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# A comparison of node-based and arc-based hop-indexed formulations for the Steiner tree problem with hop constraints

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We study the relation between the linear programming relaxation of two classes of models for the Steiner tree problem with hop constraints. One class is characterized by having hop-indexed arc variables. Although such models have proved to have a very strong linear programming bound, they are not easy to use because of the huge number of variables. This has motivated some studies with models involving fewer variables that use, instead of the hop-indexed arc variables, hop-indexed node variables.

In this paper we contextualize the linear programming relaxation of these node-based models in terms of the linear programming relaxation of known arc-based models. We show that the linear programming relaxation of a general node-based model is implied by the linear programming relaxation of a straightforward arc-based model.

## KEYWORDS

Steiner tree, hop constraint, network design, linear programming relaxation, integer programming, hop-indexed model

Consider a graph  $G = (N_0, A)$ , with node set  $N_0 = \{0, 1, \dots, n\}$  and arc set  $A$ , with a nonnegative cost  $c_a$  associated to each arc  $a \in A$ , an integer hop limit  $H$ , and a set of required terminal nodes  $R \subseteq N_0$  with  $0 \in R$ . The hop-constrained Steiner tree problem (HSTP) consists of determining an arborescence  $G' = (N', T)$  rooted at 0, spanning a subset  $N' \supseteq R$ , so that the unique path from 0 to each terminal  $r \in R$  contains at most  $H$  arcs, and the cost of selected arcs  $c(T) := \sum_{a \in T} c_a$  is minimized. This problem was first introduced by Voß [16]. Besides its natural application in telecommunications, studies on this problem and related ones have been of interest because they include state of the

art research on so-called hop-indexed and/or layered graph formulations which are topical for problems with such distance-like constraints, see, e.g., [1, 2, 4, 5, 9, 11, 12, 13, 14].

The hop-indexed models described in these works are characterized by having hop-indexed arc variables  $z_{ij}^h$  indicating whether arc  $(i, j)$  is in position  $h$  (that is, the single path from node 0 to node  $i$  contains  $h - 1$  arcs) in the solution. Although such models have proved to have a very strong linear programming bound, they are not easy to use because of the huge number of variables. This has motivated some studies with models involving fewer variables and that use node variables  $v_i^h$  indicating whether node  $i$  is in position  $h$ , instead of the arc variables  $z_{ij}^h$ . Recently, Sinnl and Ljubić [15] have presented one such model for the budget constrained hop constrained Steiner tree problem, first introduced by Costa *et al.* [3], where the objective is the maximization of the revenue.

In this paper we want to contextualize the linear programming relaxation of the node-based model in terms of the linear programming relaxation of known arc-based models. The arguments given next suggest that, in general, a node-based model has a weak linear programming bound, at least when compared with an arc-based model. First, observe that an arc variable provides more information than a node variable does; e.g., the node variable  $v_i^h$  indicates whether node  $i$  is in position  $h$  and the arc variable  $z_{ji}^h$  indicates whether node  $i$  is in position  $h$  AND the arc entering node  $i$  is coming from node  $j$ . Thus, defining a model with arc variables should be easier (or stronger in terms of the linear programming relaxation bound) than writing a valid model with node variables. We can make this argument more formal with equalities such as  $v_i^h = \sum_{(k,i) \in A} z_{ki}^h$ , relating the two sets of variables. If we add such equalities to an arc-based model, in theory we could project out the arc variables and obtain a model defined only on the node variables with an equivalent LP relaxation. In several cases, it may not be easy to find the whole set of projected inequalities, however we can obtain a subset of inequalities that still result in a valid model (although with a weaker linear programming bound). In fact, this is what happens with the pair of models,  $gBNH$  and  $AH$ , discussed later in the paper. The dominance of an arc-based model over a node-based model is a general observation: the linking equalities allow any node-based model to be rewritten as an arc model simply by using them to replace the  $v_i^h$  variables by  $z_{ij}^h$  variables and thus by simple substitution the arc model is always as strong as the node model.

In this paper we show that the linear programming relaxation of the node-based model (including a large set of generalized inequalities) is implied by the linear programming relaxation of a "simple" arc-based model that was presented formerly by Gouveia [8] for the Spanning Tree Problem and easily adapted for the more general problem studied in this paper. We have used the term "simple", because the inequalities defining this model are a weaker version of a rather small subset of a more general class of inequalities, the so-called layered graph cuts that are included in the model proposed in by Gouveia *et al.* [11]. This model is, as far we know, the strongest model known for this problem.

To simplify the notation, and before presenting the formulations, we define the following sets:  $H_1 := \{1, \dots, H\}$ ,  $H_2 := \{2, \dots, H\}$ ,  $N_1 := N_0 \setminus \{0\} = \{1, \dots, n\}$ ,  $R_1 := R \setminus \{0\}$ .

## 1 | NODE-BASED HOP-INDEXED MODEL

In this section we discuss node-based hop-indexed models for the HSTP. We classify these models either as "forward" models or "backward" models: a "forward" model is characterized by constraints forcing a node to be at distance  $h$  if there is an arc entering the node coming from a node at distance  $h - 1$ ; a "backward" model is characterized by constraints indicating that a node must be at distance  $h - 1$  if there exists an arc leaving that node to a node at distance  $h$ . We observe that although the more general models of the two classes are equivalent, there are a few relevant differences in the two modelling views. For this reason, we have divided this section into two subsections, dedicated to

each one of the two classes.

## 1.1 | Forward models

Most of the material in this section is adapted from Sinnl and Ljubić [15] where the authors proposed several node-based models for the Steiner tree problem with revenues, budget and hop-constraints. The model that we adapt here for the HSTP uses binary variables  $y_i$  to indicate if node  $i \in N_1$  belongs to  $N'$ , binary variables  $x_{ij}$  to indicate if arc  $(i, j) \in A$  belongs to  $T$ , and binary hop-indexed node variables  $v_i^h$  to indicate if node  $i \in N_1$  is at distance  $h \in H_1$  from root node 0 in  $G'$ . Consider also, the following set of constraints that all feasible solutions must satisfy

$$\sum_{(i,j) \in A} x_{ij} = y_j \quad j \in N_1, \quad (1)$$

$$\sum_{h \in H_1} v_i^h = y_i \quad i \in N_1, \quad (2)$$

$$v_j^1 = x_{0j} \quad j : (0, j) \in A, \quad (3)$$

$$v_i^{h-1} + x_{ij} \leq v_j^h + 1 \quad (i, j) \in A, i \in N_1, h \in H_2, \quad (4)$$

$$v_i^H + x_{ij} \leq 1 \quad (i, j) \in A, i \in N_1, \quad (5)$$

$$y_i = 1 \quad i \in R_1, \quad (6)$$

$$x_{ij} \in \{0, 1\} \quad (i, j) \in A, \quad (7)$$

$$y_i \in \{0, 1\} \quad i \in N_1, \quad (8)$$

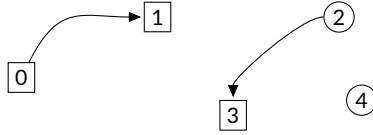
$$v_i^h \in \{0, 1\} \quad i \in N_1, h \in H_1 \quad (9)$$

Constraints (1) impose that each node, except the root node 0, has exactly one entering arc if it belongs to the set of nodes selected in the solution, or zero otherwise. Constraints (2) state that any node, other than the root, belonging to the solution is at a distance between 1 and  $H$  from the root, while constraints (3) impose that a node connected directly to the root is at distance 1 from it. Constraints (4) state that if a node  $i$  is at distance  $h - 1$  from the root, and arc  $(i, j)$  belongs to the solution, then node  $j$  is at distance  $h$  from the root. Similarly, constraints (5) forbid an arc leaving a node which is at the maximum distance  $H$  from the root. Nodes in  $R_1$  are forced to belong to the solution by constraints (6) (node 0 implicitly belongs to the solution) and constraints (7), (8) and (9) ensure that all variables are binary.

As pointed out by Sinnl and Ljubić [15], the formulation (1)-(9) is not sufficient to get a valid formulation for HSTP as connectivity between the root and the required terminal nodes is not ensured, as illustrated in the example in Figure 1. The example consists of a complete graph with 4 nodes ( $n = 3$  plus the root node) with  $H = 3$  and required nodes,  $R = \{0, 1, 3\}$ . The arcs in the example correspond to the arc variables  $x_{01} = x_{23} = 1$ . The remaining variable values are  $y_1 = v_1^1 = 1$ ,  $y_3 = v_3^2 = 1$  and zero for all other variables. This "solution" satisfies all constraints (1)-(9) but is obviously not feasible as there is no path from the root node to node 3.

To enforce connectivity, one can still follow Sinnl and Ljubić [15] and add a set of the well-known generalized cut constraints, or alternatively and as also pointed in their paper, we can add a smaller subset of generalized subtour elimination constraints of size two:

$$x_{ij} + x_{ji} \leq y_i \quad i, j \in N_1 \quad (10)$$



**FIGURE 1** Infeasible solution for the HSTP.

We refer to Theorem 1 in Sinnl and Ljubić [15] for a proof that this compact model is valid for the Steiner tree problem with revenues, budget and hop-constraint, and which also applies for the Steiner version studied in this paper.

