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# A geometric framework for asymptotic inference of principal subspaces in PCA

Dimbihery Rabenoro and Xavier Pennec

Université Côte d'Azur and Inria, Epione Project Team, Sophia Antipolis  
2004 route des Lucioles, 06902 Valbonne Cedex, France

## Abstract

Consider data assumed to be iid samples from a multivariate Gaussian distribution whose covariance matrix has repeated eigenvalues. In our model, these eigenvalues are unknown but their multiplicities are supposed to be known. In this paper, we develop an asymptotic method to infer the collection of all principal subspaces together, i.e. the eigenspaces of this covariance matrix. Our approach is based on the geometry of the flag manifold to which the collection of all principal subspaces and our estimators of it belong.

## 1 Introduction

This article deals with the asymptotic study of principal component analysis (PCA), in the light of methods from Riemannian geometry. Throughout this paper, we assume that the data are iid samples from a *Gaussian* random vector  $X$  defined on a probability space  $(\Omega, \mathcal{A}, \Pr)$  and valued in  $\mathbb{R}^d$ ,  $d \geq 1$ , whose covariance matrix  $\Sigma$  has possibly *repeated* eigenvalues. In PCA, the principal subspaces (PS's) are the eigenspaces of  $\Sigma$  associated respectively to eigenvalues in decreasing order. The sequence of multiplicities of the latter is called the *type* of  $\Sigma$ . Then, a classical problem is to estimate the PS's by deriving Central Limit Theorems (CLT's) involving the eigenvectors of sample covariance matrices. This question is addressed notably in the Anderson's celebrated paper [1]. The matrix  $\Sigma$  may have repeated eigenvalues, although the set of symmetric matrices having multiple eigenvalues is negligible. In fact, it is implicit in [1] that  $\Sigma$  is obtained by equalizing beforehand nearly equal but distinct eigenvalues of a sample covariance matrix. In [13], a principled procedure is developed to equalize such eigenvalues and thus to select the type of  $\Sigma$ .

The method of [1] yields only an estimation of each PS separately and when its dimension is equal to 1. This result is improved in [14] where other CLT's, issued from Perturbation Theory [8], provide an estimation of each PS of any dimension. However, each PS remains to be addressed separately. In contrast, our geometric method allows us notably to estimate the *collection* of all PS together, regardless of their dimensions. Thus, the main novelty of our paper is to build statistics based on the Riemannian geometry of manifolds of linear subspaces of  $\mathbb{R}^d$ . Namely, each PS belongs to a Grassmannian, while the collection of them lies in a *flag manifold*. Here, a flag is viewed as a collection of mutually orthogonal subspaces spanning  $\mathbb{R}^d$ , whose sequence of dimensions is also called the *type* of the flag. This incremental subspaces representation of a flag is equivalent to the more usual one as nested sequence of linear subspaces. In our PCA setting, we denote by  $\lambda_1 > \dots > \lambda_r$  the distinct eigenvalues of  $\Sigma$ , of respective multiplicities  $q_1, \dots, q_r$ . Thus, the collection of all PS's forms a flag of type  $\mathbf{I} = (q_i)_{1 \leq i \leq r}$ , denoted by  $F^{\mathbf{I}}(\Sigma)$ . Let  $\mathcal{F}^{\mathbf{I}}$  be the set of all flags of type  $\mathbf{I} = (q_i)_i$ , so that  $F^{\mathbf{I}}(\Sigma) \in \mathcal{F}^{\mathbf{I}}$ . It is well-known that  $\mathcal{F}^{\mathbf{I}}$  is endowed with a structure of homogeneous space for the orthogonal group  $O(d)$ : see [12]. The introduction of such manifolds in a PCA setting is notably inspired by [11], where it is proved that PCA can be rephrased as an optimization on a flag manifold. We also refer to [9] for the use of flag manifolds for studying PCA.

A first idea to estimate  $F^I(\Sigma)$  could be to exploit the structure of Riemannian manifold on  $\mathcal{F}^I$  introduced in [12] and, following [3], to establish a CLT in  $\mathcal{F}^I$  in normal coordinates i.e. defined from the Riemannian Logarithm on  $\mathcal{F}^I$ . However, the latter is *unknown* in closed form. In fact, there is no known metric on  $\mathcal{F}^I$  for which an explicit expression of the Riemannian Logarithm is available. Here, the incremental subspaces representation of a flag allows us to overcome this lack, since it provides an embedding of  $\mathcal{F}^I$  within a product of Grassmannians. Indeed, the Riemannian Logarithm is available in closed form in any Grassmannian.

In fact, our method is a wide geometric extension of that of [1]. For  $n \geq 1$ , let  $\widehat{\Sigma}_n$  be the sample covariance matrix from an iid sample of size  $n$  of  $X$ . Then, in [1], a matrix  $E_n \in O(d)$  of the form  $E_n = \Gamma' C_n$  is considered, where  $\Gamma$  and  $C_n$  are respectively matrices of eigenvectors of  $\Sigma$  and  $\widehat{\Sigma}_n$ , such that all diagonal entries of  $E_n$  are positive (See Section 2 for precise definitions). So,  $E_n$  measures the deviation between the PS's and the eigenvectors of  $\widehat{\Sigma}_n$ . Then, the asymptotic distribution of  $E_n$  is described in [1]. Thus, the diagonal blocks of size  $q_i \times q_i$  of  $E_n$  converge in distribution to a Haar measure on the set of orthogonal  $q_i \times q_i$  matrices whose diagonal entries are positive. The other blocks of  $E_n$  converge to Gaussian distributions, which provide Central Limit Theorems (CLT's) that allow to infer the PS's.

Instead of considering the usual coordinates of  $E_n$ , we introduce a *geometric splitting* of  $E_n$  into two parts, for which we recover respectively Gaussian distributions and Haar measures as  $n \rightarrow \infty$ . The first part yields CLT's from which we deduce our estimation of  $F^I(\Sigma)$ . The second part provides statistics that are valued, for all  $n \geq 1$ , in the orthogonal groups  $(O(q_i))_i$ , contrarily to the diagonal blocks of  $E_n$ . We derive such a geometric splitting of any orthogonal matrix from the structure of *Riemannian submersion* of the map which associates to a  $q$ -frame the  $q$ -linear subspace spanned by it, for any  $1 \leq q \leq d$ .

We describe briefly hereafter our method to infer  $F^I(\Sigma)$ . Thanks to the explicit expression of the Riemannian Logarithm in any Grassmannian, we establish a CLT in a Grassmannian for each PS separately. Then, contrarily to the CLT's for each PS in [14], some geometric properties of Grassmannians allow us to *concatenate* our CLT's to obtain a pivotal statistic for  $F^I(\Sigma)$ , whose limiting distribution is a  $\chi^2$  one. Namely, recall that a.s., for all  $n \geq 1$ , the eigenvalues of  $\widehat{\Sigma}_n$  are all distinct. Then, one can construct a flag of type I from the eigenvectors of  $\widehat{\Sigma}_n$  as follows: those associated to the  $q_1$  largest eigenvalues are grouped together, then those to the next  $q_2$  ones, ..., and finally those to the  $q_r$  smallest ones. Then, the linear subspaces generated by each group form a flag  $F^I(\widehat{\Sigma}_n) \in \mathcal{F}^I$  which may be compared to  $F^I(\Sigma)$ . Thus, we prove a result of the form

$$\frac{n}{4} \left[ \widetilde{\mathcal{D}}_{\widehat{\mathbb{K}}_n} \left( F^I(\Sigma), F^I(\widehat{\Sigma}_n) \right) \right]^2 \xrightarrow[n \rightarrow \infty]{d} \chi_{D^I}^2, \quad (1.1)$$

where  $\widetilde{\mathcal{D}}_{\widehat{\mathbb{K}}_n}(\cdot, \cdot)$  is a discrepancy function on  $\mathcal{F}^I$ , indexed by a statistic  $\widehat{\mathbb{K}}_n$  which is a function of the spectrum of  $\widehat{\Sigma}_n$ , and  $D^I = \frac{1}{2} (d^2 - \sum_i q_i^2)$ . The convergence in Eq.(1.1) is illustrated numerically on synthetic experiments. Finally, we derive, from Eq.(1.1), confidence regions for  $F^I(\Sigma)$  that are interpreted as confidence ellipsoids in  $\mathcal{F}^I$ , centered at  $F^I(\widehat{\Sigma}_n)$ .

Thus, our confidence regions are defined directly in the manifold  $\mathcal{F}^I$ . In contrast, the confidence regions usually met in the literature on estimation on manifolds are defined in a chart, as in [3]. Furthermore, in [14], for fixed  $1 \leq i \leq r$ , the confidence regions for the principal subspace  $P_i(\Sigma)$  are based on a  $\chi_{q_i(d-q_i)}^2$  distribution. Then, the sum of their degrees of freedom is  $\sum_i q_i(d-q_i) = 2D^I$ , so that we obtain a gain of a factor of 2.

This article is organized as follows. In Section 2, we present Anderson's results and a brief sketch of their proofs. In Section 3, we develop the background of geometry needed for the rest of the paper and we present notably the aforementioned geometric splitting of  $E_n$ . Then, in Section 4, we state our main result in Theorem 5, which provides the

limiting distribution of the statistics issued from our splitting of  $E_n$  and we deduce the estimation of the flag  $F^1(\Sigma)$ . Finally, in Section 5, we summarize our results and propose some open questions raised by them. The proof of Theorem 5 is deferred to the Appendix.

## 2 Review of Anderson's results

We keep the notations of the Introduction. We fix a matrix  $\Gamma \in O(d)$  of eigenvectors of  $\Sigma$ , associated respectively to  $\lambda_1, \dots, \lambda_r$ . Then,  $\Gamma' \Sigma \Gamma = \Delta$  where  $\Delta$  is the following  $d \times d$  block-diagonal matrix:  $\Delta := \text{Diag}(\lambda_1 I_{q_1}, \dots, \lambda_r I_{q_r})$ .

### 2.1 Preliminaries

For  $1 \leq i \leq r-1$ , set  $\bar{q}_i := \sum_{j=1}^i q_j$ . The sequence  $\mathbf{I} = (q_i)_{1 \leq i \leq r}$  induces a partition of the indices  $\{1, \dots, d\}$  into  $r$  blocks  $(\beta_i)_{1 \leq i \leq r}$  as follows:

$$\beta_1 = \{1, \dots, q_1\}, \dots, \beta_i = \{\bar{q}_i + 1, \dots, \bar{q}_{i+1}\}, \dots, \beta_r = \{\bar{q}_{r-1} + 1, \dots, d\}.$$

The sequence  $(\beta_i)_{1 \leq i \leq r}$  is called the partition of  $\{1, \dots, d\}$  wrt  $\mathbf{I}$ . Let  $A = (a_{k\ell})_{1 \leq k \leq \ell \leq d}$  be a  $d \times d$  matrix. For fixed  $1 \leq i \leq j \leq r$ , consider the submatrix  $A^{(i,j)} := (a_{k\ell})_{k \in \beta_i, \ell \in \beta_j}$ . Let  $A^{(i)}$  be the  $i$ -th block of columns of  $A$ , i.e.  $A^{(i)} = (A^{(1,i)} \dots A^{(r,i)})'$ . Thus, setting  $I_d^{(i)} := (I_d)^{(i)}$ ,

$$A^{(i)} = AI_d^{(i)} \quad \text{and} \quad A^{(i,j)} = \left(I_d^{(i)}\right)' AI_d^{(j)}. \quad (2.1)$$

**Definition 1.** Let  $\text{Sym}_d$  be the set of all  $d \times d$  symmetric matrices and  $\text{Sym}_d^\neq$  the subset of  $\text{Sym}_d$  of matrices whose eigenvalues are all distinct. For  $S \in \text{Sym}_d^\neq$ , its spectrum is denoted by  $\{\mu_1(S) > \dots > \mu_d(S)\}$ . Let  $\psi : \text{Sym}_d^\neq \rightarrow O(d)$  be the map such that for any  $1 \leq k \leq d$ , the  $k$ -th column of  $\psi(S)$  is the eigenvector of norm 1 associated to  $\mu_k(S)$  whose  $k$ -th entry is non-negative. Then, the map  $\psi$  is measurable and is called the eigenvector map.

### 2.2 Statement of Anderson's results

Assume that  $\widehat{\Sigma}_n \in \text{Sym}_d^\neq$ , which holds a.s. Set

$$E_n := \psi(T_n) \quad \text{where} \quad T_n := \Gamma' \widehat{\Sigma}_n \Gamma, \quad n \geq 1.$$

$E_n$  is well-defined, since  $T_n \in \text{Sym}_d^\neq$ . Then,  $C_n := \Gamma E_n$  is an orthogonal matrix of eigenvectors of  $\widehat{\Sigma}_n$ , also associated to eigenvalues in decreasing order. Thus,  $E_n = \Gamma' C_n$  is the matrix mentioned in the Introduction, which measures the deviation between eigenvectors of  $\Sigma$  (the PS's) and those of  $\widehat{\Sigma}_n$ . The asymptotic distribution of  $E_n = \Gamma' C_n$  is the main result of [1], stated in Theorem 1 below, for which we need the following definition.

**Definition 2.** The Haar measure is the unique probability distribution on the compact group  $O(d)$  that is invariant under right-multiplications. It is the uniform measure on  $O(d)$ . The conditional Haar invariant distribution is defined in [1] as  $2^d$  times the restriction of the Haar measure to the space of orthogonal matrices whose diagonal entries are non-negative.

**Theorem 1.** (i) For  $1 \leq i \leq r$ ,  $E_n^{(i,i)} \xrightarrow[n \rightarrow \infty]{d} E^{i,i}$ , where the distribution of  $E^{i,i}$  is the conditional Haar invariant distribution.

(ii) For  $n \geq 1$ , set  $F_n := \sqrt{n} E_n$ . Then, for all  $i \neq j$ ,  $F_n^{(i,j)} \xrightarrow[n \rightarrow \infty]{d} F^{i,j}$ , where the entries of  $F^{i,j}$  are iid rv's  $\mathcal{N}(0, \sigma_{i,j}^2)$ , of standard deviation  $\sigma_{i,j} := \frac{\sqrt{\lambda_i \lambda_j}}{|\lambda_i - \lambda_j|}$ .

(iii) The blocks  $\{E^{i,i}, F^{i,j} : 1 \leq i \leq r, 1 \leq j \leq r, i \neq j\}$  are mutually independent.

**Remark 1.** As pointed out in the Introduction, Theorem 1 allows to estimate separately each PS's, but only when its dimension is equal to 1: See Appendix B in [1].

## 2.3 Method for the proof of Theorem 1

### 2.3.1 Preliminary results

The method consists of propagating the uncertainty of the estimation of  $\widehat{\Sigma}_n$  by  $\Sigma$ , provided by Theorem 2 hereafter which is obtained in [1], to derive the limiting distribution of the deviation between their respective eigenvectors measured by  $E_n = \Gamma' C_n$ . This propagation is realized by a generalized  $\delta$ -method stated in Theorem 3 below which can be found in Section 7 of [1]. A more actual reference is section 18.11 in [15]. In fact, our version extends slightly the latter by adding Condition (2.2) in the statement, because of constraints imposed by our geometric method.

**Theorem 2.** Set  $T_n := \Gamma' \widehat{\Sigma}_n \Gamma$ . Then, the following CLT in  $\text{Sym}_d$  holds.

$$U_n := \sqrt{n} (T_n - \Delta) \xrightarrow[n \rightarrow \infty]{d} U,$$

where  $U$  is a random matrix whose distribution is characterized as follows:  $U \in \text{Sym}_d$ , the blocks  $\{U^{(i,j)} : 1 \leq i < j \leq r\}$  are mutually independent and for  $i \neq j$ , the entries of  $U^{(i,j)}$  are iid rv's  $\mathcal{N}(0, s_{i,j}^2)$ , of standard deviation  $s_{i,j} := \sqrt{\lambda_i \lambda_j}$ .

