition) on $I \subset \partial\Omega$.

In the case of the unit disk \mathbb{D} , it is known that a harmonic function $u : \mathbb{D} \rightarrow \mathbb{R}$ has a trace on \mathbb{T} almost everywhere that belongs to $L^p_{\mathbb{R}}(\mathbb{T})$ if and only if u satisfies

$$\sup_{0 < r < 1} \int_0^{2\pi} |u(re^{it})|^p dt < \infty. \quad (5)$$

For any harmonic function u in \mathbb{D} satisfying (5), the restriction to I of $\text{tr}_{\mathbb{T}}u$ is in $L^p(I)$.

For a bounded Dini-smooth domain Ω , if $u : \Omega \rightarrow \mathbb{C}$ is a harmonic function such that

$$\sup_{0 < r < 1} \int_{\Gamma_r} |u(\zeta)|^p d\mathcal{H}^1(\zeta) < \infty, \quad (6)$$

where $\Gamma_r = \psi(r\mathbb{T})$, then u has a trace on $\partial\Omega$ denoted by $\text{tr}_{\partial\Omega}u \in L^p_{\mathbb{R}}(\partial\Omega)$ (see [5, Thm]) and $(\text{tr}_{\partial\Omega}u)|_I \in L^p_{\mathbb{R}}(I)$ which solves the part (D) of the forward problem.

Remark 2 *The real Banach space V in Remark 1 is defined by*

$$V = \text{tr}_{\partial\Omega} \left(\left\{ u : \Omega \rightarrow \mathbb{R} \text{ harmonic such that } \|u\|^p := \sup_{0 < r < 1} \int_{\Gamma_r} |u(\zeta)|^p d\mathcal{H}^1(\zeta) < \infty \right\} \right).$$

Another way to find a solution to the forward problem in Dini-smooth domain is to reformulate it using complex-valued functions. Since the function u is harmonic in Ω , it is possible to construct a harmonic function \tilde{u} on Ω such that $f = u + i\tilde{u}$ is analytic on Ω (*i.e.* $\bar{\partial}f = 0$ in Ω). The function \tilde{u} is called the harmonic conjugate of u . The construction of \tilde{u} is as follows: if Ω is simply-connected, given a harmonic function $u : \Omega \rightarrow \mathbb{R}$, there is a harmonic function \tilde{u} on Ω such that u and \tilde{u} satisfy the Cauchy-Riemann equations

$$\begin{cases} \partial_x \tilde{u} &= -\partial_y u, \\ \partial_y \tilde{u} &= \partial_x u. \end{cases}$$

The harmonic conjugate is unique up to an additive constant. Whenever u is harmonic in Ω and satisfies (6), its harmonic conjugate \tilde{u} is such that $\text{tr}_{\partial\Omega}\tilde{u} \in L^p_{\mathbb{R}}(\partial\Omega)$. To get lighter notations, we still write u and \tilde{u} instead of $\text{tr}_{\partial\Omega}u$ and $\text{tr}_{\partial\Omega}\tilde{u}$. This result is known as Riesz Theorem, see [20, Ch. III, Thm. 2.3] and can be stated as follows:

$$\begin{aligned} \mathfrak{H} : L^p_{\mathbb{R}}(\partial\Omega) &\longrightarrow L^p_{\mathbb{R}}(\partial\Omega) \\ u &\longmapsto \tilde{u}, \text{ with the normalization } \int_{\partial\Omega} \tilde{u} d\mathcal{H}^1 = 0. \end{aligned}$$

is a bounded linear map on $L^p_{\mathbb{R}}(\partial\Omega)$. The operator \mathfrak{H} is called the conjugation map.

The function $f = u + i\tilde{u}$ with u and \tilde{u} satisfying the Cauchy-Riemann equations, is analytic since $\bar{\partial}f = \partial_x(u + i\tilde{u}) + i\partial_y(u + i\tilde{u}) = 0$ in Ω . It is known that if an analytic function f on Ω is such that

$$\|f\|_p^p := \sup_{0 < r < 1} \|f\|_{L^p(\Gamma_r)}^p < \infty, \quad (7)$$

where $\Gamma_r = \phi(r\mathbb{T})$, then f has a non-tangential limit on $\partial\Omega$ called trace of f and denoted by $\text{tr}_{\partial\Omega} f$ see [2, 16, 20].

