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► **To cite this version:**

José Galaz, Maria Kazolea, Antoine Rousseau. Coupling Dispersive Shallow Water Models by Deriving Asymptotic Interface Operators. Domain Decomposition Methods in Science and Engineering XXVII, Czech Technical University, Prague; VSB – Technical University of Ostrava, Institute of Mathematics of the CAS, Prague, Jul 2022, Prague (CZ), Czech Republic. hal-03851031v1

**HAL Id: hal-03851031**

**<https://inria.hal.science/hal-03851031v1>**

Submitted on 14 Nov 2022 (v1), last revised 21 Feb 2024 (v3)

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# Coupling Dispersive Shallow Water Models by Deriving Asymptotic Interface Operators

José Galaz, Maria Kazolea, Antoine Rousseau

## 1 Introduction

We are interested in coupling the linear Green-Naghdi equations (LGNE)

$$\partial_t \zeta + \partial_x V = 0 \quad (1)$$

$$\partial_t V + \partial_x \zeta = \phi \quad (2)$$

$$-\frac{\mu}{3} \partial_x^2 \phi + \phi = -\frac{\mu}{3} \partial_x^3 \zeta \quad (3)$$

and the linear shallow water equations (LSWE)

$$\partial_t \zeta + \partial_x V = 0 \quad (4)$$

$$\partial_t V + \partial_x \zeta = 0 \quad (5)$$

to represent the 1D propagation of water waves in shallow water. Here  $V(t, x)$  and  $\zeta(t, x)$  stand for the vertically-averaged velocity and the free-surface level over its state at rest;  $\phi(t, x)$  is an auxiliary variable for the elliptic part of the problem; and  $\mu > 0$  is the asymptotic parameter characterizing the wave dispersion.

In the nonlinear case, Boussinesq-type equations, such as the Green-Naghdi equations (GNE), have been coupled with the nonlinear shallow water equations

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(NSWE) to take advantage of their physical-modeling features: the dispersive terms in the GNE can be used to accurately represent the phase and amplitude of waves in the shoaling zone, while shock-capturing well-balanced finite volume schemes for the NSWE can mimic the energy dissipation of wave-breaking and provide a robust handling of vanishing water depths without ad-hoc parametrizations. However, this coupled model has been shown to be unstable unless dissipative terms are added [1].

Since this coupling is a "divide and conquer" type of problem, domain decomposition methods (DDM) can help to obtain further insights. Usually, coupling conditions have been derived for each equation on a case-by-case basis (e.g., [2, 3, 4, 5]). Here we explore a different approach, based on the steps of the derivation of the GNE and NSWE [6]. First, a DDM of the linearized Euler equations in the discrete level is defined, based on the Neumann-Dirichlet method. Then, recalling that both GNE and NSWE derive from the Euler equations, we derive transmission conditions for the children asymptotic equations by taking the vertical average of the original operators and truncating the resulting expression according to an asymptotic expansion of the velocity potential. We examine this approach in the homogeneous case first, when coupling LGNE with LGNE, and then in the heterogeneous case, coupling the LGNE with the LSWE.

## 2 A domain decomposition of the free-surface Euler equations

The linear Euler equations for an incompressible fluid and irrotational flow in one horizontal dimension can be formulated as an elliptic problem for the velocity potential  $\Phi(t, x, z)$  and two evolution equations for the free-surface function  $\zeta(t, x)$  and the trace of the potential at the surface  $\psi(t, x)$  respectively (see [6] for details). A finite-difference discretization of the equations for  $\zeta_i^n = \zeta(t^n, x_i)$  and  $\psi_i^n = \psi(t^n, x_i)$ , in a uniform grid  $x_i = i\Delta x$  and  $t^n = n\Delta t$ , with  $i \in \mathcal{N} = \{1 \dots N_x - 2\}$  and  $n = 1, 2, \dots$ , is given by

$$\frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \frac{V_i^n - V_{i-1}^n}{\Delta x} = 0 \text{ for } i \in \mathcal{N} \quad (6)$$

$$\frac{\psi_i^{n+1} - \psi_i^n}{\Delta t} + \zeta_i^n = 0 \text{ for } i \in \mathcal{N} \quad (7)$$

where  $V_i^n$  is the discrete vertically-averaged velocity given by

$$V_i^n = \sum_{j=1}^{N_z-1} \frac{\Phi_{i+1,j}^n - \Phi_{i,j}^n}{\Delta x} \Delta z \quad (8)$$