In fact, to ensure connectivity between the root node and any required node and to guarantee the validity of the forward model (1)-(9), we only need to consider the following “weaker” version of constraints (10):

$$x_{ij} \leq y_i \quad i, j \in N_1 \quad (11)$$

Observe that in the example in Figure 1, this constraint is not satisfied for  $(i, j) = (2, 3)$  and  $i = 2$ . We show next that adding these constraints to the formulation (1)-(9) guarantees the connectivity of any solution to the HSTP.

**Proposition 1** *Formulation (1), (2), (3), (4), (5), (6), (7), (8), (9) and (11) is a valid formulation for the HSTP.*

**Proof** We first observe that due to constraints (2) and (4), a solution to this model cannot contain circuits. To see this, consider a circuit  $C = \{i_1, i_2, \dots, i_k\}$ . Using constraints (4) in a circular fashion, starting from node  $i_1$  for example and moving forward, we would obtain  $v_{i_1}^h = v_{i_1}^{h+k} = 1$ , for a given value of  $h$ . But this is in contradiction with constraint (2) for node  $i_1$ .

We show next that for any node  $j$  such that  $j \in R_1$ , there exists a path  $P = \{j_1, j_2, \dots, j_k\}$  such that  $j_1 = 0$  and  $j_k = j$ . We have  $y_j = y_{j_k} = 1$  since  $j \in R_1$ . Constraints (1) guarantee that there must exist one and only one arc entering node  $j_k$ , say arc  $(j_{k-1}, j_k)$ . If  $j_{k-1} = 0$  we have found the required path. Otherwise, we have that  $y_{j_{k-1}} = 1$  either because  $j_{k-1} \in R_1$  or because of the new constraints (11). Repeating the process we find a node  $j_{k-2}$ , such that arc  $(j_{k-2}, j_{k-1}) \in A$  and either  $j_{k-2}$  is the root node or  $y_{j_{k-2}} = 1$ . By repeating this process and since  $V$  is finite and the solution cannot contain cycles we obtain a node  $y_1 = 0$  giving the required path.

Observe that this reasoning also applies to a node  $j \notin R_1$  such that  $y_j = 1$  and for which there is no arc emanating from it. However, a solution containing such a node would not be optimal. Thus, a solution to this model must contain a single path from the root node to any node in the solution.  $\square$

A strengthening of (4) and a generalization of the resulting strengthened inequality is also presented by Sinnl and Ljubić [15]. The strengthening is as follows:

$$v_i^H + v_i^{h-1} + x_{ij} \leq y_i + v_j^h \quad i \in N_1, (i, j) \in A, h \in H_2 \quad (12)$$

Observe that constraints (12) are obtained from (4) by replacing 1 by  $y_i$  on the right-hand side of (4) and adding  $v_i^H$  on the left-hand side. One question is to know whether constraints (12), in place of (4) in model (1)-(9), are sufficient to define a valid model or if, as before, we need to add extra constraints such as (11) or (10) to guarantee that any required node is connected to the root. We will argue next, that from an integer point of view, if  $H \geq 3$ , the new constraints (12) imply constraints (11), that is, if  $x_{ij} = 1$  then  $y_i = 1$ . Let us assume that for a given arc  $(i, j)$  and a position  $h$ , we have  $x_{ij} = 1$ ,  $y_i = 0$ ; then we must have  $v_j^h = 1$  for every  $h \geq 2$  for the inequalities (12) (there are at least

two) to hold. But this is inconsistent with constraint (2), since there can only be at most one  $v_j^h = 1$ . This argument leads to the next result.

**Proposition 2** *Formulation (1), (2), (3), (12), (5), (6), (7), (8), (9) is a valid formulation for the HSTP when  $H \geq 3$ .*

Note that the disconnected solution in Figure 1 shows that this result is not valid for  $H = 2$ .

The generalization of constraints (12) is obtained by considering sets of variables associated to different distance values, e.g., a specific subset of distance values,  $S \subseteq H_2$ , leading to:

$$v_i^H + \sum_{h \in S} v_i^{h-1} + x_{ij} \leq y_i + \sum_{h \in S} v_j^h \quad i \in N_1, (i, j) \in A, S \subseteq H_2 \quad (13)$$

Observe that, when  $|S| = 1$  we obtain the original inequalities (12). Also, by eliminating  $y$  variables with the help of constraints (2), we obtain

$$x_{ij} \leq \sum_{h \in H_2 \setminus S} v_i^{h-1} + \sum_{h \in S} v_j^h \quad i \in N_1, (i, j) \in A, S \subseteq H_2 \quad (14)$$

which is the form presented by Sinnl and Ljubić [15]. Although exponential in number, these inequalities can be separated in polynomial time.

## 1.2 | Backward models

Using the same set of variables, a simple version of backward inequalities would be the symmetric version of (4)

$$v_j^h + x_{ij} \leq v_i^{h-1} + 1 \quad i \in N_1, (i, j) \in A, h \in H_2 \quad (15)$$

As shown before, formulation (1)-(9) does not ensure connectivity between the root node and the required terminal nodes. In contrast, and as proven in the following proposition, the backwards modelling approach, that is replacing constraints (4) by (15), leads to a valid formulation without the need to add constraints such as (10) or (11):

**Proposition 3** *Formulation (1), (2), (3), (15), (5), (6), (7), (8), (9) is a valid formulation for the HSTP.*

**Proof** As noted before, constraints (2) and (15), guarantee that a solution to this model cannot contain circuits. The reasoning is similar to the one in the proof of Proposition 1, but this time in a backward way. Now, consider a node  $j$  in  $R_1$  and let  $(i, j)$  be the corresponding arc entering this node, that is,  $y_j = 1$  and  $x_{ij} = 1$ . Also, due to constraint (2) for node  $j$ , there exists a hop index  $h^* \in H_1$  such that  $v_j^{h^*} = 1$ .

Assume  $i \neq 0$  (if  $i = 0$ , we have a path from the root to node  $j$ ). Then constraint (15) for arc  $(i, j)$  and  $h = h^*$  becomes  $1 \leq v_i^{h^*-1}$  and thus we have  $v_i^{h^*-1} = 1$ . Constraints (2) imply that  $y_i = 1$ , and constraints (1) guarantee that there exists an arc entering node  $i$ . By repeating the reasoning above, we conclude that a solution to this model must contain a path from the root node to any node.  $\square$

Similarly to the strengthening of (4) presented in the previous section, constraints (15) can also be strengthened to take into account border effects at distance 1 and as well as including the fact that some nodes may not be included in the solution. This leads to

$$v_j^1 + v_j^h + x_{ij} \leq y_j + v_i^{h-1} \quad i \in N_1, (i, j) \in A, h \in H_2 \quad (16)$$

Finally, in the same way that (12) are generalized into (13), the inequalities (16) can be generalized into

$$v_j^1 + \sum_{h \in S'} v_j^h + x_{ij} \leq y_j + \sum_{h \in S'} v_i^{h-1} \quad i \in N_1, (i, j) \in A, S' \subseteq H_2 \quad (17)$$

As the following proposition shows, these constraints can also be shown to be equivalent to (14) by using constraints (2).

**Proposition 4** *In the presence of constraints (2), constraints (17) are equivalent to constraints (14).*

**Proof** Consider constraint (17) for a given node  $i \in N_1$ , an arc  $(i, j) \in A$  and a subset  $S' \subseteq H_2$ . After replacing  $y_j$  in (17) by the left-hand side of equality (2) for node  $j$  and cancelling equal terms we obtain constraint (14) for a subset  $S = H_2 \setminus S'$ .  $\square$

Thus, we have proved that the three sets of constraints (13), (14) and (17) are equivalent. This proposition also gives an indirect proof that the generalized backward inequalities (17) are valid.

Table 1 provides a general view of the main constraints from the two classes of models, in particular, the linking constraints between the node variables  $y_i$  and the node-hop variables  $v_i^h$  in each model. Constraints (14) stand as the bridge between the two generalized strong models. That is, constraints (13) that characterize the forward generalized strong model are shown to be equivalent to constraints (17) of the backward generalized strong model via the intermediate constraints (14).

| constraints |                    | Forward Models                    |      | Backward Models |
|-------------|--------------------|-----------------------------------|------|-----------------|
| common      |                    | (1), (2), (3), (6), (7), (8), (9) |      |                 |
| linking     | weak               | (4) <sup>1</sup>                  |      | (15)            |
|             | strong             | (12)                              |      | (16)            |
|             | generalized strong | (13)                              | (14) | (17)            |

<sup>1</sup> needs extra constraints such as (10) or (11) for validity

**TABLE 1** Valid hop-indexed node models for the HSTP: Forward and Backward

In Section 3 we show that the generalized constraints (17) (alternatively, (13) and (14)) are implied by a compact hop-indexed arc-based model.