Definition 1 *An analytic function on Ω satisfying (7) belongs to the Hardy space $H^p(\Omega)$.*

The space $H^p(\Omega)$ equipped with the norm (7) is a Banach space. The space of $\text{tr}_{\partial\Omega} f$ for $f \in H^p(\Omega)$ is a closed subspace of $L^p(\partial\Omega)$ denoted by $H^p(\partial\Omega)$ (see [16, Thm 3.3] for the case of the unit disk).

Proposition 1 *The linear map*

$$\begin{aligned} H^p(\Omega) &\longrightarrow H^p(\partial\Omega) \\ f &\longmapsto \text{tr}_{\partial\Omega} f, \end{aligned}$$

is an isometric isomorphism. As a consequence, a function $f \in H^p(\Omega)$ can be identified to its trace $\text{tr}_{\partial\Omega}(f)$.

When $f = u + i\tilde{u} \in H^p(\Omega)$, $\text{Re}(\text{tr}_{\partial\Omega} f) = u \in L^p_{\mathbb{R}}(\partial\Omega)$. Thus, the forward problem is resolved since for any $f \in H^p(\Omega)$, the restriction to I of $\text{Re}(\text{tr}_{\partial\Omega} f)$ is in $L^p_{\mathbb{R}}(I)$. Moreover, one has $\text{Re } H^p(\partial\Omega) = L^p_{\mathbb{R}}(\partial\Omega)$.

Remark 3 *The forward problem (3) with the Dirichlet boundary condition is formulated in the complex plane in terms of the operator T introduced in Section 2: for $I \subset \partial\Omega$, the operator $T : V = H^p(\partial\Omega) \longrightarrow L^p_{\mathbb{R}}(I)$ with $T(f) = \text{Re}(f)|_I$ is well defined, bounded, and surjective.*

3.1.2 Inverse problems

We saw above that any harmonic function u real-valued and satisfying (5) (for $\Omega = \mathbb{D}$) has a trace in $L^p_{\mathbb{R}}(\Omega)$. Conversely, given a function $b_D \in L^p_{\mathbb{R}}(\partial\Omega)$, it is possible to construct a function u harmonic on Ω such that u has a trace on $L^p_{\mathbb{R}}(\partial\Omega)$ which coincides with b_D on $\partial\Omega$; for $\Omega = \mathbb{D}$, u is given by

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} b_D(e^{it}) dt,$$

see [16, Chap. 1] or [20, Chap. 1.3]. This result gives a solution to the problem (3) for the (D) condition on $\partial\Omega$, and similarly with the (N) condition there.

Whenever $I \subsetneq \mathbb{T} = \partial\mathbb{D}$ with $\mathcal{H}^1(I) > 0$ and $\mathcal{H}^1(\mathbb{T} \setminus I) > 0$, if $b = b_D + i \int b_N$ on I ($b \in L^p(I)$), then there exists a sequence of $H^p(\mathbb{D})$ functions $(g_\lambda)_\lambda$ defined by

$$g_\lambda(z) = \lambda h_\lambda(z) \frac{1}{2i\pi} \int_I \frac{\overline{h_\lambda(w)} b(w)}{w - z} d\mathcal{H}^1(w)$$

where

$$h_\lambda(z) = \exp\left(\frac{-1}{4\pi} \ln(1 + \lambda) \frac{e^{i\theta} + z}{e^{i\theta} - z}\right)$$

such that the restriction to I of $(\text{tr}_{\mathbb{T}} g_\lambda)$ converges in $L^p(I)$ to b , showing that $H^p(\mathbb{D})|_I$ is dense in $L^p(I)$, see [30, Thm. 1]. In [7, Prop. 1], the authors prove the same result for $\Omega = \mathbb{T}$ and $p = 2$ using the property of $H^2(\mathbb{T})$. Precisely, let $g \in L^2(I)$ such that for all $f \in H^2(\mathbb{T})$,

$$\begin{aligned} 0 &= \langle g, f \rangle_{L^2(I)} \\ &= \langle g \vee 0, f \rangle_{L^2(\mathbb{T})} \\ &= \langle P_+(g \vee 0), f \rangle_{L^2(\mathbb{T})} \end{aligned}$$

where $g \vee 0$ is the extension of g to J by 0 and P_+ denotes the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. By taking $f = e^{in\theta} \in H^2(\mathbb{T})$, $n \geq 0$, one has $P_+(g \vee 0) = 0$ so $g \vee 0 \in (H^2(\mathbb{T}))^\perp$ which is impossible since a function in $(H^2(\mathbb{T}))^\perp$ cannot vanish on a subset of \mathbb{T} (it is a consequence of [16, Thm 2.2]). Thus $g = 0$ and $H^2(\mathbb{T})|_I$ is dense in $L^2(I)$.