**Theorem 3.** Let  $\mathbb{S}, \mathbb{T}$  be metric spaces endowed with their Borel  $\sigma$ -algebras. Let  $(\mathbb{S}_n)_{n \geq 0}$  be a sequence of measurable subsets of  $\mathbb{S}$  and for all  $n \geq 0$ ,  $g_n : \mathbb{S}_n \rightarrow \mathbb{T}$  a measurable map. Let  $(V_n)_{n \geq 1}$  be a sequence of rv's valued in  $(\mathbb{S}_n)_{n \geq 1}$  which converges in distribution to a rv  $V$  valued in  $\mathbb{S}$ , with  $\Pr(V \in \mathbb{S}_0) = 1$ . Assume that for all  $n \geq 1$ , there exists a measurable subset  $\mathbb{S}_n^*$  of  $\mathbb{S}_n$  such that

$$\Pr(V_n \in \mathbb{S}_n^*) \xrightarrow[n \rightarrow \infty]{} 1 \tag{2.2}$$

and that, for any  $v \in \mathbb{S}_0$  and all sequence  $(v_n)_{n \geq 1}$  valued in  $(\mathbb{S}_n^*)_{n \geq 1}$ ,

$$v_n \xrightarrow[n \rightarrow \infty]{} v \implies g_n(v_n) \xrightarrow[n \rightarrow \infty]{} g_0(v). \tag{2.3}$$

Then,  $g_n(V_n)$  converges in distribution to  $g_0(V)$ .

*Proof.* We prove our extended version in subsection 6.5 of the Appendix.  $\square$

## 3 Preliminaries of geometry

In subsection 3.1, we introduce the manifolds of linear subspaces in which the (collection of) PS's and their estimators lie. In subsection 3.2, we define a geometric parametrization of  $E_n$  that is an alternative to that by its coordinates in the standard basis of  $\mathbb{R}^{d \times d}$ , used in Anderson's paper [1] to describe (Theorem 1) the limiting distribution of  $E_n$ . In contrast, our parametrization of  $E_n$  provides notably confidence regions for the collection of PS's, for which geometric preliminaries are presented in subsection 3.3.

### 3.1 Grassmannians, Stiefel and flag manifolds

#### 3.1.1 Sets of linear subspaces

Let  $1 \leq q \leq d$ . The Stiefel manifold is the set  $\text{St}_q$  of all  $q$ -frames of  $\mathbb{R}^d$  i.e. of all  $q$ -tuples of orthonormal vectors of  $\mathbb{R}^d$ . Thus,

$$\text{St}_q = \{U \in \mathbb{R}^{d \times q} : U'U = I_q\}.$$

The Grassmannian  $G_q$  is the set of all  $q$ -linear subspaces of  $\mathbb{R}^d$ , identified with the orthogonal projectors of rank  $q$  i.e. such a projector  $P$  is identified with its range  $\text{rg}(P)$ :

$$G_q = \{P \in \mathbb{R}^{d \times d} : P' = P, P^2 = P, \text{rank}(P) = q\}. \quad (3.1)$$

Thus,  $G_q$  is a subset of  $\text{Sym}_d$ . Then,  $\text{St}_q$  and  $G_q$  are linked by the map  $\pi_q$  which associates to  $\mathcal{U} \in \text{St}_q$ , the (projector onto the) linear subspace spanned by  $\mathcal{U}$ , defined by

$$\pi_q : \text{St}_q \longrightarrow G_q, \quad \pi_q(\mathcal{U}) = \mathcal{U}\mathcal{U}'.$$

**Remark 2.** Let  $P \in G_q$  and  $\mathcal{U}_P$  a fixed  $q$ -frame spanning  $\text{rg}(P)$ , i.e.  $\mathcal{U}_P \in \pi_q^{-1}(P)$ . Then, any  $\mathcal{V} \in \pi_q^{-1}(P)$  is obtained from  $\mathcal{U}_P$  by a unique orthonormal base change in  $\text{rg}(P)$ . Thus, there exists a unique  $K \in O(q)$  such that  $\mathcal{V} = \mathcal{U}_P K$  i.e.  $K = \mathcal{U}_P' \mathcal{V}$ .

**Definition 3.** (i) A flag of  $\mathbb{R}^d$  is a sequence of mutually orthogonal linear subspaces spanning  $\mathbb{R}^d$ , whose sequence of respective dimensions is called the type of the flag.

(ii) A flag whose type is  $(1, 1, \dots, 1)$  is called a full flag.

(iii) For  $\Sigma \in \text{Sym}_d$ , its flag of eigenspaces is the collection of its eigenspaces associated to eigenvalues in decreasing order, whose type is the sequence of their dimensions.

(iv) Given a sequence  $\mathbf{I} = (q_i)_{1 \leq i \leq r}$  with  $\sum q_i = d$ , consider the product manifold

$$G^{\mathbf{I}} := \prod G^{q_i} \quad \text{where} \quad G^{q_i} := G_{q_i}, \quad 1 \leq i \leq r. \quad (3.2)$$

The flag manifold of type  $\mathbf{I}$  is the set  $\mathcal{F}^{\mathbf{I}}$  of all flags of type  $\mathbf{I}$ , identified with a subset of  $G^{\mathbf{I}}$ :

$$\mathcal{F}^{\mathbf{I}} = \left\{ \mathcal{P} = (P_i)_i \in G^{\mathbf{I}} : \sum P_i = \mathbb{R}^d, P_i P_j = 0, i \neq j \right\}. \quad (3.3)$$

**Remark 3.** This definition of a flag is more suitable for computing with flags of eigenspaces than the usual equivalent one as a nested sequence of linear subspaces.

In the sequel, for  $1 \leq i \leq r$ ,  $\text{St}_{q_i}$  and  $\pi_{q_i}$  are respectively denoted by  $\text{St}^i$  and  $\pi^i$ .

**Definition 4.** (i) Let  $\text{St}_{\perp}^{\mathbf{I}}$  be the set of all collections  $\overline{\mathcal{U}} = (\mathcal{U}_i)_{1 \leq i \leq r}$  of frames such that for  $1 \leq i \leq r$ ,  $\mathcal{U}_i \in \text{St}^i$  and the collection  $(\pi^i(\mathcal{U}_i))_i$  lies in  $\mathcal{F}^{\mathbf{I}}$ . Define the map  $(\pi^i)_i$  by

$$(\pi^i)_i : \text{St}_{\perp}^{\mathbf{I}} \longrightarrow \mathcal{F}^{\mathbf{I}} \quad \text{with} \quad (\pi^i)_i(\overline{\mathcal{U}}) := (\pi^i(\mathcal{U}_i))_i.$$

(ii) We say that  $\Gamma \in O(d)$  spans a flag  $\mathcal{P} = (P_i)_i \in \mathcal{F}^{\mathbf{I}}$  when for all  $1 \leq i \leq r$ ,  $\pi^i(\Gamma^{(i)}) = P_i$ .

(iii) For  $Q \in O(d)$ , the collection  $\zeta(Q) := (Q^{(i)})_{1 \leq i \leq r}$  lies in  $\text{St}_{\perp}^{\mathbf{I}}$ , which defines a diffeomorphism  $\zeta : O(d) \longrightarrow \text{St}_{\perp}^{\mathbf{I}}$ .

### 3.1.2 Actions of the orthogonal group

The group  $O(d)$  acts on vectors of  $\mathbb{R}^d$  by linear isometries, which induces an action on  $\text{St}_q$  by left-multiplication:

$$(Q, \mathcal{U}) \mapsto Q\mathcal{U}, \quad Q \in O(d), \quad \mathcal{U} \in \text{St}_q. \quad (3.4)$$

Let  $P \in G_q$  and  $\mathcal{U}_P$  a  $q$ -frame spanning  $\text{rg}(P)$ . Any  $Q \in O(d)$  sends  $\mathcal{U}_P$  to  $Q\mathcal{U}_P$ , which spans  $\text{rg}(\pi_q(Q\mathcal{U}_P)) = \text{rg}(QPQ')$ . The latter is independent of  $\mathcal{U}_P$ . So,  $O(d)$  acts on  $G_q$  by

$$(Q, P) \mapsto Q \cdot P := QPQ', \quad Q \in O(d), \quad P \in G_q. \quad (3.5)$$

The action on  $G_q$  extends to the product  $G^{\mathbf{I}}$ : for  $Q \in O(d)$  and  $\mathcal{P} = (P_i)_{1 \leq i \leq r} \in G^{\mathbf{I}}$ , set

$$Q * \mathcal{P} := (Q \cdot P_i)_{1 \leq i \leq r} \in G^{\mathbf{I}}. \quad (3.6)$$

Now, the action of  $Q \in O(d)$  preserves the mutual orthogonality of linear subspaces. Thus,  $\mathcal{P} \in \mathcal{F}^{\mathbf{I}} \implies Q * \mathcal{P} \in \mathcal{F}^{\mathbf{I}}$ . So, the action defined in (3.6) induces an action of  $O(d)$  on  $\mathcal{F}^{\mathbf{I}}$ .

**Definition 5.** For  $1 \leq i \leq r$ , the standard  $i$ -th Grassmannian is  $P_0^i \in G^i$  defined by

$$P_0^i := \pi^i(\{\epsilon_j : j \in \beta_i\}) \quad \text{i.e.} \quad P_0^i = \text{Diag}[0_{q_1}, \dots, I_{q_i}, \dots, 0_{q_r}].$$

The standard flag of type I is the flag  $\mathcal{P}_0^I := (P_0^i)_i$ . The main orbital map is the map

$$\pi^I : O(d) \longrightarrow \mathcal{F}^I, \quad \pi^I(Q) = Q * \mathcal{P}_0^I.$$

**Remark 4.** (i) It is easily obtained from (2.1) (see [12] for details) that

$$(\pi^i)_i \circ \zeta = \pi^I \quad \text{i.e.} \quad \left( \pi^i \left( Q^{(i)} \right) \right)_i = \pi^I(Q) = (QP_0^i Q')_i, \quad Q \in O(d). \quad (3.7)$$

(ii) Let  $Q \in O(d)$  which spans a flag  $\mathcal{P} \in \mathcal{F}^I$ . By definition,  $\mathcal{P} = (\pi^i(Q^{(i)}))_i$ . So, by (3.7),

$$\mathcal{P} = \pi^I(Q) := Q * \mathcal{P}_0^I. \quad (3.8)$$

### 3.1.3 Manifolds of linear subspaces

These actions of  $O(d)$  provide the manifold structures of  $\text{St}_q$ ,  $G_q$  and  $\mathcal{F}^I$ , through Proposition 1 hereafter (see [4, Proposition A.2]).

**Proposition 1.** Consider a compact Lie group  $\mathbb{G}$  which acts smoothly on a smooth manifold  $\mathbb{M}$ . For  $b_0 \in \mathbb{M}$ , let  $\mathbb{B}$  be its orbit and  $\mathbb{K}$  its isotropy group. Let  $\psi : \mathbb{G} \longrightarrow \mathbb{G}/\mathbb{K}$  be the canonical quotient map. Define the map  $\pi_0 : \mathbb{G} \longrightarrow \mathbb{B}$  by  $Q \longmapsto Q \cdot b_0$ . Then,

(i)  $\mathbb{B}$  is an embedded submanifold of  $\mathbb{M}$ .

(ii) The map  $\tilde{\pi}_0 : \mathbb{G}/\mathbb{K} \longrightarrow \mathbb{B}$  such that  $\tilde{\pi}_0 \circ \psi = \pi_0$  is a diffeomorphism.

$O(d)$  is a compact Lie group and its actions defined in subsection 3.1.2 are smooth. In [4], Proposition 1 is applied to prove that  $\text{St}_q$  (resp.  $G_q$ ) is an embedded submanifold of  $\mathbb{R}^{d \times q}$  (resp.  $\text{Sym}_d$ ) that is diffeomorphic to  $O(d)/O(d-q)$  (resp.  $O(d)/(O(q) \times O(d-q))$ ).

The manifold structure of  $\mathcal{F}^I$  is provided by the following Lemma, proved in [12].

**Lemma 1.** The main orbital map  $\pi^I : O(d) \longrightarrow \mathcal{F}^I$  with  $\pi^I(Q) = Q * \mathcal{P}_0^I$  is surjective.

By Lemma 1,  $\mathcal{F}^I$  is the orbit of  $\mathcal{P}_0^I$  under the smooth action of  $O(d)$  on  $G^I$  defined in (3.6). Now, the isotropy group of  $\mathcal{P}_0^I$  under this action is the group

$$O(I) := \{\text{Diag}(H_1, \dots, H_r) : H_i \in O(q_i), 1 \leq i \leq r\} \simeq \prod O(q_i).$$

By Proposition 1,  $\mathcal{F}^I$  is an embedded submanifold of  $G^I$  and the map  $\tilde{\pi}^I : O(d)/O(I) \longrightarrow \mathcal{F}^I$  s.t.  $\pi^I = \tilde{\pi}^I \circ \psi^I$  is a diffeomorphism, where  $\psi^I : O(d) \longrightarrow O(d)/O(I)$  is the canonical map.

### 3.1.4 Background of Riemannian geometry

We introduce Riemannian structures on these manifolds, in order to measure intrinsic distances and perform estimation on them.

More generally, let  $\mathbb{M}$  be a connected smooth manifold. A *metric* on  $\mathbb{M}$  is a collection  $g = (g_x)_{x \in \mathbb{M}}$  of inner products on the tangent spaces  $T_x \mathbb{M}$ , varying smoothly wrt  $x$ . Then,  $g$  defines the *geodesic distance*  $\rho_g$  on  $\mathbb{M}$ , where for  $x, y \in \mathbb{M}$ ,  $\rho_g(x, y)$  is the infimum of lengths (wrt  $g$ ) of all  $C^1$  curves between  $x$  and  $y$ . A *geodesic* is a smooth curve on  $\mathbb{M}$  that realizes locally this infimum. For  $x \in \mathbb{M}$  and  $v \in T_x \mathbb{M}$ , there exists a unique geodesic  $\gamma_{x,v}$  such that  $\gamma_{x,v}(0) = x$  and  $\dot{\gamma}_{x,v}(0) = v$ . By the Hopf-Rinow theorem, the metric space  $(\mathbb{M}, \rho_g)$  is complete iff all geodesics are defined on  $\mathbb{R}$ , which we assume in the sequel. In that case, for all  $x, y \in \mathbb{M}$ , there exists at least one geodesic of minimal length between  $x$  and  $y$ . For  $x \in \mathbb{M}$ , the *exponential map* at  $x$  is the map  $\text{Exp}_x^{\mathbb{M}} : T_x \mathbb{M} \longrightarrow \mathbb{M}$  such that  $\text{Exp}_x^{\mathbb{M}}(v) = \gamma_{x,v}(1)$ .

