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Improving social welfare in non-cooperative games with different types of quantum resources

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Abstract

We investigate what quantum advantages can be obtained in multipartite non-cooperative games by studying how different types of quantum resources can improve social welfare, a measure of the quality of a Nash equilibrium. We study how these advantages in quantum social welfare depend on the bias of the game, and improve upon the separation that was previously obtained using pseudo-telepathic strategies.

Two different quantum settings are analysed: a first, in which players are given direct access to an entangled quantum state, and a second, which we introduce here, in which they are only given classical advice obtained from quantum devices. For a given game G , these two settings give rise to different equilibria characterised by the sets of equilibrium correlations $Q_{\text{corr}}(G)$ and $Q(G)$, respectively. We show that $Q(G) \subseteq Q_{\text{corr}}(G)$ and, by considering explicit example games and exploiting SDP optimisation methods, provide indications of a strict separation between the social welfare attainable in the two settings. This provides a new angle towards understanding the limits and advantages of delegating quantum measurements.

2012 ACM Subject Classification Theory of computation \rightarrow Quantum computation theory; Hardware \rightarrow Quantum communication and cryptography

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1 Introduction

Nonlocality [9] is a fascinating phenomenon in quantum mechanics first pinpointed by Bell with his famous inequalities [5] which are always satisfied by correlations that satisfy a property of local causality (such as those obtainable in classical physics), but which can be violated by quantum correlations. This nonlocality also has interesting consequences in the context of games of incomplete information involving several players. In such a game, several players each receive an input unknown to the others and then produce an output by following a strategy. This output is combined with those of the others, and the resulting answer is accepted if and only if it satisfies some specific winning conditions of the game. Perhaps the most famous example of such a game is the CHSH game [10], a cooperative game where two players sharing an entangled quantum state can achieve a higher probability of winning than with any shared classical resource (but without communicating), making a direct link with Bell's inequalities. In some games, known as pseudo-telepathic games [7], quantum resources even allow for a winning probability of one, while no classical resource guarantees a win.

On the other hand, non-cooperative games, also called conflicting interest games, have been extensively studied for their many applications, particularly in economy [18]. In these games, each player has an individual payoff function, rewarding them depending on their

behaviour and that of the others. A central concept in these games is the notion of Nash equilibria, where no player has an incentive to deviate from their strategy. One way to evaluate an equilibrium is to consider the mean (or total) payoff of the players, which is often referred to as its *social welfare*. As with cooperative games, quantum entanglement can be exploited to give more strategies and better equilibria in terms of social welfare [20, 21].

Less studied, until recently, are non-cooperative multipartite games (involving more than two players) with access to quantum resources [3]. In such settings, the phenomena of entanglement and nonlocality are somewhat more complex [13, 9], and it was shown that there exist three-player games where nonlocal resources again provide advantages [6]. In a recent work it was proved that by using pseudo-telepathic solutions on graph games and by allowing for multiple repetitions, it is possible to create an unbounded separation between classical and quantum social welfare [12].

Shared quantum states are just one example of an extra resource or *advice* that one may see as being distributed amongst the players by a trusted mediator. Indeed, even in the classical study of such games, various works have looked at the capabilities afforded by providing correlated advice to the players by providing them, for instance, with shared random variables [4], belief-invariant advice [14, 17], or the capability to communicate through the mediator [3].

As non-cooperative games are a very efficient way to represent and explain real life situations such as traffic flow [22], and because there exists a potentially unlimited separation between the quantum and the classical situation [12], a better understanding of non-cooperative multipartite games with different possible uses of quantum resources may help to find new practical applications for quantum information theory.

