Boundedness of the Optimal State Estimator Rejecting Unknown Inputs

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Abstract—The Kitanidis filter is a natural extension of the Kalman filter to systems subject to arbitrary unknown inputs or disturbances. Though the optimality of the Kitanidis filter was founded for general time varying systems more than 30 years ago, its boundedness and stability analysis is still limited to time invariant systems, up to the authors' knowledge. In the framework of general time varying systems, this paper establishes upper and lower bounds of the error covariance of the Kitanidis filter, as well as upper bounds of all the auxiliary variables involved in the filter. By preventing data overflow, upper bounds are crucial for all recursive algorithms in real time applications. The upper and lower bounds of the error covariance will also serve as the basis of the Kitanidis filter stability analysis, like in the case of time varying system Kalman filter.

Index Terms—State estimation, unknown input observer, Kalman filter, stability analysis, disturbance rejection, time varying system.

I. INTRODUCTION

The Kitanidis filter [1] is a natural extension of the Kalman filter to linear time varying (LTV) stochastic systems subject to unknown inputs

$$x_{k+1} = A_k x_k + B_k u_k + E_k d_k + w_k \qquad (1a)$$
$$y_k = C_k x_k + v_k, \qquad (1b)$$

where $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^m$ the output, $u_k \in \mathbb{R}^p$ the (known) input, $d_k \in \mathbb{R}^q$ some *arbitrary unknown input* (or disturbance), $w_k \in \mathbb{R}^n$ the state noise of covariance Q_k, $v_k \in \mathbb{R}^m$ the output noise of covariance R_k, and A_k, B_k, C_k, E_k are matrices of appropriate sizes at each discrete time instant k.

Since the very beginning, the Kitanidis filter filter has been designed for optimal state estimation in the sense of minimum error variance while completely rejecting the unknown input d_k. However, after more than 30 years of existence, an important theoretic element is still incomplete: its boundedness and stability analysis.

This paper will focus on the boundedness analysis of the Kitanidis filter. By preventing data overflow, boundedness is crucial for all recursive algorithms. Moreover, it is expected that these boundedness results will found the basis of the error dynamics stability analysis of the Kitanidis filter, as in the Kalman filter theory. Unlike the case of time invariant systems, for *time varying* systems, boundedness is a prerequisite for the formulation of a valid Lyapunov function [2].

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In order to establish an upper bound of the error covariance matrix of the Kitanidis filter, the main idea of this paper is to build a non-optimal filter, for which an upper bound of the error covariance matrix can be first established. Then the optimality of the Kitanidis filter implies that its error covariance matrix cannot be larger. The difficulty resides in the construction of such a non optimal filter for the *time varying* systems considered in this paper. Unlike linear time invariant systems, keeping the eigenvalues of the error system transition matrix *always inside the unit circle* is not sufficient to ensure the stability of the *time varying* error dynamics [3].

In the Kalman filter theory, it is known that the upper bound of the error covariance matrix is essential for the *asymptotic* stability of the error dynamics. A stronger stability result, the *exponential* stability, requires also a strictly positive lower bound of the error covariance matrix [4]. In the perspective of exponential stability analysis, a lower bound of the error covariance matrix of the Kitanidis filter will also be established in this paper.

The Kitanidis filter [1] has been designed for general (time varying) LTV systems as formulated in (1), and later it has been investigated in the same framework [5]. However, when its stability is analyzed, the reported result is restricted to linear time *invariant* (LTI) systems. This stability refers to the error dynamics equation of the Kitanidis filter, which is completely decoupled from unknown inputs. Similarly, in early studies on *unknown input observers* [6]–[8], stability results are also restricted to the time invariant case.

The Kitanidis filter requires more assumptions than the classical Kalman filter in order to completely reject unknown inputs. Alternatively, H_∞ filters limit in some sense the effects of disturbances or unknown inputs on state estimation, without the specific assumptions of the Kitanidis filter [9].

Some preliminary results of this work have been reported in the conference paper [10]. The extended results in the present paper include: an analysis of the uniform complete observability for the considered state estimation problem subject to arbitrary unknown inputs, and a strictly positive lower bound of the error covariance matrix.

II. NOTATIONS AND DEFINITIONS

In this paper, lower case letters denote scalars and vectors, whereas upper case letters are reserved to matrices. The $n \times n$ identity matrix is denoted by I_n. For a vector v, its Euclidean norm is denoted by $\|v\|$. For a matrix M, its matrix-norm induced by the Euclidean vector norm is denoted by

$\|M\|$, which is equal to its largest singular value. For a real symmetric matrix M, the inequality $M > 0$ ($M \geq 0$) means that M is (semi)-positive definite. For two real symmetric matrices M, N of the same size, $M > N$ means $M - N > 0$, and $M \geq N$ means $M - N \geq 0$.

