



HAL
open science

Long time behaviour for electromagnetic waves in dissipative Lorentz media

Maxence Cassier, Patrick Joly, Luis Alejandro Rosas Martínez

► **To cite this version:**

Maxence Cassier, Patrick Joly, Luis Alejandro Rosas Martínez. Long time behaviour for electromagnetic waves in dissipative Lorentz media. Waves 2022 - 15th International Conference on Mathematical and Numerical Aspects of Wave Propagation, Jul 2022, Palaiseau, France. hal-03850494

HAL Id: hal-03850494

<https://inria.hal.science/hal-03850494>

Submitted on 13 Nov 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Long time behaviour for electromagnetic waves in dissipative Lorentz media

Maxence CASSIER¹, Patrick JOLY², Alejandro ROSAS²

¹Aix Marseille Univ, CNRS, Centrale Marseille, Institut Fresnel, Marseille, France

²POEMS, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France

Abstract

A very general class of models for describing the propagation of waves in dispersive electromagnetic media is provided by generalized Lorentz models [1]. In this work, we study the long time behaviour of the solutions of the dissipative version of these models.

1 Introduction

We are interested in Maxwell's equations

$$\begin{cases} \varepsilon_0 \partial_t \mathbf{E} - \nabla \times \mathbf{H} + \varepsilon_0 \partial_t \mathbf{P} = 0 \\ \mu_0 \partial_t \mathbf{H} + \nabla \times \mathbf{E} + \mu_0 \partial_t \mathbf{M} = 0, \end{cases} \quad (1.1)$$

where \mathbf{E} and \mathbf{H} are, respectively, the electric and magnetic fields, while \mathbf{P} (resp. \mathbf{M}) is the electric polarization (resp. magnetization). In generalized dissipative Lorentz media, these are related to the electromagnetic field via a system of ODE's (the constitutive laws of the medium)

$$\begin{cases} \mathbf{P} = \sum_{j=1}^{N_e} \Omega_{e,j}^2 \mathbf{P}_j, \quad \mathbf{M} = \sum_{\ell=1}^{N_m} \Omega_{m,\ell}^2 \mathbf{M}_\ell, \\ \partial_t^2 \mathbf{P}_j + \alpha_{e,j} \partial_t \mathbf{P}_j + \omega_{e,j}^2 \mathbf{P}_j = \mathbf{E}, \\ \partial_t^2 \mathbf{M}_\ell + \alpha_{m,\ell} \partial_t \mathbf{M}_\ell + \omega_{m,\ell}^2 \mathbf{M}_\ell = \mathbf{H}, \end{cases} \quad (1.2)$$

where $(\mathbf{P}_j, \partial_t \mathbf{P}_j, \mathbf{M}_\ell, \partial_t \mathbf{M}_\ell)$ vanish at $t = 0$. In (1.2), $(\Omega_{e,j}, \Omega_{m,\ell}) > 0$ while $(\alpha_{e,j}, \alpha_{m,\ell}) \geq 0$ and $(\omega_{e,j}, \omega_{m,\ell}) > 0$. Our goal is to analyze the long time behaviour of the solution of the Cauchy problem associated to (1.1, 1.2) and any initial data $(\mathbf{E}_0, \mathbf{H}_0)$ under the so-called weak dissipative assumption

$$\sum_{j=1}^{N_e} \alpha_{e,j} + \sum_{\ell=1}^{N_m} \alpha_{m,\ell} > 0. \quad (1.3)$$

2 Main results

In what follows $f \lesssim g$ means $f \leq C g$, for some constant $C > 0$ independent of $x, t, \mathbf{E}_0, \mathbf{H}_0$.