Boundary conditions at nodes  $i = 0$  and  $i = N_x - 1$  can be  $V_i^n = 0$  and  $\zeta_0^n = \zeta_1^n$ ,  $\zeta_{N_x-1}^n = \zeta_{N_x-2}^n$  [7]. In equation (8),  $\Phi_{i,j}^n = \Phi(t^n, x_i, z_j)$  is the discrete velocity potential computed from

$$\mu \frac{\Phi_{i+1,j}^n + \Phi_{i-1,j}^n - 2\Phi_{i,j}^n}{\Delta x^2} + \frac{\Phi_{i,j+1}^n + \Phi_{i,j-1}^n - 2\Phi_{i,j}^n}{\Delta z^2} = 0$$

for  $i, j \in \mathcal{N} \times \{1, \dots, N_z - 1\}$  (9)

on a grid  $(x_i, z_j)$  such that  $z_j = j\Delta z - 1$  and  $z_{N_z-1} = 0$  and its boundary conditions are

$$\begin{aligned} \Phi_{i,N_z-1}^n &= \psi_i^n & \Phi_{i,0} &= \Phi_{i,1} \\ \Phi_{0,j} &= \Phi_{1,j} & \Phi_{N_x-1,j} &= \Phi_{N_x-2,j} \end{aligned} \quad (10)$$

We decompose the domain in two components with a vertical interface located at  $i = l \in \mathcal{N}$ . To do this let  $\zeta_{i,s}^{n+1}$ ,  $\psi_{i,s}^{n+1}$ ,  $\Phi_{i,j,s}^{n+1}$ ,  $V_{i,s}^n$  be the values of the unknowns on each subdomain  $s = 1, 2$ . These variables are computed from equations (6), (7), (8) but with  $i$  in  $\mathcal{N}_s$  instead of  $\mathcal{N}$ , with  $\mathcal{N}_1 = \{1, \dots, l-1\}$  and  $\mathcal{N}_2 = \{l, \dots, N_x - 2\}$ ; equation (9) is solved with  $i \in \mathcal{N}_1$  for  $s = 1$  and  $i \in \mathcal{N}_2 \setminus \{l\}$  for  $s = 2$ . To obtain the same solution as the monodomain problem, equations (6) and (7) are complemented with Dirichlet transmission conditions

$$\begin{aligned} \zeta_{l,1}^{n+1} &= \zeta_{l,2}^{n+1} & V_{l,1}^{n+1} &= V_{l,2}^{n+1} \\ \zeta_{l-1,2}^{n+1} &= \zeta_{l-1,1}^{n+1} & V_{l-1,2}^{n+1} &= V_{l-1,1}^{n+1} \end{aligned} \quad (11)$$

while equation (9) uses Neumann and Dirichlet transmission conditions

$$\frac{\Phi_{l,j,1}^n - \Phi_{l-1,j,1}^n}{\Delta x} - \frac{\Delta x}{2\mu\Delta z^2} \left( \Phi_{l,j+1,1}^n + \Phi_{l,j-1,1}^n - 2\Phi_{l,j,1}^n \right) \quad (12)$$

$$\begin{aligned} &= \frac{\Phi_{l+1,j,2}^n - \Phi_{l,j,2}^n}{\Delta x} + \frac{\Delta x}{2\mu\Delta z^2} \left( \Phi_{l,j+1,2}^n + \Phi_{l,j-1,2}^n - 2\Phi_{l,j,2}^n \right) \\ &\qquad\qquad\qquad \Phi_{l,j,2}^n = \Phi_{l,j,2}^1 \end{aligned} \quad (13)$$

which include an  $O(\Delta x)$  term necessary in finite-difference schemes to preserve the monodomain solution [8]. This scheme satisfies