In this paper we study two different types of quantum advice in non-cooperative games: shared quantum states, as introduced and analysed in [3], and quantum correlations, corresponding to the case where players have only classical access to quantum correlations, either directly via a trusted mediator (who prepares and measures a distributed quantum state) or through black-box measuring devices provided to them by the mediator. These two situations induce, for any given game G , two sets of equilibrium correlations $Q(G)$ and $Q_{\text{corr}}(G)$, respectively, which fully characterise the equilibria of both settings. We study the relation between these sets, showing that $Q(G) \subseteq Q_{\text{corr}}(G)$ and providing strong evidence that, for certain games, the inclusion may be strict. We compare the quality (in terms of social welfare) of the equilibria of these two quantum settings, using techniques from semidefinite programming to place upper and lower bounds on the achievable social welfare. Our results suggest, perhaps counterintuitively, that players can reach better equilibria when they do not have the freedom that comes with each having full access to their own quantum device. On the way, we show that relaxing the pseudo-telepathic constraints in some games allows the players to reach higher social welfare, meaning that they can win more if they accept to lose in some rounds of the game.

The paper is organised as follows. In Section 2 we recall the definition of non-cooperative multipartite binary games as well as the notions of equilibria and correlated advice. In Section 3 we introduce quantum equilibria as described by [3], corresponding to situations where players share an entangled quantum state, before introducing the new class of quantum correlated advice we study in Section 4. In Section 5 we study the general relationship between these settings. Finally, in Section 6 we compare in detail three families of non-cooperative games, investigating the social welfare attainable in different settings.

Question $t_1 t_2 t_3 t_4 t_5$	Winning condition
10000	$a_5 \oplus a_1 \oplus a_2 = 0$
01000	$a_1 \oplus a_2 \oplus a_3 = 0$
00100	$a_2 \oplus a_3 \oplus a_4 = 0$
00010	$a_3 \oplus a_4 \oplus a_5 = 0$
00001	$a_4 \oplus a_5 \oplus a_1 = 0$
11111	$a_1 \oplus a_2 \oplus a_3 \oplus a_4 \oplus a_5 = 1$

■ **Table 1** Winning conditions for the $\text{NC}_{00}(C_5)$ family of games.

Question $t_1 t_2 t_3 t_4 t_5$	Winning condition
10100	$a_5 \oplus a_1 \oplus a_2 = 0$
01010	$a_1 \oplus a_2 \oplus a_3 = 0$
00101	$a_2 \oplus a_3 \oplus a_4 = 0$
10010	$a_3 \oplus a_4 \oplus a_5 = 0$
01001	$a_4 \oplus a_5 \oplus a_1 = 0$
11111	$a_1 \oplus a_2 \oplus a_3 \oplus a_4 \oplus a_5 = 1$

■ **Table 2** Winning conditions for the $\text{NC}_{01}(C_5)$ family of games.

2 Non-cooperative games and quantum equilibria

2.1 Non-cooperative multipartite games

In an n -player multipartite game G , each player i receives a type t_i from a question $t = t_1 \dots t_n$, and follows a strategy to produces an output a_i . The outputs are aggregated to form the answer $a = a_1 \dots a_n$, and each player receives a payoff depending on the question and answer. More formally:

► **Definition 1.** An n player non-cooperative binary game is defined by a set of valid questions $T \subseteq \{0, 1\}^n$, a prior probability distribution Π over T (satisfying $\forall t \in T, \Pi(t) \geq 0$ and $\sum_{t \in T} \Pi(t) = 1$), a set of valid answers $A = A_1 \times \dots \times A_n = \{0, 1\}^n$, and an individual payoff function u_i for each player i such that $u_i(a, t) \in \mathbb{R}$.

Here, we will consider games where the payoff functions have the same form for all players and are defined from a set of winning input-output pairs $\mathcal{W} \subseteq A \times T$ as

$$u_i(a, t) = \begin{cases} 0 & \text{if } (a, t) \notin \mathcal{W} \\ v_0 & \text{if } a_i = 0 \text{ and } (a, t) \in \mathcal{W} \\ v_1 & \text{if } a_i = 1 \text{ and } (a, t) \in \mathcal{W}, \end{cases} \quad (1)$$

with $v_0, v_1 > 0$. Therefore, for a given question t , if two players output the same value they receive the same payoff. It is the ratio v_0/v_1 that makes the game non-cooperative: if the payoff is unbalanced, e.g., if $v_1 > v_0$, the players might have an interest in losing some of the time, if it means answering 1 more often.