For a simply connected Dini-smooth domain Ω and $I = \partial\Omega$, for a given function $b_D \in L^p_{\mathbb{R}}(\partial\Omega)$, there is $f \in H^p(\mathbb{D})$ such that $\text{Re}(\text{tr}_{\mathbb{T}} f) = b_D \circ \psi$ and $g = f \circ \psi^{-1} \in H^p(\Omega)$ is such that $\text{Re}(\text{tr}_{\partial\Omega} g) = b_D$ on Ω . Now if $I \subsetneq \partial\Omega$ with $\mathcal{H}^1(I) > 0$ and $\mathcal{H}^1(J) > 0$, the functions $\tilde{g}_\lambda = g_\lambda \circ \psi^{-1}$ are in $H^p(\Omega)$ and $\|\text{Re}(\text{tr}_{\partial\Omega} g_\lambda \circ \psi^{-1})|_I - b_D\|_{L^p(I)} \simeq \|\text{Re}(\text{tr}_{\partial\Omega} g_\lambda)|_{\psi^{-1}(I)} - b_D \circ \psi\|_{L^p(\psi^{-1}(I))}$ converges to 0 as λ goes to $+\infty$. Thus, we get the following result:

Proposition 2 *The space $H^p(\partial\Omega)|_I$ is dense in $L^p(I)$. However, $\text{Re}(H^p(\partial\Omega))|_I = L^p_{\mathbb{R}}(I)$. Moreover,*

- (i) *if $I \subset \partial\Omega$, $T : H^p(\partial\Omega) \rightarrow L^p_{\mathbb{R}}(I)$, $g \mapsto \text{Re } g|_I$ is surjective;*
- (ii) *If $I \subsetneq \partial\Omega$ with $\mathcal{H}^1(I) > 0$ and $\mathcal{H}^1(J) > 0$, the range of $T : H^p(\partial\Omega) \rightarrow L^p(I)$, $g \mapsto g|_I$ is dense in $L^p(I)$ and the inverse problem is ill-posed.*

Remark 4 • *For $I = \partial\Omega$, for $b_D \in L^p_{\mathbb{R}}(\partial\Omega)$ a function $f \in H^p(\partial\Omega)$ such that $T(f) = \text{Re } f = b_D$ is unique up to the condition $\int_{\partial\Omega} \text{Im } f d\mathcal{H}^1 = 0$.*

• *If $I \subsetneq \partial\Omega$ with $\mathcal{H}^1(I) > 0$ and $\mathcal{H}^1(J) > 0$, if $b = b_D + i \int b_N \in L^p(I)$, $b \notin \text{Ran } T$, $T(f) = f|_I$, a sequence of functions $(f_k)_{k \geq 1}$ in $H^p(\partial\Omega)$ such that $\|f_k - b\|_{L^p(I)} \rightarrow 0$ will be such that $\|f_k\|_{L^p(J)} \rightarrow +\infty$ leading to instability in the reconstruction.*

It is possible to solve the inverse problem using the following bounded extremal problem: let $b_D \in L^p_{\mathbb{R}}(I) = W$, let $M > 0$, the (BEP) is

$$\min_{g \in B_M} \|(\text{Re } g)|_I - b_D\|_{L^p_{\mathbb{R}}(I)},$$

where:

$$B_M = \{g \in H^p(\partial\Omega), \|(\operatorname{Re} g)|_J\|_{L^p(J)} \leq M\}.$$

Proposition 3 *Let $b_D \in L^p_{\mathbb{R}}(I)$, $M > 0$. There is a unique $g_0 \in B_M$ which achieves the above minimum among such functions. Moreover, if $b_D \notin \operatorname{Re}(B_M)|_I$, then g_0 saturates the constraint in the sense that $\|(Re g_0)|_J\|_{L^p(J)} = M$.*