## 2 | ARC-BASED HOP-INDEXED MODEL

The arc-based model presented in this section was first described by Gouveia [8] for the minimum spanning tree problem with hop constraints. The main idea of the model, more precisely constraints (21) and (22) to be described next, is to show that hop-indexed arc variables can easily be used to guarantee the hop limit as well as the connectivity of the solution (by using a backwards chain reasoning from any arc to the root node). This “arc-based hop-indexed” model is easily adapted for the HSTP. It uses variables  $y_i$  and  $x_{ij}$  as in the previous model and in addition, uses binary variables  $z_{ij}^h$  to indicate if arc  $(i, j)$  is in position  $h$  in the path from 0 to  $j$ .

$$\sum_{(i,j) \in A} x_{ij} = y_j \quad j \in N_1 \quad (18)$$

$$z_{0j}^1 = x_{0j} \quad j : (0,j) \in A \quad (19)$$

$$\sum_{h \in H_2} z_{ij}^h = x_{ij} \quad i \in N_1, (i,j) \in A \quad (20)$$

$$\sum_{(k,i) \in A, k \neq 0} z_{ki}^{h-1} \geq z_{ij}^h \quad i \in N_1, (i,j) \in A, h \in H_2, h \geq 3 \quad (21)$$

$$z_{0i}^1 \geq z_{ij}^2 \quad i, j \in N_1 : (0,i), (i,j) \in A \quad (22)$$

$$y_i = 1 \quad i \in R_1, \quad (23)$$

$$x_{ij} \in \{0, 1\} \quad (i,j) \in A \quad (24)$$

$$y_i \in \{0, 1\} \quad i \in N_1 \quad (25)$$

$$z_{ij}^h \in \{0, 1\} \quad i \in N_1, (i,j) \in A, h \in H_2 \quad (26)$$

$$z_{0j}^1 \in \{0, 1\} \quad (0,j) \in A \quad (27)$$

Apart from constraints, (18), (23)-(25) that are the same as in the previous model, constraints (19) and (20) link the hop-indexed arc variables  $z_{ij}^h$  with the arc variables  $x_{ij}$ . Observe that each arc  $(0,j) \in A$  can only be in position 1 in the solution and this is the reason why variables  $z_{0j}^h$  are defined only for  $h = 1$ . The remaining arcs in  $A$  can be in any position from 2 to  $H$ , therefore, variables  $z_{ij}^h$  for  $(i,j) \in A$ , are defined for  $h \in H_2$ . Constraints (21) guarantee that, if arc  $(i,j)$  leaves node  $i \in N_1$  at position  $h \geq 3$ , then one arc  $(k,i) \neq (0,i)$  enters that same node  $i$  at position  $h-1$ . Furthermore, since 2-cycles are not allowed, we can strengthen inequalities (21) by stating that  $(k,i) \neq (j,i)$

$$\sum_{(k,i) \in A, k \neq 0, j} z_{ki}^{h-1} \geq z_{ij}^h \quad i \in N_1, (i,j) \in A, h \in H_2, h \geq 3 \quad (28)$$

Constraints (22) correspond to constraints (21) written for nodes directly connected to the root node. We denote by  $AH$  the model defined by constraints (18)-(27) and by  $sAH$  the model equivalent to  $AH$  with constraints (21) replaced by the stronger version, (28).

Constraints (19) and (20) can also be used to remove variables  $x_{ij}$  from the formulation, thus obtaining a model with fewer variables. These constraints are not needed to provide a valid formulation for the problem. They are included here in order to establish the relation proved in the next section.

### 3 | RELATIONS BETWEEN THE FORMULATIONS

In this section we compare the linear programming relaxation of the models presented in the previous sections, namely, the  $AH$  model defined by constraints (18) - (27) and the *generalized Backward Node-based Hop-indexed* ( $gBNH$ ) model presented in Section 1, defined by constraints (1), (2), (3), (17), (5), (6), (7), (8), (9).

Let  $Model_L$  be the linear programming relaxation of a given  $Model$  and  $Feas(Model_L)$  its set of feasible solutions. Also, for a given polyhedron  $Q \subseteq \mathbb{R}^{n \times m}$ , the projection of  $Q$  in the subspace  $\mathbb{R}^n$  is defined as  $proj_{(x)} Q = \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^m \text{ such that } (x, y) \in Q\}$ .

The following proposition, which is the main result of the paper, relates the  $AH_L$  model with the  $gBNH_L$  model.

For that purpose, we add the following linear equalities to model  $AH$ , defining the  $v_j^h$  variables in terms of the  $z_{ij}^h$  variables

$$v_i^h = \sum_{(k,i) \in A, k \neq 0} z_{ki}^h \quad i \in N_1, h \in H_2 \quad (29)$$

$$v_i^1 = z_{0i}^1 \quad i \in N_1 \quad (30)$$

and the  $v_i^h$  domain constraints (9). We denote by  $AH+$  the model  $AH$  augmented with these equalities. Observe that (29) and (30) are only definitional and adding them to model  $AH$  does not change the LP value. Model  $AH+$  was created to formalize a relation between the two models,  $AH_L$  and  $gBNH_L$ .

**Proposition 5** *The projection of  $Feas(AH+_L)$  onto the variable space of  $gBNH$  is contained in  $Feas(gBNH_L)$ ,*

$$proj_{(x,y,v)}(Feas(AH+_L)) \subseteq Feas(gBNH_L)$$

Moreover, this inclusion can be strict.

**Proof** We will show next that the constraints of model  $gBNH_L$  are implied by the constraints of model  $AH+_L$ , namely constraints (2), (3), (17) and (5) (the remaining constraints are straightforwardly satisfied).

- Constraints (3) are implied by constraints (19) and (30).
- For a given node  $i \in N_1$ , adding constraints (29) for all  $h \in H_2$  together with (30) results in

$$\sum_{h \in H_2} v_i^h + v_i^1 = \sum_{h \in H_2} \sum_{(k,i) \in A, k \neq 0} z_{ki}^h + z_{0i}^1$$

On the other hand, adding (20) for all  $(i,j) \in A, i \in N_1$  together with (19) and using (18) results in

$$\sum_{(i,j) \in A, i \neq 0} \sum_{h \in H_2} z_{ij}^h + z_{0j}^1 = \sum_{(i,j) \in A, i \neq 0} x_{ij} + x_{0j} = y_j$$

Thus, constraints (2) are also satisfied for every  $i \in N_1$ .

- For a given node  $i \in N_1$  and an arc  $(i,j)$ , constraints (29) together with (21) and constraints (30) together with (22) imply that  $z_{ij}^h \leq v_i^{h-1}$ ,  $h \in H_2$ . Adding  $\sum_{(k,j) \in A, k \neq 0, i} z_{kj}^h$  to both sides of these inequalities and using (29) we obtain

$$v_j^h \leq v_i^{h-1} + \sum_{(k,j) \in A, k \neq 0, i} z_{kj}^h$$

For a given set  $S \subseteq H_2$ , adding the previous inequalities for  $h \in S$  we obtain

$$\sum_{h \in S} v_j^h \leq \sum_{h \in S} v_i^{h-1} + \sum_{(k,j) \in A, k \neq 0, i} \sum_{h \in S} z_{kj}^h$$

From constraints (20), we have  $\sum_{h \in S} z_{kj}^h \leq x_{kj}$ ,  $k \in N_1, (k,j) \in A$ , therefore the right-hand side of the previous

inequality can be lifted up (using constraints (18), (19) and (30))

$$\sum_{h \in S} v_j^h \leq \sum_{h \in S} v_i^{h-1} + \sum_{(k,j) \in A, k \neq 0, i} x_{kj} = \sum_{h \in S} v_i^{h-1} + y_j - x_{ij} - x_{0j} = \sum_{h \in S} v_i^{h-1} + y_j - x_{ij} - z_{0j}^1 = \sum_{h \in S} v_i^{h-1} + y_j - x_{ij} - v_j^1$$

therefore constraint (17) is also satisfied for any  $i \in N_1$  and  $S \subseteq H_2$ .

- For a given node  $i \in N_1$  and arc  $(i, j) \in A$ , adding (21) for all  $h \in H_2, h \geq 3$  together with (22) and using (20) yields

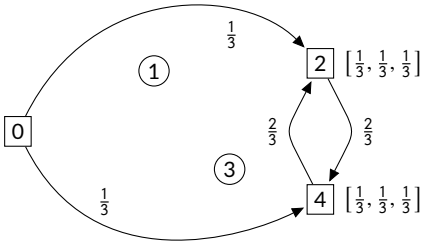
$$\sum_{h \in H_2, h \geq 3} \sum_{(k,i) \in A, k \neq 0} z_{ki}^{h-1} + z_{0i}^1 \geq \sum_{h \in H_2, h \geq 3} z_{ij}^h + z_{ij}^2 = x_{ij}$$

Note that the left-hand side can be rearranged and using the previous proof that constraints (2) are satisfied,

$$\sum_{h \in H_2, h \geq 3} \sum_{(k,i) \in A, k \neq 0} z_{ki}^{h-1} + z_{0i}^1 = \sum_{(k,i) \in A, k \neq 0} \sum_{h \in H_2} z_{ki}^h - \sum_{(k,i) \in A, k \neq 0} z_{ki}^H + z_{0i}^1 = y_i - \sum_{(k,i) \in A, k \neq 0} z_{ki}^H$$

Therefore, using equality (29) for node  $i$  and  $h = H$  we obtain  $x_{ij} \leq y_i - v_i^H \leq 1 - v_i^H$  and thus constraint (5) is satisfied and we conclude the proof of inclusion.  $\square$

To show that this inclusion can be strict, consider an example (see Figure 2) consisting of a complete graph with four nodes and the root node, with required nodes 2 and 4, and  $H = 3$ . A feasible solution to  $gBNH_L$  may be represented by the subgraph where the values close to the arcs represent the  $x_{ij}$  values, the values close to the required nodes represent the  $v_i^h$  values for  $h = 1, 2, 3$ , respectively, and  $y_2 = y_4 = 1$ . All other variables have a zero value.