Definition 1 (State transition matrix): Given the matrix sequence $A_k \in \mathbb{R}^{n \times n}$ involved in system (1), the state transition matrix $\Phi_{l|k}$ is defined, for integers $l > k \geq 0$, as

$$\Phi_{l|k} \triangleq A_{l-1} A_{l-2} \cdots A_k, \tag{2}$$

and for $l = k$, as

$$\Phi_{k|k} \triangleq I_n. \tag{3}$$

Definition 2 (UCO): A matrix sequence pair $[A_k, C_k]$, with $A_k \in \mathbb{R}^{n \times n}$ and $C_k \in \mathbb{R}^{m \times n}$ for $k = 0, 1, 2, \ldots$ is uniformly completely observable (UCO[1]), if there exist two positive constants $\alpha_2 > \alpha_1 > 0$ and a positive integer h such that, for all $k = 0, 1, 2, \ldots$,

$$\alpha_1 I_n \leq \sum_{j=k}^{k+h-1} \Phi_{j|k}^T C_j^T C_j \Phi_{j|k} \leq \alpha_2 I_n. \tag{4}$$

Definition 3 (UCC): A matrix sequence pair $[A_k, B_k]$, with $A_k \in \mathbb{R}^{n \times n}$ and $B_k \in \mathbb{R}^{n \times p}$ for $k = 0, 1, 2, \ldots$, is uniformly completely controllable (UCC), if there exist two positive constants $\beta_2 > \beta_1 > 0$ and a positive integer h such that, for all $k = h, h+1, \ldots$,

$$\beta_1 I_n \leq \sum_{j=k-h+1}^{k} \Phi_{k+1|j+1} B_j B_j^T \Phi_{k+1|j+1}^T \leq \beta_2 I_n. \tag{5}$$

These UCO and UCC definitions follow [4]. See also [11], [12] for similar definitions.

III. PROBLEM FORMULATION AND ASSUMPTIONS

Consider LTV stochastic systems as formulated in (1). Among all recursive linear filters of the form

$$\hat{x}_{k+1} = A_k \hat{x}_k + B_k u_k$$
$$+ L_{k+1}(y_{k+1} - C_{k+1} A_k \hat{x}_k - C_{k+1} B_k u_k), \tag{6}$$

with the state estimate $\hat{x}_k \in \mathbb{R}^n$, the filter gain matrix $L_{k+1} \in \mathbb{R}^{n \times m}$, and the state estimation error

$$\tilde{x}_k \triangleq x_k - \hat{x}_k, \tag{7}$$

the Kitanidis filter is the unbiased minimum variance filter, characterized by an optimal gain matrix sequence

$$L_{1:(k+1)}^* = (L_1^*, L_2^*, \ldots, L_{k+1}^*). \tag{8}$$

More specifically, at instant $k + 1$, L_{k+1}^* is determined by solving the optimization problem

$$L_{k+1}^* = \arg \min_{L_{k+1}} \mathsf{Trace}\, \mathsf{Cov}(\tilde{x}_{k+1} | L_{1:k}^*, L_{k+1}) \tag{9}$$

[1] In this paper, the abbreviation "UCO" will be used both as a noun for "uniform complete observability" and as an adjective for "uniformly completely observable". The abbreviation "UCC" introduced later will be used similarly.

subject to the unbiasedness constraint

$$\mathsf{E}(\tilde{x}_{k+1} | L_{1:(k+1)}^*) = 0, \tag{10}$$

where the dependence of the filter error $\tilde{x}_{k+1}$ on the gain sequence is indicated in the notations of *error covariance* $\mathsf{Cov}(\tilde{x}_{k+1} | \cdot)$ and *error mean* $\mathsf{E}(\tilde{x}_{k+1} | \cdot)$.

It is shown in [1] that the constraint (10) is equivalent to a linear constraint on $L_{1:(k+1)}^*$. See also Lemma 2 presented later in this section.

The unbiasedness (10) holds in spite of the unknown input $d_k \in \mathbb{R}^q$, which is totally unknown and arbitrary.

The Kitanidis filter [1] is given by

$$\hat{x}_{k+1} = A_k \hat{x}_k + B_k u_k$$
$$+ L_{k+1}^*(y_{k+1} - C_{k+1} A_k \hat{x}_k - C_{k+1} B_k u_k) \tag{11}$$

with the optimal gain L_{k+1}^* recursively computed as

$$P_{k+1|k} = A_k P_{k|k} A_k^T + Q_k \tag{12a}$$
$$\Sigma_{k+1} = C_{k+1} P_{k+1|k} C_{k+1}^T + R_{k+1} \tag{12b}$$
$$\Gamma_{k+1} = E_k - P_{k+1|k} C_{k+1}^T \Sigma_{k+1}^{-1} C_{k+1} E_k \tag{12c}$$
$$\Xi_{k+1} = E_k^T C_{k+1}^T \Sigma_{k+1}^{-1} C_{k+1} E_k \tag{12d}$$
$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} C_{k+1}^T \Sigma_{k+1}^{-1} C_{k+1} P_{k+1|k}$$
$$+ \Gamma_{k+1} \Xi_{k+1}^{-1} \Gamma_{k+1}^T \tag{12e}$$
$$L_{k+1}^* = P_{k+1|k} C_{k+1}^T \Sigma_{k+1}^{-1}$$
$$+ \Gamma_{k+1} \Xi_{k+1}^{-1} E_k^T C_{k+1}^T \Sigma_{k+1}^{-1}. \tag{12f}$$

At the initial instant $k = 0$, this filter is initialized as $\hat{x}_0 = \bar{x}_0$, $P_{0|0} = P_0$, with $\bar{x}_0 \in \mathbb{R}^n$ and $P_0 \in \mathbb{R}^{n \times n}$ respectively the mean and the covariance matrix of the initial state x_0.

