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Set-membership estimation based on ellipsoid bundles for discrete-time LPV descriptor systems [★]

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Abstract

This paper considers set-membership estimation for discrete-time linear parameter-varying descriptor systems. Ellipsoid bundle, a new set representation tool combining certain characteristics of ellipsoids and zonotopes, is used to design a set-membership estimation method for the considered systems. We use the L_∞ technique to design the estimator parameters in order to optimize estimation accuracy. This paper also studies the stability problem of the proposed method and establishes an offline sufficient stability condition. Finally, simulation results are presented to illustrate the effectiveness of the proposed method.

Key words: Set-membership estimation; Ellipsoid bundle; Descriptor systems; LPV systems; zonotope.

1 Introduction

State estimation is an important task for many engineering systems. In most of the existing results on state estimation, uncertainties are usually assumed to be random but with known probability distributions, typically Gaussian. This assumption may be less representative when dealing with unknown deterministic behaviors (Combastel, 2015). Instead of requiring probability distributions, set-membership estimation only assumes that uncertainties are bounded. Recently, set-membership estimation has received considerable attention. Different set representations have been used to design set-membership estimation methods, for instance intervals, ellipsoids, polytopes and zonotopes. A

pioneering work of set-membership estimation based on ellipsoids appeared in Schweppe (1968). Due to the low computational complexity of the ellipsoid-based method, it has attracted considerable attention. Especially, Kurzhanski has made significant contributions to the field of ellipsoid-based methods for state estimation and reachability analysis (Kurzhanski & Varaiya, 2000, 2002, 2006). Kurzhanski also has done substantial work on applying the ellipsoid-based methods to the motion control field (Kurzhanski & Mesyats, 2012, 2014). The polytope-based method (Blanchini & Miani, 2008) has the most accurate estimation results but suffers from large computational burden. One of the most important problems for set-membership estimation is to find a tradeoff between estimation accuracy and computational complexity. Since zonotopes can achieve a good tradeoff between estimation accuracy and computational complexity, zonotope-based methods have received considerable attention (Le, Stoica, Alamo, Camacho & Dumur, 2013; Combastel, 2015; Tang, Wang, Zhang & Shen, 2020a) recently. However, zonotope-based methods may not be suitable for the systems whose uncertainties are usually bounded by ellipsoids, for instance, the uncertainties of distances. Recently, a new set representation tool, ellipsoid bundle, has been proposed in Tang, Zhang, Wang & Shen (2020b), which combines certain characteristics of ellipsoids and zonotopes. In fact, ellipsoids and zonotopes are both particular

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instances of ellipsoid bundles. Therefore, ellipsoid bundles can be applied to a wider range of systems. Tang et al. (2020b) also proposed a set-membership estimation method based on ellipsoid bundles. However, only regular linear systems were considered in Tang et al. (2020b).

Set-membership estimation for nonlinear systems is a challenging task. Fortunately, the linear parameter-varying (LPV) model provides an effective approach to handle nonlinear systems. The descriptor system model is also widely used in many practical systems, for instance in power systems (Marszalek & Trzaska, 2004) and mechanical systems (Blajer & Kołodziejczyk, 2004). There are only limited results on set-membership estimation for descriptor systems (Wang, Puig & Cembrano, 2018a; Wang, Wang, Puig & Cembrano, 2019; Tang, Wang, Zhang & Shen, 2020a). In Wang et al. (2019), a zonotopic estimation method was proposed for discrete-time LPV descriptor systems. However, Wang et al. (2019) has not optimized all the coupled observer parameters, which may cause some conservatism.

This paper investigates the set-membership estimation for discrete-time LPV descriptor systems based on ellipsoid bundles. In Tang et al. (2020b), the ellipsoid bundle tool was proposed with a new set-membership estimation method for regular linear systems. The original contributions of this paper lie in the following aspects. First, the ellipsoid bundle tool is presented in a new concise form and a novel reduction method is proposed. Second, the set-membership estimation based on ellipsoid bundles is extended to LPV descriptor systems. The L_∞ technique is introduced in the robust design of estimator parameters to increase estimation accuracy. The parameters are obtained offline via solving a set of linear matrix inequalities (LMIs). Third, the approximation error caused by the reduction operator is analyzed quantitatively. Then, we investigate the stability problem of the proposed method and obtain an offline sufficient stability condition, which is weakened with respect to the existing result. Finally, simulation results demonstrate the performance of the proposed method.

2 Preliminaries and ellipsoid bundles

The following notations are standard in this paper. $\mathbb{R}^{m \times n}$ and \mathbb{R}^n denote the $m \times n$ and n dimensional Euclidean spaces, respectively. I_n represents the $n \times n$ dimensional identity matrix. A boldface letter represents a set in the rest of this paper. The symbols \leq and \geq are understood element-wise. For a real matrix $A \in \mathbb{R}^{m \times n}$, $\sigma_{max}(A)$ and $\sigma_{min}(A)$ represent the maximal and minimal singular values of A , respectively. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and $\|x\|_1$ denotes its 1-norm. Given a symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P \succ 0$ ($P \prec 0$) indicates that P is a positive (negative) definite matrix. And $P \succeq 0$ ($P \preceq 0$) indicates that P is

a positive (negative) semidefinite matrix. Given a series of matrices M_i ($i = 1, \dots, m$), $N = \text{diag}(M_1, \dots, M_m)$ returns a block diagonal matrix with its i -th diagonal component matrix equal to M_i . In a block matrix, $*$ represents a term that can be introduced by symmetry. Given a negative definite matrix $P \in \mathbb{R}^{n \times n}$, the inequality $\text{tr}(M^T P M) \leq 0$ holds for any matrix $M \in \mathbb{R}^{n \times m}$. Given a positive definite matrix $W \in \mathbb{R}^{n \times n}$ and a matrix $R \in \mathbb{R}^{n \times p}$, then $\|R\|_{F,W} = \sqrt{\text{tr}(R^T W R)}$ denotes the weighted Frobenius norm of R . The following definitions are fundamental for this paper:

Definition 1 An m -order zonotope is an affine transformation of the hypercube $\mathbf{B}^m = [-1, 1]^m$ as follows.

$$\mathcal{Z} = p \oplus H\mathbf{B}^m = \{x \in \mathbb{R}^n : x = p + Hz, z \in \mathbf{B}^m\},$$

where \oplus denotes the Minkowski sum, $p \in \mathbb{R}^n$ is the center of \mathcal{Z} and $H \in \mathbb{R}^{n \times m}$ defines its shape and size. For simplicity, we denote $\mathcal{Z} = \langle p, H \rangle$.

A common definition of ellipsoids is $\mathbf{E} = \{x \in \mathbb{R}^n : (x-c)^T P^{-1} (x-c) \leq 1\}$, where $c \in \mathbb{R}^n$ is the center of \mathbf{E} and $P \in \mathbb{R}^{n \times n}$ is a positive definite matrix. However, it cannot represent degenerate ellipsoids, for instance, an ellipse in a plane of 3-dimensional space. To cover both the non-degenerate and degenerate cases, we adopt the following definition (Durieu, Walter & Polyak, 2001).

Definition 2 An ellipsoid is an affine transformation of a ball, i.e.

$$\mathbf{E}(c, M) = \{x \in \mathbb{R}^n : x = c + Mz, z \in \mathbb{R}^m, z^T z \leq 1\}, \quad (1)$$

where $c \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times m}$.

Let $P = MM^T$. If P is positive definite, (1) can be converted into the common definition of ellipsoids. In fact, the matrix MM^T do not have to be invertible. According to Durieu et al. (2001), we can denote $\mathbf{E}^d(c, P) = \mathbf{E}(c, M)$ with $P = MM^T$ no matter P is invertible or not. Definition 2 is used in the rest of the paper. A single ellipsoid can hardly describe a set with complex shape. In set-membership estimation, using a single ellipsoid to over-approximate the complex reachable set of state may cause large conservatism. Zonotopes have more flexibility to describe complex sets, but they are not suitable in some cases where the uncertainties are bounded by ellipsoids. These drawbacks motivated us to design a new set representation tool, ellipsoid bundle, which combines certain characteristics of ellipsoids and zonotopes.