We recall that, for all  $x \in \mathbb{M}$ , any geodesic starting at  $x$  is locally length-minimizing. The *cut locus* of  $x \in \mathbb{M}$  is the set  $\text{Cut}(x)$  of points  $y$  such that the geodesics starting at  $x$  cease to be length-minimizing beyond  $y$ . Thus, on a sphere, the geodesics are the great circles, so that two antipodal points are each other's cut locus. Let  $x \in \mathbb{M}$  and  $y \in \mathbb{M} \setminus \text{Cut}(x)$ . Then, there exists a *unique* geodesic of minimal length between  $x$  and  $y$ , whose initial velocity is thus the smallest tangent vector  $v \in T_x\mathbb{M}$  such that  $\text{Exp}_x^{\mathbb{M}}(v) = y$ . This vector is called the *Riemannian Logarithm* of  $y$  at  $x$ , denoted by  $\text{Log}_x^{\mathbb{M}}(y)$ , related to the geodesic distance by:

$$\rho_g(x, y) = \left\| \text{Log}_x^{\mathbb{M}}(y) \right\|_x. \quad (3.9)$$

For any  $x \in \mathbb{M}$ , there exists an open set  $\text{ID}_x \subset T_x\mathbb{M}$ , called the *injectivity domain* at  $x$ , such that the restriction of  $\text{Exp}_x^{\mathbb{M}}$  to  $\text{ID}_x$  is a diffeomorphism onto  $\mathbb{M} \setminus \text{Cut}(x)$ , whose inverse is a restriction of the map  $\text{Log}_x^{\mathbb{M}}(\cdot)$ . Finally, we present results on *Riemannian submersions*.

**Definition 6.** Let  $(\mathbb{E}, g^{\mathbb{E}})$  and  $(\mathbb{B}, g^{\mathbb{B}})$  be Riemannian manifolds. Let  $\pi : \mathbb{E} \rightarrow \mathbb{B}$  be a smooth submersion, i.e. for all  $x \in \mathbb{E}$ ,  $d_x\pi$  is a surjective linear map.

- (i) For  $x \in \mathbb{E}$ , the vertical space at  $x$  is  $V_x\mathbb{E} := \ker(d_x\pi) \subset T_x\mathbb{E}$ . The horizontal space at  $x$  is its orthogonal complement in  $T_x\mathbb{E}$  wrt the inner product  $g_x^{\mathbb{E}}$ , denoted by  $H_x\mathbb{E} := (V_x\mathbb{E})^{\perp}$ .
- (ii) Let  $y \in \mathbb{B}$  and  $v \in T_y\mathbb{B}$ . By (i), for all  $x \in \mathbb{E}$ , there exists a unique vector  $v_x^{\sharp} \in H_x\mathbb{E}$  such that  $(d_x\pi)(v_x^{\sharp}) = v$ . The vector  $v_x^{\sharp}$  is called the horizontal lift wrt  $\pi$  of  $v$  at  $x$ .
- (iii) The map  $\pi$  is a Riemannian submersion when for all  $x \in \mathbb{E}$ , the restriction of  $d_x\pi$  to  $H_x\mathbb{E}$  is a linear isometry between  $(H_x\mathbb{E}, g_x^{\mathbb{E}})$  and  $(T_{\pi(x)}\mathbb{B}, g_{\pi(x)}^{\mathbb{B}})$ .

**Proposition 2.** Let  $\pi : \mathbb{E} \rightarrow \mathbb{B}$  be a Riemannian submersion.

- (i) Let  $\gamma$  be a geodesic in  $\mathbb{B}$ . Then, for all  $x \in \mathbb{E}$ , there exists a unique geodesic  $\hat{\gamma}^x$  in  $\mathbb{E}$  through  $x$  which is horizontal, i.e. all its tangent vectors are horizontal, and which projects to  $\gamma$ , i.e.  $\pi(\hat{\gamma}^x) = \gamma$ . The geodesic  $\hat{\gamma}^x$  is called the horizontal lift wrt  $\pi$  of  $\gamma$  through  $x$ .
- (ii) The geodesics in  $\mathbb{B}$  are the images by  $\pi$  of the horizontal geodesics in  $\mathbb{E}$ .

### 3.1.5 Metrics on Stiefel manifolds and Grassmannians

We recall hereafter the description of the tangent spaces of  $\text{St}_q$  and  $G_q$ : See [4]. For any  $\mathcal{U} \in \text{St}_q$  and  $P \in G_q$ ,

$$T_{\mathcal{U}}\text{St}_q = \{\mathcal{D} \in \mathbb{R}^{d \times q} : \mathcal{U}'\mathcal{D} = -\mathcal{D}'\mathcal{U}\} \quad \text{and} \quad T_P G_q = \{\Delta \in \text{Sym}_d : \Delta P + P\Delta = \Delta\}. \quad (3.10)$$

In the sequel, we endow  $\text{St}_q$  with the Euclidean metric  $g^{\text{St}}$  defined by

$$g_{\mathcal{U}}^{\text{St}}(\mathcal{D}_1, \mathcal{D}_2) = \text{tr}(\mathcal{D}_1^T \mathcal{D}_2), \quad \mathcal{U} \in \text{St}_q \text{ and } \mathcal{D}_1, \mathcal{D}_2 \in T_{\mathcal{U}}\text{St}_q.$$

Then,  $G_q$  is endowed with the metric  $g^G$  (see [4] for a discussion on it) defined by

$$g_P^G(\Delta_1, \Delta_2) = \frac{1}{2} \text{tr}(\Delta_1 \Delta_2), \quad P \in G_q \text{ and } \Delta_1, \Delta_2 \in T_P G_q.$$

The Riemannian manifold  $(G_q, g^G)$  has a rich structure, notably of *symmetric space*. So, many Riemannian operations in  $G_q$  are available in closed form: see paragraph 3.2.3. Moreover, the following result holds: see [12].

**Lemma 2.** The map  $\pi_q$  is a Riemannian submersion from  $(\text{St}_q, g^{\text{St}})$  onto  $(G_q, g^G)$ .



### 3.2 Geodesic parametrization of a full flag

With the notations of Section 2, let  $F^I(\Sigma)$  be the flag of eigenspaces of  $\Sigma$  (I denotes its type) and  $\widehat{F}_n$  that of  $\widehat{\Sigma}_n$ . Thus,  $F^I(\Sigma)$  is the flag of PS's and  $\widehat{F}_n$  is a.s. a full flag. As in Anderson's paper [1],  $\widehat{F}_n$  is represented by the matrix  $C_n \in O(d)$  of eigenvectors of  $\widehat{\Sigma}_n$ . When  $F^I(\Sigma)$  is estimated by  $\widehat{F}_n$ , the main difficulty to measure the deviation between them is that their types are different in general.

To solve this problem, we define a parametrization of (almost) any full flag, represented by an orthogonal matrix, which determines its relative position wrt the pair formed by a given flag and an orthogonal matrix that spans the latter. Then, we prove that such a parametrization of  $C_n$  wrt  $(F^I(\Sigma), \Gamma)$  is equivalent to that of  $E_n := \Gamma' C_n$  wrt  $(\mathcal{P}_0^I, I_d)$ .

#### 3.2.1 Geodesic projection of a frame

Fix  $R \in G_q$  and set

$$\mathbb{V}_R := \pi_q^{-1}(G_q \setminus \text{Cut}(R)) = \{\mathcal{U} \in \text{St}(q, d) : \pi_q(\mathcal{U}) \notin \text{Cut}(R)\}.$$

For  $\mathcal{U} \in \mathbb{V}_R$ , set  $P := \pi_q(\mathcal{U})$ . Let  $\gamma_{(P,R)}$  be the minimal geodesic between  $P$  and  $R$  in  $G_q$  and  $\widehat{\gamma}_{(P,R)}^{\mathcal{U}}$  its horizontal lift wrt  $\pi_q$  through  $\mathcal{U}$ . Then, define the  $q$ -frame  $\mathcal{H}_{(P,R)}^q(\mathcal{U})$  as the endpoint of  $\widehat{\gamma}_{(P,R)}^{\mathcal{U}}$ . By (ii) of Proposition 2,  $\pi_q(\mathcal{H}_{(P,R)}^q(\mathcal{U})) = R$ .

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\widehat{\gamma}_{(P,R)}^{\mathcal{U}}} & \mathcal{H}_{(P,R)}^q(\mathcal{U}) \\ \pi_q \downarrow & & \downarrow \pi_q \\ P & \xrightarrow{\gamma_{(P,R)}} & R \end{array}$$

**Remark 5.** Among all  $q$ -frames generating  $\text{rg}(R)$ ,  $\mathcal{H}_{(P,R)}^q(\mathcal{U})$  minimizes the geodesic distance in  $\text{St}_q$  to  $\mathcal{U}$ , denoted by  $d_g^{\text{St}}$ : See Lemma 26.11 in [10]. Formally,

$$d_g^{\text{St}}(\mathcal{U}, \mathcal{H}_{(P,R)}^q(\mathcal{U})) = \min \{d_g^{\text{St}}(\mathcal{U}, \mathcal{W}) : \pi^{\text{SG}}(\mathcal{W}) = R\}. \quad (3.11)$$

So,  $\mathcal{H}_{(P,R)}^q(\mathcal{U})$  is interpreted as the orthogonal projection of  $\mathcal{U}$  on  $\text{rg}(R)$  wrt the metric  $g^{\text{St}}$ .

**Definition 7.** The  $q$ -frame  $\mathcal{H}_{(P,R)}^q(\mathcal{U})$  is called the geodesic projection of  $\mathcal{U}$  onto  $\text{rg}(R)$ . For  $v \in T_P G_q$ , its horizontal lift wrt  $\pi_q$  at  $\mathcal{U}$  is denoted by  $v_{\mathcal{U}}^{\sharp}$ . By definition,

$$\mathcal{H}_{(P,R)}^q(\mathcal{U}) = \text{Exp}_{\mathcal{U}}^{\text{St}_q} \left( \left( \text{Log}_P^{G_q}(R) \right)_{\mathcal{U}}^{\sharp} \right). \quad (3.12)$$

#### 3.2.2 Geodesic decomposition of a frame

Fix  $R \in G_q$  and  $\mathcal{W}_R \in \pi_q^{-1}(R)$ .

**Lemma 3.** (i) Define the map  $\theta_R : \mathbb{V}_R \rightarrow \pi_q(\mathbb{V}_R) \times \pi_q^{-1}(R)$  by

$$\theta_R(\mathcal{U}) = \left( \pi_q(\mathcal{U}), \mathcal{H}_{(\pi_q(\mathcal{U}), R)}^q(\mathcal{U}) \right). \quad (3.13)$$

Then, the map  $\theta_R$  is a diffeomorphism, whose inverse is defined by

$$\theta_R^{-1} : \pi_q(\mathbb{V}_R) \times \pi_q^{-1}(R) \rightarrow \mathbb{V}_R \quad \text{with} \quad (P, \mathcal{V}) \mapsto \mathcal{H}_{(R,P)}^q(\mathcal{V}).$$

(ii) Let  $\text{ID}_R$  be the injectivity domain at  $R$ . Define the map  $\xi_R^{\mathcal{W}_R} : \mathbb{V}_R \rightarrow \text{ID}_R \times O(q)$  by

$$\xi_R^{\mathcal{W}_R}(\mathcal{U}) = \left( \text{Log}_R^{G_q}(\pi_q(\mathcal{U})), (\mathcal{W}_R)' \mathcal{H}_{(\pi_q(\mathcal{U}), R)}^q(\mathcal{U}) \right). \quad (3.14)$$

Then, the map  $\xi_R^{\mathcal{W}_R}$  is a diffeomorphism.

*Proof.* One checks that the map  $\theta_R^{-1}$  is indeed the inverse of  $\theta_R$ : see the diagram in Definition 7. Their smoothness follows from properties of Riemannian submersions. This proves (i). Then, (ii) follows readily from (i) and Remark 2.  $\square$

**Definition 8.** For  $\mathcal{U} \in \mathbb{V}_R$ , the pair  $\theta_R(\mathcal{U})$  is called the geodesic decomposition of  $\mathcal{U}$  wrt  $R$  and  $\xi_R^{\mathcal{W}_R}(\mathcal{U})$  its effective geodesic decomposition wrt  $(R, \mathcal{W}_R)$ . By Lemma 3, any  $\mathcal{U} \in \mathbb{V}_R$  is uniquely determined by these decompositions.

**Remark 6.** The effective geodesic decomposition lies in a product of a vector space and a group. Thus, it provides a handleable measure of the relative position of any  $q$ -frame  $\mathcal{U} \in \mathbb{V}_R$  wrt a given  $q$ -linear subspace  $R$  and a  $q$ -frame spanning  $\text{rg}(R)$ .

### 3.2.3 Explicit formulas

The explicit formulas in  $G_q$  provide those for the (effective) geodesic decomposition of a frame. First, we describe the cut locus in  $G_q$ : see [4].

**Lemma 4.** (i) Let  $P \in G_q$ . Then, writing  $P = YY'$ ,

$$\text{Cut}(P) = \{R = ZZ' \in G_q : \text{rank}(Y'Z) < q\}.$$

(ii) Thus, the following symmetry holds:  $R \in \text{Cut}(P) \iff P \in \text{Cut}(R)$ .

(iii) The cut locus is compatible with the action of  $O(d)$  on  $G_q$ , i.e. for all  $Q \in O(d)$ ,

$$R \in \text{Cut}(P) \iff Q \cdot R \in \text{Cut}(Q \cdot P).$$

Now, a closed form for the Riemannian Logarithm in  $G_q$  is presented hereafter: see [2].

**Theorem 4.** Let  $P \in G_q$  and  $R \in G_q \setminus \text{Cut}(P)$ . Then,

$$\text{Log}_P^{G_q}(R) = [\Omega, P] \quad \text{where} \quad \Omega = \frac{1}{2} \log_m((I_d - 2R)(I_d - 2P)), \quad (3.15)$$

and  $\log_m$  denotes the matrix logarithm.

**Corollary 1.** Let  $P, R \in G_q$  and  $Q \in O(d)$ . If  $R \notin \text{Cut}(P)$ , then

$$\text{Log}_{Q \cdot P}^{G_q}(Q \cdot R) = Q \left( \text{Log}_P^{G_q}(R) \right) Q'.$$

*Proof.* By the properties of the matrix logarithm,

$$\begin{aligned} \log_m((I_d - 2QRQ')(I_d - 2QPQ')) &= \log_m(Q(I_d - 2R)(I_d - 2P)Q') \\ &= Q \log_m((I_d - 2R)(I_d - 2P)) Q'. \end{aligned}$$

So, with the notations of (3.15),  $\text{Log}_{Q \cdot P}^{G_q}(Q \cdot R) = [Q\Omega Q', QPQ'] = Q[\Omega, P]Q'$ .  $\square$

Then, we establish from (3.12) an explicit expression of  $\mathcal{H}_{(P,R)}^q(\mathcal{U})$ , proved in [12].

**Proposition 3.** For  $\mathcal{U} \in \mathbb{V}_R$ , set  $P := \pi_q(\mathcal{U})$  and  $\Delta := \text{Log}_P^{G_q}(R)$ . Then, setting also  $\mathcal{V} := (\mathcal{U} \quad \Delta\mathcal{U}) \in \mathbb{R}^{n \times 2q}$  and  $C := \mathcal{U}'\Delta^2\mathcal{U}$ ,

$$\mathcal{H}_{(P,R)}^q(\mathcal{U}) = \mathcal{V} \left( \exp_m \begin{pmatrix} 0 & -C \\ I_q & 0 \end{pmatrix} \right) \begin{pmatrix} I_q \\ 0_q \end{pmatrix}. \quad (3.16)$$

We deduce hereafter the effect on the geodesic decomposition of the action of  $O(d)$ .

**Corollary 2.** (i) Let  $R \in G_q$  and  $\mathcal{U} \in \mathbb{V}_R$ . Then, for all  $Q \in O(d)$ ,

$$Q\mathcal{U} \in \mathbb{V}_{Q \cdot R} \quad \text{and} \quad \text{Log}_{Q \cdot R}^{G_q}(\pi_q(Q\mathcal{U})) = Q \left( \text{Log}_R^{G_q}(\pi_q(\mathcal{U})) \right) Q'. \quad (3.17)$$

(ii) For the geodesic projections, we have that

$$\mathcal{H}_{(\pi_q(Q\mathcal{U}), Q \cdot R)}(Q\mathcal{U}) = Q\mathcal{H}_{(\pi_q(\mathcal{U}), R)}(\mathcal{U}).$$

(iii) For any  $\mathcal{W}_R \in \pi_q^{-1}(R)$ , the effective geodesic decomposition of  $\mathcal{U}$  wrt  $(R, \mathcal{W}_R)$  is fully determined by that of  $Q\mathcal{U}$  wrt  $(Q \cdot R, Q\mathcal{W}_R)$ .