The linear filter (6) is said bounded if the filter gain sequence L_k is bounded and the covariance matrix of the resulting state estimation error $\tilde{x}_k$ is bounded. In particular, if the the Kitanidis filter is bounded, the auxiliary variables and the optimal gain computed in (12) must be bounded.

The purpose of this paper is to study the boundedness of the Kitanidis filter, mainly the existence of upper bounds for the error covariance matrix $P_{k|k} = \mathsf{Cov}(\tilde{x}_k | L_{1:k}^*)$ and for the Kitanidis gain matrix L_k^*, as well as the existence of a strictly positive lower bound of $P_{k|k}$. These boundedness properties are prerequisites to define a Lyapunov function involving $P_{k|k}$, like in the classical Kalman filter stability analysis.

Basic assumptions

(i) A_k, B_k, C_k, E_k are bounded matrix sequences for all $k \geq 0$.

(ii) The initial state $x_0 \in \mathbb{R}^n$ is a random vector of mean $\bar{x}_0 \in \mathbb{R}^n$ and covariance $P_0 > 0$.

(iii) w_k and v_k are zero mean white noises independent of each other and of x_0, with bounded covariance matrices $\mathsf{E}(w_k w_k^T) = Q_k$ and $\mathsf{E}(v_k v_k^T) = R_k$ for all $k \geq 0$. The inverse matrix R_k^{-1} exists and is bounded for all $k \geq 0$.

Unknown input subspace assumption

(iv) For all $k \geq 0$, the matrix product $C_{k+1} E_k$ has a full column rank and a bounded Moore-Penrose inverse.

In the Kitanidis filter, the matrix product $C_{k+1}E_k$ is involved in (12d) and its inverse in (12e). The full column rankness of $C_{k+1}E_k$ means that, whatever is the (non zero) arbitrary unknown input vector d_k, its effect on the state x_{k+1} through the vector $E_k d_k$ alters at least one of the output sensors. In this case, the Moore-Penrose inverse of $C_{k+1}E_k$ is $[(C_{k+1}E_k)^T(C_{k+1}E_k)]^{-1}(C_{k+1}E_k)^T$, which allows to define the following two bounded matrix sequences, for all integer $k \geq 0$,

$$G_{k+1} \triangleq E_k[(C_{k+1}E_k)^T(C_{k+1}E_k)]^{-1}(C_{k+1}E_k)^T \quad (13)$$
$$\bar{A}_k \triangleq (I_n - G_{k+1}C_{k+1})A_k. \quad (14)$$

It follows from Assumptions (i) and (iv) that G_{k+1} and $\bar{A}_k$ are both bounded.

Observability and controllability assumptions

(v) The matrix sequence pair $[\bar{A}_k, C_k]$ is uniformly completely observable (UCO, see Definition 2).
(vi) The matrix sequence pair $[A_k, Q_k^{\frac{1}{2}}]$ is uniformly completely controllable (UCC, see Definition 3), where $Q_k^{\frac{1}{2}}$ is a symmetric matrix square root of Q_k.

The UCC in Assumption (vi) is the same as in the classical Kalman filter theory [4], but the UCO in Assumption (v) is slightly different, involving $[\bar{A}_k, C_k]$ instead of $[A_k, C_k]$. By reverting to the classical UCO, Assumptions (i)-(vi) would not be sufficient due to arbitrary unknown inputs, as demonstrated by the counterexample

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (15)$$

See Appendix 1 for the details about this example.

On the other hand, the following lemma justifies the relevance of the UCO of $[\bar{A}_k, C_k]$ as in Assumption (v).

Lemma 1: In the noise-free case, the UCO of $[\bar{A}_k, C_k]$ stated in Assumption (v) ensures that there exists a positive integer h such that, for all $k = 0, 1, 2, \ldots$, the state x_k is fully determined from $u_k, u_{k+1}, \ldots, u_{k+h-2}$ and $y_k, y_{k+1}, \ldots, y_{k+h-1}$, whatever is the arbitrary unknown input d_k. $\square$

See Appendix 2 for a proof of this lemma.

Let us end this section with a basic fact about the unbiasedness constraint.