As an example, in Tables 1 and 2 we give the winning conditions of two (families of) 5-player games defined in [12], and which we analyse in more detail in Section 6. These games are an adaption of multipartite binary graph games [2] which associate games to graphs and are based on the C_5 cycle graph; $\text{NC}_{01}(C_5)$ is a symmetrised version of $\text{NC}_{00}(C_5)$. Throughout this paper, we will always consider Π to be the uniform distribution on T , so that the questions are drawn uniformly at random.

In a game, the strategy of a player influences their payoff but also that of other players. Naturally, each player seeks to maximise their own payoff. Some configurations of strategies are stable, meaning that no player can increase their payoff by unilaterally deviating from their strategy, and these configurations are called equilibria. Furthermore, when players are allowed to coordinate with some correlation or advice, they may reach different equilibria. In the next section formalise these concepts.

2.2 Correlated strategies and equilibria

We consider a setting where the players can coordinate their strategies via a shared correlation C over two sets $R = R_1 \times \dots \times R_n$ and $S = S_1 \times \dots \times S_n$. The standard interpretation of this correlation is that of a trusted mediator communicating with each party. Each party sends an input bit $r_i \in R_i$ to the mediator, who computes a correlation $C(s_1 \dots s_n | r_1 \dots r_n)$ before sending back the advice $s_i \in S_i$ to each party. Allowing the players to access, through the mediator, different types of correlations may give them different capabilities to obtain good equilibria.

We now formalise the notions of strategies and equilibria. Throughout, for any string $t = t_1 \dots t_n$ we will use the shorthand notation that $t_{-i} = t_1 \dots t_{i-1} t_{i+1} \dots t_n$ and $t_i t_{-i} = t_{-i} t_i = t$. The simplest type of local strategy a player may use is a pure, or deterministic strategy.

► **Definition 2** (Pure strategy). *A pure strategy for a player i is given by a pair of functions $f_i : T_i \rightarrow R_i$ and $g_i : T_i \times S_i \rightarrow A_i$, where $T_i = \{0, 1\}$, so that $T \subseteq T_1 \times \dots \times T_n$.*

More generally a player may use a probabilistic strategy obtained by mixing pure ones by marginalising over a set of latent variables Λ_i .

► **Definition 3** (Mixed strategy). *A mixed strategy for a player i is given by a probability distribution $\pi_i : \Lambda_i \rightarrow \mathbb{R}$ and two functions $f_i : T_i \times \Lambda_i \rightarrow R_i$ and $g_i : T_i \times S_i \times \Lambda_i \rightarrow A_i$.*

We can now define solutions, which describe the strategies of each player as well as the shared correlations.

► **Definition 4** (Solution). *A solution is a tuple (f, g, π, C) where $\pi = (\pi_1, \dots, \pi_n)$, $f = (f_1, \dots, f_n)$ and $g = (g_1, \dots, g_n)$ define the mixed strategy of each player i , and C is the shared correlation. Then, a solution induces a probability distribution on $A \times T$ as*

$$P(a|t) = \sum_{\lambda, s} C(s|f(t, \lambda)) \pi(\lambda) \delta_{g(t, s, \lambda), a}, \quad (2)$$

where δ is the Kronecker delta.

Via a slight abuse of notation, we will use (f, g, C) to denote the less general case of a pure solution, with each player following a pure strategy. The expression of its induced probability distribution is then simplified to

$$P(a|t) = \sum_s C(s|f(t)) \delta_{g(t, s), a}. \quad (3)$$

Note that different solutions can induce the same probability distribution.

A solution is a *Nash equilibrium* if no player can improve their payoff by unilaterally deviating from their strategy. For a player i , a deviation is expressed by the use of a new mixed strategy. By linearity, however, it is sufficient to verify that for any fixed type t_i the payoff decreases under any deviation to a pure strategy. That is, it is sufficient to check that it decreases for any alternative input r_i sent by the player to the mediator, and any deterministic function $\mu_i : T_i \times S_i \rightarrow A_i$.