*Proof.* (i) and (ii) follow from Lemma 4, Corollary 1 and Proposition 3. Then, (iii) is deduced from (i) and (ii).  $\square$

### 3.2.4 Geodesic parametrization of an orthogonal matrix

In this paragraph, we consider a fixed flag  $\mathcal{R} = (R_i)_i \in \mathcal{F}^I$  and a matrix  $\Gamma \in O(d)$  which spans  $\mathcal{R}$ . Now, set

$$\mathbb{W}_{\mathcal{R}} := \left\{ Q \in O(d) : \forall 1 \leq i \leq r, \pi^i(Q^{(i)}) \notin \text{Cut}(R_i) \text{ i.e. } QP_0^i Q' \notin \text{Cut}(R_i) \right\}. \quad (3.18)$$

For  $C \in \mathbb{W}_{\mathcal{R}}$  identified with  $\zeta(C) := (C^{(i)})_i$ , consider the collection of geodesic decompositions of  $C^{(i)}$  wrt  $R_i$ , illustrated hereafter, where (3.7) justifies that  $\pi^I(C) = (\pi^i(C^{(i)}))_i$ .

$$\begin{array}{ccccc} C & \xrightarrow{\zeta} & \zeta(C) := (C^{(i)})_i & \cdots \cdots \cdots & \left( \mathcal{H}_{(\pi^i(C^{(i)}), R_i)}(C^{(i)}) \right)_i \\ & \searrow \pi^I & \downarrow (\pi^i)_i & & \downarrow (\pi^i)_i \\ & & \pi^I(C) = (\pi^i(C^{(i)}))_i & \cdots \cdots \cdots & \mathcal{R} = (R_i)_i \end{array}$$

**Definition 9.** The geodesic parametrization of  $C \in \mathbb{W}_{\mathcal{R}}$  (or of the associated full flag) wrt  $(\mathcal{R}, \Gamma)$  is defined as the collection of effective geodesic decompositions of  $C^{(i)}$  wrt  $(R_i, \Gamma^{(i)})$ , for  $1 \leq i \leq r$ . This parametrization determines  $C \in \mathbb{W}_{\mathcal{R}}$  uniquely.

**Lemma 5.** For  $C \in \mathbb{W}_{\mathcal{R}}$ , set  $E := \Gamma' C$ . Then,  $E \in \mathbb{W}_{\mathcal{P}_0^I}$  and the geodesic parametrization of  $C$  wrt  $(\mathcal{R}, \Gamma)$  is fully determined by that of  $E$  wrt  $(\mathcal{P}_0^I, I_d)$ .

*Proof.* We apply Corollary 2 with  $Q = \Gamma'$ . Thus, the geodesic parametrization of  $C$  wrt  $(\mathcal{R}, \Gamma)$  is fully determined by that of  $E := \Gamma' C$  wrt  $(\Gamma' * \mathcal{R}, \Gamma' \Gamma) = (\Gamma^{-1} * \mathcal{R}, I_d)$ . Now,  $\Gamma \in O(d)$  spans the flag  $\mathcal{R}$ . So, by (3.8),  $\mathcal{R} = \Gamma * \mathcal{P}_0^I$ , which implies that  $\Gamma^{-1} * \mathcal{R} = \mathcal{P}_0^I$ .  $\square$

### 3.2.5 Tangent spaces at standard Grassmannians

A geodesic parametrization wrt  $(\mathcal{P}_0^I, I_d)$  involves the maps  $\text{Log}_{\mathcal{P}_0^I}^{G^i}$ , for  $1 \leq i \leq r$ , each valued in the tangent space  $T_{\mathcal{P}_0^I} G^i$ . Thus, we describe hereafter the structure of the latter.

**Lemma 6.** For  $1 \leq i \leq r$ , set  $\mathbb{T}_0^i := T_{\mathcal{P}_0^I} G^i$ . Then,

$$\begin{aligned} \mathbb{T}_0^1 &= \left\{ \begin{pmatrix} 0_{q_1} & A_{>}^1 \\ (A_{>}^1)' & 0 \end{pmatrix} : A_{>}^1 \in \mathbb{R}^{q_1 \times (q_2 + \dots + q_r)} \right\}, \quad \mathbb{T}_0^r = \left\{ \begin{pmatrix} 0 & (A_{<}^r)' \\ A_{<}^r & 0_{q_r} \end{pmatrix} : A_{<}^r \in \mathbb{R}^{q_r \times (q_1 + \dots + q_{r-1})} \right\} \\ \mathbb{T}_0^i &= \left\{ \begin{pmatrix} 0 & (A_{<}^i)' & 0 \\ A_{<}^i & 0_{q_i} & A_{>}^i \\ 0 & (A_{>}^i)' & 0 \end{pmatrix} : A_{<}^i \in \mathbb{R}^{q_i \times (q_1 + \dots + q_{i-1})}, A_{>}^i \in \mathbb{R}^{q_i \times (q_{i+1} + \dots + q_r)} \right\}, \quad i \notin \{1, r\}. \end{aligned}$$

*Proof.* By (3.10), for any  $P \in G(q, d)$ ,  $T_P G(q, d) = \{\Delta \in \text{Sym}_d : \Delta P + P\Delta = \Delta\}$ . So,

$$\Delta \in \mathbb{T}_0^i \iff \Delta \in \text{Sym}_d \text{ and } \Delta P_0^i + P_0^i \Delta = \Delta. \quad (3.19)$$

We readily check that the sets described in the rhs' in the statement of this Lemma satisfy (3.19). Thus, each of these sets is a linear subspace of  $\mathbb{T}_0^i$ , and clearly, its dimension is equal to that of  $\mathbb{T}_0^i$ . This concludes the proof.  $\square$

**Corollary 3.** For any  $d \times d$  matrix  $A = (a_{k\ell})_{1 \leq k \leq \ell \leq d}$ , its Frobenius norm is

$$\|A\|_F := \text{tr}(A'A) = \sqrt{\sum (a_{k\ell})^2}. \quad (3.20)$$

Let  $1 \leq i \leq r$ . Then, for all  $\mathcal{S} \in \mathbb{T}_0^i$ ,

$$\sum_{i=1}^r \|\mathcal{S}\|_F^2 = 4 \sum_{1 \leq i < j \leq r} \|\mathcal{S}^{(i,j)}\|_F^2. \quad (3.21)$$

*Proof.* This identity is an easy consequence of Lemma 6 and (3.20).  $\square$

### 3.3 Preliminaries for estimating the flag of PS's

The flag  $F^I(\Sigma)$  of PS's is estimated by the *flag of eigenprojections*  $F^I(\widehat{\Sigma}_n)$ , of type I, introduced in paragraph 3.3.1 below. Thus, we need a distance on  $\mathcal{F}^I$  to measure the deviation between flags of type I. However, there is no known metric on  $\mathcal{F}^I$  for which the geodesic distance on  $\mathcal{F}^I$  is available in closed form. To overcome this lack, we introduce in (3.24) below, an *extrinsic distance*  $\mathfrak{D}_{\text{ex}}$  on  $\mathcal{F}^I$  i.e. that is the restriction to  $\mathcal{F}^I$  of a distance on  $G^I$ , which has an explicit expression.

#### 3.3.1 Flag of eigenprojections

Let  $\Sigma \in \text{Sym}_d$ , whose flag of eigenspaces, denoted by  $F^I(\Sigma)$ , is of type I :=  $(q_i)_{1 \leq i \leq r}$ . For  $S \in \text{Sym}_d^{\neq}$ , let  $\Gamma$  and  $C$  be respective orthogonal matrices of eigenvectors of  $\Sigma$  and  $S$ , associated to eigenvalues in decreasing order.

For all  $1 \leq i \leq r$ ,  $\pi^i(C^{(i)})$  is the projector onto a  $q_i$ -linear subspace spanned by eigenvectors of  $C$ . This projector is independent of the choice of  $C$  i.e. depends only on  $S$ . Thus it is called the *i-th eigenprojection* of  $S$  wrt I, denoted by  $P_i(S)$ . Then, the collection  $F^I(S) := (P_i(S))_i$  form a flag, called the *flag of eigenprojections* of  $S$  wrt I. By (3.7),

$$F^I(S) = (P_i(S))_i = \left( \pi^i \left( C^{(i)} \right) \right)_i = \pi^I(C). \quad (3.22)$$

Assume that  $C \in \mathbb{W}_{F^I(\Sigma)}$ . Set  $E := \Gamma' C$ . Since  $\Gamma$  spans  $F^I(\Sigma)$ , by Lemma 5, the geodesic parametrization of  $C$  wrt  $(F^I(\Sigma), \Gamma)$  is fully determined by that of  $E$  wrt  $(\mathcal{P}_0^I, I_d)$ , for which

$$\pi^i(E^{(i)}) = \Gamma' \cdot \pi^i(C^{(i)}) = \Gamma' \cdot P_i(S), \quad 1 \leq i \leq r. \quad (3.23)$$

#### 3.3.2 Extrinsic distance

Let  $\mathcal{R} = (R_i)_i \in \mathcal{F}^I$  and  $\mathbb{W}_{\mathcal{R}}$  defined in (3.18). Then,

$$\pi^I(\mathbb{W}_{\mathcal{R}}) = \{\mathcal{P} = (P_i)_i \in \mathcal{F}^I : \forall 1 \leq i \leq r, P_i \notin \text{Cut}(R_i)\}.$$

Now, for  $\mathcal{P} \in \pi^I(\mathbb{W}_{\mathcal{R}})$ , set

$$\mathfrak{D}_{\text{ex}}(\mathcal{P}, \mathcal{R}) := \left( \sum_{i=1}^r \delta^i(P_i, R_i)^2 \right)^{1/2} \quad \text{where } \delta^i(P_i, R_i) := \left\| \text{Log}_{P_i}^{G_i}(R_i) \right\|_F. \quad (3.24)$$

Thus,  $\delta^i$  is the geodesic distance on  $G^i$ . The distance  $\mathfrak{D}_{\text{ex}}$  is *extrinsic*, i.e. it is the restriction to  $\mathcal{F}^I$  of a distance on  $G^I$ , and has an *explicit expression*, provided by that of  $\text{Log}_{P_0^i}^{G^i}$ . Clearly, the action of  $O(d)$  on  $\mathcal{F}^I$  preserves  $\mathfrak{D}_{\text{ex}}$ , i.e. for all  $Q \in O(d)$ ,

$$\mathfrak{D}_{\text{ex}}(Q * \mathcal{P}, Q * \mathcal{R}) = \mathfrak{D}_{\text{ex}}(\mathcal{P}, \mathcal{R}). \quad (3.25)$$

In particular, with the notations of paragraph 3.2.4, we deduce from (3.25) that for  $E := \Gamma' C$ ,

$$\mathfrak{D}_{\text{ex}}(\pi^I(E), \mathcal{P}_0^I) = \mathfrak{D}_{\text{ex}}(\Gamma' * \pi^I(C), \Gamma' * F^I(\Sigma)) = \mathfrak{D}_{\text{ex}}(F^I(S), F^I(\Sigma)).$$

### 3.3.3 A discrepancy between flags

In Section 4, to obtain a pivotal statistic for the flag  $F^I(\Sigma)$  of PS's, we need to standardize some Gaussian rv's. Due to such renormalizations, the resulting confidence regions for  $F^I(\Sigma)$  are expressed in terms of a *discrepancy* between  $F^I(\Sigma)$  and  $F^I(\widehat{\Sigma}_n)$ , instead of the extrinsic distance  $\mathfrak{D}_{\text{ex}}$  between them. In fact, this discrepancy, defined in (3.27) below, is a *deformation* of  $\mathfrak{D}_{\text{ex}}$ . This means that, without these renormalizations, this discrepancy is equal to  $\mathfrak{D}_{\text{ex}}$ . Let  $\mathcal{D}(I)$  be the set of all block-diagonal matrices whose structure agrees with the type I, i.e.

$$M \in \mathcal{D}(I) \iff M = \text{Diag}\left(M^{(1,1)}, \dots, M^{(i,i)}, \dots, M^{(r,r)}\right).$$

In this paragraph,  $\Gamma$  denotes a variable matrix in  $O(d)$ . Let  $\mathbb{K} = (K^i)_{1 \leq i \leq r}$  be a collection of matrices such that for all  $1 \leq i \leq r$ ,  $K^i \in \mathcal{D}(I)$ . Let  $\mathcal{P} = (P_i)_i \in \mathcal{F}^I$ . By Lemma 4,

$$\Gamma \in \mathbb{W}_{\mathcal{P}} \iff \forall 1 \leq i \leq r, \Gamma' P_i \Gamma \notin \text{Cut}(P_0^i). \quad (3.26)$$

Then, for  $\Gamma \in \mathbb{W}_{\mathcal{P}}$ , set

$$\mathfrak{D}_{\mathbb{K}}(\Gamma, \mathcal{P}) := \sqrt{\sum_{i=1}^r (\delta_{K^i}^i(\Gamma, P_i))^2} \text{ where } \delta_{K^i}^i(\Gamma, P_i) := \left\| K^i \text{Log}_{P_0^i}^{G^i}(\Gamma' \cdot P_i) K^i \right\|_F. \quad (3.27)$$

**Lemma 7.** *Let  $\mathcal{P} = (P_i)_i \in \mathcal{F}^I$  and  $\Gamma \in \mathbb{W}_{\mathcal{P}}$ . Then,  $\mathfrak{D}_{\mathbb{K}}(\Gamma, \mathcal{P})$  is independent of the class of  $\Gamma$  modulo  $O(I)$ , i.e. for all  $H \in O(I)$ ,*

$$\Gamma H \in \mathbb{W}_{\mathcal{P}} \text{ and } \mathfrak{D}_{\mathbb{K}}(\Gamma H, \mathcal{P}) = \mathfrak{D}_{\mathbb{K}}(\Gamma, \mathcal{P}). \quad (3.28)$$

*Proof.* Recall that  $O(I)$  is the isotropy group of  $\mathcal{P}_0^I$  under the action of  $O(d)$  on  $\mathcal{F}^I$ , i.e. for all  $H \in O(I)$  and  $1 \leq i \leq r$ ,  $H \cdot P_0^i = P_0^i$ . So, by (3.26), for all  $H \in O(I)$ ,

$$\Gamma \in \mathbb{W}_{\mathcal{P}} \iff \Gamma H \in \mathbb{W}_{\mathcal{P}}. \quad (3.29)$$

Then, by Corollary 1, for any  $\mathcal{P} = (P_i)_i \in \mathcal{F}^I$ ,  $\Gamma \in \mathbb{W}_{\mathcal{P}}$  and  $H \in O(I)$ ,

$$\text{Log}_{P_0^i}^{G^i}((\Gamma H)' \cdot R_i) = H' \left( \text{Log}_{P_0^i}^{G^i}(\Gamma' \cdot R_i) \right) H.$$

This, combined to Remark 7 below, implies that for all  $1 \leq i \leq r$ ,

$$\delta_{K^i}^i(\Gamma H, P_i) = \left\| K^i \left( H' \text{Log}_{P_0^i}^{G^i}(\Gamma' \cdot P_i) H \right) K^i \right\|_F = \left\| H' \left( K^i \text{Log}_{P_0^i}^{G^i}(\Gamma' \cdot P_i) K^i \right) H \right\|_F.$$

Finally, by the properties of  $\|\cdot\|_F$ , the rhs hereabove is equal to  $\delta_{K^i}^i(\Gamma, P_i)$ .  $\square$

**Remark 7.** *For all  $H \in O(I)$  and  $K \in \mathcal{D}(I)$ ,  $H$  and  $K$  commute, i.e.  $HK = KH$ .*

Now, we may introduce a discrepancy between flags of type I as follows.