Definition 3 An m -order ellipsoid bundle $\mathcal{E}(c, \mathbf{H}) \subset \mathbb{R}^n$ is the Minkowski sum of the vector $c \in \mathbb{R}^n$ and m ellipsoids centered at the origin as follows.

$$\mathcal{E}(c, \mathbf{H}) = c \oplus \bigoplus_{i=1}^m \mathbf{E}(0, M_i),$$

where $\mathbf{H} = \{M_1, \dots, M_m\}$ is not a block matrix but an ordered set of a group of matrices and $M_i \in \mathbb{R}^{n \times p_i}$, $i = 1, \dots, m$. \mathbf{H} is called the generator of $\mathcal{E}(c, \mathbf{H})$.

The Minkowski sum and linear transformation of ellipsoid bundles can be converted into simple matrix operations.

Property 1 Minkowski sum: Given two ellipsoid bundles, $\mathcal{E}(c_1, \mathbf{H}_1) \subset \mathbb{R}^n$ and $\mathcal{E}(c_2, \mathbf{H}_2) \subset \mathbb{R}^n$, their Minkowski sum satisfies $\mathcal{E}_1 \oplus \mathcal{E}_2 = \mathcal{E}(\tilde{c}, \tilde{\mathbf{H}})$, where $\tilde{c} = c_1 + c_2$ and $\tilde{\mathbf{H}} = \{\mathbf{H}_1, \mathbf{H}_2\}$.

Property 2 Linear transformation: Given an m -order ellipsoid bundle $\mathcal{E}(c, \mathbf{H}) \subset \mathbb{R}^n$ and a matrix $L \in \mathbb{R}^{r \times n}$, we have $L\mathcal{E}(c, \mathbf{H}) = \mathcal{E}(Lc, L\mathbf{H})$, where $L\mathbf{H} = \{LM_1, \dots, LM_m\}$.

Remark 1. An m -order zonotope $\mathcal{Z} = \langle p, H \rangle$ can also be expressed as an ellipsoid bundle: $\mathcal{Z} = \mathcal{E}(p, \mathbf{H})$, where $\mathbf{H} = \{l_1, \dots, l_m\}$ and l_i is equal to the i -th column of matrix H .

Lemma 1 (Durieu et al., 2001) Given M ellipsoids $\mathbf{E}_k^d(c_k, P_k)$ ($k = 1, \dots, K$), define $\mathbf{M}_K = \bigoplus_{k=1}^K \mathbf{E}_k^d(c_k, P_k)$. Let \mathcal{D}^{+*} be the convex set of all vectors $\alpha \in \mathbb{R}^K$ with all $\alpha_k > 0$ and $\sum_{k=1}^K \alpha_k = 1$. For all $\alpha \in \mathcal{D}^{+*}$, the ellipsoid $\mathbf{E}_\alpha = \mathbf{E}^d(c^*, P_\alpha)$, with $c^* = \sum_{k=1}^K c_k$ and $P_\alpha = \sum_{k=1}^K \alpha_k^{-1} P_k$, contains \mathbf{M}_K .

To apply the ellipsoid bundle tool to set-membership estimation, a reduction operator is necessary, which encloses a high order ellipsoid bundle with a lower one. The reduction operator can be described by the following lemma.

Lemma 2 Given an m -order ellipsoid bundle $\mathcal{E}(c, \mathbf{H}) \subset \mathbb{R}^n$, where $\mathbf{H} = \{M_1, \dots, M_m\}$. Reorder M_i in decreasing order of $\|M_i\|_{F,W}$, where $W \in \mathbb{R}^{n \times n}$ is a positive definite matrix. The reordered set is denoted as $\tilde{\mathbf{H}} = \{M_{\sigma(1)}, \dots, M_{\sigma(m)}\}$, where $\sigma(i)$ is a permutation of $i = 1, \dots, m$. Given a fix integer s ($1 < s < m$), we can obtain an s -order ellipsoid bundle $\mathcal{E}(c, \mathcal{R}_{s,W}(\mathbf{H}))$ such that $\mathcal{E}(c, \mathbf{H}) \subseteq \mathcal{E}(c, \mathcal{R}_{s,W}(\mathbf{H}))$, where

$$\mathcal{R}_{s,W}(\mathbf{H}) = \{M_{\sigma(1)}, \dots, M_{\sigma(s-1)}, \tilde{M}\}, \quad \tilde{M}\tilde{M}^T = \tilde{P},$$

$$\tilde{P} = \sum_{i=s}^m \|M_{\sigma(i)}\|_{F,W} \sum_{j=s}^m \frac{P_{\sigma(j)}}{\|M_{\sigma(j)}\|_{F,W}}, \quad (2)$$

where $P_{\sigma(j)} = M_{\sigma(j)} M_{\sigma(j)}^T$ and \tilde{M} can be obtained by Cholesky decomposition if \tilde{P} is positive definite, otherwise by the singular value decomposition.

PROOF. The reordered ellipsoid bundle $\mathcal{E}(c, \tilde{\mathbf{H}}) = \mathcal{E}(c, \mathbf{H}_1) \oplus \mathcal{E}(0, \mathbf{H}_2)$, where $\mathbf{H}_1 = \{M_{\sigma(1)}, \dots, M_{\sigma(s-1)}\}$

and $\mathbf{H}_2 = \{M_{\sigma(s)}, \dots, M_{\sigma(m)}\}$. According to Definition 3, $\mathcal{E}(c, \mathbf{H}_2)$ satisfies $\mathcal{E}(0, \mathbf{H}_2) = \bigoplus_{i=\sigma(s)}^{\sigma(m)} \mathbf{E}(0, M_{\sigma(i)}) = \bigoplus_{i=\sigma(s)}^{\sigma(m)} \mathbf{E}^d(0, P_{\sigma(i)})$, where $P_{\sigma(i)} = M_{\sigma(i)} M_{\sigma(i)}^T$. From (2), we have $\tilde{P} = \sum_{k=1}^{m-s+1} \alpha_k^{-1} P_{\sigma(k+s-1)}$, where $\alpha_k = \frac{\|M_{\sigma(k+s-1)}\|_{F,W}}{\sum_{i=s}^m \|M_{\sigma(i)}\|_{F,W}}$. Then, according to Lemma 1, $\mathbf{E}^d(0, \tilde{P}) \supseteq \mathcal{E}(0, \mathbf{H}_2)$. In addition, $\mathbf{E}^d(0, \tilde{P}) = \mathbf{E}(0, \tilde{M}) = \mathcal{E}(0, \{\tilde{M}\})$. Therefore, $\mathcal{E}(0, \mathbf{H}_2) \subseteq \mathcal{E}(0, \{\tilde{M}\})$. It follows that $\mathcal{E}(c, \tilde{\mathbf{H}}) \subseteq \mathcal{E}(c, \mathbf{H}_1) \oplus \mathcal{E}(0, \{\tilde{M}\}) = \mathcal{E}(c, \mathcal{R}_{s,W}(\mathbf{H}))$. Since the reordering does not change the set, we have $\mathcal{E}(c, \mathbf{H}) \subseteq \mathcal{E}(c, \mathcal{R}_{s,W}(\mathbf{H}))$. \square

Remark 2. The matrix \tilde{M} satisfying $\tilde{M}\tilde{M}^T = \tilde{P}$ is not unique, but all the possible matrices define the same ellipsoid since all the ellipsoids defined by any possible \tilde{M} have the same support function for any vector $l \in \mathbb{R}^n$ and therefore they are the same set (Schweppe, 1968).