► **Definition 5** (Nash equilibrium). *A solution (f, g, π, C) is a Nash equilibrium for a game G if for all players i , for all t_i, r_i and deterministic functions $\mu_i : T_i \times S_i \rightarrow A_i$:*

$$\begin{aligned} & \sum_{t_{-i}, \lambda, s} u_i(g(t_i t_{-i}, s, \lambda), t_i t_{-i}) C(s|f(t_i t_{-i}, \lambda)) \Pi(t_i t_{-i}) \pi(\lambda) \\ & \geq \sum_{t_{-i}, \lambda, s} u_i(\mu_i(t_i, s_i) g_{-i}(t_{-i}, s_{-i}, \lambda_{-i}), t_i t_{-i}) C(s|r_i f_{-i}(t_{-i}, \lambda_{-i})) \Pi(t_i t_{-i}) \pi(\lambda), \end{aligned} \quad (4)$$

where we recall the convention that $t_i t_{-i} = t$.

The advantage provided by correlated advice may depend on the characteristics of the correlation C . Thus, it is natural to distinguish different families of correlations, such as arbitrary correlations allowing total communication, or local correlations, corresponding to a random variable shared between the players. These families and others will be introduced later. For a family of correlations \mathcal{C} , we introduce the notion of \mathcal{C} -Nash equilibrium.

► **Definition 6** (*\mathcal{C} -Nash equilibrium*). *Let \mathcal{C} be a family of correlations and G a game. Then, a solution (f, g, π, C) is a \mathcal{C} -Nash equilibrium for G if and only if it is a Nash equilibrium for G and $C \in \mathcal{C}$.*

We furthermore introduce the concept of a canonical solution to simplify the manipulation of equilibria and ease calculations. The idea is to delegate all the ‘‘computation’’ of a strategy to the mediator, so that the players send directly their type to the mediator and simply output the advice they receive in return. Then, without loss of generality, the sets T_i (A_i) and R_i (resp. S_i) can be taken to be identical.

► **Definition 7** (*Canonical solution*). *Let (f, g, π, C) be a solution and P its induced probability distribution on $A \times T$ as per Eq. (2). Then, the pure solution (id_T, id_A, P) is called its canonical solution, where id_T (resp. id_A) represents the identity function on T (resp. A).*

By definition, a solution and its corresponding canonical solution both induce the same probability distribution P .

► **Remark 8**. It is important to note that for a generic correlation family \mathcal{C} , the canonical form (id_T, id_A, P) of a \mathcal{C} -Nash equilibrium is not necessarily a \mathcal{C} -Nash equilibrium itself. Indeed, P is not necessarily included in \mathcal{C} , which is required by Definition 6.

The conditions for a canonical solution to be a Nash equilibrium can be simplified using the facts that it is a pure solution and that we can identify each T_i with R_i and A_i with S_i .

► **Proposition 9**. *A canonical solution (id_T, id_A, P) is a Nash equilibrium for a game G , if and only if for all players i , all t_i, r_i , and all deterministic functions $\mu_i : T_i \times A_i \rightarrow A_i$:*

$$\sum_{t_{-i}, a} u_i(a, t_i t_{-i}) P(a | t_i t_{-i}) \Pi(t_i t_{-i}) \geq \sum_{t_{-i}, a} u_i(\mu_i(t_i, a_i) a_{-i}, t_i t_{-i}) P(a | r_i t_{-i}) \Pi(t_i t_{-i}). \quad (5)$$

A standard way to compare different equilibria is to consider the mean or total payoff of all the players [15] (we consider here the former measure). This quantity is called the *social welfare*, and is obtained by considering the induced probability distribution of a given solution.

► **Definition 10** (*Social welfare*). *For a game G , the social welfare of a solution (f, g, π, C) is defined as a function of its induced probability distribution P by:*

$$SW_G(P) = \frac{1}{n} \sum_i \sum_{a, t} u_i(a, t) P(a | t) \Pi(t). \quad (6)$$

Throughout the paper, we will speak interchangeably of the social welfare of a solution or of a probability distribution, implying in the later case that there exists a solution inducing the probability distribution.

Note that two solutions inducing the same probability distribution therefore have the same social welfare. This is notably the case for a solution and its canonical form.

2.3 Different types of correlations

For a game G , the equilibria that players can access depend on the type of correlations they can share. Here we present different families of correlations, and for a family \mathcal{C} , we associate to its set of \mathcal{C} -Nash equilibria a corresponding set of *induced equilibrium correlations*.