If $p = 2$, g_0 is solution of the implicit equation

$$P_+(\chi_I g_0) + \gamma P_+(\chi_J \operatorname{Re} g_0) = P_+(b_D \vee 0), \text{ for some } \gamma > 0, \quad (8)$$

where $b_D \vee 0$ coincides with b on I and vanishes on $J = \partial\Omega \setminus I$ and P_+ denotes the projection from $L^2(\partial\Omega)$ onto $H^2(\partial\Omega)$ defined by

$$P_+(f) = \frac{1}{2}(I + i\mathfrak{N})f + \frac{1}{2}L(f), \quad L(f) = \int_{\partial\Omega} f(t) d\mathcal{H}^1(t).$$

Remark 5 *For $\Gamma \subset \partial\Omega$, the operator \mathcal{T}_Γ acting on $H^2(\partial\Omega)$ and defined by $\mathcal{T}_\Gamma(f) = P_+(\chi_\Gamma f)$ is the Toeplitz operator with symbol $\phi = \chi_\Gamma$.*

Actually, the above proposition is a corollary of the following one, from [7]. Let $b = b_D + i \int b_N \in L^p(I) = W$, let $M > 0$. Another version of the (BEP) is

$$\min_{g \in B_M} \|g|_I - b\|_{L^p(I)},$$

where:

$$B_M = \{g \in H^p(\partial\Omega), \|g|_J\|_{L^p(J)} \leq M\}.$$

Proposition 4 *Let $b \in L^p(I)$, $M > 0$. There is a unique $g_0 \in B_M$ which achieves the above minimum among such functions. Moreover, if $b \notin (B_M)|_I$, then g_0 saturates the constraint in the sense that $\|(g_0)|_J\|_{L^p(J)} = M$.*

If $p = 2$, g_0 is solution of the implicit equation

$$P_+(\chi_I g_0) + \gamma P_+(\chi_J g_0) = P_+(b \vee 0), \text{ for some } \gamma > 0,$$

where $b \vee 0$ coincides with b on I and vanishes on J .

The existence of a solution for the bounded extremal problem follows from the use of a standard weak compactness argument, see [14, Lem 2.1]. The saturation of the constraint by g_0 is a consequence of the density of the range of T . The uniqueness of g_0 relies on the strict convexity of the norm. In the case $p = 2$, the implicit equation comes from the minimization of the form $\psi(g) = \|g|_I - b\|_{L^2(I)}^2$ under the constraint $\phi(g) = \|(g_0)|_J\|_{L^2(J)}^2 = M$ using a Lagrange multiplier, see [19, Prop.2] or [14, Thm.2.1].

Proposition 3 is generalized to the elliptic operator $\operatorname{div}(\sigma \nabla \cdot)$ in Section 3.2.2.

3.2 Conductivity PDE, generalized Hardy spaces

In this section, we consider the case when $\sigma \in W_{\mathbb{R}}^{1,r}(\Omega)$, $r > 2$ with $\sigma \neq 1$ for the initial problem (3). To solve the forward and the inverse problems, we transpose them in the complex plane. When $\sigma = 1$, the key ingredient to rewrite the Laplace equation as a complex partial differential equation is the existence of a harmonic conjugate. Precisely, for $\Omega = \mathbb{D}$ and $\sigma = 1$ and u solution to $\Delta u = 0$ there is a real-valued function v defined on \mathbb{D} such that $u + iv$ is analytic in \mathbb{D} (i.e. $\bar{\partial}f = 0$). This result can be extended to a simply connected Dini-smooth domain by use of a conformal map. The existence of a harmonic conjugate remains true in an annular domain and thus to a finitely-connected Dini-smooth domain by use of a conformal map.

When σ is no longer constant and Ω is a simply-connected domain, if $u \in L_{\mathbb{R}}^{\gamma}(\Omega)$, $\gamma > \frac{r}{r-1}$ satisfies $\operatorname{div}(\sigma \nabla u) = 0$ in Ω , the differential form $a = (\sigma \partial_y u)dx + (\sigma \partial_x u)dy$ is closed in a simply connected domain which implies that a is exact. Thus, there exists $v \in L_{\mathbb{R}}^{\gamma}$ such that $a = (\partial_x v)dx + (\partial_y v)dy$. Then, u and v satisfy the generalized Cauchy-Riemann equations in Ω

$$\begin{cases} \partial_x v &= -\sigma \partial_y u \\ \partial_y v &= \sigma \partial_x u \end{cases}$$

and v is solution to $\operatorname{div}\left(\frac{1}{\sigma} \nabla v\right) = 0$. For such u and v satisfying the generalized Cauchy-Riemann equations, the complex-valued function $f = u + iv$ satisfies in the distributional sense