3 Problem formulation

Consider the following discrete-time LPV descriptor systems:

$$\begin{cases} E x_{k+1} = A(\rho_k) x_k + B(\rho_k) u_k + D_w(\rho_k) w_k \\ y_k = C(\rho_k) x_k + D_v(\rho_k) v_k \end{cases} \quad (3)$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ and $y_k \in \mathbb{R}^{n_y}$ are the vectors of state, input and measurement output, respectively. $w_k \in \mathbb{R}^{n_w}$ and $v_k \in \mathbb{R}^{n_v}$ are the process disturbance and measurement noise. $E \in \mathbb{R}^{n_e \times n_x}$ can be singular or even not square. $A(\rho_k) \in \mathbb{R}^{n_e \times n_x}$, $B(\rho_k) \in \mathbb{R}^{n_e \times n_u}$, $D_w(\rho_k) \in \mathbb{R}^{n_e \times n_w}$, $C(\rho_k) \in \mathbb{R}^{n_y \times n_x}$ and $D_v(\rho_k) \in \mathbb{R}^{n_y \times n_v}$ are parameter-varying matrices scheduled by the measurable vector ρ_k satisfying $A(\rho_k) = \sum_{i=1}^q h_i(\rho_k) A_i$, $B(\rho_k) = \sum_{i=1}^q h_i(\rho_k) B_i$, $D_w(\rho_k) = \sum_{i=1}^q h_i(\rho_k) D_{wi}$, $C(\rho_k) = \sum_{i=1}^q h_i(\rho_k) C_i$ and $D_v(\rho_k) = \sum_{i=1}^q h_i(\rho_k) D_{vi}$, where $A_i \in \mathbb{R}^{n_e \times n_x}$, $B_i \in \mathbb{R}^{n_e \times n_u}$, $D_{wi} \in \mathbb{R}^{n_e \times n_w}$, $C_i \in \mathbb{R}^{n_y \times n_x}$ and $D_{vi} \in \mathbb{R}^{n_y \times n_v}$ are known matrices. $h_i(\rho_k)$ ($i = 1, \dots, q$) is the weighting function satisfying $\sum_{i=1}^q h_i(\rho_k) = 1$ and $h_i(\rho_k) \geq 0$ for all $i = 1, \dots, q$. For simplicity, denote $\rho = \rho_k$ and $\rho_+ = \rho_{k+1}$. In fact, the system (3) is a polytopic LPV system.

Assumption 1 The initial state x_0 , the process disturbance w_k and the measurement noise v_k are assumed to be unknown but belong to some known bounded sets:

$$x_0 \in \mathcal{E}(\hat{x}_0, \mathbf{H}_0), \quad w_k \in \mathcal{E}(0, \mathbf{H}_w), \quad v_k \in \mathcal{E}(0, \mathbf{H}_v), \quad (4)$$

where $\hat{x}_0 \in \mathbb{R}^{n_x}$, $\mathbf{H}_0 = \{M_1^0, \dots, M_{m_0}^0\}$ ($M_i^0 \in \mathbb{R}^{n_x \times n_{x_i}}$), $\mathbf{H}_w = \{M_1^w, \dots, M_{m_w}^w\}$ ($M_i^w \in \mathbb{R}^{n_w \times n_{w_i}}$) and $\mathbf{H}_v = \{M_1^v, \dots, M_{m_v}^v\}$ ($M_i^v \in \mathbb{R}^{n_v \times n_{v_i}}$).

Assumption 2 For system (3), we assume that

$$\text{rank} \begin{bmatrix} E \\ C_i \end{bmatrix} = n_x, \quad i = 1, \dots, q. \quad (5)$$

In this paper, we aim to estimate a bounded set \mathbf{X}_k such that $x_k \in \mathbf{X}_k$ for all $k \geq 0$. In the rest of this paper, we propose a set-membership estimation method for system (3) based on ellipsoid bundles. To obtain accurate estimation, a robust design method for the estimator parameters is proposed, which can be converted into solving a set of LMIs. We also investigate the stability condition of the proposed set-membership estimation method.

4 Set-membership estimation based on ellipsoid bundles

4.1 Set-membership estimator

Consider the following observer structure, which is modified from Wang, Shen, Zhang & Wang (2012).

$$\begin{aligned} \hat{x}_{k+1} = & T(\rho_+)A(\rho)\hat{x}_k + T(\rho_+)B(\rho)u_k + Ny_{k+1} \\ & + L(\rho)(y_k - C(\rho)\hat{x}_k), \end{aligned} \quad (6)$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the single state trajectory estimation, $T(\rho_+) = \sum_{i=1}^q h_i(\rho_+)T_i$ and $L(\rho) = \sum_{i=1}^q h_i(\rho)L_i$. $T_i \in \mathbb{R}^{n_x \times n_e}$, $L_i \in \mathbb{R}^{n_x \times n_y}$, $i = 1, \dots, q$ and $N \in \mathbb{R}^{n_x \times n_y}$ are the estimator parameters to be designed. In addition, $T(\rho_+)$ and N should satisfy the following equality constraint:

$$T(\rho_+)E + NC(\rho_+) = I_{n_x}, \quad (7)$$

which is equivalent to $\sum_{i=1}^q h_i(\rho_+)(T_iE + NC_i) = I_{n_x}$ and a sufficient condition for it is $T_iE + NC_i = I_{n_x}$, $i = 1, \dots, q$. It can be rewritten as $[T_1, \dots, T_q, N] \Theta =$

Π , where $\Theta = \begin{bmatrix} I_q \otimes E \\ \bar{C} \end{bmatrix}$, $\bar{C} = [C_1, \dots, C_q]$ and $\Pi =$

$\mathbf{1}_q \otimes I_{n_x}$. Thereinto, \otimes denotes Kronecker product and $\mathbf{1}_q$ is the q -dimensional row vector with its every component equal to 1. From (5), Θ has full column rank. According to Wang, Lim & Shen (2018b), the general solutions to T_i and N are as follows.

$$\begin{aligned} T_i = & \Pi\Theta^\dagger A_i + S\Omega A_i, \quad i = 1, \dots, q \\ N = & \Pi\Theta^\dagger A_{q+1} + S\Omega A_{q+1}, \quad \Omega = I_{q \times n_x + n_y} - \Theta\Theta^\dagger, \end{aligned}$$

where $S \in \mathbb{R}^{n_x \times (q \times n_x + n_y)}$ is an arbitrary matrix and

$$A_i = \begin{bmatrix} 0_{(i-1)n_x \times n_x} \\ I_{n_x} \\ 0_{((q-i)n_x + n_y) \times n_x} \end{bmatrix}, \quad A_{q+1} = \begin{bmatrix} 0_{(q \times n_x) \times n_y} \\ I_{n_y} \end{bmatrix}.$$

Based on the observer structure (6), we propose a set-membership estimation method based on ellipsoid bundles for system (3). The proposed method can be described by the following theorem.

Theorem 1 For system (3), if $x_k \in \mathcal{E}(\hat{x}_k, \mathbf{H}_k)$ and \mathbf{H}_k has m component matrices, then we can obtain an ellipsoid bundle $\mathcal{E}(\hat{x}_{k+1}, \mathbf{H}_{k+1})$ such that $x_{k+1} \in \mathcal{E}(\hat{x}_{k+1}, \mathbf{H}_{k+1})$, where \hat{x}_{k+1} satisfies (6) and \mathbf{H}_{k+1} is obtained from

$$\mathbf{H}_{k+1} = \{ \bar{A}(\rho)\bar{\mathbf{H}}_k, T(\rho_+)D_w(\rho)\mathbf{H}_w, -L(\rho)D_v(\rho)\mathbf{H}_v, -ND_v(\rho_+)\mathbf{H}_v \} \quad (8)$$

where $\bar{A}(\rho) = T(\rho_+)A(\rho) - L(\rho)C(\rho)$. $\bar{\mathbf{H}}_k = \mathcal{R}_{s, Q(\rho)}(\mathbf{H}_k)$ when $m > s$ and $\bar{\mathbf{H}}_k = \mathbf{H}_k$ when $m \leq s$, where $Q(\rho)$ is a sequence of positive definite matrices to be designed.