No feasible solution to  $AH+L$  can be obtained from this solution  $(x, y, v)$  from  $Feas(gBNH_L)$ . In fact, from constraints (19) and (20) we have  $z_{02}^1 = \frac{1}{3}$ ,  $z_{04}^1 = \frac{1}{3}$ ,  $z_{24}^2 + z_{24}^3 = \frac{2}{3}$  and  $z_{42}^2 + z_{42}^3 = \frac{2}{3}$ . On the other hand, from constraints (21) we have  $z_{24}^3, z_{42}^3 \leq 0$ , therefore  $z_{24}^2 = z_{42}^2 = \frac{2}{3}$  which violates constraints (22) for the root arcs

$$\frac{1}{3} = z_{02}^1 \geq z_{24}^2 = \frac{2}{3} \quad \text{and} \quad \frac{1}{3} = z_{04}^1 \geq z_{42}^2 = \frac{2}{3}$$

$\square$

**FIGURE 2** Feasible solution to  $gBNH_L$ .

One relevant observation is that this result shows that we obtained strict dominance over the node model  $gBNH$  using the weaker arc model  $AH$ . Thus, one other question is to know what can be obtained by using the stronger model  $sAH$  defined by constraints (18) - (20), (28), (22) - (27). The next result gives a partial answer to this question by showing that the generalized subtour elimination constraints of size two (10) are implied by the stronger model  $sAH$ .

**Proposition 6** *The generalized subtour elimination constraints of size two (10) are redundant if included in the  $sAH$  model.*

**Proof** For a given arc  $(i, j)$ , we start by adding the term  $z_{ji}^{h-1}$  to both sides of the inequalities (28), the "missing" term on the summation corresponding to arc  $(j, i)$ , leading to the following constraints that might be viewed as a kind of

hop-indexed subtour elimination constraint of size 2,

$$\sum_{(k,i) \in A, k \neq 0} z_{ki}^{h-1} \geq z_{ij}^h + z_{ji}^{h-1}, \quad h \in H_2, h \geq 3$$

Next, we add these inequalities for all  $h \in H_2, h \geq 3$  together with constraint (22) for arc  $(i, j)$  leading to,

$$z_{0i}^1 + \sum_{(k,i) \in A, k \neq 0} \sum_{h \in H_2, h \geq 3} z_{ki}^{h-1} \geq z_{ij}^2 + \sum_{h \in H_2, h \geq 3} z_{ij}^h + \sum_{h \in H_2, h \geq 3} z_{ji}^{h-1}$$

Using the equality constraints (19) and (20) on the above inequality we obtain

$$x_{0i} + \sum_{(k,i) \in A, k \neq 0} (x_{ki} - z_{ki}^H) \geq x_{ij} + (x_{ji} - z_{ji}^H)$$

Finally, using the indegree constraint (18) we obtain constraint (10) for the set  $S = \{i, j\}$ ,

$$y_i \geq x_{ij} + x_{ji} - z_{ji}^H + \sum_{(k,i) \in A, k \neq 0} z_{ki}^H = x_{ij} + x_{ji} + \sum_{(k,i) \in A, k \neq 0, j} z_{ki}^H \geq x_{ij} + x_{ji}$$

□

Other inequalities of interest can probably be derived from the stronger model  $sAH$ . This is an open question that we leave for the future work.

## 4 | COMPUTATIONAL RESULTS

In this section, we compare some of the models introduced in the previous sections in terms of the Linear Programming (LP) relaxation bounds and CPU times to obtain the optimal integer solution. The tests were performed on a PC Intel Core i5-9400, 2.90 GHz with 8 GB of RAM. All models were implemented using ILOG CPLEX Optimization Studio 12.9.

A summary of the models that were tested is the following:

*BNH*: Backward Node Hop-indexed model defined by constraints (1), (2), (3), (15), (5), (6), (7), (8) and (9).

*sBNH*: strong Backward Node Hop-indexed model defined by constraints (1), (2), (3), (16), (5), (6), (7), (8) and (9).

*gBNH*: generalized Backward Node Hop-indexed model defined by constraints (1), (2), (3), (17), (5), (6), (7), (8) and (9).

*AH*: Arc Hop-indexed model defined by constraints (18), (19), (20), (21), (22), (23), (24), (25), (26) and (27).

*sAH*: strong Arc Hop-indexed model defined by constraints (18), (19), (20), (28), (22), (23), (24), (25), (26) and (27).

And also for three of these models, *sBNH*, *gBNH* and *AH*, we tested a version including the Generalized Subtour Elimination constraints of size two (10), *sBNH\**, *gBNH\** and *AH\**, respectively.

For this experiment we used a set of graphs already used in the computational experience in several previous works for the HSTP, e.g., in the work by Gouveia [7]. In these graphs, the coordinates of  $(n + 1)$  points were first randomly generated in a square grid. The cost of a candidate edge is then taken as the integer part of the Euclidean distance between the points defining the endpoints of the edge. The edge set  $E$  of the graph was then defined as

follows: i) all the edges incident to the root node were included in  $E$  (this ensures that the problem has at least one feasible solution) and ii) the  $m$  least cost candidate edges not incident to the root, were also included in  $E$ . Thus, for each graph, we have  $|E| = n + m$  for appropriate values of  $n$  and  $m$ , leading to fairly sparse graphs, which is typical in telecommunications networks. The original set contained two classes of graphs, depending on the location of the root node on the square grid: either located at the center (TC class) or at the corner (TE class). The two classes were used in our experiments.

The arc set is then build by considering every arc  $(0, j), j \in \{1, \dots, n\}$ , and arcs  $(i, j), (j, i)$ , for every  $\{i, j\}$  in the edge set. In order to reduce the size of each instance, we used a standard arc elimination test (as far as we know first used by Gouveia [6]), that consists in removing every arc  $(i, j), i \neq 0$ , such that  $c_{ij} \geq c_{0j}$ . Table 2 shows the different values for  $n, m, |E|, |A|$  and the number of arcs after the elimination test for the TC and TE classes (note that this reduction is small, due to the sparsity of the graphs and the way they were built).

| $n$                       | 60  | 80  | 100 | 120 | 160 |
|---------------------------|-----|-----|-----|-----|-----|
| $m$                       | 150 | 200 | 250 | 300 | 400 |
| $ E $                     | 210 | 280 | 350 | 420 | 560 |
| $ A $                     | 360 | 480 | 600 | 720 | 960 |
| <i>reduced</i> $ A $ (TC) | 340 | 449 | 571 | 705 | 956 |
| <i>reduced</i> $ A $ (TE) | 355 | 476 | 595 | 720 | 959 |

**TABLE 2** Instances graph sizes

For each graph, we tested four values for the number of required nodes: 25%, 50%, 75% and 100%, respectively of the total number of nodes (this last case corresponds to a hop-constrained spanning tree problem) and for the hop limit we tested, as in previous works, the following values  $H = 3, 4, 5$ , thus obtaining a total of 60 instances in each class.

#### 4.1 | The LP performance of the models

Table 3 presents the gaps for the linear programming relaxations of the eight models. The format of the table is as follows: the first three columns define the instance size in terms of the number of nodes beside the root node ( $n$ ), the number of required nodes beside the root node ( $|R_1|$ ) and the hop limit ( $H$ ). The following eight columns contain the LP gaps (in percentage) for the TC class of instances and the next eight columns contain the LP gaps for the TE class of instances. Figures 3 and 4 report the same results in the form of a performance profile graph for TC and TE instances, respectively. For each model, a curve represents the number of instances for which the gap is lower than a given value. The higher the curve, the better. A few observations can be derived from the reported results.

- The performance profile graphs clearly show that we can cluster the models in three groups: the arc models  $sAH$ ,  $AH^*$  and  $AH$ , in this order, are the best ones, a second group is composed of  $gBNH^*$ ,  $gBNH$  and  $sBNH^*$ , and finally  $sBNH$  and  $BNH$  are clearly the worst models with regards to LP bounds.
- As expected, TE instances have worse gaps than the TC instances although this observation is less evident in larger instances. This difference in the two classes is also seen later, when we report the CPU times to obtain the optimal solutions (see Table 4).