Lemma 2: A bounded gain matrix sequence $L_k \in \mathbb{R}^{n \times m}$ leads to an unbiased filter (6), *i.e.*, the filter error satisfies $\mathsf{E}(\tilde{x}_k|L_{1:k}) = 0$, if and only if

$$(I_n - L_{k+1}C_{k+1})E_k = 0 \quad (16)$$

for all $k = 0, 1, 2, \ldots$, in spite of any arbitrary unknown input d_k affecting system (1). $\square$

This result is due to the fact that any filter gain satisfying (16) leads to an error dynamics completely decoupled from d_k [1]. See also the proof of Theorem 1 in [13].

IV. UPPER BOUNDEDNESS OF THE KITANIDIS FILTER

To establish an upper bound of the error covariance matrix $P_{k|k} = \mathsf{Cov}(\tilde{x}_k|L_{1:k}^*)$ of the *optimal* filter, the main idea is to build a *non-optimal* bounded gain sequence $L_{1:k} = \bar{L}_{1:k}$, such

that the resulting filter (6) is unbiased and has a bounded error covariance $\mathsf{Cov}(\tilde{x}_k|\bar{L}_{1:k})$. It will bound the optimal Kitanidis filter, whose error covariance cannot be larger.

A. A non-optimal filter and its upper bound

The non optimal gain $\bar{L}_k$ built below will consist of two parts. The first part is simply G_k as defined in (13). It will ensure the unbiasedness of the filter. In order to build the second part stabilizing the filter error dynamics, an *auxiliary* stochastic system will be designed such that the error dynamics of its classical Kalman filter is equivalent to the error dynamics of the linear filter (6) specified with $L_k = \bar{L}_k$. Then Assumption (v) ensures the stability of this error dynamics. The auxiliary system is

$$\bar{x}_{k+1} = \bar{A}_k \bar{x}_k + \bar{w}_k \quad (17a)$$
$$\bar{y}_k = C_k \bar{x}_k + \bar{v}_k \quad (17b)$$

where $\bar{x}_k \in \mathbb{R}^n$ is the state, $\bar{y}_k \in \mathbb{R}^m$ the output, $\bar{A}_k$ is as defined in (14), $\bar{w}_k \in \mathbb{R}^n$ and $\bar{v}_k \in \mathbb{R}^m$ are mutually independent white noises with

$$\mathsf{Cov}(\bar{w}_k) = I_n \quad (18a)$$
$$\mathsf{Cov}(\bar{v}_k) = I_m. \quad (18b)$$

Apply the *classical* Kalman filter to the auxiliary LTV system (17), resulting in

$$\hat{x}_{k+1|k} = \bar{A}_k \hat{x}_{k|k} \quad (19a)$$
$$\tilde{y}_{k+1} = y_{k+1} - C_{k+1}\hat{x}_{k+1|k} \quad (19b)$$
$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\tilde{y}_{k+1}. \quad (19c)$$

The Kalman gain K_k is recursively computed as

$$\bar{P}_{k+1|k} = \bar{A}_k \bar{P}_{k|k} \bar{A}_k^T + I_n \quad (20a)$$
$$\bar{\Sigma}_{k+1} = C_{k+1}\bar{P}_{k+1|k}C_{k+1}^T + I_m \quad (20b)$$
$$K_{k+1} = \bar{P}_{k+1|k}C_{k+1}^T \bar{\Sigma}_{k+1}^{-1} \quad (20c)$$
$$\bar{P}_{k+1|k+1} = (I_n - K_{k+1}C_{k+1})\bar{P}_{k+1|k}. \quad (20d)$$

Remark that the notations $\hat{x}_{k+1|k}$ and $\hat{x}_{k|k}$ with double indexes are used for state estimates in this Kalman filter, whereas the single indexed notation $\hat{x}_k$ is reserved for the Kitanidis filter. A similar remark will apply to the state estimation error notations $\tilde{x}_{k+1|k}$ and $\tilde{x}_{k|k}$.

The *non-optimal* filter gain is then built as

$$\bar{L}_k \triangleq G_k + K_k - K_k C_k G_k, \quad (21)$$

with G_k as defined in (13).

Proposition 1: Under Assumptions (i)-(v), the time varying filter gain matrix $\bar{L}_k$ defined in (21) is bounded for all $k \geq 0$. $\square$

Proof.

The boundedness of the first term G_k, which is defined in (13), is trivially due to Assumptions (i) and (iv), whereas the boundedness of K_k relies on the classical Kalman filter theory, as shortly recalled in the following.

Under Assumptions (i) and (iv), the boundedness of G_k implies that the matrix $\bar{A}_k$ defined in (14) is also bounded.