Let $\mathcal{E}(t)$ be the electromagnetic energy

$$\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^3} (\varepsilon_0 |\mathbf{E}|^2 + \mu_0 |\mathbf{H}|^2)(x, t) dx \quad (2.1)$$

The generic result (cf. Remark 3) is as follows

Theorem 1 *For \mathbf{E}_0 and \mathbf{H}_0 in $L^2(\mathbb{R}^3)^3$ such that $\nabla \cdot \mathbf{E}_0 = \nabla \cdot \mathbf{H}_0 = 0$, the energy $\mathcal{E}(t)$ tends to 0 when $t \rightarrow +\infty$. Moreover if for some integers $m \geq 0$ and $p \geq 0$,*

$$\begin{cases} (\mathbf{E}_0, \mathbf{H}_0) \in H^m(\mathbb{R}^3)^3 \times H^m(\mathbb{R}^3)^3, \\ |x|^p (\mathbf{E}_0, \mathbf{H}_0) \in L^1(\mathbb{R}^3)^3 \times L^1(\mathbb{R}^3)^3, \\ \int_{\mathbb{R}^3} x^\alpha \mathbf{E}_0 dx = 0, \int_{\mathbb{R}^3} x^\alpha \mathbf{H}_0 dx = 0, \quad \forall |\alpha| < p \end{cases} \quad (2.2)$$

one has a polynomial decay rate

$$\mathcal{E}(t) \leq \frac{C_r^m(\mathbf{E}_0, \mathbf{H}_0)}{t^m} + \frac{C_i^p(\mathbf{E}_0, \mathbf{H}_0)}{t^{p+\frac{3}{2}}} \quad (2.3)$$

where the above constants satisfy

$$\begin{cases} C_r^m(\mathbf{E}_0, \mathbf{H}_0) \lesssim \|\mathbf{E}_0\|_{H^m(\mathbb{R}^3)}^2 + \|\mathbf{H}_0\|_{H^m(\mathbb{R}^3)}^2, \\ C_i^p(\mathbf{E}_0, \mathbf{H}_0) \lesssim \|\mathbf{E}_0\|_{L^{1,p}(\mathbb{R}^3)}^2 + \|\mathbf{H}_0\|_{L^{1,p}(\mathbb{R}^3)}^2 \end{cases}$$

and $\|u\|_{L^{1,p}(\mathbb{R}^3)} := \|(1 + |x|^p) u\|_{L^1(\mathbb{R}^3)}$.

Remark 2 *The upper bounds in (2.3) are sharp in the sense that similar lower bounds can be obtained for well chosen initial data.*

Remark 3 *The statement of the theorem has to be slightly modified if one of the $\alpha_{e,j}$ vanishes while $\alpha_{\ell,m} = 0$ for all m (or vice versa). More precisely, with the same assumptions, one has to replace in the estimate (2.3) t^{-m} by $t^{-m/2}$.*

3 Comparison with the literature

The law (1.2) enters a more general class of non local (in time) constitutive laws of the form

$$\begin{cases} \mathbf{P}(x, \cdot) = \chi_e \star \mathbf{E}(x, \cdot), \\ \mathbf{M}(x, \cdot) = \chi_m \star \mathbf{H}(x, \cdot) \end{cases} \quad (3.1)$$

where (χ_e, χ_m) are time convolution causal kernels. The long time behaviour of solutions of (1.1, 3.1) has been investigated in many papers such as in [2] for bounded domains of propagation (using abstract results from semi-group theory). In these works, polynomial stability, i.e. time decay estimates of the type

$$\mathcal{E}(t) \lesssim (1+t)^{-p}, \quad \text{for some } p > 0, \quad (3.2)$$

is proven under direct assumptions on the kernels (χ_e, χ_m) . In the case of Lorentz media, the assumptions used in [2] are satisfied only under the strong dissipativity assumption, i. e.

$$\alpha_{e,j}, \alpha_{m,\ell} > 0, \quad \forall j, \ell, \quad (3.3)$$

Another major difference is in the method of proof: we use here a more physically oriented method based on a modal/spectral approach.