| $n$ | $ R_1 $ | $H$ | TC instances |        |          |        |          |      |        |       | TE instances |        |          |        |          |      |        |       |
|-----|---------|-----|--------------|--------|----------|--------|----------|------|--------|-------|--------------|--------|----------|--------|----------|------|--------|-------|
|     |         |     | $BNH$        | $sBNH$ | $sBNH^*$ | $gBNH$ | $gBNH^*$ | $AH$ | $AH^*$ | $sAH$ | $BNH$        | $sBNH$ | $sBNH^*$ | $gBNH$ | $gBNH^*$ | $AH$ | $AH^*$ | $sAH$ |
| 60  | 3       | 3   | 51           | 51     | 17       | 15     | 14       | 10   | 8      | 8     | 83           | 83     | 61       | 54     | 54       | 41   | 35     | 35    |
|     | 7       | 4   | 52           | 52     | 18       | 26     | 18       | 18   | 14     | 13    | 82           | 82     | 57       | 65     | 57       | 50   | 47     | 45    |
|     | 5       | 5   | 53           | 53     | 18       | 26     | 18       | 22   | 17     | 15    | 81           | 81     | 53       | 64     | 53       | 53   | 48     | 46    |
|     | 3       | 3   | 55           | 55     | 27       | 24     | 23       | 19   | 16     | 16    | 70           | 70     | 51       | 49     | 49       | 27   | 25     | 23    |
|     | 15      | 4   | 54           | 54     | 25       | 32     | 25       | 23   | 19     | 19    | 69           | 69     | 47       | 54     | 47       | 39   | 36     | 34    |
|     | 5       | 5   | 53           | 53     | 23       | 31     | 22       | 26   | 20     | 19    | 67           | 67     | 41       | 52     | 41       | 41   | 37     | 35    |
|     | 3       | 4   | 41           | 41     | 26       | 23     | 23       | 12   | 11     | 11    | 57           | 57     | 44       | 44     | 43       | 20   | 18     | 15    |
|     | 30      | 4   | 36           | 36     | 19       | 21     | 18       | 12   | 11     | 9     | 52           | 52     | 35       | 41     | 35       | 25   | 22     | 19    |
|     | 5       | 5   | 33           | 33     | 14       | 19     | 14       | 13   | 10     | 9     | 49           | 44     | 29       | 37     | 29       | 27   | 23     | 20    |
|     | 3       | 3   | 23           | 23     | 23       | 23     | 23       | 6    | 6      | 4     | 40           | 39     | 39       | 39     | 39       | 14   | 14     | 10    |
|     | 60      | 4   | 19           | 19     | 16       | 17     | 16       | 8    | 7      | 5     | 38           | 38     | 32       | 34     | 31       | 19   | 18     | 14    |
|     | 5       | 5   | 14           | 14     | 10       | 13     | 10       | 7    | 7      | 3     | 32           | 32     | 25       | 31     | 25       | 20   | 18     | 15    |
| 80  | 3       | 3   | 63           | 63     | 29       | 24     | 24       | 19   | 16     | 16    | 82           | 82     | 66       | 62     | 62       | 43   | 38     | 37    |
|     | 10      | 4   | 61           | 61     | 28       | 32     | 27       | 24   | 21     | 19    | 81           | 81     | 62       | 68     | 62       | 53   | 51     | 48    |
|     | 5       | 5   | 60           | 60     | 26       | 31     | 26       | 27   | 23     | 22    | 80           | 80     | 60       | 68     | 60       | 59   | 55     | 53    |
|     | 3       | 3   | 49           | 49     | 28       | 24     | 24       | 15   | 13     | 13    | 71           | 71     | 55       | 52     | 52       | 31   | 27     | 26    |
|     | 20      | 4   | 46           | 46     | 26       | 30     | 25       | 21   | 18     | 18    | 67           | 67     | 48       | 53     | 48       | 39   | 36     | 34    |
|     | 5       | 5   | 44           | 44     | 23       | 29     | 23       | 23   | 20     | 19    | 67           | 67     | 47       | 53     | 47       | 45   | 42     | 40    |
|     | 3       | 3   | 39           | 39     | 27       | 26     | 26       | 11   | 10     | 9     | 60           | 60     | 48       | 48     | 47       | 24   | 22     | 20    |
|     | 40      | 4   | 35           | 35     | 22       | 25     | 22       | 16   | 15     | 13    | 55           | 55     | 39       | 44     | 39       | 29   | 27     | 24    |
|     | 5       | 5   | 31           | 31     | 18       | 22     | 18       | 16   | 15     | 13    | 52           | 52     | 35       | 42     | 35       | 32   | 29     | 26    |
|     | 3       | 3   | 23           | 23     | 23       | 23     | 23       | 7    | 7      | 6     | 45           | 45     | 45       | 45     | 45       | 17   | 17     | 14    |
|     | 80      | 4   | 17           | 17     | 17       | 17     | 17       | 9    | 9      | 8     | 39           | 39     | 35       | 37     | 35       | 21   | 20     | 17    |
|     | 5       | 5   | 12           | 12     | 12       | 12     | 12       | 9    | 8      | 6     | 34           | 34     | 29       | 34     | 29       | 23   | 22     | 18    |
| 100 | 3       | 3   | 67           | 67     | 38       | 33     | 31       | 26   | 22     | 22    | 81           | 81     | 64       | 60     | 60       | 42   | 37     | 32    |
|     | 12      | 4   | 64           | 64     | 34       | 38     | 33       | 30   | 27     | 26    | 80           | 80     | 60       | 66     | 60       | 51   | 47     | 44    |
|     | 5       | 5   | 63           | 63     | 31       | 36     | 31       | 32   | 28     | 27    | 80           | 80     | 58       | 65     | 58       | 57   | 52     | 50    |
|     | 3       | 3   | 50           | 50     | 33       | 30     | 29       | 19   | 16     | 15    | 71           | 71     | 55       | 54     | 54       | 32   | 29     | 26    |
|     | 25      | 4   | 46           | 46     | 28       | 32     | 27       | 23   | 20     | 19    | 67           | 67     | 46       | 52     | 46       | 37   | 34     | 31    |
|     | 5       | 5   | 42           | 42     | 22       | 29     | 22       | 23   | 19     | 18    | 67           | 67     | 45       | 53     | 45       | 44   | 40     | 38    |
|     | 3       | 3   | 37           | 37     | 26       | 24     | 24       | 11   | 10     | 9     | 60           | 60     | 46       | 46     | 46       | 23   | 21     | 18    |
|     | 50      | 4   | 30           | 30     | 19       | 21     | 19       | 12   | 11     | 10    | 55           | 55     | 37       | 42     | 37       | 28   | 25     | 23    |
|     | 5       | 5   | 25           | 25     | 14       | 17     | 13       | 12   | 10     | 8     | 52           | 52     | 32       | 39     | 32       | 30   | 26     | 24    |
|     | 3       | 3   | 22           | 22     | 22       | 22     | 22       | 7    | 7      | 5     | 47           | 47     | 47       | 47     | 47       | 17   | 17     | 14    |
|     | 100     | 4   | 15           | 15     | 15       | 15     | 15       | 9    | 9      | 7     | 43           | 43     | 38       | 40     | 38       | 22   | 22     | 18    |
|     | 5       | 5   | 10           | 10     | 10       | 10     | 10       | 8    | 8      | 6     | 36           | 36     | 30       | 35     | 30       | 23   | 22     | 18    |
| 120 | 3       | 3   | 76           | 76     | 55       | 51     | 51       | 37   | 32     | 31    | 86           | 86     | 73       | 70     | 70       | 51   | 45     | 43    |
|     | 15      | 4   | 77           | 77     | 52       | 60     | 52       | 46   | 43     | 40    | 86           | 86     | 72       | 77     | 71       | 64   | 61     | 57    |
|     | 5       | 5   | 77           | 77     | 51       | 61     | 51       | 52   | 47     | 44    | 85           | 85     | 67       | 74     | 67       | 66   | 62     | 59    |
|     | 3       | 3   | 71           | 69     | 47       | 44     | 43       | 29   | 24     | 22    | 82           | 79     | 66       | 64     | 63       | 41   | 37     | 32    |
|     | 30      | 4   | 69           | 69     | 42       | 51     | 42       | 34   | 31     | 29    | 80           | 80     | 61       | 68     | 61       | 50   | 46     | 43    |
|     | 5       | 5   | 68           | 68     | 39       | 51     | 39       | 41   | 34     | 32    | 78           | 78     | 58       | 66     | 58       | 56   | 52     | 49    |
|     | 3       | 3   | 57           | 55     | 43       | 41     | 41       | 20   | 19     | 17    | 72           | 70     | 61       | 60     | 60       | 32   | 30     | 25    |
|     | 60      | 4   | 55           | 55     | 35       | 43     | 35       | 26   | 24     | 21    | 68           | 68     | 54       | 59     | 54       | 41   | 38     | 34    |
|     | 5       | 5   | 54           | 53     | 31       | 42     | 31       | 31   | 26     | 23    | 66           | 66     | 49       | 57     | 49       | 46   | 43     | 40    |
|     | 3       | 3   | 36           | 35     | 35       | 35     | 35       | 13   | 13     | 10    | 49           | 48     | 48       | 48     | 48       | 20   | 20     | 14    |
|     | 120     | 4   | 35           | 35     | 29       | 31     | 28       | 17   | 16     | 13    | 46           | 46     | 42       | 44     | 42       | 27   | 26     | 23    |
|     | 5       | 5   | 31           | 31     | 23       | 29     | 23       | 19   | 17     | 13    | 43           | 43     | 37       | 42     | 37       | 31   | 30     | 27    |
| 160 | 3       | 3   | 80           | 79     | 62       | 59     | 58       | 36   | 31     | 30    | 85           | 81     | 69       | 66     | 66       | 39   | 35     | 30    |
|     | 20      | 4   | 77           | 77     | 55       | 61     | 53       | 43   | 39     | 34    | 84           | 84     | 69       | 73     | 68       | 56   | 52     | 45    |
|     | 5       | 5   | 77           | 76     | 52       | 61     | 52       | 50   | 45     | 42    | 82           | 82     | 64       | 70     | 64       | 58   | 55     | 49    |
|     | 3       | 3   | 77           | 76     | 61       | 60     | 59       | 36   | 32     | 30    | 83           | 80     | 70       | 68     | 68       | 40   | 37     | 34    |
|     | 40      | 4   | 74           | 74     | 54       | 61     | 54       | 44   | 40     | 35    | 81           | 81     | 67       | 72     | 67       | 54   | 51     | 45    |
|     | 5       | 5   | 73           | 72     | 50       | 60     | 50       | 48   | 43     | 39    | 78           | 78     | 61       | 69     | 61       | 57   | 53     | 48    |
|     | 3       | 3   | 65           | 62     | 51       | 51     | 50       | 24   | 21     | 19    | 72           | 71     | 64       | 63     | 63       | 31   | 29     | 24    |
|     | 80      | 4   | 63           | 63     | 44       | 51     | 43       | 33   | 29     | 21    | 73           | 73     | 61       | 65     | 61       | 47   | 44     | 38    |
|     | 5       | 5   | 61           | 61     | 42       | 52     | 41       | 38   | 33     | 25    | 69           | 69     | 54       | 61     | 54       | 48   | 46     | 41    |
|     | 3       | 3   | 46           | 43     | 43       | 43     | 43       | 13   | 13     | 9     | 55           | 53     | 53       | 53     | 53       | 18   | 17     | 12    |
|     | 160     | 4   | 50           | 50     | 41       | 45     | 41       | 26   | 24     | 17    | 59           | 59     | 54       | 56     | 54       | 36   | 35     | 28    |
|     | 5       | 5   | 45           | 45     | 35       | 43     | 35       | 29   | 26     | 18    | 54           | 54     | 47       | 53     | 47       | 40   | 38     | 32    |