PROOF. From (3) and (7), we have

$$\begin{aligned} x_{k+1} = & (T(\rho_+)E + NC(\rho_+))x_{k+1} \\ = & T(\rho_+)Ex_{k+1} + Ny_{k+1} - ND_v(\rho_+)v_{k+1} \\ = & T(\rho_+)A(\rho)x_k + T(\rho_+)B(\rho)u_k \\ & + T(\rho_+)D_w(\rho)w_k + Ny_{k+1} - ND_v(\rho_+)v_{k+1} \end{aligned} \quad (9)$$

Define the estimation error as $e_k = x_k - \hat{x}_k$. If $x_k \in \mathcal{E}(\hat{x}_k, \mathbf{H}_k)$, we have $x_k \in \mathcal{E}(\hat{x}_k, \bar{\mathbf{H}}_k)$ according to Lemma 2. It follows that $e_k \in \mathcal{E}(0, \bar{\mathbf{H}}_k)$. Subtracting (6) from (9) yields the following error system:

$$\begin{aligned} e_{k+1} = & \bar{A}(\rho)e_k + T(\rho_+)D_w(\rho)w_k - L(\rho)D_v(\rho)v_k \\ & - ND_v(\rho_+)v_{k+1}. \end{aligned} \quad (10)$$

Then, from (4), we have

$$\begin{aligned} e_{k+1} \in & \bar{A}(\rho)\mathcal{E}(0, \bar{\mathbf{H}}_k) \oplus T(\rho_+)D_w(\rho)\mathcal{E}(0, \mathbf{H}_w) \\ & \oplus -L(\rho)D_v(\rho)\mathcal{E}(0, \mathbf{H}_v) \oplus -ND_v(\rho_+)\mathcal{E}(0, \mathbf{H}_v), \end{aligned}$$

which implies $e_{k+1} \in \mathcal{E}(0, \mathbf{H}_{k+1})$ according to Property 1 and Property 2. Since $x_{k+1} = \hat{x}_{k+1} + e_{k+1}$, we have $x_{k+1} \in \mathcal{E}(\hat{x}_{k+1}, \mathbf{H}_{k+1})$. \square

After obtaining the estimation set $\mathcal{E}(\hat{x}_k, \mathbf{H}_k)$ with $\mathbf{H}_k = \{M_{1,k}, \dots, M_{m,k}\}$, an interval estimation $[\underline{x}_k, \bar{x}_k]$ can be achieved based on the following Lemma derived from Theorem 4 in Tang et al. (2020b).

Lemma 3 (Tang et al., 2020b) If $x_k \in \mathcal{E}(\hat{x}_k, \mathbf{H}_k) = \hat{x}_k \oplus \bigoplus_{i=1}^m \mathbf{E}^d(0, \bar{P}_i^k)$ with $P_i^k = M_{i,k} M_{i,k}^T$, we can obtain an interval vector $[\underline{x}_k, \bar{x}_k]$ such that $\underline{x}_k \leq x_k \leq \bar{x}_k$ with

$$\begin{cases} \bar{x}_k = \hat{x}_k + \bar{e}_k \\ \underline{x}_k = \hat{x}_k + \underline{e}_k \end{cases}$$

where $\underline{e}_k = -\bar{e}_k$ and $\bar{e}_k(j) = \sum_{i=1}^m \sqrt{P_i^k(j, j)}$, $j = 1, \dots, n_x$.

4.2 Robust design of the estimator parameters

In order to obtain accurate estimation, we should design T_i , N and L_i ($i = 1, \dots, q$) such that estimation error is robust against the process disturbance and measurement noise. The error system (10) can be rewritten as

$$e_{k+1} = \bar{A}(\rho)e_k + \bar{B}(\rho)d_k, \quad (11)$$

where $d_k = [w_k^T, v_k^T, v_{k+1}^T]^T$ and $\bar{B}(\rho) = [T(\rho_+)D_w(\rho), -L(\rho)D_v(\rho), -ND_v(\rho_+)]$.

We aim to design T_i , N and L_i ($i = 1, \dots, q$) such that the error system satisfies the following L_∞ index:

$$\|e_k\| < \gamma \sqrt{\alpha^k V_0 + \|d\|_\infty^2}, \quad (12)$$

where $\gamma > 0$, $0 < \alpha < 1$, $V_0 = e_0^T Q(\rho_0) e_0$, $Q(\rho_0) \in \mathbb{R}^{n_x \times n_x}$ is positive definite and $\|d\|_\infty = \max_{k \geq 0} \|d_k\|$, called the L_∞ norm of d_k . The following theorem gives the conditions for $T(\rho_+)$, N and $L(\rho)$ to satisfy the L_∞ index (12).

Theorem 2 Given $\gamma > 0$, $0 < \alpha < 1$, if there exist a sequence of positive definite matrices, $Q(\rho) \in \mathbb{R}^{n_x \times n_x}$, and an invertible matrix $G \in \mathbb{R}^{n_x \times n_x}$ such that the following inequalities hold, then (12) is satisfied.

$$\begin{bmatrix} -\alpha Q(\rho) & 0 & 0 & 0 & * \\ 0 & -\beta I_{n_w} & 0 & 0 & * \\ 0 & 0 & -\beta I_{n_v} & 0 & * \\ 0 & 0 & 0 & -\beta I_{n_v} & * \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & Q(\rho_+) - G - G^T \end{bmatrix} < 0, \quad (13)$$

$$\frac{1}{\gamma^2} I_{n_x} - Q(\rho) < 0, \quad (14)$$

where $\beta = 1 - \alpha$, $Q(\rho) = \sum_{i=1}^q h_i(\rho) Q_i$ and

$$\begin{aligned} \Gamma_1 &= GT(\rho_+)A(\rho) - GL(\rho)C(\rho), & \Gamma_2 &= GT(\rho_+)D_w(\rho), \\ \Gamma_3 &= -GL(\rho)D_v(\rho), & \Gamma_4 &= -GND_v(\rho_+). \end{aligned} \quad (15)$$

PROOF. The inequality (13) can be rewritten as

$$\begin{bmatrix} -\alpha Q(\rho) & * & * \\ 0 & -\beta I_{n_w+2n_v} & * \\ G\bar{A}(\rho) & G\bar{B}(\rho) & Q(\rho_+) - G - G^T \end{bmatrix} < 0. \quad (16)$$

Pre-multiply and post-multiply (16) with the following matrix

$$\begin{bmatrix} I_{n_x} & 0 & \bar{A}(\rho)^T \\ 0 & I_{n_w+2n_v} & \bar{B}(\rho)^T \end{bmatrix}$$

and its transpose respectively. Then we have

$$\begin{bmatrix} -\alpha Q(\rho) + \bar{A}(\rho)^T Q(\rho_+) \bar{A}(\rho) & * \\ \bar{B}(\rho)^T Q(\rho_+) \bar{A}(\rho) & M_{22} \end{bmatrix} < 0, \quad (17)$$

where $M_{22} = -\beta I_{n_w+2n_v} + \bar{B}(\rho)^T Q(\rho_+) \bar{B}(\rho)$. By pre-multiplying and post-multiplying (17) with $[e_k^T, d_k^T]$ and its transpose respectively, we have

$$\begin{aligned} & -\alpha e_k^T Q(\rho) e_k + e_k^T \bar{A}(\rho)^T Q(\rho_+) \bar{A}(\rho) e_k \\ & + e_k^T \bar{A}(\rho)^T Q(\rho_+) \bar{B}(\rho) d_k + d_k^T \bar{B}(\rho)^T Q(\rho_+) \bar{A}(\rho) e_k \\ & - \beta d_k^T d_k + d_k^T \bar{B}(\rho)^T Q(\rho_+) \bar{B}(\rho) d_k \leq 0. \end{aligned} \quad (18)$$

Define a Lyapunov function: $V_k = e_k^T Q(\rho) e_k$. From (11) and (18), we have $V_{k+1} \leq \alpha V_k + (1 - \alpha) d_k^T d_k$, which implies

$$\begin{aligned} V_k &\leq \alpha^k V_0 + (1 - \alpha) \sum_{i=0}^{k-1} \alpha^i d_{k-1-i}^T d_{k-1-i} \\ &\leq \alpha^k V_0 + (1 - \alpha) \sum_{i=0}^{k-1} \alpha^i \|d\|_\infty^2 \\ &< \alpha^k V_0 + \|d\|_\infty^2. \end{aligned} \quad (19)$$

Pre-multiplying and post-multiplying (14) with e_k^T and e_k yield $e_k^T e_k = \|e_k\|^2 \leq \gamma^2 e_k^T Q(\rho) e_k = \gamma^2 V_k$. Then, from (19), we have $\|e_k\|^2 < \gamma^2 (\alpha^k V_0 + \|d\|_\infty^2)$, which is equivalent to (12). \square

Note that there exist couplings between the matrix G and the estimator parameters. It is difficult to solve the inequalities (13) and (14). In order to obtain the estimator parameters, we propose a design method based on solving a set of LMIs, which can be described by the following theorem.