The matrix K_k is the gain of the *classical* Kalman filter applied to the auxiliary system (17), as expressed in (19) and (20). According to [4], the boundedness of the Kalman filter is ensured by the boundedness of the involved matrices, under UCO and UCC (regarding the state noise) conditions. In the case of the auxiliary system (17), the UCO of the matrix pair $[\bar{A}_k, C_k]$ is ensured by Assumption (v), whereas the UCC of the matrix pair $[\bar{A}_k, I_n]$ holds trivially due to the unitary state noise covariance $\mathsf{Cov}(\bar{w}_k) = I_n$. Then the Kalman gain K_k is bounded, according to [4]. The matrix $\bar{L}_k$ defined in (21) is thus also bounded, since C_k is assumed bounded in Assumption (i). $\qquad\square$

Proposition 2: The gain matrix $\bar{L}_k$ defined in (21) leads to an unbiased filter (6), *i.e.*, the filter error satisfies $\mathsf{E}(\tilde{x}_k|\bar{L}_{1:k}) = 0$ for all $k = 1, 2, \ldots$. $\qquad\square$

Proof.

It is straightforward to check that

$$I_n - \bar{L}_{k+1}C_{k+1}$$
$$= I_n - (G_{k+1} + K_{k+1} - K_{k+1}C_{k+1}G_{k+1})C_{k+1}$$
$$= (I_n - K_{k+1}C_{k+1})(I_n - G_{k+1}C_{k+1}). \qquad (22)$$

On the other hand, the definition of G_{k+1} in (13) leads immediately to

$$(I_n - G_{k+1}C_{k+1})E_k = 0, \qquad (23)$$

hence

$$(I_n - \bar{L}_{k+1}C_{k+1})E_k = 0. \qquad (24)$$

Proposition 2 is then proved by applying Lemma 2. $\qquad\square$

Proposition 3: Under Assumptions (i)-(v), the linear filter (6) with the gain matrix $L_k = \bar{L}_k$ as defined in (21) has a bounded error covariance matrix $\mathsf{Cov}(\tilde{x}_k|\bar{L}_{1:k})$. $\qquad\square$

Proof.

The main step of this proof will show that the *error system* of the linear filter (6) with $L_k = \bar{L}_k$ is an *exponentially stable* system driven by a white noise of bounded covariance. For this purpose, it will be shown that the error system of the linear filter (6) is equivalent to the error system of the Kalman filter designed for the auxiliary system (17), in order to rely the analysis on the stability of this Kalman filter.

Let us first study the error system of the *classical* Kalman filter (19) applied to the auxiliary system (17). Let

$$\tilde{x}_{k|k} \triangleq \bar{x}_k - \hat{x}_{k|k} \qquad (25)$$

where $\bar{x}_k$ is governed by (17a) and $\hat{x}_{k|k}$ computed with (19), both recursively. After some computations combining (17) and (19), the recursive equation governing $\tilde{x}_{k|k}$ writes:

$$\tilde{x}_{k+1|k+1} = (I_n - K_{k+1}C_{k+1})\bar{A}_k\tilde{x}_{k|k}$$
$$+ (I_n - K_{k+1}C_{k+1})\bar{w}_k - K_{k+1}\bar{v}_{k+1}. \qquad (26)$$

According to [4], under the UCO of $[\bar{A}_k, C_k]$ (ensured by Assumption (v)) and the UCC of $[\bar{A}_k, I_n]$ (trivially satisfied

with $\mathsf{Cov}(\bar{w}_k) = I_n$), the error system (26) is exponentially stable, in the sense that there exist two positive constants α and β such that

$$\|\check{A}_{l-1}\check{A}_{l-2}\cdots\check{A}_k\| \leq \alpha e^{-\beta(l-k)}, \qquad (27)$$

with the notation

$$\check{A}_k \triangleq (I_n - K_{k+1}C_{k+1})\bar{A}_k. \qquad (28)$$

Now consider the error system of the linear filter (6). It is straightforward to check from (1) and (6) that the filter error $\tilde{x}_k$, as defined in (7), satisfies

$$\tilde{x}_{k+1} = (I_n - L_{k+1}C_{k+1})A_k\tilde{x}_k + (I_n - L_{k+1}C_{k+1})E_kd_k$$
$$+ (I_n - L_{k+1}C_{k+1})w_k - L_{k+1}v_{k+1}. \qquad (29)$$

With $L_k = \bar{L}_k$ satisfying (16) (see Lemma 2 and Proposition 2), the term involving the unknown input d_k disappears,

$$\tilde{x}_{k+1} = (I_n - \bar{L}_{k+1}C_{k+1})A_k\tilde{x}_k$$
$$+ (I_n - \bar{L}_{k+1}C_{k+1})w_k - \bar{L}_{k+1}v_{k+1}. \qquad (30)$$

It turns out that the state transition matrix $(I_n - \bar{L}_{k+1}C_{k+1})A_k$ of this error system is equal to the matrix $\check{A}_k$ defined in (28). To check this fact, substitute (14) into (28), then

$$\check{A}_k = (I_n - K_{k+1}C_{k+1})(I_n - G_{k+1}C_{k+1})A_k. \qquad (31)$$

It then follows from (22) and (31) that

$$\check{A}_k = (I_n - \bar{L}_{k+1}C_{k+1})A_k.$$

Hence the state transition matrix of (30) is indeed $\check{A}_k$.