- $\Phi_{i,j,*}^n = \Phi_{i,j}^n$ , with  $\Phi_{i,j,*}^n = \Phi_{i,j,1}^n$  if  $i \leq l$  and  $\Phi_{i,j,*}^n = \Phi_{i,j,2}^n$  if  $i > l$ . This means that the solution  $\Phi_{i,j,*}^n$  formed by both subdomains will be equal to the monodomain solution  $\Phi_{i,j}^n$
- A parallel or alternating method to solve (9) with (13) will be convergent if  $L_1 < L_2$ , with  $L_1 = l\Delta x$  and  $L_2 = (N_x - l)\Delta x$  (see [9] for details)

### 3 Asymptotic domain-decomposition method

We first need an asymptotic expansion  $\Phi_{i,j} = \Phi_{i,j}^0 + \mu\Phi_{i,j}^1 + \dots + \mu^k\Phi_{i,j}^k$  of the solution to the Laplace equation (9). We do this for the discrete Laplace equation instead of the continuous one to be consistent with the DDM that is discrete too. At first order,  $\Phi_{i,j} = \Phi_{i,j}^0 + O(\mu)$ , which substituted on equation (9) and after discarding  $O(\mu)$  terms leads to the recurrence equation  $\Phi_{i,j+1}^0 + \Phi_{i,j-1}^0 - 2\Phi_{i,j}^0 = 0$  whose solution is  $\Phi_{i,j}^0 = \psi_i$  so the first order expansion is

$$\Phi_{i,j} = \psi_i + O(\mu) \quad (14)$$

Similarly, at second order, after replacing  $\Phi_{i,j} = \Phi_{i,j}^0 + \mu\Phi_{i,j}^1 + O(\mu^2)$  back into equation (9), the discrete recurrence equation reads

$$\frac{\Phi_{i,j+1}^1 + \Phi_{i,j-1}^1 - 2\Phi_{i,j}^1}{\Delta x^2} = -\frac{\psi_{i+1} + \psi_{i-1} - 2\psi_i}{\Delta x^2} \quad (15)$$

whose solution gives us the second order expansion

$$\Phi_{i,j} = \psi_i - \mu \frac{\Delta z^2}{2\Delta x^2} (\psi_{i+1} + \psi_{i-1} - 2\psi_i)(j^2 - j - (N_z - 1)(N_z - 2)) + O(\mu^2) \quad (16)$$

Substituting (14) into (8) and using that  $(N_z - 1)\Delta z = 1$  one obtains that at first order

$$\frac{\psi_{i+1} - \psi_i}{\Delta x} = V_i + O(\mu) \quad (17)$$

From (16) we can proceed similarly to obtain the second order expansion for  $V_i$

$$V_i = \frac{\psi_{i+1} - \psi_i}{\Delta x} - \nu T \left( \frac{\psi_{i+1} - \psi_i}{\Delta x} \right) + O(\mu^2) \quad (18)$$

where  $TV_i = -(V_{i+1} + V_{i-1} - 2V_i)/(3\Delta x^2)$  and  $\nu = \mu(1 - \Delta z/2)(1 - \Delta z)$ . We can now substitute the first order approximation (17) into (18) and isolate  $(\psi_{i+1} - \psi_i)/\Delta x$  to obtain

$$\frac{\psi_{i+1} - \psi_i}{\Delta x} = V_i + \nu TV_i + O(\mu^2) \quad (19)$$

To substitute this into (7) let us apply a forward finite-difference in  $x$  ( $i$  index) to equation (7). Introducing the notation  $D_x^+ f_i = (f_{i+1} - f_i)/\Delta x$ ,  $D_z^+ f_j = (f_{j+1} - f_j)/\Delta z$  and  $D_z^- f_j = (f_j - f_{j-1})/\Delta z$ , so  $V_i = \sum_{j=1}^{N_z-1} D_x^+ \Phi_{i,j} \Delta z$ . Using the asymptotic expansion (19), discarding terms of size  $O(\mu^2)$  and rearranging one finally obtains the discrete momentum equation of the LGNE

$$\begin{aligned} \frac{V_i^{n+1} - V_i^n}{\Delta t} + \frac{\zeta_{i+1}^n - \zeta_i^n}{\Delta x} &= \phi_i \\ (1 + \nu T)\phi_i &= \nu T(D_x^+ \zeta_i^n) \end{aligned} \quad (20)$$

with  $\phi_i$  an auxiliary variable for the new elliptic problem. And discarding all  $O(\mu)$  terms the discrete momentum equation of the LSWE reads