TABLE 3 LP gaps (%) for the TC and TE instances (LP CPU times are less than 2 sec.)

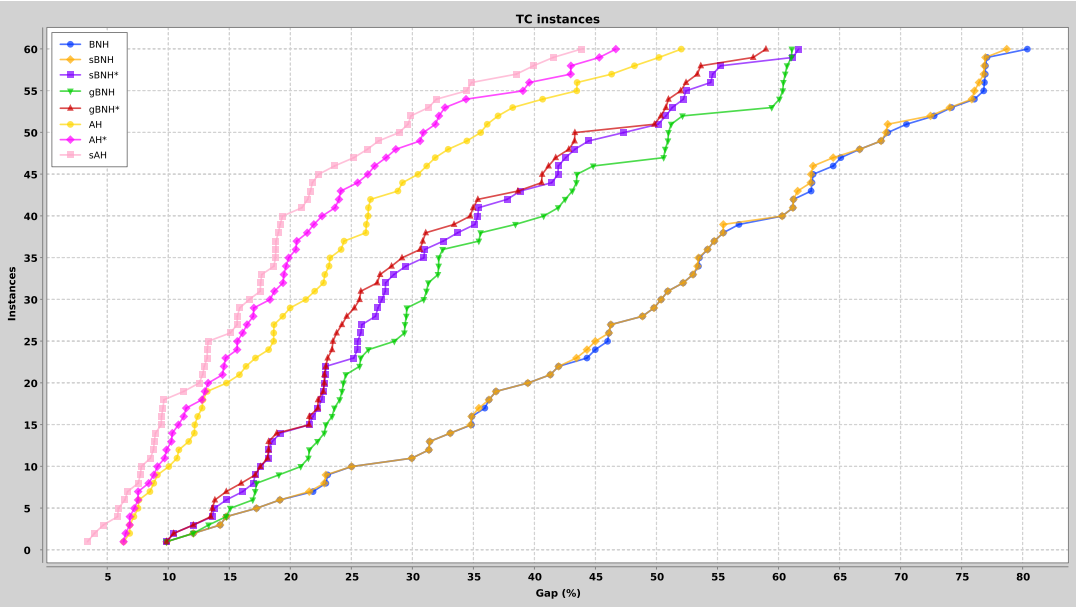


FIGURE 3 LP gaps (%) for the TC instances

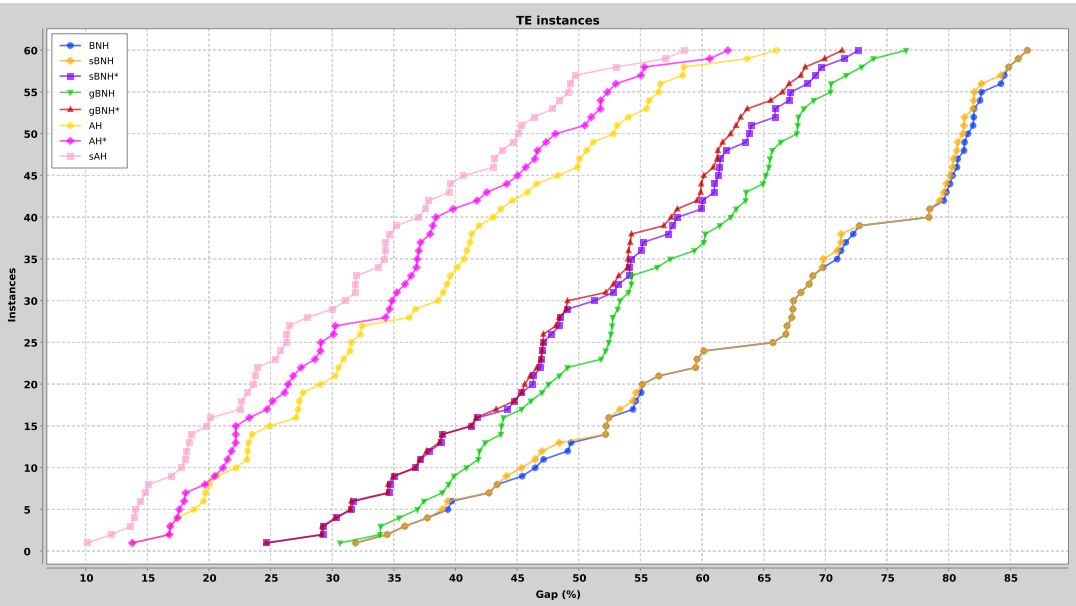


FIGURE 4 LP gaps (%) for the TE instances

- As expected, the node-based models produce LP bounds that are worse than the ones produced by the arc-based models. We observe that there is a clear improvement in LP bounds from the  $sBNH$  model to the  $gBNH$  model (on average 14% and 10% decrease on the TC and TE instances, respectively). This improvement is more significant in the case where not all of the nodes are required nodes. The inclusion of the generalized subtour elimination constraints of size two (10) in these models further reduces the gaps, but in this case the differences between the lower bounds of the enhanced models,  $sBNH^*$  and  $gBNH^*$ , are smaller indicating that the effect of these inequalities is more effective on the weaker model.
- Comparing the node-based *versus* arc-based models, the weakest arc-based model,  $AH$ , outperforms the strongest node-based model,  $gBNH$ , in every instance (on average 11% and 17% decrease on the TC and TE instances, respectively). This observation still remains when we add the constraints (10) to these two models and it is more relevant for the larger instances in both classes, TC and TE.
- When comparing the two arc-based models, we observe that the stronger one outperforms the weaker one in every instance, with a decrease of 5% on average, most significantly for the larger instances. Adding constraints (10) to the weakest arc-based model reduces the average gap difference between the two models. Note also that the corresponding CPU times and the results of the next section indicate that even after this reduction, the strong arc model is preferable to the weak arc model with the subtour elimination constraints (10).

## 4.2 | Obtaining the optimal (integer) solutions

Although the LP values are important indicators for an overall comparison between all the models in our experiment, they are not sufficient to allow us to assess what is the best (faster) model to obtain the optimal integer solution. Since other factors need also to be considered, e.g., the size of the models as well as unknown factors of the ILP package used to solve the instances. In Table 4 we provide the CPU times (in seconds) to obtain the optimal integer solutions taken from all the models described before (the time limit was set to one hour). The format of the table is identical to the one in Table 3. The designation "O.M." indicates a model that could not be solved due to an "Out of Memory" issue and the designation "T.L." indicates a model that reached the "Time Limit" of one hour, before obtaining an optimal solution (or proving the optimality of the best found solution). Performance profile graphs for the same results are presented in Figures 5 and 6, showing the number of instances solved within a given CPU time (on a logarithmic time scale). Again, a higher curve corresponds to a better model.

As in previous works, the TC class instances are easier to solve rather than the TE class instances. Since the size of the models strongly depends on the hop limit, it is also expected that the CPU times increase when the value of  $H$  increases. Also, although for  $H = 3$ , the CPU times are insignificant, for  $H = 5$  the CPU times are significantly larger, specially for the node-based models.

From the performance profile graphs in Figure 5 and 6, we can observe the following interesting fact: the addition of inequalities (10) to a model ( $sBNH^*$ ,  $gBNH^*$  or  $AH^*$ ), although contributing to a reasonable improvement in the corresponding LP gaps, does not necessarily lead to models with better CPU times to obtain the optimal solutions. In fact, the performance profile graphs for CPU times show that the line corresponding to an enhanced model ( $sBNH^*$ ,  $gBNH^*$  and  $AH^*$ ) is below the line corresponding to the original model without enhancements most of the time (this is more evident for models  $AH$  and  $AH^*$ ). However, when comparing an enhanced model versus the corresponding original model, the increase in CPU times is usually not greater than 5 minutes and there are even some cases where the original model did not find the optimal solution within the time limit whereas the enhanced model did.