The error dynamics equation (30) is then rewritten as

$$\tilde{x}_{k+1} = \check{A}_k\tilde{x}_k + \mu_k, \qquad (32)$$

with

$$\mu_k \triangleq (I_n - \bar{L}_{k+1}C_{k+1})w_k - \bar{L}_{k+1}v_{k+1}, \qquad (33)$$

which is a white noise with a bounded covariance matrix.

The error dynamics system (32) is exponentially stable, according to (27). It is driven by a white noise μ_k of bounded covariance. Hence the covariance matrix of $\tilde{x}_k$ governed by (32) is bounded, according to the Lemma 1 in the appendix of [10]. Since (32) is a shorter rewrite of (30), the boundedness of $\mathsf{Cov}(\tilde{x}_k|\bar{L}_{1:k})$ is then established. $\qquad\square$

B. Upper Bounds of the optimal filter

Theorem 1: Under Assumptions (i)-(v), the error covariance matrix $P_{k|k} = \mathsf{Cov}(\tilde{x}_k|L^*_{1:k})$ of the Kitanidis filter (11) is upper bounded for all $k > 0$, so are the Kitanidis gain matrix L^*_k and all the auxiliary variables involved in the filter recursions (12), namely, $P_{k+1|k}, \Sigma_{k+1}, \Gamma_{k+1}, \Xi_{k+1}$. $\qquad\square$

The proof of this result is immediate from Proposition 3 and the optimality of the Kitanidis filter [13].

V. LOWER BOUND OF THE ERROR COVARIANCE

In the classical Kalman filter theory, the *exponential* stability of the Kalman filter error dynamics is based on both upper and lower bounds of the error covariance matrix [2], [4], [11], [12]. This fact motivates the study on the lower bound for the Kitanidis filter.

Theorem 2: Under Assumptions (i)-(vi), the error covariance matrix $P_{k|k} = \mathsf{Cov}(\tilde{x}_k|L_{1:k}^*)$ of the Kitanidis filter (11)-(12) has a strictly positive lower bound, *i.e.*, there exists a constant $\eta > 0$ such that $P_{k|k} \geq \eta I_n$. $\square$

Proof.

With any gain matrix L_k, the filter error $\tilde{x}_k$ is governed by (29). In particular, with the optimal gain $L_k = L_k^*$ satisfying (16) (see [1]), the error equation (29) becomes

$$\tilde{x}_{k+1} = (I_n - L_{k+1}^* C_{k+1}) A_k \tilde{x}_k$$
$$+ (I_n - L_{k+1}^* C_{k+1}) w_k - L_{k+1}^* v_{k+1}. \quad (34)$$

In this equation, $\tilde{x}_k$, w_k and v_{k+1} are pairwise independent.

Take the mathematical expectation of the square of both sides of (34) by noticing that $P_{k|k} = \mathsf{E}(\tilde{x}_k \tilde{x}_k^T | L_{1:k}^*)$, then

$$P_{k+1|k+1} = (I_n - L_{k+1}^* C_{k+1}) A_k P_{k|k} A_k^T (I_n - L_{k+1}^* C_{k+1})^T$$
$$+ (I_n - L_{k+1}^* C_{k+1}) Q_k (I_n - L_{k+1}^* C_{k+1})^T$$
$$+ L_{k+1}^* R_{k+1} L_{k+1}^{*T} \quad (35)$$
$$= (I_n - L_{k+1}^* C_{k+1})(A_k P_{k|k} A_k^T + Q_k)$$
$$\times (I_n - L_{k+1}^* C_{k+1})^T + L_{k+1}^* R_{k+1} L_{k+1}^{*T}. \quad (36)$$

Apply Lemma 3 (see Appendix 3) with $L = L_{k+1}^*$, $C = C_{k+1}$, $M = (A_k P_{k|k} A_k^T + Q_k)$ and $R = R_{k+1}$, then

$$P_{k+1|k+1} \geq \left(1 + \left\|C_{k+1}^T R_{k+1}^{-1} C_{k+1}\right\| \left\|(A_k P_{k|k} A_k^T + Q_k)\right\|\right)^{-1}$$
$$\times \left(A_k P_{k|k} A_k^T + Q_k\right). \quad (37)$$

The matrices C_{k+1}, R_{k+1}^{-1}, A_k and Q_k are all bounded according to Assumptions (i) and (iii), so is $P_{k|k}$ due to Theorem 1. Hence there exists a constant $\nu \in (0, 1)$ such that