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} + \frac{\zeta_{i+1}^n - \zeta_i^n}{\Delta x} = 0 \quad (21)$$

We can proceed in a similar fashion to derive asymptotic versions of the Neumann boundary condition. To do this let us multiply equation (13) by  $\Delta z$  and sum it up from  $j = 1$  to  $j = N_z - 2$ . Using the formula for  $V_i$  and simplifying, one obtains that (13) can be written as

$$\begin{aligned} & V_{l-1,1}^n - D_x^+ \Phi_{l-1, N_z-1,1}^n \Delta z - \frac{\Delta x}{2\mu} D_z^- \Phi_{l, N_z-1,1}^n \\ &= V_{l,2}^n - D_x^+ \Phi_{l, N_z-1,2}^n \Delta z + \frac{\Delta x}{2\mu} D_z^- \Phi_{l, N_z-1,2}^n \end{aligned} \quad (22)$$

To further simplify the remaining  $\Phi$  terms we will use that  $\partial_z \Phi_{z=0}/\mu = \partial_x V + O(\varepsilon)$  [6], and substitute the discrete derivatives to finally obtain  $1/\mu D_z^- \Phi_{i, N_z-1} = -D_x^p V_i + O(\varepsilon, \Delta z/\mu, \Delta x^p)$ , where  $D_x^p V_i = \partial_x V|_{x=x_i} + O(\Delta x^p)$  is a finite-difference operator to be defined. If we replace this back into the equation, use equation (14) written as  $D_x \Phi_{i, N_z-1} = V_i + O(\mu)$ , and replace  $D_x^p$  with the backward finite difference  $D_x^-$  in the left-hand-side of the equation and a forward finite difference  $D_x^+$  in the right-hand-side, we can write the original Neumann boundary condition as

$$\frac{1}{2}(V_{l,1}^n + V_{l-1,1}^n) - V_{l-1,1}^n \Delta z = \frac{1}{2}(3V_{l,2}^n - V_{l+1,2}^n) - V_{l,2}^n \Delta z \quad (23)$$

Substituting the auxiliary variable for the LGNE,  $\phi_{i,s}^n = D_t^+ V_{i,s}^n + D_x^+ \zeta_{i,s}^n$ ,

$$\begin{aligned} & \frac{1}{2}(\phi_{l,1}^n + \phi_{l-1,1}^n) - \phi_{l-1,1}^n \Delta z = \frac{1}{2}(3\phi_{l,2}^n - \phi_{l+1,2}^n) - \phi_{l,2}^n \Delta z \\ & \quad - \left( \frac{3\Delta x^2}{2} T(D_x^+ \zeta_l^n) - (D_x^+ \zeta_l^n - D_x^+ \zeta_{l-1}^n) \Delta z \right) \end{aligned} \quad (24)$$

Summarizing, for the homogeneous case, the domain decomposition of the LGNE reads, for each subdomain  $s = 1, 2$ ,

$$\begin{aligned} & \frac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \frac{V_i^n - V_{i-1}^n}{\Delta x} = 0 \text{ for } i \in \mathcal{N}_s \\ & \frac{V_i^{n+1} - V_i^n}{\Delta t} + \frac{\zeta_{i+1}^n - \zeta_i^n}{\Delta x} = \phi_i \text{ for } x \in \mathcal{N}_s \\ & \phi_{i,s}^n + \nu T \phi_{i,s}^n = \nu T (D_x^+ \zeta_i) \text{ for } i \in \mathcal{N}_s \\ & \quad \phi_{0,1}^n = \phi_{Left} \\ & \quad \phi_{N_x-1,2}^n = \phi_{Right} \end{aligned} \quad (25)$$

and at the interface  $i = l$  equation (11) holds for  $V_{l,s}^n$  and  $\zeta_{l,s}^n$ , while for  $\phi_{l,s}^n$

$$\begin{aligned} \frac{1}{2}(\phi_{l,1}^n + \phi_{l-1,1}^n) - \phi_{l-1,1}^n \Delta z &= \frac{1}{2}(3\phi_{l,2}^n - \phi_{l+1,2}^n) - \phi_{l,2}^n \Delta z \\ &\quad - \left( \frac{3\Delta x^2}{2} T(D_x^+ \zeta_l^n) - (D_x^+ \zeta_l^n - D_x^+ \zeta_{l-1}^n) \Delta z \right) \\ \phi_{l,2}^n &= \phi_{l,1}^n \end{aligned} \quad (26)$$