**Definition 10.** Let  $\mathbb{K} = (K^i)_i \in (\mathcal{D}(\mathbb{I}))^r$ . Let  $\mathcal{P}, \mathcal{R} \in \mathcal{F}^{\mathbb{I}}$  such that  $\mathcal{R} \in \pi^{\mathbb{I}}(\mathbb{W}_{\mathcal{P}})$ . Then, by Lemma 7, we may define the  $\mathbb{K}$ -discrepancy between  $\mathcal{R}$  and  $\mathcal{P}$  as follows.

$$\tilde{\mathfrak{D}}_{\mathbb{K}}(\mathcal{R}, \mathcal{P}) := \mathfrak{D}_{\mathbb{K}}(\Gamma, \mathcal{P}), \quad (3.30)$$

for any  $\Gamma \in O(d)$  such that  $\pi^{\mathbb{I}}(\Gamma) = \mathcal{R}$ , which is defined modulo  $O(\mathbb{I})$ . In particular, it is easy to check that, for  $\mathbb{I} := (I_d, \dots, I_d) \in (\mathcal{D}(\mathbb{I}))^r$ , we have that  $\tilde{\mathfrak{D}}_{\mathbb{I}}(\mathcal{R}, \mathcal{P}) = \mathfrak{D}_{\text{ex}}(\mathcal{R}, \mathcal{P})$ . Thus, the  $\mathbb{K}$ -discrepancy  $\tilde{\mathfrak{D}}_{\mathbb{K}}$  is interpreted as a deformation of the extrinsic distance  $\mathfrak{D}_{\text{ex}}$ .

**Remark 8.** In general,  $\tilde{\mathfrak{D}}_{\mathbb{K}}$  is not a distance on  $\mathcal{F}^{\mathbb{I}}$ . However, we readily prove that  $\tilde{\mathfrak{D}}_{\mathbb{K}}$  fulfills the following properties. (i) The separation property holds, i.e. for all  $\mathcal{R}, \mathcal{P} \in \mathcal{F}^{\mathbb{I}}$

$$\tilde{\mathfrak{D}}_{\mathbb{K}}(\mathcal{R}, \mathcal{P}) = 0 \iff \mathcal{R} = \mathcal{P}. \quad (3.31)$$

(ii) The  $\mathbb{K}$ -discrepancy preserves the action of  $O(d)$ , i.e. for all  $Q \in O(d)$ ,

$$\tilde{\mathfrak{D}}_{\mathbb{K}}(Q * \mathcal{R}, Q * \mathcal{P}) = \tilde{\mathfrak{D}}_{\mathbb{K}}(\mathcal{R}, \mathcal{P}), \quad \mathcal{R}, \mathcal{P} \in \mathcal{F}^{\mathbb{I}}.$$

## 4 Limit theorem and estimation of the flag of PS's

If  $E_n \in \mathbb{W}_{\mathcal{P}_0^{\mathbb{I}}}$ , then we establish in Theorem 5 the limiting distribution of its geodesic parametrization wrt  $(\mathcal{P}_0^{\mathbb{I}}, I_d)$ . In fact,  $\Pr(E_n \notin \mathbb{W}_{\mathcal{P}_0^{\mathbb{I}}}) \rightarrow 0$  as  $n \rightarrow \infty$ : see (i) of Proposition 7 in Appendix. Thus, we obtain Gaussian and Haar limiting distributions, as in Theorem 1, where  $E_n$  is parametrized by its usual coordinates. Then, we derive from (i) of Theorem 5, confidence regions for the flag  $F^{\mathbb{I}}(\Sigma)$  of PS's, in terms of a discrepancy (in the sense of Definition 10) between  $F^{\mathbb{I}}(\Sigma)$  and the flag  $F^{\mathbb{I}}(\hat{\Sigma}_n)$  of eigenprojections of  $\hat{\Sigma}_n$ , both of type I.

### 4.1 Statement of main theorem

First, for  $1 \leq i \leq r$ , we define the limit function  $g_0^i$  valued in  $\mathbb{T}_0^i$  involved in (4.1) below. For  $u \in \text{Sym}_d$ , set

$$\mathcal{G}^{i,j}(u) := \frac{1}{\lambda_i - \lambda_j} u^{(i,j)}, \quad 1 \leq i, j \leq r \text{ with } i \neq j.$$

Then, consider the matrices  $\mathcal{G}_{<}^i(u)$  and  $\mathcal{G}_{>}^i(u)$  of respective sizes  $q_i \times (q_1 + \dots + q_{i-1})$  and  $q_i \times (q_{i+1} + \dots + q_r)$  defined by

$$\mathcal{G}_{<}^i(u) := (\mathcal{G}^{i,1}(u) \quad \dots \quad \mathcal{G}^{i,i-1}(u)) \quad \text{and} \quad \mathcal{G}_{>}^i(u) := (\mathcal{G}^{i,i+1}(u) \quad \dots \quad \mathcal{G}^{i,r}(u))$$

Finally, define the map  $g_0^i : \text{Sym}_d \rightarrow \mathbb{T}_0^i$  by

$$g_0^1(u) = \begin{pmatrix} 0_{q_1} & \mathcal{G}_{>}^1(u) \\ (\mathcal{G}_{>}^1(u))' & 0 \end{pmatrix}, \quad g_0^r(u) = \begin{pmatrix} 0 & (\mathcal{G}_{<}^r(u))' \\ \mathcal{G}_{<}^r(u) & 0_{q_r} \end{pmatrix}$$

$$g_0^i(u) = \begin{pmatrix} 0 & (\mathcal{G}_{<}^i(u))' & 0 \\ \mathcal{G}_{<}^i(u) & 0_{q_i} & \mathcal{G}_{>}^i(u) \\ 0 & (\mathcal{G}_{>}^i(u))' & 0 \end{pmatrix}, \quad i \neq 1, r.$$

Now, we can state our main result hereafter, where the rv  $U$  is defined in Theorem 2.

**Theorem 5.** (i) For any  $1 \leq i \leq r$ , the following CLT holds in  $\mathbb{T}_0^i$ .

$$G_n^i := \sqrt{n} \text{Log}_{\mathcal{P}_0^{\mathbb{I}}}^{G^i} \left( \pi^i \left( E_n^{(i)} \right) \right) \xrightarrow[n \rightarrow \infty]{d} g_0^i(U). \quad (4.1)$$

(ii) For any  $1 \leq i \leq r$ , the following convergence holds in  $O(q_i)$ .

$$H_n^i := \left( I_d^{(i)} \right)' \left( \mathcal{H}_{\left( \pi^i \left( E_n^{(i)} \right), P_0^i \right)}^{q_i} \left( E_n^{(i)} \right) \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{E}^{i,i}(U), \quad (4.2)$$

where the distribution of  $\mathcal{E}^{i,i}(U)$  is the conditional Haar invariant distribution.

**Remark 9.** In conclusion, we conjecture a possible application of (4.2).

## 4.2 Convergence to a $\chi^2$ distribution

In Corollary 4, we derive a *pivotal statistic* for the matrix  $\Gamma$ , i.e. which depends only on the sample, and whose limit distribution does not depend on any unknown parameter. Thus, Lemma 8 allows to *concatenate the CLT's* of (4.1) for  $1 \leq i \leq r$ , providing the convergence of a pivotal statistic to a  $\chi^2$  distribution. Namely, by Theorem 2, for  $i \neq j$ , the entries of  $(g_0^i(U))^{(i,j)} = \frac{1}{\lambda_i - \lambda_j} U^{(i,j)}$  are real i.i.d rv's  $\mathcal{N}(0, \sigma_{i,j}^2)$ , of standard deviation  $\sigma_{i,j} = \frac{\sqrt{\lambda_i \lambda_j}}{|\lambda_i - \lambda_j|}$ . We normalize these entries by setting

$$\bar{g}_0^i(U) := K^i g^i(U) K^i \quad \text{where} \quad K^i := \text{Diag} \left( \frac{1}{\sigma_{i,1}} I_{q_1}, \dots, I_{q_i}, \dots, \frac{1}{\sigma_{i,r}} I_{q_r} \right). \quad (4.3)$$

**Lemma 8.** (i) The non-null entries of the matrix  $\bar{g}_0^i(U)$  are real iid rv's  $\mathcal{N}(0, 1)$ .

(ii) The blocks  $\left\{ (\bar{g}_0^i(U))^{(i,j)} : 1 \leq i < j \leq r \right\}$  are mutually independent.

(iii) The real rv  $\frac{1}{4} \sum_{i=1}^r \|\bar{g}_0^i(U)\|_F^2$  is distributed as a  $\chi_{\mathbb{D}^I}^2$ , where

$$\mathbb{D}^I := \frac{1}{2} \left( d^2 - \sum_{i=1}^r q_i^2 \right).$$

*Proof.* Clearly,  $\bar{g}_0^i(U)$  is a random matrix valued in  $\mathbb{T}_0^i$  and for  $i \neq j$ ,

$$(\bar{g}_0^i(U))^{(i,j)} = \frac{1}{\sigma_{i,j}} \left( \frac{1}{\lambda_i - \lambda_j} U^{(i,j)} \right). \quad (4.4)$$

This proves (i) and (ii). Then, (iii) holds, by Corollary 3 applied with  $\mathcal{S} = \bar{g}_0^i(U)$ .  $\square$

**Proposition 4.** For  $(K^i)_{1 \leq i \leq r}$  defined in (4.3),

$$\frac{n}{4} \sum_{i=1}^r \left\| K^i \text{Log}_{P_0^i}^{G^i} \left( \Gamma' \cdot P_i \left( \hat{\Sigma}_n \right) \right) K^i \right\|_F^2 \xrightarrow[n \rightarrow \infty]{d} \chi_{\mathbb{D}^I}^2, \quad (4.5)$$

*Proof.* By (i) of Theorem 5 and (iii) of Lemma 8,

$$\frac{n}{4} \sum_{i=1}^r \left\| K^i \text{Log}_{P_0^i}^{G^i} \left( \pi^i \left( E_n^{(i)} \right) \right) K^i \right\|_F^2 \xrightarrow[n \rightarrow \infty]{d} \frac{1}{4} \sum_{i=1}^r \|\bar{g}_0^i(U)\|_F^2 \sim \chi_{\mathbb{D}^I}^2. \quad (4.6)$$

Now, by (3.23) applied to  $S = \hat{\Sigma}_n \in \text{Sym}_d^\neq$  a.s., we have that  $\pi^i \left( E_n^{(i)} \right) = \Gamma' \cdot P_i \left( \hat{\Sigma}_n \right)$  a.s. So, by injecting this relation in the lhs of (4.6), we deduce that (4.5) holds.  $\square$

However, in the lhs of (4.5), the matrix  $K^i$  still depends on the unknown parameters  $(\lambda_i)_i$ . Thus, in  $K_i$ , we replace  $\lambda_i$  by  $\hat{\lambda}_n^i$ , where (with the notations of paragraph 2.1), denoting by  $(\mu_k \left( \hat{\Sigma}_n \right))_{1 \leq k \leq d}$  the eigenvalues of  $\hat{\Sigma}_n$  and by  $(\beta_i)_i$  the partition of  $\{1, \dots, d\}$  wrt I,

$$\hat{\lambda}_n^i := \frac{1}{q_i} \sum_{k \in \beta_i} \mu_k \left( \hat{\Sigma}_n \right) \xrightarrow[n \rightarrow \infty]{a.s.} \lambda_i.$$

**Corollary 4.** *The statistic  $\widehat{S}_n$  defined hereafter is a pivotal statistic for  $\Gamma$ .*

$$\widehat{S}_n := \frac{n}{4} \sum_{i=1}^r \left\| \widehat{K}_n^i \text{Log}_{P_0^i}^{G^i} \left( \Gamma' \cdot P_i \left( \widehat{\Sigma}_n \right) \right) \widehat{K}_n^i \right\|_F^2 \xrightarrow[n \rightarrow \infty]{d} \chi_{D^I}^2, \quad (4.7)$$

where, for  $1 \leq i \leq r$ ,  $\widehat{K}_n^i$  is derived from  $K^i$  by replacing  $\lambda_i$  by  $\widehat{\lambda}_n^i$ .

*Proof.* Since  $\widehat{K}_n^i \rightarrow K^i$  a.s. as  $n \rightarrow \infty$ , this follows from Proposition 4.  $\square$

### 4.3 Confidence regions

For  $n \geq 1$ , set  $\widehat{\mathbb{K}}_n := \left( \widehat{K}_n^i \right)_{1 \leq i \leq r}$ . Thus, by definition,

$$\widehat{S}_n = \frac{n}{4} \left[ \mathfrak{D}_{\widehat{\mathbb{K}}_n} \left( \Gamma, F^I \left( \widehat{\Sigma}_n \right) \right) \right]^2, \quad (4.8)$$

where  $\mathfrak{D}_{\widehat{\mathbb{K}}_n}$  is defined in (3.27). Thus,  $\widetilde{\mathfrak{D}}_{\widehat{\mathbb{K}}_n}$  is the  $\widehat{\mathbb{K}}_n$ -discrepancy, in the sense of Definition 10. Since  $\pi^I(\Gamma) = F^I(\Sigma)$ , Corollary 4 and (4.8) imply that

$$\widehat{S}_n = \frac{n}{4} \left[ \widetilde{\mathfrak{D}}_{\widehat{\mathbb{K}}_n} \left( F^I(\Sigma), F^I \left( \widehat{\Sigma}_n \right) \right) \right]^2 \xrightarrow[n \rightarrow \infty]{d} \chi_{D^I}^2. \quad (4.9)$$

We deduce hereafter confidence regions for  $F^I(\Sigma)$  of the desired form.

**Proposition 5.** *For any  $\alpha \in (0, 1)$ , let  $\chi_{D^I}^2(1 - \alpha)$  be the quantile of order  $(1 - \alpha)$  of the  $\chi_{D^I}^2$  distribution. For  $n \geq 1$ , set*

$$R_{n,\alpha} := \left\{ \mathcal{P} \in \mathcal{F}^I : \frac{n}{4} \left[ \widetilde{\mathfrak{D}}_{\widehat{\mathbb{K}}_n} \left( \mathcal{P}, F^I \left( \widehat{\Sigma}_n \right) \right) \right]^2 \leq \chi_{D^I}^2(1 - \alpha) \right\}.$$

Then, by (4.9),  $R_{n,\alpha}$  is a confidence region for  $F^I(\Sigma)$  of asymptotic level  $(1 - \alpha)$ . By (3.31),

$$\widetilde{\mathfrak{D}}_{\widehat{\mathbb{K}}_n} \left( \mathcal{P}, F^I \left( \widehat{\Sigma}_n \right) \right) = 0 \iff Q = F^I \left( \widehat{\Sigma}_n \right).$$

So,  $R_{n,\alpha}$  is interpreted as a full deformed ellipsoid in  $\mathcal{F}^I$ , whose center is  $F^I \left( \widehat{\Sigma}_n \right)$ .

**Corollary 5.** *Fix  $Q_0 \in O(d)$ . Consider the following null hypothesis assumption.  $H_0 : \pi^I(Q_0) = F^I(\Sigma)$ . For any  $\alpha \in (0, 1)$ , consider the test which accepts  $H_0$  when  $\pi^I(Q_0) \in R_{n,\alpha}$  and rejects  $H_0$  else. Then, this test is of asymptotic level  $\alpha$ .*

### 4.4 Simulation

Figure 1 illustrates the convergence in distribution of  $\widehat{T}_n$  to a  $\chi^2$  distribution. The parameters for this simulation are  $d = 4$  and  $I = (1, 1, 1, 1)$ . Then,  $D^I = 6$ . For the sample size, we take  $n = 10000$ . The histogram in blue represents the distribution of  $\widehat{T}_n$  and the curve in red is that of the probability distribution function of the  $\chi_6^2$  distribution. We see that the distribution of  $\widehat{T}_n$  is indeed very close to that of the  $\chi_6^2$  one.