$$P_{k+1|k+1} \geq \nu \left(A_k P_{k|k} A_k^T + Q_k\right). \quad (38)$$

Recursively apply (38), with $\Phi_{l|k}$ as defined in (2), then

$$P_{k+1|k+1} \geq \nu \left(\Phi_{k+1|k} P_{k|k} \Phi_{k+1|k}^T + Q_k\right) \quad (39)$$
$$\geq \nu \Phi_{k+1|k} \nu \left(\Phi_{k|k-1} P_{k-1|k-1} \Phi_{k|k-1}^T + Q_{k-1}\right) \Phi_{k+1|k}^T$$
$$+ \nu Q_k \quad (40)$$
$$\vdots \quad (41)$$
$$\geq \nu^h \Phi_{k+1|k-h+1} P_{k-h+1|k-h+1} \Phi_{k+1|k-h+1}^T$$
$$+ \sum_{j=k-h+1}^{k} \nu^{k-j+1} \Phi_{k+1|j+1} Q_j \Phi_{k+1|j+1}^T \quad (42)$$
$$\geq \nu^h \sum_{j=k-h+1}^{k} \Phi_{k+1|j+1} Q_j \Phi_{k+1|j+1}^T \quad (43)$$
$$\geq \nu^h \beta_1 I_n, \quad (44)$$

where the last inequality is due to the UCC in Assumption (vi). The proof is thus completed. $\square$

VI. CONCLUSION

It has been established in this paper that the error covariance of the Kitanidis filter is upper bounded, so are the Kitanidis gain matrix and all the auxiliary variables involved in the filter recursive computations. Such boundedness results are of prime importance for real time applications, in which it is crucial to prevent data overflow. A strictly positive lower bound of the error covariance has also been established. Hopefully, like in the classical Kalman filter theory, these boundedness results will found the basis for a complete stability analysis of the Kitanidis filter.

APPENDIX

1. COUNTEREXAMPLE OF THE CLASSICAL UCO

The example with the matrices A, C, E given in (15) satisfies Assumptions (i)-(vi) except Assumption (v), but it does fulfill the classical UCO of $[A, C]$. To see that this system is not observable in the sense of the "unknown input observer problem", write the component-wise noise-free equations:

$$x_{k+1}^{(1)} = x_k^{(2)} + d_k \quad (45)$$
$$x_{k+1}^{(2)} = x_k^{(2)} \quad (46)$$
$$y_k = x_k^{(1)}. \quad (47)$$

In this noise-free system, the second state component $x_k^{(2)}$ is constant, hence always equal to the initial state value $x_0^{(2)}$, which is unknown. Given an arbitrary constant $c \in \mathbb{R}$, replace the initial state value $x_0^{(2)}$ with $x_0^{(2)} + c$, and replace also the unknown input sequence d_k with $d_k - c$, then the trajectory of $x_k^{(2)}$ is modified, but $y_k = x_k^{(1)}$ remains unchanged. Therefore, it is not possible to distinguish different trajectories of the state vector x_k from the observation sequence y_k. This example confirms that the UCO of $[A_k, C_k]$ is not sufficient.

2. PROOF OF LEMMA 1

In the noise-free case ($w_k = 0$ and $v_k = 0$), (1) becomes

$$x_{k+1} = A_k x_k + B_k u_k + E_k d_k \quad (48a)$$
$$y_k = C_k x_k. \quad (48b)$$

It will be shown below that, if the pair $[\bar{A}_k, C_k]$ is UCO as in Assumption (v), then x_k is fully determined from $u_k, u_{k+1}, \ldots, u_{k+h-2}$ and $y_k, y_{k+1}, \ldots, y_{k+h-1}$ as

$$x_k = \left(\sum_{j=k}^{k+h-1} \bar{\Phi}_{j|k}^T C_j^T C_j \bar{\Phi}_{j|k}\right)^{-1} \sum_{j=k}^{k+h-1} \bar{\Phi}_{j|k}^T C_j^T$$
$$\times \left(y_j - C_j \sum_{s=k}^{j-1} \bar{\Phi}_{j|s+1}(\bar{B}_s u_s + G_{s+1} y_{s+1})\right), \quad (49)$$

with

$$\bar{\Phi}_{j|k} \triangleq \bar{A}_{j-1} \bar{A}_{j-2} \cdots \bar{A}_k \quad (50)$$
$$\bar{B}_k \triangleq (I_n - G_{k+1} C_{k+1}) B_k. \quad (51)$$

First, rewrite (14) as

$$A_k = \bar{A}_k + G_{k+1}C_{k+1}A_k. \qquad (52)$$

Accordingly, in (48a), break the term $A_k x_k$ into two parts,

$$x_{k+1} = \bar{A}_k x_k + G_{k+1}C_{k+1}A_k x_k + B_k u_k + E_k d_k \qquad (53)$$

so that this equation is related to the UCO of $[\bar{A}_k, C_k]$ through $\bar{A}_k$. Recursively apply equation (53) for $j = k+1, k+2, \ldots,$ as if $G_{k+1}C_{k+1}A_k x_k + B_k u_k + E_k d_k$ was a single input term,

$$x_j = \bar{\Phi}_{j|k} x_k + \sum_{s=k}^{j-1} \bar{\Phi}_{j|s+1}(G_{s+1}C_{s+1}A_s x_s + B_s u_s + E_s d_s). \qquad (54)$$

Then (48b) leads to

$$\begin{aligned} y_j &= C_j \bar{\Phi}_{j|k} x_k \\ &+ C_j \sum_{s=k}^{j-1} \bar{\Phi}_{j|s+1}(G_{s+1}C_{s+1}A_s x_s + B_s u_s + E_s d_s). \end{aligned} \qquad (55)$$

This expression of y_j will be inserted into the right hand side of (49).