$$\begin{aligned} \bar{\partial}f &= \partial_x u + i\partial_x v + i\partial_y u - \partial_y v \\ &= (1 - \sigma)(\partial_x u + i\partial_y u) \end{aligned}$$

while

$$\begin{aligned} \bar{\partial}f &= \overline{\partial_x u - i\partial_x v + i\partial_y u + \partial_y v} \\ &= (1 + \sigma)(\partial_x u + i\partial_y u), \end{aligned}$$

which implies that $\bar{\partial}f = \nu \bar{\partial}f$ with $\nu = \frac{1+\sigma}{1-\sigma}$. Now, if u is solution to $\operatorname{div}(\sigma \nabla u) = 0$ and

$$\operatorname{ess\,sup}_{0 < r < 1} \|u\|_{L^p(\Gamma_r)} < \infty, \text{ where } \Gamma_r = \phi(r\mathbb{T}), \quad (9)$$

then u has a trace on $\partial\Omega$ and $\operatorname{tr}_{\partial\Omega} u \in L_{\mathbb{R}}^p(\partial\Omega)$ and its harmonic conjugate v has also a trace in $L_{\mathbb{R}}^p(\partial\Omega)$ making the linear operator

$$\begin{aligned} \mathfrak{H}_{\nu} : L_{\mathbb{R}}^p(\partial\Omega) &\longrightarrow L_{\mathbb{R}}^p(\partial\Omega) \\ u &\longmapsto v, \text{ with the normalization } \int_{\partial\Omega} v \, d\mathcal{H}^1 = 0. \end{aligned}$$

bounded on $L^p_{\mathbb{R}}(\partial\Omega)$, see [8, Cor.4.4.2.1(2)] and [18, Cor.1]. The operator \mathfrak{J}_ν is called the generalized conjugation map. For a Dini-smooth finitely-connected domain Ω , a function u satisfying a condition like (9) ($\phi(r\mathbb{T})$ being replaced by $\partial\Delta_n$ which is a finite union of rectifiable Jordan curves of uniformly bounded length) has a harmonic conjugate if and only if $\int_\Gamma \sigma \partial_n u = 0$ for any smooth-curve $\Gamma \subset \Omega$, see [5, Lem.5.2].

The authors in [5, 18] proved the following result for a Dini-smooth multi-connected domain Ω , $\sigma \in W^{1,r}_{\mathbb{R}}(\Omega)$, $r > 2$. Let $\gamma > \frac{r}{r-1}$.

Proposition 5 *Let $\nu = \frac{1-\sigma}{1+\sigma} \in W^{1,r}_{\mathbb{R}}(\Omega)$ with $\|\nu\|_{L^\infty(\Omega)} \leq \kappa < 1$. Then, the following assertions are equivalent*

(i) $f = u + iv \in L^\gamma(\Omega)$ satisfies the Beltrami conjugate equation

$$\bar{\partial}f = \nu \bar{\partial}f \tag{10}$$

(ii) $u, v \in L^\gamma_{\mathbb{R}}(\Omega)$ satisfy respectively $\operatorname{div}(\sigma \nabla u) = 0$ and $\operatorname{div}(\frac{1}{\sigma} \nabla v) = 0$;

(iii) $u, v \in L^\gamma_{\mathbb{R}}(\Omega)$ are solutions of the generalized Cauchy-Riemann equations

$$\begin{cases} \partial_x v &= -\sigma \partial_y u \\ \partial_y v &= \sigma \partial_x u \end{cases}$$

(iv) $w = \mathcal{J}_{\nu,\alpha} f = \frac{f - \nu \bar{f}}{\sqrt{1-\nu^2}} = \sigma^{1/2} u + i\sigma^{-1/2} v$ is solution of $\bar{\partial}w = \alpha \bar{w}$ for $\alpha = -\frac{\bar{\partial}\nu}{1-\nu^2} = \bar{\partial} \log \sigma^{1/2}$.

Definition 2 *The generalized Hardy space of functions on Ω denoted by $H^p_\nu(\Omega)$ is the set of functions f satisfying (10) such that*

$$\|f\|_{p,\nu}^p = \operatorname{ess\,sup}_{0 < r < 1} \|f\|_{L^p(\Gamma_r)}^p < \infty,$$

where $\Gamma_r = \psi(r\mathbb{T})$. In the same way, $G^p_\alpha(\Omega)$ denotes the space of functions w satisfying $\bar{\partial}w = \alpha \bar{w}$ and $\|w\|_{p,\alpha}^p = \operatorname{ess\,sup}_{0 < r < 1} \|w\|_{L^p(\Gamma_r)}^p < \infty$.