Theorem 3 Given $\gamma > 0$ and $0 < \alpha < 1$, if there exist a sequence of positive definite matrices $Q_i \in \mathbb{R}^{n_x \times n_x}$

($i = 1, \dots, q$), matrices $Y \in \mathbb{R}^{n_x \times (q \times n_x + n_y)}$, $W_i \in \mathbb{R}^{n_x \times n_y}$ ($i = 1, \dots, q$) and an invertible matrix $G \in \mathbb{R}^{n_x \times n_x}$ such that the following LMIs hold, then the L_∞ index is satisfied.

$$\begin{cases} \Psi_{ijj} \prec 0, & i, j \in 1, \dots, q, \\ \Psi_{ijl} + \Psi_{ilj} \prec 0, & j < l, \quad i, j, l \in 1, \dots, q, \\ \delta I_{n_x} - Q_i \prec 0 & i = 1, \dots, q, \end{cases} \quad (20)$$

where $\delta = \frac{1}{\gamma^2}$ and

$$\Psi_{ijl} = \begin{bmatrix} -\alpha Q_l & * & * & * & * \\ 0 & -\beta I_{n_w} & * & * & * \\ 0 & 0 & -\beta I_{n_v} & * & * \\ 0 & 0 & 0 & -\beta I_{n_v} & * \\ \Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 & Q_i - G - G^T \end{bmatrix}, \quad (21)$$

$$\begin{aligned} \Phi_1 &= G\Pi\Theta^\dagger \Lambda_i A_j + Y\Omega \Lambda_i A_j - W_j C_l, \\ \Phi_2 &= G\Pi\Theta^\dagger \Lambda_i D_{wj} + Y\Omega \Lambda_i D_{wj} \\ \Phi_3 &= -W_j D_{vl}, \\ \Phi_4 &= -G\Pi\Theta^\dagger \Lambda_{q+1} D_{vi} - Y\Omega \Lambda_{q+1} D_{vi}, \end{aligned} \quad (22)$$

and T_i , N , L_i ($i = 1, \dots, q$) are obtained from the following equations:

$$\begin{aligned} T_i &= \Pi\Theta^\dagger \Lambda_i + G^{-1}Y\Omega \Lambda_i, & L_i &= G^{-1}W_i, \\ N &= \Pi\Theta^\dagger \Lambda_{q+1} + G^{-1}Y\Omega \Lambda_{q+1}. \end{aligned} \quad (23)$$

PROOF. By pre-multiplying (23) with G and substituting it into (22), we have

$$\begin{aligned} \Phi_1 &= G(T_i A_j - L_j C_l), & \Phi_2 &= G T_i D_{wj}, \\ \Phi_3 &= -G L_j D_{vl}, & \Phi_4 &= -G N D_{vi} \end{aligned} \quad (24)$$

From (15), (21) and (24), the inequality (13) is equivalent to

$$\begin{aligned} &\sum_{i=1}^q \sum_{j=1}^q \sum_{l=1}^q h_i(\rho_+) h_j(\rho) h_l(\rho) \Psi_{ijl} = \\ &= \sum_{i=1}^q \sum_{j=1}^q h_i(\rho_+) h_j(\rho)^2 \Psi_{ijj} \\ &+ \sum_{i=1}^q \sum_{j=1}^q \sum_{l=j+1}^q h_i(\rho_+) h_j(\rho) h_l(\rho) (\Psi_{ijl} + \Psi_{ilj}) \prec 0. \end{aligned}$$

Therefore, (20) implies (13) and (14). Then, according to Theorem 2, the L_∞ index is satisfied. \square

Remark 3. Define an ellipsoid $\tilde{\mathbf{E}}_k = \{x: x^T Q(\rho)x \leq \alpha^k V_0 + \|d\|_\infty^2\}$. According to (19), $V_k = e_k^T Q(\rho) e_k < \alpha^k V_0 + \|d\|_\infty^2$, which implies $e_k \in \tilde{\mathbf{E}}_k$. Define an ellipsoid $\mathbf{E} = \{x: x^T Q(\rho)x \leq \|d\|_\infty^2\}$. Since $0 < \alpha < 1$, we have $\lim_{k \rightarrow \infty} \alpha^k V_0 = 0$, which implies $\lim_{k \rightarrow \infty} \tilde{\mathbf{E}}_k = \mathbf{E}$. Therefore, the size of the ellipsoid \mathbf{E} can be considered as an estimation accuracy criterion when k is big enough, which is reasonable since $\alpha^k V_0$ decreases exponentially. To increase estimation accuracy, the ellipsoid \mathbf{E} should be as small as possible. Since $Q(\rho)$ determines the size of \mathbf{E} and $2/\sqrt{\sigma_{\min}(Q(\rho))}$ is its diameter (Boyd, El Ghaoui, Feron & Balakrishnan, 1994), a direct idea to increase the estimation accuracy is to maximize $\sigma_{\min}(Q(\rho))$. However, it is not easy to maximize $\sigma_{\min}(Q(\rho))$. Note that $\sigma_{\min}(Q(\rho)) > \frac{1}{\gamma^2} = \delta$ according to (14). Therefore, a reasonable way to increase the estimation accuracy is to solve the following optimization problem:

$$\begin{aligned} &\max \delta, \\ &s.t. (20). \end{aligned} \quad (25)$$

If δ is maximized, γ is minimized since $\gamma = \sqrt{\frac{1}{\delta}}$. In addition, (20) implies (12). Therefore, the L_∞ index (12) is also optimized when the optimization problem (25) is solved.

5 Stability analysis

In practical applications, the stability of the estimation is important for the safety of systems. However, many existing results on set-membership estimation have not considered the stability problem. In Martinez, Loukkas & Meslem (2020), H_∞ design is introduced into the set-membership observer design for discrete-time LPV systems. Observer parameters are designed based on solving a finite number of offline LMI conditions to make the set-membership estimation stable. Then, the ellipsoidal robustly positive invariant (RPI) sets for the estimation error dynamics are obtained. The stable and robust set-membership estimation is obtained by combining the state estimation with its corresponding estimation error bounds. In Combastel (2015), an offline stability condition of the set-membership estimation based on zonotopes is established. The reduction operator on zonotopes will introduce approximation error. Combastel (2015) has proved that the approximation error is bounded and can be controlled by setting the reduction order. If the order of the reduced zonotope remains large enough, then the set-membership estimation is bounded, i.e. stable (see Theorem 12 in Combastel (2015)). The

reduction operator in this paper will also cause approximation error. Fortunately, the approximation error is bounded and can be controlled by choosing a big enough reduction order s . Define a size criterion of $\mathcal{E}(c, \mathbf{H})$ as $J(\mathbf{H}) = \sum_{i=1}^m \|M_i\|_{F,W}^2$. Then, we have the following conclusion:

Theorem 4 *Given an m -order ellipsoid bundle $\mathcal{E}(c, \mathbf{H}) \subset \mathbb{R}^n$ and applying the reduction operator $\mathcal{R}_{s,W}(\mathbf{H})$ to it, the following inequality holds:*

$$J(\mathcal{R}_{s,W}(\mathbf{H})) - J(\mathbf{H}) \leq \frac{d(d+1)}{s+d} J(\mathbf{H}), \quad d = m - s.$$

PROOF. Since W is positive definite, there certainly exists a matrix $S \in \mathbb{R}^{n \times n}$ such that $S^T S = W$. From (2), we have

$$S\tilde{M}\tilde{M}^T S^T = \sum_{i=s}^m \|M_{\sigma(i)}\|_{F,W} \sum_{j=s}^m \frac{SM_{\sigma(j)}M_{\sigma(j)}^T S^T}{\|M_{\sigma(j)}\|_{F,W}}. \quad (26)$$

According to the basic linear algebra, we have that $\text{tr}(S\tilde{M}\tilde{M}^T S^T) = \text{tr}(\tilde{M}^T S^T S\tilde{M}) = \text{tr}(\tilde{M}^T W\tilde{M}) = \|\tilde{M}\|_{F,W}^2$. Similarly, $\text{tr}(SM_{\sigma(j)}M_{\sigma(j)}^T S^T) = \|M_{\sigma(j)}\|_{F,W}^2$, $j = s, \dots, m$. Then, from (26), we have

$$\begin{aligned} \|\tilde{M}\|_{F,W}^2 &= \sum_{i=s}^m \|M_{\sigma(i)}\|_{F,W} \sum_{j=s}^m \frac{\|M_{\sigma(j)}\|_{F,W}^2}{\|M_{\sigma(j)}\|_{F,W}} \\ &= \left(\sum_{i=s}^m \|M_{\sigma(i)}\|_{F,W} \right)^2 \end{aligned} \quad (27)$$