Notice that (51) implies

$$B_s - \bar{B}_s = G_{s+1}C_{s+1}B_s, \qquad (56)$$

and that (13) implies

$$E_s = G_{s+1}C_{s+1}E_s. \qquad (57)$$

Then

$$\begin{aligned} &(G_{s+1}C_{s+1}A_s x_s + B_s u_s + E_s d_s) - (\bar{B}_s u_s + G_{s+1}y_{s+1}) \\ &= G_{s+1}C_{s+1}A_s x_s + G_{s+1}C_{s+1}B_s u_s + G_{s+1}C_{s+1}E_s d_s \\ &\quad - G_{s+1}y_{s+1} \qquad (58) \\ &= G_{s+1}C_{s+1}(A_s x_s + B_s u_s + E_s d_s) - G_{s+1}y_{s+1} \qquad (59) \\ &= G_{s+1}C_{s+1}x_{s+1} - G_{s+1}y_{s+1} \qquad (60) \\ &= G_{s+1}y_{s+1} - G_{s+1}y_{s+1} \qquad (61) \\ &= 0. \qquad (62) \end{aligned}$$

where the last equalities were based on (48a) and then on (48b).

Continuing from (55) and using (62),

$$\begin{aligned} &y_j - C_j \sum_{s=k}^{j-1} \bar{\Phi}_{j|s+1}(\bar{B}_s u_s + G_{s+1}y_{s+1}) \qquad (63) \\ &= C_j \bar{\Phi}_{j|k} x_k \\ &+ C_j \sum_{s=k}^{j-1} \bar{\Phi}_{j|s+1}\big[(G_{s+1}C_{s+1}A_s x_s + B_s u_s + E_s d_s) \\ &\qquad\qquad - (\bar{B}_s u_s + G_{s+1}y_{s+1})\big] \qquad (64) \\ &= C_j \bar{\Phi}_{j|k} x_k. \qquad (65) \end{aligned}$$

The equality presumed in (49) is then confirmed. The UCO of $[\bar{A}_k, C_k]$ ensures the well-definedness of the inverse matrix in (49). □

3. A MATRIX INEQUALITY LEMMA

Lemma 3: Let $C \in \mathbb{R}^{m \times n}$, $R \in \mathbb{R}^{m \times m}$, $M \in \mathbb{R}^{n \times n}$ be matrices such that R is symmetric positive definite and M is symmetric positive semidefinite. Then for any $L \in \mathbb{R}^{n \times m}$,

$$\begin{aligned} &(I_n - LC)M(I_n - LC)^T + LRL^T \\ &\qquad \geq (1 + \|C^T R^{-1}C\| \|M\|)^{-1}M. \end{aligned} \qquad (66)$$

□

Proof.

$$\begin{aligned} &(I_n - LC)M(I_n - LC)^T + LRL^T \\ &= M - LCM - MC^T L^T + L(CMC^T + R)L^T \\ &= M - LCM - MC^T L^T + LSL^T \end{aligned}$$

with

$$S \triangleq CMC^T + R > 0.$$

Then,

$$\begin{aligned} &(I_n - LC)M(I_n - LC)^T + LRL^T \\ &= M + (L - MC^T S^{-1})S(L - MC^T S^{-1})^T \\ &\quad - MC^T S^{-1}CM \\ &\geq M - MC^T S^{-1}CM \\ &= M - MC^T(CMC^T + R)^{-1}CM \\ &= M^{\frac{1}{2}}(I_n - (CM^{\frac{1}{2}})^T(CM^{\frac{1}{2}}(CM^{\frac{1}{2}})^T + R)^{-1}CM^{\frac{1}{2}})M^{\frac{1}{2}} \end{aligned}$$

where $M^{1/2}$ is a symmetric square root of M.

Using the Woodbury matrix identity,

$$\begin{aligned} &(I_n - LC)M(I_n - LC)^T + LRL^T \\ &\geq M^{1/2}(I_n + (CM^{\frac{1}{2}})^T R^{-1}(CM^{\frac{1}{2}}))^{-1}M^{1/2} \\ &= M^{1/2}(I_n + M^{1/2}C^T R^{-1}CM^{1/2})^{-1}M^{1/2} \\ &\geq M^{1/2}\left(\|I_n + M^{1/2}C^T R^{-1}CM^{1/2}\|^{-1}I_n\right)M^{1/2} \\ &\geq \left(1 + \|C^T R^{-1}C\|\|M\|\right)^{-1}M, \end{aligned}$$

where, as defined at the beginning of Section II, $\|\cdot\|$ denotes the matrix norm induced by the Euclidean vector norm, which is equal to the largest singular value of the matrix. The lemma is then established. □

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