The comparison between the models  $AH$ ,  $AH^*$  and  $sAH$  is also interesting since it allows us to provide some model improvement insights. As specified before, the model  $AH^*$  is obtained from the model  $AH$  by adding a set of

| $n$ | $ R_1 $ | $H$ | TC instances |        |          |        |          |      |        |       | TE instances |        |          |        |          |      |        |       |
|-----|---------|-----|--------------|--------|----------|--------|----------|------|--------|-------|--------------|--------|----------|--------|----------|------|--------|-------|
|     |         |     | $BNH$        | $sBNH$ | $sBNH^*$ | $gBNH$ | $gBNH^*$ | $AH$ | $AH^*$ | $sAH$ | $BNH$        | $sBNH$ | $sBNH^*$ | $gBNH$ | $gBNH^*$ | $AH$ | $AH^*$ | $sAH$ |
| 60  | 7       | 3   | 1            | 0      | 0        | 0      | 0        | 0    | 1      | 0     | 1            | 1      | 3        | 1      | 1        | 0    | 1      | 0     |
|     |         | 4   | 1            | 1      | 1        | 1      | 1        | 0    | 0      | 0     | 10           | 12     | 15       | 13     | 28       | 1    | 6      | 1     |
|     |         | 5   | 1            | 1      | 1        | 2      | 1        | 1    | 1      | 0     | 85           | 45     | 41       | 110    | 79       | 19   | 26     | 28    |
|     | 15      | 3   | 1            | 1      | 1        | 1      | 0        | 1    | 1      | 0     | 2            | 2      | 3        | 2      | 2        | 1    | 1      | 1     |
|     |         | 4   | 3            | 3      | 2        | 3      | 2        | 1    | 1      | 1     | 35           | 21     | 30       | 36     | 27       | 12   | 13     | 2     |
|     |         | 5   | 7            | 6      | 3        | 18     | 8        | 3    | 4      | 0     | 55           | 73     | 64       | 195    | 112      | 28   | 102    | 25    |
|     | 30      | 3   | 1            | 1      | 1        | 1      | 1        | 1    | 1      | 1     | 2            | 2      | 2        | 2      | 2        | 1    | 1      | 1     |
|     |         | 4   | 2            | 3      | 3        | 3      | 4        | 1    | 1      | 1     | 6            | 4      | 14       | 15     | 16       | 2    | 3      | 1     |
|     |         | 5   | 10           | 7      | 4        | 13     | 20       | 1    | 1      | 0     | 57           | 57     | 30       | 59     | 135      | 14   | 13     | 5     |
|     | 60      | 3   | 1            | 1      | 1        | 1      | 1        | 1    | 1      | 0     | 2            | 1      | 1        | 2      | 1        | 1    | 1      | 1     |
|     |         | 4   | 5            | 2      | 4        | 4      | 8        | 1    | 1      | 1     | 3            | 2      | 2        | 5      | 5        | 2    | 2      | 1     |
|     |         | 5   | 6            | 4      | 5        | 18     | 18       | 2    | 1      | 0     | 40           | 19     | 14       | 93     | 138      | 16   | 23     | 5     |
| 80  | 10      | 3   | 1            | 1      | 2        | 1      | 1        | 1    | 1      | 1     | 1            | 1      | 2        | 1      | 1        | 1    | 1      | 1     |
|     |         | 4   | 4            | 5      | 2        | 3      | 4        | 1    | 1      | 1     | 8            | 8      | 30       | 85     | 25       | 2    | 3      | 1     |
|     |         | 5   | 10           | 11     | 9        | 18     | 9        | 3    | 4      | 0     | 53           | 48     | 31       | 135    | 55       | 5    | 69     | 21    |
|     | 20      | 3   | 1            | 1      | 2        | 1      | 1        | 1    | 1      | 1     | 1            | 1      | 5        | 3      | 3        | 1    | 1      | 1     |
|     |         | 4   | 3            | 2      | 6        | 4      | 7        | 1    | 1      | 1     | 12           | 17     | 105      | 70     | 45       | 4    | 4      | 2     |
|     |         | 5   | 13           | 12     | 25       | 33     | 16       | 6    | 4      | 0     | 292          | 157    | 160      | 235    | 132      | 68   | 240    | 37    |
|     | 40      | 3   | 2            | 1      | 2        | 3      | 3        | 1    | 1      | 1     | 3            | 2      | 3        | 2      | 3        | 1    | 1      | 1     |
|     |         | 4   | 10           | 25     | 32       | 63     | 69       | 2    | 2      | 1     | 12           | 16     | 29       | 41     | 27       | 3    | 3      | 2     |
|     |         | 5   | 32           | 93     | 63       | 98     | 331      | 11   | 14     | 0     | 126          | 216    | 250      | 359    | 543      | 16   | 63     | 75    |
|     | 80      | 3   | 3            | 2      | 2        | 2      | 1        | 1    | 1      | 1     | 2            | 2      | 1        | 1      | 2        | 1    | 1      | 1     |
|     |         | 4   | 6            | 7      | 7        | 16     | 41       | 3    | 2      | 2     | 13           | 12     | 4        | 22     | 31       | 3    | 3      | 3     |
|     |         | 5   | 43           | 10     | 29       | 166    | 447      | 10   | 11     | 0     | 29           | 42     | 59       | 758    | 1229     | 72   | 16     | 8     |
| 100 | 12      | 3   | 2            | 4      | 4        | 9      | 8        | 1    | 1      | 1     | 2            | 3      | 7        | 5      | 8        | 1    | 1      | 1     |
|     |         | 4   | 29           | 76     | 79       | 75     | 57       | 15   | 14     | 2     | 119          | 523    | 499      | 715    | 287      | 5    | 19     | 10    |
|     |         | 5   | 541          | 434    | 325      | 433    | 401      | 195  | 171    | 24    | T.L.         | T.L.   | 2059     | 3255   | 2048     | T.L. | 3479   | 418   |
|     | 25      | 3   | 2            | 6      | 6        | 6      | 6        | 1    | 1      | 1     | 7            | 6      | 23       | 8      | 10       | 2    | 2      | 1     |
|     |         | 4   | 23           | 88     | 86       | 84     | 82       | 6    | 5      | 2     | 102          | 413    | 413      | 230    | 1059     | 14   | 7      | 3     |
|     |         | 5   | 203          | 126    | 101      | 86     | 78       | 29   | 27     | 4     | O.M.         | T.L.   | T.L.     | T.L.   | T.L.     | 3571 | T.L.   | 1438  |
|     | 50      | 3   | 5            | 5      | 5        | 6      | 6        | 1    | 1      | 1     | 5            | 6      | 11       | 11     | 9        | 3    | 2      | 2     |
|     |         | 4   | 21           | 76     | 76       | 53     | 50       | 5    | 4      | 2     | 174          | 1358   | 1385     | 956    | 608      | 18   | 6      | 4     |
|     |         | 5   | 73           | 187    | 138      | 166    | 156      | 15   | 14     | 3     | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 1132 | T.L.   | 138   |
|     | 100     | 3   | 4            | 3      | 3        | 5      | 5        | 2    | 2      | 2     | 7            | 4      | 5        | 9      | 8        | 2    | 2      | 2     |
|     |         | 4   | 31           | 30     | 31       | 16     | 17       | 7    | 7      | 4     | 82           | 22     | 22       | 107    | 30       | 38   | 17     | 25    |
|     |         | 5   | 2750         | 1098   | 974      | T.L.   | T.L.     | 70   | 60     | 36    | 544          | 385    | 387      | 1824   | 400      | 933  | 1713   | 178   |
| 120 | 15      | 3   | 3            | 3      | 4        | 10     | 12       | 1    | 2      | 1     | 3            | 2      | 5        | 19     | 20       | 2    | 2      | 2     |
|     |         | 4   | 40           | 64     | 86       | 141    | 266      | 4    | 9      | 3     | 1379         | 1665   | 2054     | T.L.   | T.L.     | 9    | 16     | 6     |
|     |         | 5   | 2274         | 2160   | 1188     | 2153   | T.L.     | 190  | 709    | 145   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 506  | T.L.   | 128   |
|     | 30      | 3   | 5            | 8      | 10       | 15     | 27       | 2    | 2      | 1     | 13           | 11     | 26       | 67     | 46       | 2    | 5      | 2     |
|     |         | 4   | 168          | 460    | 433      | T.L.   | 660      | 8    | 31     | 5     | 118          | 1175   | 2811     | T.L.   | 2621     | 15   | 205    | 18    |
|     |         | 5   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 316  | T.L.   | 128   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 918  | T.L.   | 182   |
|     | 60      | 3   | 6            | 9      | 9        | 25     | 17       | 2    | 2      | 2     | 8            | 4      | 11       | 46     | 27       | 3    | 2      | 2     |
|     |         | 4   | 464          | 124    | 310      | 1853   | 1609     | 6    | 16     | 5     | 321          | 448    | 265      | 2787   | 3533     | 18   | 41     | 6     |
|     |         | 5   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 719  | T.L.   | 219   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | T.L. | 2076   | 474   |
|     | 120     | 3   | 6            | 5      | 4        | 15     | 32       | 2    | 3      | 2     | 12           | 6      | 7        | 42     | 32       | 3    | 3      | 3     |
|     |         | 4   | 183          | 94     | 44       | 392    | 188      | 10   | 21     | 6     | 38           | 43     | 26       | 98     | 138      | 18   | 40     | 27    |
|     |         | 5   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 511  | 1714   | 98    | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | T.L. | T.L.   | T.L.  |
| 160 | 20      | 3   | 8            | 4      | 8        | 25     | 21       | 3    | 4      | 2     | 12           | 8      | 23       | 48     | 55       | 3    | 4      | 2     |
|     |         | 4   | 1035         | 361    | 442      | 700    | 398      | 9    | 50     | 6     | 42           | 54     | 1020     | 622    | 641      | 31   | 61     | 6     |
|     |         | 5   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 3017 | T.L.   | 190   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 200  | 293    | 56    |
|     | 40      | 3   | 107          | 154    | 346      | 443    | 446      | 4    | 8      | 3     | 20           | 24     | 31       | 99     | 76       | 4    | 4      | 3     |
|     |         | 4   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 327  | 1154   | 94    | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 359  | 1126   | 171   |
|     |         | 5   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 2787 | T.L.   | 493   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | T.L. | T.L.   | 285   |
|     | 80      | 3   | 212          | 111    | 181      | 305    | 458      | 14   | 21     | 6     | 151          | 94     | 111      | 224    | 394      | 8    | 10     | 7     |
|     |         | 4   | 746          | T.L.   | T.L.     | T.L.   | T.L.     | 300  | 518    | 103   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 66   | 147    | 182   |
|     |         | 5   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | T.L. | T.L.   | T.L.  | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | T.L. | T.L.   | T.L.  |
|     | 160     | 3   | 47           | 50     | 45       | 111    | 129      | 7    | 7      | 13    | 39           | 37     | 53       | 84     | 109      | 15   | 17     | 10    |
|     |         | 4   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | 329  | T.L.   | 564   | 651          | 455    | 750      | T.L.   | 2379     | 72   | 258    | 77    |
|     |         | 5   | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | T.L. | T.L.   | 1554  | T.L.         | T.L.   | T.L.     | T.L.   | T.L.     | T.L. | T.L.   | T.L.  |