## 5 Conclusion

Given a Gaussian random vector  $X$  whose covariance matrix  $\Sigma$  has possibly repeated eigenvalues, we develop a geometric method to estimate the flag  $F^I(\Sigma)$  of its PS's, for which we provide confidence regions and implementable tests. These results open many questions, among which the following ones, which we present with their motivations.



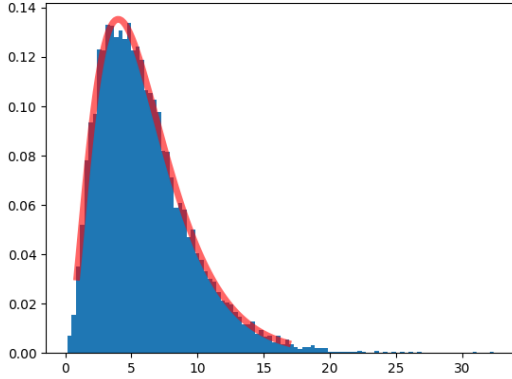


Figure 1: Illustration of the convergence of  $\hat{T}_n$  to the  $\chi_6^2$  distribution

(i) Are our geometric CLT's of Theorem 5 valid when the distribution of  $X$  is elliptic? If so, then, for such a distribution, the estimation of  $F^1(\Sigma)$  would be derived as in paragraphs 4.2 and 4.3, which are only based on such CLT's. This question is motivated by [14]. Therein, when  $X$  is elliptic, a CLT for each  $P_i(\hat{\Sigma}_n)$  is obtained, but not of the form of Eq.(4.1)

(ii) Does  $H_n^i$  converge in  $O(q_i)$  to  $E^{i,i}$  wrt the Kullback-Leibler (KL) divergence?

By Remark 5, among all  $q_i$ -frames generating  $\text{rg}(P_0^i)$ , we have that  $\mathcal{H}_{(\Gamma'P_i(\hat{\Sigma}_n)\Gamma, P_0^i)}^{q_i}(E_n^{(i)})$

minimizes the geodesic distance in  $\text{St}_{q_i}$  to  $E_n^{(i)}$ . We conjecture that this geometric optimality should imply a stronger mode of convergence of  $H_n^i$  to  $E^{i,i}$ , i.e. wrt the KL divergence in the compact group  $O(q_i)$ . Our conjecture is motivated by the case of iid convolutions on compact groups, for which the convergence in distribution to the Haar measure has been first obtained: See [6] and references therein. Later, it was proved in [5] or [7], by information theoretic methods, that they converge wrt the KL divergence.

## 6 Appendix: Method for the proof of Theorem 5

This Appendix is devoted to the proof of Theorem 5 and to some tools developed for it. The order of appearance of the proofs does not follow that of the preceding sections. Instead, the most technical parts are presented at the end. Thus, the proof of the generalized  $\delta$ -method of Theorem 3 is deferred to the end of this Appendix, i.e. to subsection 6.5.

Now, we describe the proof of Theorem 5. By Theorem 2,  $U_n$  converges in distribution to  $U$ . Thus, we prove Theorem 5 by expressing, in (6.7), the lhs' of (4.1) and (4.2), i.e.  $G_n^i$  and  $H_n^i$ , as functions of  $U_n$  and then by deriving, from the generalized  $\delta$ -method of Theorem 3, that these functions of  $U_n$  converge to functions of  $U$  which are the rhs' of (4.1) and (4.2). However,  $G_n^i$  and  $H_n^i$  are defined only when  $\pi^i(E_n^{(i)}) \notin \text{Cut}(P_0^i)$ . So, instead of applying Theorem 3 with  $U_n$ , we apply it with *truncations* of  $U_n$ , in the sense of paragraph 6.2.2.

### 6.1 Preliminaries from Anderson's proof

First, we present results from the proof of Theorem 1 which will be used later for ours. Consider the open subset of  $\text{Sym}_d$  defined by

$$\mathbb{S}_0 := \left\{ S \in \text{Sym}_d : S^{(i,i)} \in \text{Sym}_{q_i}^\neq, 1 \leq i \leq r \right\}. \quad (6.1)$$

Let  $(u_n)_{n \geq 1}$  be a sequence valued in  $\left(\mathbb{S}_n^A\right)_{n \geq 1}^*$  which converges to  $u \in \mathbb{S}_0$ . Set

$$\mathcal{E}_n := e_n(u_n). \quad (6.2)$$

**Proposition 6.** For  $1 \leq i, j \leq r$  with  $i \neq j$ , we have that  $\mathcal{E}_n^{(i,i)}$  and  $\sqrt{n}\mathcal{E}_n^{(i,j)}$  converge to finite limits denoted respectively by  $\mathcal{E}^{i,i}(u)$  and  $\mathcal{F}^{i,j}(u)$  which satisfy

$$\lambda_i \mathcal{G}^{i,j}(u) + \lambda_j \mathcal{L}^{i,j}(u) = u^{(i,j)} \quad \text{and} \quad \mathcal{G}^{i,j}(u) + \mathcal{L}^{i,j}(u) = 0, \quad (6.3)$$

with  $\mathcal{G}^{i,j}(u) := \mathcal{E}^{i,i}(u) (\mathcal{F}^{j,i}(u))'$  and  $\mathcal{L}^{i,j}(u) := \mathcal{F}^{i,j}(u) (\mathcal{E}^{j,j}(u))'$ . Thus, by (6.3),

$$\mathcal{G}^{i,j}(u) = \frac{1}{\lambda_i - \lambda_j} u^{(i,j)}. \quad (6.4)$$

Recall that  $U$  is the limit rv of Theorem 2. Then, the following Lemma is obtained in [1].

**Lemma 9.**  $\mathcal{E}^{i,i}(U)$  is distributed as  $E^{i,i}$  in (i) of Theorem 1 and  $\mathcal{F}^{i,j}(U)$  is distributed as  $F^{i,j}$  in (ii) of Theorem 1.

## 6.2 Expression of $G_n^i$ and $H_n^i$ as functions of a truncation of $U_n$

### 6.2.1 Expression of $G_n^i$ and $H_n^i$ as functions of $U_n$

First, in (6.5), we express  $E_n$  in function of  $U_n$ . Recall that  $E_n = \psi(T_n)$ , where  $\psi : \text{Sym}_d^\neq \rightarrow O(d)$  is the eigenvector map of Definition 1. In view of truncations, we extend the map  $\psi$  as follows: see Remark 10.

**Definition 11.** Let  $\bar{\psi} : \text{Sym}_d^\neq \cup \{\Delta\} \rightarrow O(d)$  be the map such that for  $S \in \text{Sym}_d^\neq$ ,  $\bar{\psi}(S) = \psi(S)$  and  $\bar{\psi}(\Delta) = I_d$ . Then,  $\bar{\psi}$  is called the extended eigenvector map.

(i) By definition,  $U_n := \sqrt{n}(T_n - \Delta)$ . Then,  $T_n = \phi_n(U_n)$ , where

$$\phi_n(u) = \Delta + n^{-1/2}u, \quad u \in \text{Sym}_d.$$

So,  $E_n = \psi(\phi_n(U_n))$  provided that  $U_n \in \phi_n^{-1}(\text{Sym}_d^\neq)$ . Thus, consider the sets

$$\left(\mathbb{S}_n^A\right)^* := \phi_n^{-1}(\text{Sym}_d^\neq) \quad \text{and} \quad \mathbb{S}_n^A := \left(\mathbb{S}_n^A\right)^* \cup \{0\}.$$

Since  $\phi_n(0) = \Delta$ , we may define the map  $e_n : \mathbb{S}_n^A \rightarrow O(d)$  by  $e_n(u) = \bar{\psi}(\phi_n(u))$ . Then,

$$U_n \in \left(\mathbb{S}_n^A\right)^* \implies E_n = e_n(U_n). \quad (6.5)$$

(ii) Now, for  $1 \leq i \leq r$ , we express  $G_n^i$  and  $H_n^i$  as functions of  $U_n$ . Define the map  $p_n^i$  by

$$p_n^i : \mathbb{S}_n^A \rightarrow G^i \quad \text{and} \quad p_n^i(u) = \pi^i(e_n(u)^{(i)}).$$

If  $p_n^i(u) \notin \text{Cut}(P_0^i)$ , then  $g_n^i(u)$  and  $h_n^i(u)$  introduced hereafter are well-defined.

$$g_n^i(u) := \sqrt{n} \text{Log}_{P_0^i}^{G^i}(p_n^i(u)) \in \mathbb{T}_0^i \quad \text{and} \quad h_n^i(u) := \mathcal{H}_{(p_n^i(u), P_0^i)}^i(e_n(u)^{(i)}) \in \text{St}^i \quad (6.6)$$

Thus, we consider the sets

$$\mathbb{W}_n^i := \{u \in \text{Sym}_d : p_n^i(u) \notin \text{Cut}(P_0^i)\} \quad \text{and} \quad \left(\mathbb{S}_n^i\right)^* := \left(\mathbb{S}_n^A\right)^* \cap \mathbb{W}_n^i.$$

Now, Equation (6.5) implies that  $\pi^i(E_n^{(i)}) = p_n^i(U_n)$  provided that  $U_n \in \left(\mathbb{S}_n^A\right)^*$ . So,

$$U_n \in \left(\mathbb{S}_n^i\right)^* \implies G_n^i = g_n^i(U_n) \quad \text{and} \quad H_n^i = \left(I_d^{(i)}\right)' h_n^i(U_n). \quad (6.7)$$

### 6.2.2 Truncations of rv's

Let  $M$  be a metric space with Borel  $\sigma$ -algebra  $\mathcal{B}$ .

**Definition 12.** Let  $Z : (\Omega, \mathcal{A}) \rightarrow (M, \mathcal{B})$  be a rv and  $A \in \mathcal{A}$ . If  $M$  is a vector space, let  $Z\mathbf{1}_A^+$  be the rv such that  $(Z\mathbf{1}_A^+)(\omega) = Z(\omega)$  if  $\omega \in A$  and  $(Z\mathbf{1}_A^+)(\omega) = 0 \in M$  else. If  $M$  is a subgroup of  $GL_q(\mathbb{R})$ , denote by  $Z\mathbf{1}_A^\times$  the rv such that,  $(Z\mathbf{1}_A^\times)(\omega) = Z(\omega)$  if  $\omega \in A$  and  $(Z\mathbf{1}_A^\times)(\omega) = I_q$  else. Then, we say that the rv's  $Z\mathbf{1}_A^+$  and  $Z\mathbf{1}_A^\times$  are truncations of  $Z$  wrt  $A$ .

**Lemma 10.** Let  $(X_n)_{n \geq 1}$  be a sequence of rv's valued in the metric space  $M$  and  $(A_n)_{n \geq 1}$  a sequence of events in  $\mathcal{A}$ . Assume that  $X_n \mathbf{1}_{A_n}^+ \xrightarrow{d} X$  or  $X_n \mathbf{1}_{A_n}^\times \xrightarrow{d} X$  as  $n \rightarrow \infty$ , where  $X$  is a rv valued in  $M$ . If  $P(A_n) \xrightarrow[n \rightarrow \infty]{} 1$ , then  $X_n \xrightarrow[n \rightarrow \infty]{d} X$  also.

*Proof.* The proof follows readily from the definitions.  $\square$

### 6.2.3 Expression of $G_n^i$ and $H_n^i$ as functions of $V_n^i$

The rv  $U_n$  is valued in the vector space  $\text{Sym}_d$ . So, for  $1 \leq i \leq r$ , we define below a truncation of  $U_n$ . Thus, set

$$V_n^i := U_n \mathbf{1}_{A_n^i}^+ \quad \text{where } A_n^i := \left\{ U_n \in (\mathbb{S}_n^i)^* \right\}.$$

So,  $V_n^i$  is valued in the set  $\mathbb{S}_n^i$  defined hereafter, which holds for all  $\omega \in \Omega$  and not only a.s.

$$\mathbb{S}_n^i := (\mathbb{S}_n^i)^* \bigcup \{0\}.$$

**Remark 10.** By definition of the map  $\bar{\psi}$ ,  $e_n(0) = \bar{\psi}(\phi_n(0)) = \bar{\psi}(\Delta) = I_d$ , so that  $p_n^i(0) = \pi^i \left( I_d^{(i)} \right) = P_0^i$ . So,  $g_n^i(u)$  and  $h_n^i(u)$  are also well-defined when  $u = 0$ , for which

$$g_n^i(0) = 0 \in \mathbb{T}_0^i \quad \text{and} \quad \left( I_d^{(i)} \right)' h_n^i(0) = I_{q_i}. \quad (6.8)$$

So, we may consider the maps  $g_n^i : \mathbb{S}_n^i \rightarrow \mathbb{T}_0^i$  and  $h_n^i : \mathbb{S}_n^i \rightarrow \text{St}^i$  defined by (6.6). Then, by (6.7) and (6.8), we express hereafter  $G_n^i \mathbf{1}_{A_n^i}^+$  and  $H_n^i \mathbf{1}_{A_n^i}^\times$  as functions of  $V_n^i$ :

$$G_n^i \mathbf{1}_{A_n^i}^+ = g_n^i(V_n^i) \quad \text{and} \quad H_n^i \mathbf{1}_{A_n^i}^\times = \left( I_d^{(i)} \right)' h_n^i(V_n^i). \quad (6.9)$$

## 6.3 Proof of Theorem 5

We prove Theorem 5 by applying Theorem 3. Thus, we define in Table 1 the elements to which we apply Theorem 3. In Table 1,  $\mathbb{S}_0$  is the subset defined in (6.1) and the map  $\mathcal{E}^{i,i} : \mathbb{S}_0 \rightarrow O(q_i)$  is defined in Proposition 6.

	proof of (4.1)	proof of (4.2)
Metric spaces $\mathbb{S}$ and $\mathbb{T}$	$\mathbb{S} = \text{Sym}_d$ and $\mathbb{T} = \mathbb{T}_0^i$	$\mathbb{S} = \text{Sym}_d$ and $\mathbb{T} = O(q_i)$
Function $g_n$	$g_n^i : \mathbb{S}_n^i \rightarrow \mathbb{T}_0^i$	$\left( I_d^{(i)} \right)' h_n^i : \mathbb{S}_n^i \rightarrow O(q_i)$
Domain $\mathbb{S}_n$ and subset $\mathbb{S}_n^*$	$\mathbb{S}_n^i$ and $(\mathbb{S}_n^i)^*$	
Random variable $V_n$	$V_n^i = U_n \mathbf{1}_{\{\mathbf{U}_n \in (\mathbb{S}_n^i)^*\}}$	
Limit $V$ of $V_n$	$V_n^i \xrightarrow[n \rightarrow \infty]{d} U$	
Limit function $g_0$	$g_0^i : \mathbb{S}_0 \rightarrow \mathbb{T}_0^i$	$\mathcal{E}^{i,i} : \mathbb{S}_0 \rightarrow O(q_i)$

Table 1: Elements to which Theorem 3 is applied

### 6.3.1 Conditions to apply Theorem 3

We prove hereafter that all the conditions of Theorem 3 hold with the elements of Table 1. Firstly, by (ii) of Proposition 7 below, for all  $1 \leq i \leq r$ ,  $V_n^i \xrightarrow{d} U$  as  $n \rightarrow \infty$ . Then, by (iii) of Proposition 7, Assumption (2.2) holds. Finally, by Propositions 8 and 9 below, Assumption (2.3) holds.