Define $\Sigma_1 = \sum_{i=1}^{s-1} \|M_{\sigma(i)}\|_{F,W}^2$ and $\Sigma_2 = \sum_{i=s}^m \|M_{\sigma(i)}\|_{F,W}^2$, then $J(\mathbf{H}) = J(\tilde{\mathbf{H}}) = \Sigma_1 + \Sigma_2$. In addition, according to the mean value inequality, (27) implies $\|\tilde{M}\|_{F,W}^2 \leq (d+1)\Sigma_2$. Since $\|M_{\sigma(i)}\|_{F,W} \geq \|M_{\sigma(j)}\|_{F,W}$ if $i < j$, we have $\Sigma_1 \geq (s-1)\|M_{\sigma(s-1)}\|_{F,W}^2$ and $\Sigma_2 \leq (d+1)\|M_{\sigma(s-1)}\|_{F,W}^2$, which imply $\Sigma_2 \leq (d+1)/(s-1)\Sigma_1$. In addition, $\Sigma_1 = J(\mathbf{H}) - \Sigma_2$, then we have $\Sigma_2 \leq (d+1)/(s-1)(J(\mathbf{H}) - \Sigma_2)$. It follows that $\Sigma_2 \leq (d+1)/(s+d)J(\mathbf{H})$. Since $J(\mathcal{R}_{s,W}(\mathbf{H})) = \Sigma_1 + \|\tilde{M}\|_{F,W}^2$ and $\|\tilde{M}\|_{F,W}^2 \leq (d+1)\Sigma_2$, we have $J(\mathcal{R}_{s,W}(\mathbf{H})) - J(\mathbf{H}) = \|\tilde{M}\|_{F,W}^2 - \Sigma_2 \leq \frac{d(d+1)}{s+d} J(\mathbf{H})$. \square

In Combastel (2015), the choice of the reduction order to guarantee the estimation stability is dependent on the weighting matrix W_k . If $\sigma_{max}(W_k)/\sigma_{min}(W_k)$ is too big, the reduction order to guarantee the stability will be very large. Thanks to Theorem 4, the stability condition of the proposed method is not influenced by the weighting matrix, as shown in the following theorem.

Theorem 5 *The observer parameters of the proposed method have been designed to satisfy the conditions of Theorem 3. Then, the set-membership estimation obtained by the proposed method is stable if the reduction order s ($s < m$) is big enough such that*

$$\alpha\eta < 1, \quad \eta = \frac{s+r(r+2)}{s+r}, \quad r = m_w + 2m_v.$$

PROOF. The size of the ellipsoid bundle $\mathcal{E}(\hat{x}_k, \mathbf{H}_k)$ depends on $\mathbf{H}_k = \{M_{1,k}, \dots, M_{m,k}\}$. And we have $J(\mathbf{H}_k) = \sum_{i=1}^m \|M_{i,k}\|_{F,Q(\rho)}^2 = \sum_{i=1}^m \text{tr}(M_{i,k}^T Q(\rho) M_{i,k})$. Since $Q(\rho)$ is positive definite, $M_{i,k}$ ($i = 1, \dots, m$) is bounded if $J(\mathbf{H}_k)$ is bounded, which implies the size of $\mathcal{E}(\hat{x}_k, \mathbf{H}_k)$ is bounded.

Denote the block matrix corresponding to the set $\tilde{\mathbf{H}}_k$ as $\tilde{H}_k = [\tilde{M}_{1,k}, \dots, \tilde{M}_{s,k}]$. Similarly, we obtain the block matrices H_w and H_v . Define a block diagonal matrix $U = \text{diag}(\tilde{H}_k, H_d)$, where $H_d = \text{diag}(H_w, H_v, H_v)$.

Pre-multiplying and post-multiplying (17) with U^T and U respectively and considering the trace, we have

$$\begin{aligned} & -\alpha \text{tr}(\tilde{H}_k^T Q(\rho) \tilde{H}_k) + \text{tr}(\tilde{H}_k^T \bar{A}(\rho)^T Q(\rho_+) \bar{A}(\rho) \tilde{H}_k) \\ & + 2\text{tr}(\tilde{H}_k^T \bar{A}(\rho)^T Q(\rho_+) \bar{B}(\rho) H_d) - \beta \text{tr}(H_d^T H_d) \\ & + \text{tr}(H_d^T \bar{B}(\rho)^T Q(\rho_+) \bar{B}(\rho)) \leq 0. \end{aligned} \quad (28)$$

Denote the block matrix corresponding to the set \mathbf{H}_{k+1} as H_{k+1} . From (8), we have $H_{k+1} = [\bar{A}(\rho) \tilde{H}_k, \bar{B}(\rho) H_d]$. Then, (28) can be reformulated as

$$\begin{aligned} & -\alpha \text{tr}(\tilde{H}_k^T Q(\rho) \tilde{H}_k) + \text{tr}(H_{k+1}^T Q(\rho_+) H_{k+1}) \\ & - \beta \text{tr}(H_d^T H_d) \leq 0 \end{aligned} \quad (29)$$

Note that $\text{tr}(\tilde{H}_k^T Q(\rho) \tilde{H}_k) = \sum_{i=1}^s \|\tilde{M}_{i,k}\|_{F,Q(\rho)}^2 = J(\tilde{\mathbf{H}}_k)$. Similarly, we have $\text{tr}(H_{k+1}^T Q(\rho_+) H_{k+1}) = J(\mathbf{H}_{k+1})$. Therefore, (29) can be reformulated as

$$J(\mathbf{H}_{k+1}) \leq \alpha J(\tilde{\mathbf{H}}_k) + \beta \phi, \quad (30)$$

where $\phi = \text{tr}(H_d^T H_d)$.

When $m > s$, $\tilde{\mathbf{H}}_k = \mathcal{R}_{s,Q(\rho)}(\mathbf{H}_k)$. According to Theorem 4, $J(\tilde{\mathbf{H}}_k) = J(\mathcal{R}_{s,Q(\rho)}(\mathbf{H}_k)) \leq \frac{s+d(d+2)}{s+d} J(\mathbf{H}_k)$. It is easy to prove that the function $f(d) = \frac{s+d(d+2)}{s+d}$ is increasing. In addition, $d \leq m_w + 2m_v = r$ for the proposed method. Therefore, $\frac{s+d(d+2)}{s+d} \leq \frac{s+r(r+2)}{s+r} = \eta$. Note that $\tilde{\mathbf{H}}_k = \mathbf{H}_k$ when $m \leq s$, which implies $J(\tilde{\mathbf{H}}_k) = J(\mathbf{H}_k)$. In addition, $\eta > 1$. Therefore, for any m and s , we

have $J(\bar{\mathbf{H}}_k) \leq \eta J(\mathbf{H}_k)$. Define $\theta = \alpha\eta$. Then, from (30), we have $J(\mathbf{H}_{k+1}) \leq \theta J(\mathbf{H}_k) + \beta\phi$. Therefore, if $\theta < 1$, $J(\mathbf{H}_k)$ is bounded, which implies that the set-membership estimation is stable. \square

Remark 4. The considered system (3) is a time-varying system. Combastel (2015) presents a γ -detectability condition for the detectability assumption of time-varying systems. For the considered system (3), α is corresponding to γ in Combastel (2015). Similar to the detectability definition in Assumption 9 of Combastel (2015), we assume that there exist $T(\rho_+)$ and $L(\rho)$ such that $x_{k+1} = \bar{A}(\rho)x_k$ is α -stable for any possible ρ , i.e., there exists positive definite matrices $Q(\rho)$ and $Q(\rho_+)$ such that

$$\begin{bmatrix} \alpha Q(\rho) & \bar{A}(\rho)^T Q(\rho_+) \\ Q(\rho_+) \bar{A}(\rho) & Q(\rho_+) \end{bmatrix} \succ 0. \quad (31)$$

According to the Schur complement lemma, (31) is equivalent to $-\alpha Q(\rho) + \bar{A}(\rho)^T Q(\rho_+) \bar{A}(\rho) \prec 0$. In this paper, the observer parameters are designed under the conditions of Theorem 3 to make the estimation error satisfy the L_∞ index. According to the proofs of Theorem 2 and Theorem 3, the inequality (17) is the key condition for satisfying the L_∞ index. Note that $-\alpha Q(\rho) + \bar{A}(\rho)^T Q(\rho_+) \bar{A}(\rho)$ is only the first diagonal element matrix of the block matrix in (17). Therefore, (31) is only a necessary condition for (17). The condition (17) is more restrictive than (31). However, the observer satisfying the condition (17) may obtain more accurate estimation since (17) also considers the attenuation of disturbances. In addition, the reduction operation in the set-membership estimation will cause over-approximations. In order to guarantee the stability of the set-membership estimation, Theorem 5 poses additional constraints on the reduction order s to control the effect of over-approximations.