TABLE 4 CPU times (sec.) for solving the IP models

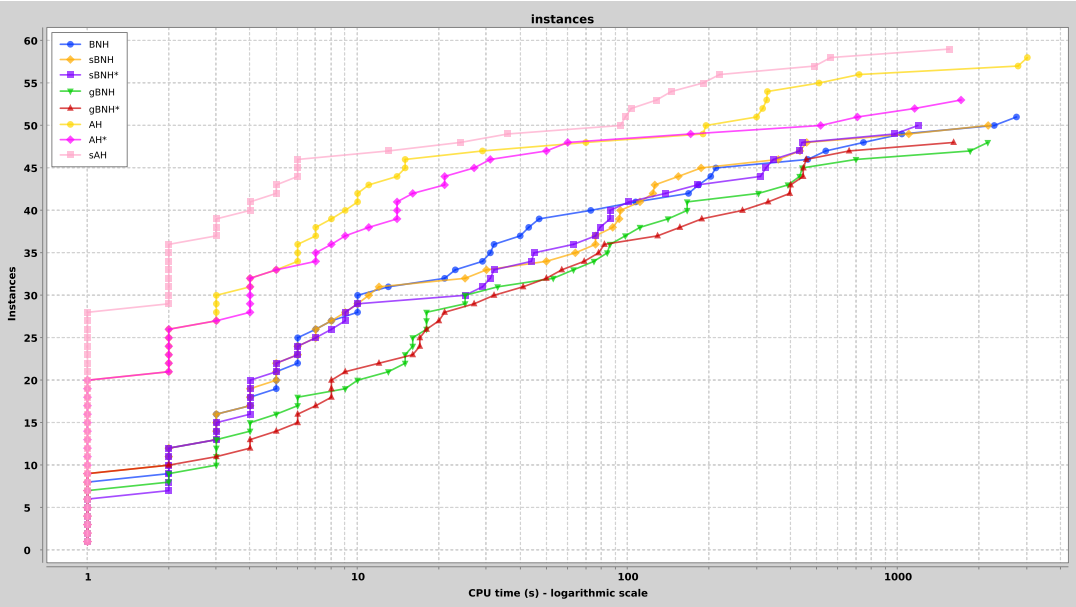


FIGURE 5 CPU times (sec.) for the TC instances

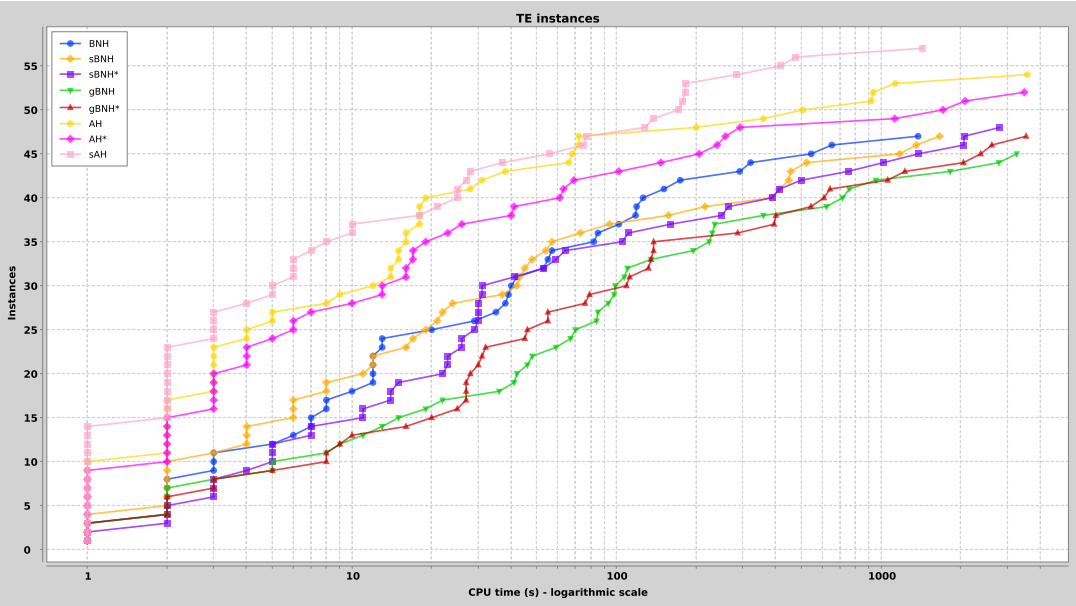


FIGURE 6 CPU times (sec.) for TE instances

constraints ((10) in this case). The model  $sAH$  is obtained from strengthening (lifting) a set of inequalities in model  $AH$ . As proved in Proposition 6, the model  $sAH$  implies the extra constraints in model  $AH^*$ . This explains the improvements in the lower bounds when going from model  $AH$  to model  $AH^*$  and then to model  $sAH$ . However, these relations are not necessarily "propagated" to the integer CPU times since model  $AH^*$  has more constraints than model  $AH$ , while model  $sAH$  is of the same size as model  $AH$ . The main conclusion from this analysis is that whenever possible, strengthening a constraint (which implies the original constraint plus a new set), besides strengthening the LP bounds, leads to a more efficient model to be solved, at least when compared with the original model plus the extra set of constraints.

As a conclusion, we observe that the best three models are the arc-based models with  $sAH$  clearly dominating, and  $AH$  better than  $AH^*$  despite the better bounds of  $AH^*$ . It is also noteworthy to observe that  $gBNH^*$ , despite being the best node model in terms of LP gap, becomes the worst one for TC instances, and the next to worst one for TE instances, with respect to CPU times for solving the problem to optimality.

## 5 | CONCLUSIONS

In this paper we have contextualized the linear programming relaxation of a *hop-indexed node-based model* (and also of a variant including a large set of generalized inequalities) in terms of a simple *hop-indexed arc-based model*. More precisely we have shown that the linear programming relaxation of the first model is implied by the linear programming relaxation of the second model. We observe that the result "arc model implies node model" is not surprising due to the equalities relating the arc variables with the node variables. This theoretical dominance was then evaluated, in practice, with the results taken from a computational experiment. The results indicate that despite using more variables, the arc-based models might be preferable to the node-based models when solving instances of the HTSP. This might be explained by the difference in gap values reported in the computational experiment. There are two points worth discussing. First, the time-dependent models can be viewed as models in a layered graph where different layers correspond to different positions. Also the inequalities (28) defining the strong arc model can be viewed as "simple" cut-set inequalities in the layered graph (see, e.g., the work by Gouveia, Leitner and Ruthmair [10]) and one wonders what inequalities in the space of the node variables are implied by the more general cut-set inequalities. Second, it would be interesting to try to "enlarge" the relations established in this paper by adding relations or non-dominance relations with other node-based models such as, for instance, the "weak" LP based Miller-Tucker-Zemlin model. Despite having a weak LP bound, this model is very compact and can allow the determination of optimal integer solutions with reasonable computing times, with current ILP packages, in cases where theoretically stronger models might fail, due to a large number of variables (such as in the time-dependent model).

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