**Proposition 7.** (i) For all  $1 \leq i \leq r$ ,  $\Pr(A_n^i) \rightarrow 1$  as  $n \rightarrow \infty$ , where we recall that  $A_n^i := \{U_n \in (\mathbb{S}_n^i)^*\}$ . In particular, we deduce that

$$\Pr(E_n \in \mathbb{W}_{\mathcal{P}_0^i}) = \Pr\left(\forall 1 \leq i \leq r, U_n \in (\mathbb{S}_n^i)^*\right) \xrightarrow{n \rightarrow \infty} 0.$$

(ii) For all  $1 \leq i \leq r$ , the sequence  $(V_n^i)_n$  converges in distribution to  $U$  as  $n \rightarrow \infty$ .

(iii) For all  $1 \leq i \leq r$ ,  $\Pr(V_n^i \in (\mathbb{S}_n^i)^*) \rightarrow 1$  as  $n \rightarrow \infty$ .

*Proof.* See paragraph 6.4.1. □

**Proposition 8.** Let  $(u_n)_{n \geq 1}$  be a sequence valued in  $\left((\mathbb{S}_n^i)^*\right)_{n \geq 1}$  which converges to  $u \in \mathbb{S}_0$ . Then,  $g_n^i(u_n) \rightarrow g_0^i(u)$  as  $n \rightarrow \infty$ .

*Proof.* See paragraph 6.4.5. □

**Proposition 9.** Let  $(u_n)_{n \geq 1}$  be as in Proposition 8. Then,  $(I_d^{(i)})' h_n^i(u_n) \rightarrow \mathcal{E}^{i,i}(u)$  as  $n \rightarrow \infty$ , where  $\mathcal{E}^{i,i}(u)$  is defined in Proposition 6.

*Proof.* See paragraph 6.4.3. □

### 6.3.2 End of proof of Theorem 5

We deduce that we may apply Theorem 3, from which (6.9) implies that

$$G_n^i \mathbf{1}_{A_n^i}^+ = g_n^i(V_n^i) \xrightarrow[n \rightarrow \infty]{d} g_0^i(U) \quad \text{and} \quad H_n^i \mathbf{1}_{A_n^i}^\times = (I_d^{(i)})' h_n^i(V_n^i) \xrightarrow[n \rightarrow \infty]{d} \mathcal{E}^{i,i}(U). \quad (6.10)$$

Finally, by Proposition 7,  $\Pr(A_n^i) \rightarrow 1$  as  $n \rightarrow \infty$ . So, by Lemma 10 and (6.10), we conclude that (4.1) and (4.2) hold, which proves Theorem 5.

## 6.4 Proof of Proposition 7, 8 and 9

### 6.4.1 Proof of Proposition 7

*Proof.* (i) By definition,  $U_n \in (\mathbb{S}_n^i)^*$  iff  $U_n \in (\mathbb{S}_n^A)^*$  and  $p_n^i(U_n) \notin \text{Cut}(P_0^i)$ . Recall that  $U_n \in (\mathbb{S}_n^A)^* \implies p_n^i(U_n) = \pi^i(E_n^{(i)})$  and that  $U_n \in (\mathbb{S}_n^A)^*$  a.s. So, since  $P_0^i = I_d^{(i)} (I_d^{(i)})'$ ,

$$\Pr(U_n \in (\mathbb{S}_n^i)^*) = \Pr\left(\pi^i(E_n^{(i)}) \notin \text{Cut}(P_0^i)\right) = \Pr\left(E_n^{(i)} (E_n^{(i)})' \notin \text{Cut}\left(I_d^{(i)} (I_d^{(i)})'\right)\right).$$

Now, by the description of the cut locus in Grassmannians given in Lemma 4,

$$E_n^{(i)} (E_n^{(i)})' \notin \text{Cut}(P_0^i) \iff \text{rk}\left(\left(I_d^{(i)}\right)' E_n^{(i)}\right) = q_i \iff \text{rk}\left(E_n^{(i,i)}\right) = q_i.$$

So, (i) holds provided that  $\Pr\left(\text{rk}\left(E_n^{(i,i)}\right) = q_i\right) \rightarrow 1$  as  $n \rightarrow \infty$ , which we derive by applying Lemma 11 below. We check hereafter that its assumptions hold. First, by Theorem

1,  $E_n^{(i,i)}$  converges in distribution to  $E^{i,i}$ , where  $\text{rk}(E^{i,i}) = q_i$  a.s. Furthermore, for all  $n \geq 1$ ,  $\text{rk}(E_n^{(i,i)}) \leq q_i$ . So, the assumptions of Lemma 11 hold, from which (i) is deduced.

(ii) and (iii): By definition of  $(\mathbb{S}_n^i)^*$  and  $V_n^i$ ,

$$\Pr\left(U_n \in (\mathbb{S}_n^i)^*\right) \leq \Pr\left(V_n^i = U_n \text{ and } U_n \in (\mathbb{S}_n^i)^*\right). \quad (6.11)$$

On the one hand, the rhs of (6.11) is bounded by  $\Pr(V_n^i = U_n)$ . So, by (i),  $\Pr(V_n^i = U_n) \rightarrow 1$  as  $n \rightarrow \infty$ . Since  $(U_n)$  converges in distribution to  $U$ , we deduce that (ii) holds. On the other hand, the rhs of (6.11) is bounded by  $\Pr(V_n^i \in (\mathbb{S}_n^i)^*)$ . So (i) implies that (iii) holds.  $\square$

The following Lemma is used in the proof of Proposition 7. It is proved in [14].

**Lemma 11.** *Let  $B_n$  and  $B$  be random matrices of same size. Assume that  $B_n \xrightarrow[n \rightarrow \infty]{d} B$ , that, a.s.,  $\text{rk}(B) = b$  and that  $\Pr(\text{rk}(B_n) \leq b) \xrightarrow[n \rightarrow \infty]{} 1$ . Then,  $\Pr(\text{rk}(B_n) = b) \xrightarrow[n \rightarrow \infty]{} 1$ .*

#### 6.4.2 Preliminaries for the proofs of Propositions 8 and 9

Let  $(u_n)_{n \geq 1}$  be a sequence valued in  $\left((\mathbb{S}_n^i)^*\right)_{n \geq 1}$  which converges to  $u \in \mathbb{S}_0$ , where  $\mathbb{S}_0$  is defined in (6.1). Set

$$\mathcal{E}_n := e_n(u_n) \quad \text{and} \quad \mathcal{P}_n^i := p_n^i(u_n), \quad n \geq 1, 1 \leq i \leq r. \quad (6.12)$$

Since  $(\mathbb{S}_n^i)^* \subset (\mathbb{S}_n^A)^*$ , Proposition 6 still holds with this sequence  $(u_n)_{n \geq 1}$ .

**Lemma 12.** *For  $n \geq 1$ , there exist matrices  $\mathcal{A}_n$  and  $\mathcal{B}_n$  such that*

$$\mathcal{E}_n = \mathcal{A}_n + n^{-1/2} \mathcal{B}_n, \quad (6.13)$$

where  $\mathcal{A}_n \in \mathcal{D}(\mathbf{I})$  i.e.  $\mathcal{A}_n = \text{Diag}\left(\mathcal{A}_n^{(1,1)}, \dots, \mathcal{A}_n^{(i,i)}, \dots, \mathcal{A}_n^{(r,r)}\right)$  and the sequences  $(\mathcal{A}_n)_{n \geq 1}$  and  $(\mathcal{B}_n)_{n \geq 1}$  converge respectively to finite limits  $\mathcal{A}(u)$  and  $\mathcal{B}(u)$  such that

$$\mathcal{A}(u) \in \mathcal{D}(\mathbf{I}), \quad \mathcal{A}(u)^{(i,i)} = \mathcal{E}^{i,i}(u) \quad \text{and} \quad \mathcal{B}(u)^{(i,j)} = \mathcal{F}^{i,j}(u). \quad (6.14)$$

*Proof.* Set  $\mathcal{A}_n := \text{Diag}\left(e_n^1(u_n), \dots, e_n^i(u_n), \dots, e_n^r(u_n)\right) \in \mathcal{D}(\mathbf{I})$  and  $\mathcal{B}_n$  defined by:  $\mathcal{B}_n^{(i,j)} = f_n^{i,j}(u_n)$  if  $i \neq j$  and  $\mathcal{B}_n^{(i,i)} = 0$ . So,  $e_n^i(u_n) = \mathcal{E}_n^{(i,i)}$  and for  $j \neq i$ , we have that  $f_n^{i,j}(u_n) = \sqrt{n} \mathcal{E}_n^{(i,j)}$ . By Proposition 6,  $\mathcal{A}_n$  and  $\mathcal{B}_n$  converge to limits satisfying (6.14).  $\square$

**Corollary 6.** *With the notations of Lemma (12), we have that*

$$\mathcal{E}_n^{(i)} \xrightarrow[n \rightarrow \infty]{} (0 \quad \dots \quad \mathcal{E}^{i,i}(u) \quad \dots \quad 0)'. \quad (6.15)$$

Recall that we have set  $\mathcal{P}_n^i := p_n^i(u_n)$ . Then,

$$\mathcal{P}_n^i = \left(\mathcal{E}_n^{(i)}\right) \left(\mathcal{E}_n^{(i)}\right)' = \mathcal{A}_n^{(i)} \left(\mathcal{A}_n^{(i)}\right)' + n^{-1/2} \Gamma_n^i + n^{-1} \Phi_n^i, \quad (6.16)$$

where  $\Gamma_n^i := \mathcal{A}_n^{(i)} \left(\mathcal{B}_n^{(i)}\right)' + \mathcal{B}_n^{(i)} \left(\mathcal{A}_n^{(i)}\right)'$  and  $\Phi_n^i := \mathcal{B}_n^{(i)} \left(\mathcal{B}_n^{(i)}\right)'$ . Furthermore,

$$\mathcal{P}_n^i \rightarrow P_0^i \quad \text{as } n \rightarrow \infty. \quad (6.17)$$

*Proof.* (6.15) holds by Lemma 12 and (6.16) by (6.13). Finally, by (6.16), since  $\mathcal{A}(u) \in \mathcal{D}(\mathbf{I})$ ,

$$\mathcal{P}_n^i \xrightarrow[n \rightarrow \infty]{} \left(\mathcal{A}(u)^{(i)}\right) \left(\mathcal{A}(u)^{(i)}\right)' = \text{Diag}\left(0_{q_1}, \dots, \left(\mathcal{A}(u)^{(i,i)}\right) \left(\mathcal{A}(u)^{(i,i)}\right)', \dots, 0_{q_r}\right). \quad (6.18)$$

By (6.14) and (6.3),  $\mathcal{A}(u)^{(i,i)} = \mathcal{E}^{i,i}(u) \in O(q_i)$ , so that the rhs of (6.18) is equal to  $P_0^i$ .  $\square$

### 6.4.3 Proof of Proposition 9

We compute the limit of  $h_n^i(u_n)$ . By definition,

$$h_n^i(u_n) = \mathcal{H}_{(\mathcal{P}_n^i, P_0^i)}^i(\mathcal{E}_n^{(i)}) = \begin{pmatrix} \mathcal{E}_n^{(i)} & \Delta_n^i \mathcal{E}_n^{(i)} \end{pmatrix} \left( \exp_m \begin{pmatrix} 0 & -C_n^i \\ I_{q_i} & 0 \end{pmatrix} \right) \begin{pmatrix} I_{q_i} \\ 0_{q_i} \end{pmatrix}, \quad (6.19)$$

where  $\Delta_n^i := \text{Log}_{\mathcal{P}_n^i}^{G^i}(P_0^i)$  and  $C_n^i := \left( \mathcal{E}_n^{(i)} \right)' (\Delta_n^i)^2 \left( \mathcal{E}_n^{(i)} \right)$ . By (6.17),  $\Delta_n^i$  converges to  $\text{Log}_{P_0^i}^{G^i}(P_0^i) = 0 \in \mathbb{T}_0^i$ , i.e.  $C_n^i$  converges to 0. So, setting  $h^i(u) := (0 \ \dots \ \mathcal{E}^{i,i}(u) \ \dots \ 0)' \in \text{St}^i$ ,

$$h_n^i(u_n) \xrightarrow{n \rightarrow \infty} \begin{pmatrix} h^i(u) & 0 \end{pmatrix} \left( \exp_m \begin{pmatrix} 0 & 0 \\ I_{q_i} & 0 \end{pmatrix} \right) \begin{pmatrix} I_{q_i} \\ 0_{q_i} \end{pmatrix}. \quad (6.20)$$

We notice that the rhs of (6.20) is equal to  $\mathcal{H}_{(P_0^i, P_0^i)}^i(h^i(u)) = h^i(u)$ . Therefore, by (6.20),  $\left( I_d^{(i)} \right)' h_n^i(u_n)$  converges to  $\left( I_d^{(i)} \right)' h^i(u) = \mathcal{E}^{i,i}(u)$  as  $n \rightarrow \infty$ . This concludes the proof.

### 6.4.4 Technical preliminaries for the proof of Proposition 8

**Lemma 13.** *Let  $1 \leq i \leq r$ . Then, for all  $A, B, C \in \mathcal{D}_0^i(\mathbb{1})$  and  $M \in \text{M}_d(\mathbb{R})$ ,*

$$[A, P_0^i] = 0 \quad \text{and} \quad [BMC, P_0^i] = 0.$$

*Proof.* This follows from elementary calculations.  $\square$

**Lemma 14.**  *$\text{M}_d(\mathbb{R})$  is endowed with a norm  $\|\cdot\|$  such that, for  $A, B \in \text{M}_d(\mathbb{R})$ ,  $\|AB\| \leq \|A\| \cdot \|B\|$ . Let  $(u_n)_{n \geq 1}$  be a sequence valued in  $\text{M}_d(\mathbb{R})$  and  $m \geq 1$ .*

(i) *If  $u_n = o(1)$ , then  $\sum_{k=m}^{\infty} (u_n)^k = o(1)$  as  $n \rightarrow \infty$ .*

(ii) *Assume that  $u_n = O(a_n)$ , where  $(a_n)_{n \geq 1}$  is a sequence valued in  $(0, \infty)$  such that  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then,  $\sum_{k=m}^{\infty} (u_n)^k = O((a_n)^m)$  as  $n \rightarrow \infty$ .*

*Proof.* This follows readily from the properties of sums of geometric series.  $\square$

### 6.4.5 Proof of Proposition 8

*Proof.* Set  $\mathcal{P}_n^i := p_n^i(u_n)$ . Then,  $g_n^i(u_n) = \sqrt{n} \text{Log}_{P_0^i}^{G^i}(\mathcal{P}_n^i)$ . By the explicit formula for the Riemannian Logarithm in Grassmannians given by (3.15),

$$\text{Log}_{P_0^i}^{G^i}(\mathcal{P}_n^i) = \frac{1}{2} [\log_m((I_d - 2\mathcal{P}_n^i) \mathcal{I}_i), P_0^i], \quad \text{where } \mathcal{I}_i := (I_d - 2P_0^i). \quad (6.21)$$

By Corollary 6,  $\mathcal{P}_n^i \rightarrow P_0^i$  as  $n \rightarrow \infty$ . So, by (6.21),  $\text{Log}_{P_0^i}^{G^i}(\mathcal{P}_n^i)$  converges to 0. Thus, we need to prove that the rate of this convergence is in  $O(n^{-1/2})$ . Since  $\mathcal{I}_i^2 = I_d$ , we have that

$$(I_d - 2\mathcal{P}_n^i) \mathcal{I}_i = I_d + \mathcal{K}_n^i, \quad \text{where } \mathcal{K}_n^i = o(1). \quad (6.22)$$