Remark 6 • When $\sigma = 1$ ($\nu = 0$, $\alpha = 0$), the space $H^p_0(\Omega) = G^p_0(\Omega)$ corresponds to the Hardy space $H^p(\Omega)$ and the norms coincide.

• The spaces H^p_ν and $G^p_\alpha(\Omega)$ equipped respectively with the norm $\|\cdot\|_{p,\nu}$ and $\|\cdot\|_{p,\alpha}$ are Banach spaces. The map $\mathcal{J}_{\nu,\alpha}$ in Proposition 5 is an isomorphism between the two Banach spaces.

The spaces $G^p_\alpha(\Omega)$ also called generalized Hardy spaces are related to the spaces $H^p(\Omega)$ through the following factorization theorem called Bers similarity principle, see [2, Lem. 3.1].

Proposition 6 *Let $\alpha \in L^r(\Omega)$ and $w \in L^\gamma_{loc}(\Omega)$, with $\gamma > \frac{r}{r-1}$. Then, w can be factorized as $w = e^s F$, where F is analytic in Ω and $s \in W^{1,r}(\Omega)$ with*

$$\|s\|_{W^{1,r}(\Omega)} \leq C \|\alpha\|_{L^r(\Omega)}, \tag{11}$$

the constant C depending only on r and Ω .

Remark 7 *The Bers similarity principle remains true if Ω is a bounded domain of \mathbb{C} without specific regularity on the boundary.*

A consequence of this result is that $w = e^s F \in G_\alpha^p(\Omega)$ if and only if $F \in H^p(\Omega)$. The factorization combined with Equation (11) permits to establish properties of $G_\alpha^p(\Omega)$ functions from the spaces $H^p(\Omega)$. Using the isomorphism $\mathcal{J}_{\nu,\alpha}$ between $(H_\nu^p(\Omega), \|\cdot\|_{p,\nu})$ and $(G_\alpha^p(\Omega), \|\cdot\|_{p,\alpha})$, the properties will be extended to $H_\nu^p(\Omega)$ -functions.

The authors in [2] give a converse of the similarity principle in the case of the unit disk for $r = 2$. We give a version of this result for $r > 2$ and a Dini-smooth domain Ω :

Proposition 7 *Let $\alpha \in L^r(\Omega)$, $r > 2$ and F be an analytic function on Ω . Fix $\psi \in W^{1-1/r,r}(\partial\Omega)$ and $\lambda \in \mathbb{R}$. There is a unique $s \in W^{1,r}(\Omega)$ such that $\text{Re}(tr_{\partial\Omega}s) = \psi$ and $\int_{\partial\Omega} \text{Im}(s)d\mathcal{H}^1 = \lambda$ and s satisfies (11).*

Let $F \in H^p(\Omega)$ and $\alpha \in L^r(\Omega)$, $r > 2$. There is $s \in W^{1,r}(\Omega)$ such that $w = e^s F$. The map $s \mapsto e^s$ is continuous from $W^{1,r}(\Omega)$ to $W^{1,r}(\Omega)$, see [2, Section 2].

3.2.1 Forward problem

The following Proposition gives a condition on the existence of a boundary value on $\partial\Omega$ for a function satisfying (10).

Proposition 8

- *A function $f \in H_\nu^p(\Omega)$ (resp. $w \in G_\alpha^p(\Omega)$) has a trace on $\partial\Omega$, $tr_{\partial\Omega}f \in L^p(\partial\Omega)$ (resp. $tr_{\partial\Omega}w \in L^p(\partial\Omega)$); moreover, there exist $C_{i,\nu}$, $C_{i,\alpha} > 0$, $i = 1, 2$, such that:*

$$C_{1,\nu}\|tr_{\partial\Omega}f\|_{L^p(\partial\Omega)} \leq \|f\|_{H_\nu^p(\Omega)} \leq C_{2,\nu}\|tr_{\partial\Omega}f\|_{L^p(\partial\Omega)},$$

and

$$C_{1,\alpha}\|tr_{\partial\Omega}w\|_{L^p(\partial\Omega)} \leq \|w\|_{G_\alpha^p(\Omega)} \leq C_{2,\alpha}\|tr_{\partial\Omega}w\|_{L^p(\partial\Omega)}.$$