6 Simulation results

In this section, two simulation examples are provided to demonstrate the performance of the proposed method.

Example 1. A vehicle lateral dynamics system from Varrier, Koenig & Martinez (2014) is considered, which has the following model:

$$\begin{bmatrix} \dot{\beta}(t) \\ \dot{r}(t) \end{bmatrix} = \begin{bmatrix} -\frac{c_{\alpha V} + c_{\alpha H}}{mv(t)} & \frac{l_H c_{\alpha H} - l_V c_{\alpha V}}{mv(t)^2} - 1 \\ \frac{l_H c_{\alpha H} - l_V c_{\alpha V}}{I_z} & \frac{l_V^2 c_{\alpha V} + l_H^2 c_{\alpha H}}{I_z v(t)} \end{bmatrix} \begin{bmatrix} \beta(t) \\ r(t) \end{bmatrix} + \begin{bmatrix} \frac{c_{\alpha V}}{mv(t)} \\ \frac{l_V c_{\alpha V}}{I_z} \end{bmatrix} u_L(t),$$

where $\beta(t)$ is the slide slip angle, $r(t)$ is the yaw rate, $u_L(t)$ is the relative steering wheel angle and $v(t)$ denotes the speed of the vehicle. The related parameter values are $c_{\alpha V} = 57117\text{N/rad}$, $c_{\alpha H} = 81396\text{N/rad}$, $m = 1621\text{kg}$, $l_H = 1.38\text{m}$, $l_V = 1.15\text{m}$, $I_z = 1975\text{kg.m}^2$. The measurement output is chosen as $y(t) = a_y(t) - \frac{c_{\alpha V}}{mv} u_L(t) = -\frac{c_{\alpha V} + c_{\alpha H}}{m} \beta(t) + \frac{l_H c_{\alpha H} - l_V c_{\alpha V}}{mv(t)^2} r(t)$, where $a_y(t)$ is the vehicle lateral acceleration. The continuous-time model is discretized by the first order Euler method with the sampling time $T_d = 0.02\text{s}$. Define the scheduling vector as $\rho_k = [\rho_{1,k} \ \rho_{2,k}]^T$ where $\rho_{1,k} = \frac{1}{v(t_k)}$ and $\rho_{2,k} = \frac{1}{v(t_k)^2}$ ($t_k = kT_d$). In addition, the process disturbance and measurement noise are also taken into consideration. The previous continuous-time model is converted into a discrete-time LPV system as follows.

$$\begin{cases} x_{k+1} = \begin{bmatrix} 1 - 1.7090\rho_{1,k} & 0.5755\rho_{2,k} - 0.02 \\ 0.4723 & 1 - 2.3347\rho_{1,k} \end{bmatrix} x_k \\ \quad + \begin{bmatrix} 0.7047\rho_{1,k} \\ 0.6652 \end{bmatrix} u_k + \begin{bmatrix} \rho_{1,k} & 0.3 \\ 0 & 0.2 \end{bmatrix} w_k, \\ y_k = [-85.4491 \ 28.7736\rho_{1,k}] x_k + 0.1v_k \end{cases} \quad (32)$$

where $x_k = [\beta_k \ r_k]^T$ and $u_k = u_L(t_k)$. The speed of the vehicle, $v(t_k)$, is assumed to be time-varying between 2m/s and 4m/s. In simulation, $v(t_k) = 1/(0.5 - 0.005k)$ when $0 \leq k \leq 50$ and $v(t_k) = 4$ when $k > 50$. Then, the scheduling variables $\rho_{1,k} \in [0.25, 0.5]$ and $\rho_{2,k} \in [0.0625, 0.25]$ (unit omitted). The time-varying scheduling vector can be bounded by the triangle with vertices as $(\rho_{1,k}^{\min}, \rho_{2,k}^{\min})$, $(\rho_{1,k}^{\max}, \rho_{2,k}^{\min})$ and $(\rho_{1,k}^{\max}, \rho_{2,k}^{\max})$ (Varrier et al., 2014). Therefore, system (32) can be converted into a polytopic LPV model form with parameters as follows.

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.5728 & 0.0160 \\ 0.4723 & 0.4163 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0.1455 & 0.0160 \\ 0.4723 & -0.1673 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0.1455 & 0.1239 \\ 0.4723 & -0.1673 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0.1762 \\ 0.6652 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0.3524 \\ 0.6652 \end{bmatrix}, & B_3 &= \begin{bmatrix} 0.3524 \\ 0.6652 \end{bmatrix}, & D_{w1} &= \begin{bmatrix} 0.25 & 0.3 \\ 0 & 0.2 \end{bmatrix}, \\ D_{w2} &= \begin{bmatrix} 0.5 & 0.3 \\ 0 & 0.2 \end{bmatrix}, & D_{w3} &= \begin{bmatrix} 0.5 & 0.3 \\ 0 & 0.2 \end{bmatrix}, \\ C_1 &= [-85.4491 \ 1.7983], & C_2 &= [-85.4491 \ 1.7983], \\ C_3 &= [-85.4491 \ 7.1934], & D_{v1} &= D_{v2} = D_{v3} = 0.1. \end{aligned}$$

The initial state $x_0 = [0.05 \ 0.03]^T$ and the input $u_k = 0.1 \sin(0.3k)$. The process disturbance and

measurement noise are random and uniformly distributed, satisfying $w_k \in \mathbf{E}(0, 0.1I_2)$ and $|v_k| \leq 0.1$. For the proposed method, we set $\hat{x}_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $\mathbf{H}_0 = \{0.1I_2\}$. For zonotope enclosure, we choose $\langle 0, 0.1I_2 \rangle$ and $\langle 0, 0.1I_2 \rangle$ to enclose x_0 and w_k respectively. The considered example can be seen as a descriptor system with $E = I_2$. Choosing $\alpha = 0.3$ and solving the optimization problem (25), we obtain the estimator parameters for the proposed method as follows:

$$\begin{aligned} T_1 &= \begin{bmatrix} -0.0273 & 0.0216 \\ -0.3365 & 1.0071 \end{bmatrix}, T_3 = \begin{bmatrix} -0.0273 & 0.0865 \\ -0.3365 & 1.0283 \end{bmatrix}, \\ N &= \begin{bmatrix} -0.0120 \\ -0.0039 \end{bmatrix}, L_1 = \begin{bmatrix} -0.0004 \\ -0.0055 \end{bmatrix}, L_2 = \begin{bmatrix} -0.0002 \\ -0.0050 \end{bmatrix}, \\ T_2 &= T_1, L_3 = \begin{bmatrix} -0.0005 \\ -0.0053 \end{bmatrix}, Q_1 = \begin{bmatrix} 81.2223 & -8.4559 \\ -8.4559 & 16.9576 \end{bmatrix}, \\ Q_2 &= \begin{bmatrix} 71.3655 & -3.6273 \\ -3.6273 & 16.2059 \end{bmatrix}, Q_3 = \begin{bmatrix} 71.6492 & -4.7226 \\ -4.7226 & 16.2520 \end{bmatrix}. \end{aligned}$$