4 Method of proof

One first writes the problem as a generalized Schrödinger equation of the form

$$\frac{d\mathcal{U}}{dt} + i\mathcal{A}(\nabla)\mathcal{U} = 0 \quad (4.1)$$

where $\mathcal{U} := (\mathbf{E}, \mathbf{H}, \mathbf{P}_j, \partial_t \mathbf{P}_j, \mathbf{M}_\ell, \partial_t \mathbf{M}_\ell) \in \mathbb{R}^N$ with $N = 3(2 + 2N_e + 2N_m)$ and $\mathcal{A}(\nabla)$ is a first order differential operator in space. We apply the space Fourier transform

$$\mathcal{U}(x, t) \longrightarrow \mathbf{U}(\mathbf{k}, t),$$

so that $\mathbf{U}(\mathbf{k}, t)$ satisfies

$$\frac{d\mathbf{U}}{dt}(\mathbf{k}, t) + i\mathcal{A}(\mathbf{k})\mathbf{U}(\mathbf{k}, t) = 0, \quad (4.2)$$

where $\{\mathcal{A}(\mathbf{k}), \mathbf{k} \in \mathbb{R}^3\}$ is a family of *non-normal* $N \times N$ matrices. We next derive a priori estimates for $\mathbf{U}(\mathbf{k}, t)$ before coming back to space domain via Plancherel's theorem.

We have developed two approaches to obtain estimates in the \mathbf{k} -space.

(I) *Via frequency dependent Lyapunov functions.*

This approach is *more direct* but *limited* to the strict dissipativity assumption (3.3).

We construct a $|\mathbf{k}|$ -dependent and positive quadratic functional $\mathcal{L}_{m,p}(|\mathbf{k}|; \mathbf{U})$ such that

$$\frac{d}{dt} \mathcal{L}_{m,p}(|\mathbf{k}|; \mathbf{U}) + \Phi(|\mathbf{k}|) \mathcal{L}_{m,p}(|\mathbf{k}|; \mathbf{U}) \leq 0,$$

for some function $\Phi(r) > 0$ ($r \in \mathbb{R}_*^+$). Then combining Grönwall's lemma with a careful examination of the behaviour of $\Phi(|\mathbf{k}|)$ for small and large values of $|\mathbf{k}|$ leads to (2.3).

(II) *Via spectral decomposition.*

We use the spectrum $\{\omega_n(\mathbf{k})\}$ of $\mathcal{A}(\mathbf{k})$ which is made of the solutions of a *dispersion relation* that satisfy $\text{Im } \omega_n(\mathbf{k}) < 0$. We then use the associated spectral decomposition of $\mathcal{A}(\mathbf{k})$ to represent the solution $\mathbf{U}(\mathbf{k}, t)$ of (4.2) as

$$\mathbf{U}(\mathbf{k}, t) = \sum e^{-i\omega_n(\mathbf{k})t} P_n(\mathbf{k}, t) \mathbf{U}_0(\mathbf{k}), \quad (4.3)$$

where the operators $P_n(\mathbf{k}, t)$ are directly related to the spectral projectors of $\mathcal{A}(\mathbf{k})$. We then split the analysis in three parts:

- (i) estimates for low (space) frequencies $|\mathbf{k}| < m$,
- (ii) estimates for high frequencies $|\mathbf{k}| > M$,
- (iii) estimates for mid-frequencies $m \leq |\mathbf{k}| < M$.

While step (iii) provides a uniform (in \mathbf{k}) exponential decay, the polynomial decay results from the estimates (i) and (ii) which rely on

- (a) the asymptotic behaviour of each $\omega_n(\mathbf{k})$ for small and large values of $|\mathbf{k}|$,
- (b) uniform bounds for the operators $P_n(\mathbf{k}, t)$.

For the above, we use the implicit function theorem and the holomorphic functional calculus.

References

- [1] M. Cassier, P. Joly, and M. Kachanovska. Mathematical models for dispersive electromagnetic waves: An overview. *Computers & Mathematics with Applications*, 74(11):2792–2830, dec 2017.
- [2] S. Nicaise and C. Pignotti. Asymptotic behavior of dispersive electromagnetic waves in bounded domains. *Zeitschrift für angewandte Mathematik und Physik*, 71(3), apr 2020.