- *The space $tr_{\partial\Omega}(H_\nu^p(\Omega))$ and $tr_{\partial\Omega}(G_\alpha^p(\Omega))$ are closed subspaces of $L^p(\partial\Omega)$ denoted by $H_\nu^p(\partial\Omega)$ and $G_\alpha^p(\partial\Omega)$.*
- *The spaces $(H_\nu^p(\Omega), \|\cdot\|_{p,\nu})$ and $(H_\nu^p(\partial\Omega), \|\cdot\|_p)$ are isomorphic; in the same way, the spaces $(G_\alpha^p(\Omega), \|\cdot\|_{p,\alpha})$ and $(G_\alpha^p(\partial\Omega), \|\cdot\|_p)$ are isomorphic.*

The forward problem in the complex plane that consists in the existence of a trace for a function in $H_\nu^p(\Omega)$ is solved: the operator $T : H_\nu^p(\Omega) \rightarrow L_{\mathbb{R}}^p(\partial\Omega)$ given by $T(f) = \text{Re}(tr_{\partial\Omega}f)$ is well defined, linear and bounded. It implies that the forward problem for $\text{div}(\sigma\nabla u) = 0$ on Ω with $\text{tr}_{\partial\Omega}u = b_D \in L_{\mathbb{R}}^p(\partial\Omega)$ and the operator $\tilde{T} : \text{Re}(H_\nu^p(\Omega)) \rightarrow L_{\mathbb{R}}^p(\partial\Omega)$ given by $\tilde{T}(u) = \text{tr}_{\partial\Omega}u$ is well defined, linear and bounded.

For the Neumann boundary condition (N), if $u \in W_{\mathbb{R}}^{1,p}(\mathbb{D})$ is such that $\text{div}(\sigma\nabla u) = 0$ with ∇u satisfying the condition (5) has a trace $\text{tr}_{\mathbb{T}}u \in W^{1,p}(\mathbb{T})$ and $\partial_n u \in L_{\mathbb{R}}^p(\mathbb{T})$ with $\int_{\mathbb{T}} \sigma \partial_n u = 0$, see [8, Cor.4.4.3.1] which is generalization of [17].

3.2.2 Inverse problems

Assume that $I = \Omega$. Given a function $b \in L^p_{\mathbb{R}}(\partial\Omega)$, the inverse problem asks the existence of a function $f \in H^p_{\nu}(\Omega)$ such that $\operatorname{Re}(\operatorname{tr}_{\partial\Omega} f) = b$ on $\partial\Omega$. This question is solved in [8] in the case of the unit disk for $r = \infty$, [5] in the case of a multi-connected Dini-smooth domain for $r > 2$ and [2] for $r \geq 2$ in the case of the unit disk. We collect the results in the following Theorem:

Theorem 1 *Let $\alpha \in W^{1,r}(\Omega)$ and $b \in L^p_{\mathbb{R}}(\partial\Omega)$. There exists $f \in H^p_{\nu}(\Omega)$ such that $\operatorname{Re}(\operatorname{tr}_{\partial\Omega} f) = b$ on $\partial\Omega$ and $\int_{\partial\Omega} \operatorname{Im}(\operatorname{tr}_{\partial\Omega} f) d\mathcal{H}^1 = 0$. Moreover, there is $C_{p,\alpha,\nu} > 0$ such that:*

$$\|\operatorname{tr}_{\partial\Omega} f\|_p \leq C_{p,\alpha,\nu} \|b\|_p \quad (12)$$

Remark 8 *Under the same hypothesis, there is a unique function $u \in \operatorname{Re}(H^p_{\nu}(\Omega))$ defined on Ω such that $\operatorname{tr}_{\partial\Omega} u = b$.*

Thus T is surjective and bijective when restricted to:

$$H^p_{\nu}(\Omega) / \left\{ f \in H^p_{\nu}(\Omega) : \int_{\partial\Omega} \operatorname{tr}_{\partial\Omega} f d\mathcal{H}^1 = 0 \right\}.$$

Given a function $g \in L^p_{\mathbb{R}}(\mathbb{T})$ such that $\int_{\mathbb{T}} \sigma \partial_n(u) = 0$, there exists $u \in W^{1,p}_{\mathbb{R}}(\mathbb{D})$ unique up to an additive constant satisfying $\operatorname{div}(\sigma \nabla u) = 0$ in \mathbb{D} with ∇u satisfying (5) with $\partial_n(u) = g$ on \mathbb{T} .