To demonstrate the effectiveness of the proposed method, it is compared with different methods, including the ellipsoid state-bounding-based set-membership estimation (ES-SME) method proposed in Liu, Zhao & Wu (2016) and the Zonotopic Kalman Filter (ZKF) in Combastel (2015). To make a fair comparison, we also consider the proposed estimator using zonotopes and ZKF using ellipsoid bundles, which has the exactly same observer parameters with ZKF. For ellipsoid bundles, the reduction order is set as $s = 10$. For Theorem 5, $r = 3$. Then, we have $\alpha\eta = 0.5769 < 1$. Therefore, the set-membership estimation of the proposed method is stable. In fact, all the component matrices of the ellipsoid bundle generator have 20 columns totally. Therefore, to make a fair comparison, the reduction order for zonotopes is set as 20. Consider three indices. First, the root mean square error (RMSE), i.e. $(\frac{1}{N+1} \sum_{k=0}^N \frac{1}{n_x} \|e_k\|^2)^{\frac{1}{2}}$. Second, since the topic of this paper on set-membership estimation is related to error upper bounds, another index, mean interval width sum (MIWS) is more relevant. It is defined as $\frac{1}{N+1} \sum_{k=0}^N \|\bar{x}_k - \underline{x}_k\|_1$. Third, the average running time (ART) for 5000 iterations using MATLAB R2018b under Intel(R) Core(TM) i7-8750H CPU @ 2.20GHz. The three indices for all the considered methods are shown in Table 1. It shows that the proposed method (Method 1) has the highest accuracy in MIWS, and less obviously in RMSE. Comparing Method 1 with 2, Method 5 with 4, we can see that replacing zonotopes with ellipsoid bundles decreases the MIWS indices, which is corresponding to the increasing of set-membership estimation accuracy. Method 1, 4 and 5 have longer ART indices. Method 1 and 5 need to deal with the decomposition of an $n_x \times n_x$ dimensional

matrix, Method 4 involves computing the inverse of a matrix. And they both have the reduction operators. The computational complexity of the matrix decomposition is $\mathcal{O}(n_x^3)$. The Average Running Time (ART) reported in Table 1 below shows that the methods based on ellipsoid bundles (methods 1 and 5) take more time, but all the tested methods have ART values at the same order of magnitude. Based on the modern computer processing ability, it is able to implement the proposed method in real-time for many applications. Since ZKF only uses the last measurement which is available before possibly applying an observer-based real-time control, it performs worse than the other methods using both y_k and y_{k+1} .

Table 1
Simulation results of Example 1

Methods	MIWS	RMSE	ART(ms)
Method 1	0.0428	0.0049	0.0854
Method 2	0.0595	0.0049	0.0366
Method 3	0.1255	0.0052	0.0348
Method 4	0.1894	0.0152	0.0796
Method 5	0.1468	0.0152	0.0978

Method 1: proposed method, Method 2: proposed estimator using zonotopes, Method 3: ES-SME, Method 4: ZKF, Method 5: ZKF using ellipsoid bundles.

Example 2. An example of a truck-trailer system from Wang et al. (2019) is considered, which has the following parameters:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.025 & 0 & 0 & 0 \\ -0.218 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0.06 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1.05 & 0 & 0 & 0 \\ -0.436 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0.12 & 0 & 1 \end{bmatrix}, \\ E &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} -0.025 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} -0.05 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ D_{w1} &= \begin{bmatrix} 0 \\ 0 \\ -0.12 \\ 0 \end{bmatrix}, D_{w2} = 2D_{w1}, C_1 = C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

and $D_{v1} = D_{v2} = 0.2I_3$. In the simulation, $x_0 = [-0.1745, 0.0873, 3, -0.0057]^T$ and the input $u_k = Kx_k$, where $K = [-3.1091, 3.2882, -0.6461, 0]$ making the system stable. The scheduling variables are set as $h_1(\rho_k) = 0.5 + 0.5 \sin(0.0386k)$ and $h_2(\rho_k) =$

$0.5 - 0.5 \sin(0.0386k)$. The disturbance and measurement noise are random and uniformly distributed, satisfying $|w_k| \leq 0.03$ and $v_k \in \mathbf{E}(0, 0.01I_3)$. For the method in Wang et al. (2019), we set $\hat{x}_0 = [-0.1, 0.08, 3, 0]^T$ and use $\langle \hat{x}_0, H_0 \rangle, \langle 0, 0.01I_3 \rangle$ to enclose x_0, v_k , respectively, where $H_0 = 0.1I_4$. For the proposed method, \mathbf{H}_0 is the set comprised of the columns of H_0 . Choosing $\alpha = 0.63$ and solving the optimization problem (25), we obtain the estimator parameters for the proposed method as follows:

$$\begin{aligned}
T_1 &= \begin{bmatrix} 1 & 6.6219 & 0.0078 & 0.8226 \\ 0 & 0.4229 & 0.0094 & -0.1111 \\ 0 & 0.1243 & 0.2829 & -0.2921 \\ 0 & -0.1736 & -0.2627 & 0.3136 \end{bmatrix}, \\
T_2 &= \begin{bmatrix} 1 & 6.6219 & 0.0078 & 0.2882 \\ 0 & 0.4229 & 0.0094 & -0.0973 \\ 0 & 0.1243 & 0.2829 & -0.2874 \\ 0 & -0.1736 & -0.2627 & 0.3073 \end{bmatrix}, \\
N &= \begin{bmatrix} -6.6219 & -0.0078 & 0.0078 \\ 0.5771 & -0.0094 & 0.0094 \\ -0.1243 & 0.7171 & 0.2829 \\ 0.1736 & 0.2627 & 0.7373 \end{bmatrix}, \\
L_1 &= \begin{bmatrix} -1.7110 & -0.1375 & 0.3339 \\ 0.2497 & 0.0038 & -0.0354 \\ 0.0646 & 0.1866 & 0.0246 \\ -0.0927 & -0.1729 & 0.0157 \\ -0.6885 & 0.3612 & 0.1308 \end{bmatrix}, \\
L_2 &= \begin{bmatrix} 0.2811 & -0.0181 & -0.0359 \\ 0.0635 & 0.1721 & 0.0169 \\ -0.0949 & -0.1556 & 0.0092 \end{bmatrix}, \\
Q_1 &= \begin{bmatrix} 0.1572 & 1.8290 & 0.0460 & -0.0509 \\ 1.8290 & 37.5258 & 0.9228 & -1.0143 \\ 0.0460 & 0.9228 & 7.4789 & 0.1073 \\ -0.0509 & -1.0143 & 0.1073 & 2.0537 \end{bmatrix}, \\
Q_2 &= \begin{bmatrix} 0.1467 & 1.5692 & -0.0969 & 0.0308 \\ 1.5692 & 33.6828 & 0.1443 & -1.8114 \\ -0.0969 & 0.1443 & 8.9239 & -1.4967 \\ 0.0308 & -1.8114 & -1.4967 & 4.0069 \end{bmatrix}.
\end{aligned}$$

The proposed method is compared with three different methods, including the method based on zonotopes proposed in Wang et al. (2019), the proposed estimator

using zonotopes and the method in Wang et al. (2019) using ellipsoid bundles. For ellipsoid bundles, the reduction order is set as $s = 20$. Then, we have $\alpha\eta = 0.9587 < 1$. Therefore, according to Theorem 5, the proposed method is stable. In fact, all the component matrices of the ellipsoid bundle generator have about 50 columns totally. To make a fair comparison, the reduction order for zonotopes is set as 50. Table 2 presents the three indices of the compared methods for Example 2. The proposed method (Method 1) has the highest estimation accuracy in MIWS and RMSE. Comparing Method 1 with 2, Method 4 with 3, we can see that using ellipsoid bundles instead of zonotopes can increase estimation accuracy under the same observer parameters. Method 1 and 4 have longer ART values (5000 iterations) since they need to deal with the decomposition of an $n_x \times n_x$ dimensional matrix, but they are still affordable for real-time computation and increase the estimation accuracy in the same time.

Table 2
Simulation results of Example 2

Methods	MIWS	RMSE	ART(ms)
Method 1	0.1005	0.0027	0.1140
Method 2	0.1179	0.0027	0.0494
Method 3	0.5887	0.0030	0.0618
Method 4	0.4650	0.0030	0.1248

Method 1: proposed method, Method 2: proposed estimator using zonotopes, Method 3: method in Wang et al. (2019), Method 4: method in Wang et al. (2019) using ellipsoid bundles.

7 Conclusion

In this paper, we investigate the set-membership estimation for discrete-time LPV descriptor systems. A novel set-membership estimation method based on ellipsoid bundles is proposed for the considered systems. The L_∞ technique is applied to the robust design of the estimator parameters, which can be converted into solving a set of LMIs. This paper also investigates the stability problem of the proposed method and establishes an offline sufficient stability condition. Finally, the simulation results have demonstrated the performance of the proposed method.

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