To test this asymptotic domain decomposition method with an additive iterative scheme we compare the monodomain solution with the DDM solution for  $x \in (0, 1)$ ,  $\phi_{Left} = 0$ ,  $\phi_{Right} = -0.5$ ,  $\partial_x^3 \zeta^n = -1$ ,  $\mu = 3$ ,  $\Delta x = 0.05$ ,  $\Delta z = \Delta x^2$ . These parameters are convenient to avoid overflow in intermediate calculations in the formula of the analytical solution. Figure 1 (left) shows the  $L^2$  distance between the discrete monodomain solution and each subdomain solution at each iteration for an interface located at  $x = 0.4$  where we can see that the DDM diverges.

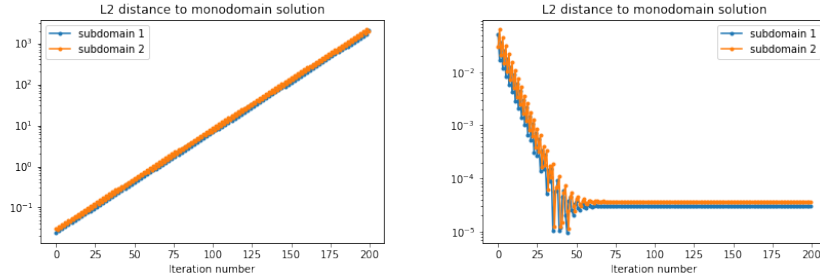


Fig. 1:  $L^2$  distance to monodomain of the solution on each subdomain as a function of the iteration number when using (26) and (27) respectively.

To fix this situation we can subtract  $\phi_{l,1}^n$  from each side of the first equation of (26), use that  $\phi_{l,1}^n = \phi_{l,2}^n$ , and multiply the equation by  $-\frac{2}{\Delta x}$ . The coupling conditions become

$$\begin{aligned} \frac{\phi_{l,1}^n - \phi_{l-1,1}^n}{\Delta x} + 2\frac{\Delta z}{\Delta x} \phi_{l-1,1}^n &= \frac{\phi_{l+1,2}^n - \phi_{l,2}^n}{\Delta x} + 2\frac{\Delta z}{\Delta x} \phi_{l,2}^n \\ &\quad + 3\Delta x T(D_x^+ \zeta_l^n) - 2\frac{(D_x^+ \zeta_l^n - D_x^+ \zeta_{l-1}^n)}{\Delta x} \Delta z \\ \phi_{l,2}^n &= \phi_{l,1}^n \end{aligned} \quad (27)$$

As before, Figure 1 (right) shows the  $L^2$  distance to the monodomain solution at each iteration of the DDM. In contrast with the previous case now the algorithm converges. This happens because (27) is also consistent at  $O(\Delta x)$  with a Neumann boundary condition. To see this notice that  $3\Delta x T(D_x^+ \zeta_l^n) = O(\Delta x)$  and  $(D_x^+ \zeta_l^n -$

$D_x^+ \zeta_{l-1}^n \Delta z / \Delta x = O(\Delta z)$ , so if  $\Delta z = O(\Delta x^2)$ , when taking the limit  $\Delta x \rightarrow 0$ , equation (27) will satisfy  $\partial_x \phi_1^n = \partial_x \phi_2^n + O(\Delta x)$  at  $x = x_l$ , with  $\phi_s^n$  the limit of  $\phi_{i,s}^n$  when  $\Delta x \rightarrow 0$ . However, solutions of the subdomains are different from the monodomain discrete solution, since the  $O(\Delta x)$  terms in equation (27) lead to a linear system that is not equivalent to the monodomain's.

Defining  $\phi_i^*$  as  $\phi_i^* = \phi_i^1$  if  $i \leq l$  and  $\phi_i^* = \phi_i^2$  if  $i > l$ , we conclude that in the homogeneous case the asymptotic domain decomposition method:

- Similarly to its parent DDM, it also corresponds to the Neumann-Dirichlet method, so the additive scheme will be convergent when  $L_1 < L_2$ , with  $L_1 = l\Delta x$  and  $L_2 = (N_x - l)\Delta x$ .
- The monodomain solution  $\phi_i$  is different than  $\phi_i^*$ , the solution formed by each subdomain solution. This is because the asymptotic boundary condition includes  $O(\Delta x)$  terms that induce a different linear system than the monodomain. Additional  $O(\Delta x)$  terms must be added to fix this situation.