Then, we expand the  $\log_m(\cdot)$  in (6.21). Thus, by (6.22), for all  $n$  large enough,

$$\text{Log}_{P_0^i}^{G^i}(\mathcal{P}_n^i) = \frac{1}{2} [\log_m(I_d + \mathcal{K}_n^i), P_0^i] = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} [(\mathcal{K}_n^i)^k, P_0^i]. \quad (6.23)$$

In the rhs of (6.23), for  $k \geq 1$ , the rate of convergence to 0 of  $(\mathcal{K}_n^i)^k = o(1)$  is not controlled. This explains why we needed a *power series* for the  $\log_m(\cdot)$  and not only a

Taylor expansion. To prove Proposition 8, we need to compute the limit of  $g_n^i(u_n)$ . By (6.23),

$$g_n^i(u_n) = \sqrt{n} \text{Log}_{P_0^i}^{G^i}(\mathcal{P}_n^i) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} [(\mathcal{K}_n^i)^k, P_0^i] \quad (6.24)$$

where  $\mathcal{P}_n^i := p_n^i(u_n)$  and by (6.16), after calculations,

$$\mathcal{K}_n^i := (I_d - 2P_n^i)(I_d - 2P_0^i) - I_d = \mathfrak{A}_n^i - 2n^{-1/2}\Gamma_n^i \mathcal{I}_i - 2n^{-1}\Phi_n^i \mathcal{I}_i, \quad (6.25)$$

with  $\mathfrak{A}_n^i := -2\text{Diag}\left(0_{q_1}, \dots, I_{q_i} - \left(\mathcal{A}_n^{(i,i)}\right) \left(\mathcal{A}_n^{(i,i)}\right)', \dots, 0_{q_r}\right) = o(1)$ . We split the sum in (6.24) by isolating the term for  $k=1$ . Thus, by Lemma 15 below,

$$\text{Log}_{P_0^i}^{G^i}(\mathcal{P}_n^i) = \frac{1}{2} [\mathcal{K}_n^i, P_0^i] + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} [(\mathcal{K}_n^i)^k, P_0^i] = \frac{1}{2} [\mathcal{K}_n^i, P_0^i] + o(n^{-1/2}). \quad (6.26)$$

Now, we deal with the first term in the rhs of (6.26). By Lemma 16 below,

$$\frac{\sqrt{n}}{2} [\mathcal{K}_n^i, P_0^i] = g_0^i(u) + O(n^{-1/2}) = g_0^i(u) + o(1). \quad (6.27)$$

Finally, by (6.26) and (6.27),  $g_n^i(u_n) = \sqrt{n} \text{Log}_{P_0^i}^{G^i}(\mathcal{P}_n^i) = g_0^i(u) + o(1)$ .  $\square$

#### 6.4.6 Auxiliary results for the proof of Proposition 8

**Lemma 15.** *Recall that the sequence  $(\mathcal{K}_n^i)_n$  is defined in (6.25). Then,*

$$\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} [(\mathcal{K}_n^i)^k, P_0^i] = o(n^{-1/2}). \quad (6.28)$$

*Proof. First part:* By the expression of  $\mathcal{K}_n^i$  in (6.25),

$$\mathcal{K}_n^i = \mathfrak{A}_n^i + \mathcal{M}_n^i, \quad \text{where } \mathfrak{A}_n^i \in \mathcal{D}_0^i(\mathbb{I}), \quad \text{with } \mathfrak{A}_n^i = o(1) \text{ and } \mathcal{M}_n^i = O(n^{-1/2}). \quad (6.29)$$

In the rhs of (6.24),  $(\mathcal{K}_n^i)^k$  still contains some  $(\mathfrak{A}_n^i)^\ell = o(1)$ , whose rate of convergence to 0 is not controlled. Lemma 13 provides simplifications. Thus, by (6.29), for all  $k \geq 2$ ,

$$(\mathcal{K}_n^i)^k = (\mathfrak{A}_n^i)^k + \left( \sum_{\ell=0}^{k-1} (\mathfrak{A}_n^i)^\ell (\mathcal{M}_n^i)^{k-\ell} \right) + \left( \sum_{\ell=1}^k (\mathcal{M}_n^i)^\ell (\mathfrak{A}_n^i)^{k-\ell} \right) + \mathcal{R}_n^i, \quad (6.30)$$

where  $\mathcal{R}_n^i$  is a sum of terms of the form  $(\mathfrak{A}_n^i)^\ell (\mathcal{M}_n^i)^{\ell'} (\mathfrak{A}_n^i)^{\ell''}$ , with  $\ell + \ell' + \ell'' = k$  and  $\ell > 1, \ell'' > 1$ . Since  $\mathfrak{A}_n^i \in \mathcal{D}_0^i(\mathbb{I})$ , Lemma 13 implies that

$$\left[ (\mathfrak{A}_n^i)^k, P_0^i \right] = 0 \quad \text{and} \quad [\mathcal{R}_n^i, P_0^i] = 0. \quad (6.31)$$

By combining (6.30) and (6.31),

$$\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} [(\mathcal{K}_n^i)^k, P_0^i] = \frac{1}{2} [\alpha_n^i, P_0^i] + \frac{1}{2} [\beta_n^i, P_0^i], \quad (6.32)$$

where, setting  $\epsilon_k := \frac{(-1)^{k+1}}{k}$ ,

$$\alpha_n^i = \sum_{k=2}^{\infty} \sum_{\ell=0}^{k-1} \epsilon_k (\mathfrak{A}_n^i)^\ell (\mathcal{M}_n^i)^{k-\ell} \quad \text{and} \quad \beta_n^i = \sum_{k=2}^{\infty} \sum_{\ell=1}^k \epsilon_k (\mathcal{M}_n^i)^\ell (\mathfrak{A}_n^i)^{k-\ell}.$$

*Second part:* We estimate  $\alpha_n^i$  and  $\beta_n^i$  as  $n \rightarrow \infty$ . In the double sum defining  $\alpha_n^i$ , we wish to factorize  $(\mathfrak{A}_n^i)^\ell$  in sums indexed by  $k$ , by inverting the summation indices. Namely, we invert on the following triangular domain  $\tau := \{(k, \ell) \in \mathbb{N}^2 : k \geq 2, 1 \leq \ell \leq k-1\}$ . Thus, for fixed  $k$ , we split the sum in  $\ell$  into two parts: the terms corresponding to  $\ell = 0$  and those to  $\ell \geq 1$ :

$$\alpha_n^i = \left( \sum_{k=2}^{\infty} \epsilon_k (\mathcal{M}_n^i)^k \right) + \left( \sum_{k=2}^{\infty} \sum_{\ell=1}^{k-1} \epsilon_k (\mathfrak{A}_n^i)^\ell (\mathcal{M}_n^i)^{k-\ell} \right). \quad (6.33)$$

Then, in the second part of the rhs in (6.33), we invert the indices, which lie in  $\tau$ .

$$\sum_{k=2}^{\infty} \sum_{\ell=1}^{k-1} \epsilon_k (\mathfrak{A}_n^i)^\ell (\mathcal{M}_n^i)^{k-\ell} = \sum_{\ell=1}^{\infty} \sum_{k=\ell+1}^{\infty} \epsilon_k (\mathfrak{A}_n^i)^\ell (\mathcal{M}_n^i)^{k-\ell} = \sum_{\ell=1}^{\infty} (\mathfrak{A}_n^i)^\ell \left( \sum_{k'=\ell+1}^{\infty} \epsilon_{\ell+k'} (\mathcal{M}_n^i)^{k'} \right). \quad (6.34)$$

Since  $\mathcal{M}_n^i = O(n^{-1/2})$  and  $\mathfrak{A}_n^i = o(1)$ , Lemma 14 implies that

$$\sum_{k'=\ell+1}^{\infty} \epsilon_{\ell+k'} (\mathcal{M}_n^i)^{k'} = O(n^{-1/2}), \quad \text{uniformly in } \ell, \quad \text{and} \quad \sum_{\ell=1}^{\infty} (\mathfrak{A}_n^i)^\ell = o(1). \quad (6.35)$$

Therefore, by combining (6.34) and (6.35),

$$\sum_{k=2}^{\infty} \sum_{\ell=1}^{k-1} \epsilon_k (\mathfrak{A}_n^i)^\ell (\mathcal{M}_n^i)^{k-\ell} = o(n^{-1/2}). \quad (6.36)$$

By Lemma 14, since  $\mathcal{M}_n^i = O(n^{-1/2})$ , we obtain for the first part of the rhs in (6.33):

$$\sum_{k=2}^{\infty} \epsilon_k (\mathcal{M}_n^i)^k = O\left(\left(n^{-1/2}\right)^2\right) = O(n^{-1}). \quad (6.37)$$

By (6.33), (6.36) and (6.37), we deduce that  $\alpha_n^i = o(n^{-1/2})$ . Similarly,  $\beta_n^i = o(n^{-1/2})$ .  $\square$

**Lemma 16.** *We have that*

$$\frac{\sqrt{n}}{2} [\mathcal{K}_n^i, P_0^i] = g_0^i(u) + O(n^{-1/2}). \quad (6.38)$$

*Proof.* Since  $[\mathfrak{A}_n^i, P_0^i] = 0$ , the expression of  $\mathcal{K}_n^i$  in (6.25) yields that

$$\frac{1}{2} [\mathcal{K}_n^i, P_0^i] = \frac{1}{2} \left[ \mathfrak{A}_n^i - 2n^{-1/2} \Gamma_n^i \mathcal{I}_i - 2n^{-1} \Phi_n^i \mathcal{I}_i, P_0^i \right] = -n^{-1/2} [\Gamma_n^i \mathcal{I}_i, P_0^i] + O(n^{-1}). \quad (6.39)$$

Thus, it is enough to prove that  $-[\Gamma_n^i \mathcal{I}_i, P_0^i] \rightarrow g_0^i(u)$  as  $n \rightarrow \infty$ . Consider the matrices

$$N_{<}^i := \left( \mathcal{A}_n^{(i,i)} \left( \mathcal{B}_n^{(1,i)} \right)' \quad \dots \quad \mathcal{A}_n^{(i,i)} \left( \mathcal{B}_n^{(i-1,i)} \right)' \right) \quad \text{and} \quad N_{>}^i := \left( \mathcal{A}_n^{(i,i)} \left( \mathcal{B}_n^{(i+1,i)} \right)' \quad \dots \quad \mathcal{A}_n^{(i,i)} \left( \mathcal{B}_n^{(r,i)} \right)' \right)$$

of respective sizes  $q_i \times (q_1 + \dots + q_{i-1})$  and  $q_i \times (q_{i+1} + \dots + q_r)$ . After calculations:

$$\begin{aligned} -[\Gamma_n^1 \mathcal{I}_1, P_0^1] &= \begin{pmatrix} 0_{q_1} & N_{>}^1 \\ (N_{>}^1)' & 0 \end{pmatrix}, & -[\Gamma_n^r \mathcal{I}_r, P_0^r] &= \begin{pmatrix} 0 & (N_{<}^r)' \\ N_{<}^r & 0_{q_r} \end{pmatrix} \\ -[\Gamma_n^i \mathcal{I}_i, P_0^i] &= \begin{pmatrix} 0 & (N_{<}^i)' & 0 \\ N_{<}^i & 0_{q_i} & N_{>}^i \\ 0 & (N_{>}^i)' & 0 \end{pmatrix}, \quad i \neq 1, r. \end{aligned}$$

By (6.14) in Lemma 12, for  $j \neq i$ ,

$$\mathcal{A}_n^{(i,i)} \left( \mathcal{B}_n^{(j,i)} \right)' \xrightarrow{n \rightarrow \infty} \mathcal{E}^{i,i}(u) (\mathcal{F}^{j,i}(u))' =: \mathcal{G}^{i,j}(u) = \frac{1}{\lambda_i - \lambda_j} u^{(i,j)}. \quad (6.40)$$

By definition of the functions  $g_0^i$ , this concludes the proof.  $\square$



## 6.5 Proof of Theorem 3

*Proof.* It is enough to prove that for any closed set  $T \subset \mathbb{T}$ ,

$$\overline{\lim} \Pr(V_n \in g_n^{-1}(T)) \leq \Pr(V \in g_0^{-1}(T)). \quad (6.41)$$

Indeed, if (6.41) holds, then the Portmanteau Theorem allows to conclude the proof. First,

$$\overline{\lim} \Pr(V_n \in g_n^{-1}(T)) \leq \overline{\lim} \Pr(V_n \in g_n^{-1}(T) \cap \mathbb{S}_n^*) + \overline{\lim} \Pr(V_n \in g_n^{-1}(T) \cap (\mathbb{S}_n^*)^c). \quad (6.42)$$

By (2.2), the second term of the rhs in (6.42) vanishes. So,

$$\overline{\lim} \Pr(V_n \in g_n^{-1}(T)) \leq \overline{\lim} \Pr(V_n \in g_n^{-1}(T) \cap \mathbb{S}_n^*). \quad (6.43)$$

Consider the following sequence of sets  $(R_m)_{m \geq 1}$  and the set  $R^*$  defined by

$$R_m := \bigcup_{n \geq m} \{g_n^{-1}(T) \cap \mathbb{S}_n^*\}, \quad m \geq 1 \quad \text{and} \quad R^* := \bigcap_{m \geq 1} \overline{R_m}.$$

Fix  $m \geq 1$ . Then, for all  $n \geq m$ ,  $\{g_n^{-1}(T) \cap \mathbb{S}_n^*\} \subset \overline{R_m}$ . Since  $V_n \xrightarrow{d} V$ ,

$$\overline{\lim} \Pr(V_n \in g_n^{-1}(T) \cap \mathbb{S}_n^*) \leq \overline{\lim} \Pr(V_n \in \overline{R_m}) \leq \Pr(V \in \overline{R_m}). \quad (6.44)$$

Since the lhs of (6.44) is independent of  $m$ , taking the limit as  $m \rightarrow \infty$  in (6.44) yields that

$$\overline{\lim} \Pr(V_n \in g_n^{-1}(T) \cap \mathbb{S}_n^*) \leq \Pr(V \in R^*). \quad (6.45)$$

Indeed, the sequence  $(R_m)_{m \geq 1}$  is *decreasing*. Now, we claim that

$$R^* \subset g_0^{-1}(T) \cup (\mathbb{S}_0)^c. \quad (6.46)$$

Indeed, let  $v \in R^*$ . Then, by definition of  $R^*$ , we can construct a sequence  $(v_m)_{m \geq 1}$  valued in  $(R_m)_{m \geq 1}$  which converges to  $v$ . Now, by definition of  $R_m$ , we may build a subsequence  $(v_{\phi(m)})_{m \geq 1}$  of  $(v_m)_m$  such that for all  $m \geq 1$ ,  $v_{\phi(m)} \in g_{\phi(m)}^{-1}(T) \cap \mathbb{S}_{\phi(m)}^*$ . Thus,  $g_{\phi(m)}(v_{\phi(m)}) \in T$  and, since  $v_{\phi(m)} \in \mathbb{S}_{\phi(m)}^*$ , assumption (2.3) implies that  $g_{\phi(m)}(v_{\phi(m)}) \rightarrow g_0(v)$  as  $m \rightarrow \infty$ . Since  $T$  is closed, we deduce that  $v \in g_0^{-1}(T)$ . This proves the claim (6.46). Thus, by (6.46) and the assumption  $\Pr(V \in \mathbb{S}_0) = 1$ ,

$$\Pr(V \in R^*) \leq \Pr(V \in g_0^{-1}(T) \cup (\mathbb{S}_0)^c) \leq \Pr(V \in g_0^{-1}(T)). \quad (6.47)$$

We combine (6.43), (6.45) and (6.47) to conclude that (6.41) holds.  $\square$

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