Assume now that $I \subsetneq \partial\Omega$ with $\mathcal{H}^1(I) > 0$ and $\mathcal{H}^1(J) > 0$, $J = \partial\Omega \setminus I$.

Proposition 9 *The space $(H^p_{\nu}(\partial\Omega))|_I$ is dense in $L^p(I)$. Moreover, the range of T is dense in $L^p_{\mathbb{R}}(\partial\Omega)$ and when $b_D \notin \operatorname{Ran} T$ the inverse problem is ill-posed.*

In particular, $(\operatorname{Re}(H^p_{\nu}(\partial\Omega)))|_I$ is dense in $L^p_{\mathbb{R}}(I)$. Because of the ill-posedness of the inverse problem, one can use a bounded extremal problem (BEP) formulated in the more general setting of generalized Hardy spaces:

$$\min_{g \in B_M} \|\operatorname{Re}(g)|_I - b\|_{L^p_{\mathbb{R}}(I)}, \quad (13)$$

where:

$$B_M = \{g \in H^p_{\nu}(\partial\Omega), \|\operatorname{Re}(g)|_J\|_{L^p_{\mathbb{R}}(J)} \leq M\}.$$

Proposition 10 *Let $b \in L^p_{\mathbb{R}}(I)$, $M > 0$. There exists a unique $g_0 \in B_M$ such that g_0 solves (13). Moreover, if $b \notin (B_M)|_I$, then g_0 saturates the constraint: $\|\operatorname{Re}(g_0)\|_{L^p_{\mathbb{R}}(I)} = M$.*

If $p = 2$ and $\Omega = \mathbb{D}$, g_0 satisfies the implicit equation

$$\mathcal{P}_{\nu}(\chi_I g_0) + \gamma \mathcal{P}_{\nu}(\chi_J \operatorname{Re}(g_0)) = \mathcal{P}_{\nu}(b \vee 0),$$

where $\mathcal{P}_{\nu} : L^2(\mathbb{T}) \rightarrow H^p_{\nu}(\mathbb{T})$ defined by

$$\mathcal{P}_{\nu}(f) = \frac{1}{2}(I + i\mathfrak{H}_{\nu})f + \frac{1}{2}L(f), \quad L(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt.$$

Remark 9 *The operator \mathcal{P}_{ν} generalizes P_+ (see Remark 5) and thus extends the definition of a Toeplitz operator with symbol $\phi = \chi_I$ on generalized Hardy spaces: $\mathcal{T}_{\Gamma} : L^p(\partial\Omega) \rightarrow H^p_{\nu}(\partial\Omega)$, with $\mathcal{T}_{\Gamma}(f) = P_{\nu}(\chi_I f)$, $f \in L^p(\partial\Omega)$.*

4 Conclusion

The results for the forward and the inverse problems can be extended to finitely-connected bounded Dini-smooth domains [5], with some variations of the (BEP) in terms of the constraint: norm constraint on the imaginary part of the function g , see [28].

We focused to inverse problems with partial boundary data: this situation occurs when physical quantities are measured on the boundary of a bounded domain and the source of such measurements is located inside the domain, like the EEG source estimation problem described in Section 1. In some situations, the measures are taken inside the domain and the source is defined on the boundary. For example, in geosciences and planetary sciences, the inverse magnetization problem is to estimate the magnetization μ from measurements of the normal component of the magnetic field measured within a domain Ω , far from the support of μ , see [29] in \mathbb{R}^2 , [3, 6] in \mathbb{R}^3 : find μ supported on $S \subset \partial\Omega$ (model for a magnetized thin rock sample),

$$\begin{cases} \Delta u = \operatorname{div} \mu, \\ (\partial_n u)|_K \text{ given on } K \subset \Omega. \end{cases}$$

Such problems has been studied in [29] were $\Omega = \Pi_+$ the upper-half plane, that corresponds to the two-dimensional analogous of the physical three-dimensional situation (see [3, 4, 6]), and to the Hilbertian setting where $p = 2$. This problem and related issues of course make sense in more general domains Ω and for $p \neq 2$. It is our belief that the corresponding properties and results remain valid for $1 < p < \infty$ and in Dini-smooth domains Ω .

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