**The heterogeneous case** Now we want to use (27) to couple the LGNE with the LSWE. To do this we can add the constraint  $\phi_i^2 = 0$  for  $i \geq l + 1$  to impose the LSWE on the right side of the domain, and  $\phi_i^1$  satisfying the third equation of system (25), to impose the LGNE on the left side of the domain. If we write the Neumann boundary condition (27) as  $(\phi_{l,1}^n - \phi_{l-1,1}^n) / \Delta x = (\phi_{l+1,2}^n - \phi_{l,2}^n) / \Delta x + O(\Delta x)$ , use the LSWE  $\phi_{l+1,2}^n = 0$  and the Dirichlet boundary condition  $\phi_{l,2}^n = \phi_{l,1}^n$ , we arrive to the condition  $\phi_{l,1}^n = \frac{1}{2}\phi_{l-1,1}^n + O(\Delta x^2)$ . An interpretation of this formula is that the Neumann-Dirichlet condition becomes a linear interpolation between  $\phi_{l-1,1}^n$  and  $\phi_{l+1,2}^n = 0$  plus the  $O(\Delta x)$  term.

To test this heterogeneous DDM we manufacture the solitary wave of the GNE into the LGNE and LSWE  $\zeta(t, x) = \text{sech}^2(x - t + 3)$ ,  $V(t, x) = \zeta(t, x) / (1 + \varepsilon \zeta(t, x))$  with  $\varepsilon = 0.2$ , for  $(x, t) \in (-11, 11) \times (-3, 3)$ , and an interface located at  $x = 0$ . By definition any change on its shape must be due to the influence of the asymptotic transmission conditions. Figure 2 shows the results for  $\Delta x = 0.05, 0.02, 0.01$ , when half of the solitary wave has crossed the interface at  $t = 0$  and later at  $t = 3$ . We see that the interface boundary conditions have introduced oscillations and a discontinuity at the interface whose amplitudes grow as  $\Delta x$  decreases. This is similar to the results reported by [1].

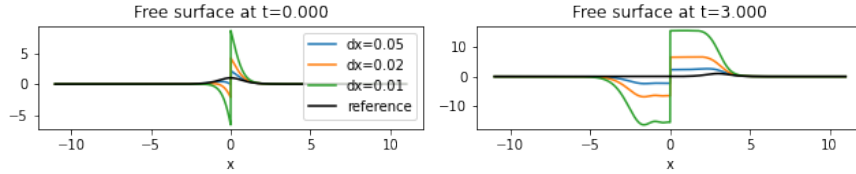


Fig. 2: Comparison of the free surface of a solitary wave calculated with the asymptotic heterogeneous DDM at two time steps and different grids.



## 4 Conclusions

Transmission conditions have been derived for the LGNE and LSWE by taking a vertical average and truncating an asymptotic expansion of the transmission conditions of a DDM of the discrete linear free-surface Euler equations. This DDM uses the Neumann-Dirichlet method on the elliptic problem for the velocity potential. In the homogeneous case, when the LGNE are solved on both sides of the interface, we recover the Neumann-Dirichlet method of the elliptic part of the LGNE, plus an  $O(\Delta x)$  term. The method has the same convergence property as its parent method but the  $O(\Delta x)$  terms make the limit of the subdomain iterations different from the monodomain solution, even though this was imposed on the parent DDM. In the heterogeneous case the Neumann-Dirichlet method corresponds to a linear interpolation of the elliptic variable between its last value in the LGNE domain and 0, the condition that defines the LSWE, plus an  $O(\Delta x^2)$  term. Numerical results show that this induces an unstable scheme, due to oscillations and discontinuities in the interface that grow in amplitude as  $\Delta x$  decreases. The next steps could be the introduction of a free parameter in the boundary conditions to optimize the convergence of the method, for example through a Robin boundary condition, and the analysis of this approach in the continuous case.

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