► **Definition 11** (Set of induced equilibrium correlations). *For a correlation family \mathcal{C} , the set of induced equilibrium correlations is defined as the set of probability distributions*

$$\{P : \text{there exists a } \mathcal{C}\text{-Nash equilibrium } (f, g, \pi, C) \text{ inducing } P \text{ as per Eq. (2)}\}. \quad (7)$$

Because the social welfare of a solution is characterised by its induced probability distribution, these sets are useful to compare the different types of advice and the quality of equilibria they allow one to reach.

The first set of correlations introduced corresponds to a situation where the advice carries no information. Said otherwise, when it can be factorised as a product of n -uncorrelated local distributions L_i .

► **Definition 12** (Factorisable correlations). *A factorisable correlation C is a probability distribution that can be written*

$$C(s|r) = L_1(s_1|r_1) \cdots L_n(s_n|r_n). \quad (8)$$

As such, the correlated advice can be absorbed into each player's local strategy and this situation is equivalent to a setting without correlated advice. The equilibria obtained are the basic Nash equilibria and the corresponding induced equilibrium correlations are denoted, for a given game G , $\text{Nash}(G)$.

The situation where players can share a correlated random variable, corresponds to a family of correlations called local correlations.

► **Definition 13** (Local correlations). *A local correlation C is a probability distribution that can be written*

$$C(s|r) = \sum_{\gamma} V(\gamma) L_1(s_1|r_1, \gamma) L_2(s_2|r_2, \gamma) \cdots L_n(s_n|r_n, \gamma), \quad (9)$$

where $V(\gamma)$ is a probability distribution over the shared random variable γ .

In local correlations, any correlation between the players is solely due to γ ; conditioned on its value, the advice a player receives from the mediator $L_i(s_i|r_i, \gamma)$ is uncorrelated from that of the other players. Equilibria obtained using local correlations correspond to those intuitively obtainable using “classical” resources and we denote by $\text{Corr}(G)$ the corresponding set of induced equilibrium correlations.

A more general family of correlations that has been considered in this context is that of belief invariant, or non-signalling, correlations [14]. With these correlations, the action of the mediator may depend on the inputs of all parties, but in such a way that the knowledge of (s_i, r_i) does not give information about the other inputs r_j . This can be interpreted as a privacy guarantee which can be valuable in conflicting interest games.

To state these correlations, let us denote, for a subset $I \subset [n] := \{1, \dots, n\}$, $R_I = \times_{i \in I} R_i$ (and likewise for S_I).

► **Definition 14** (Belief invariant (non-signalling) correlations [14]). *A correlation C is belief invariant or non-signalling if, for any set of players $I \in [n]$, with $J = [n] \setminus I$ we have*

$$\forall s_I \in S_I, \forall r_I \in R_I, \forall r_J, r'_J \in R_J \quad \sum_{s_J \in S_J} C(s_I s_J | r_I r_J) = \sum_{s_J \in S_J} C(s_I s_J | r_I r'_J). \quad (10)$$

We denote the corresponding set of induced equilibrium correlations $\text{B.I.}(G)$.

Finally, the case where players are allowed unlimited communication corresponds to the situation where $C(s|r)$ may be an arbitrary probability distribution. We denote the corresponding induced equilibrium correlations $\text{Comm}(G)$.

The inclusion structure of these families of correlations leads to a corresponding structure on the induced equilibrium correlations, namely [3, 17]:

$$\text{Nash}(G) \subset \text{Corr}(G) \subset \text{B.I.}(G) \subset \text{Comm}(G). \quad (11)$$

When a family of correlations \mathcal{C} is closed under pre- and post-processing, the set of induced equilibrium correlations is generated by its canonical \mathcal{C} -Nash equilibria. This characterisation is applicable to belief-invariant and communication correlations, but not for local correlations [3].

3 Quantum strategies

The settings discussed above are classical insofar as the players have only classical capabilities: they interact with their advice classically and can essentially only choose to follow it, or ignore it and play some other strategy.

Quantum mechanics, however, allows for another natural possibility: for the players to receive intrinsically quantum advice. This type of advice has been previously studied [16, 20] and here we follow the same presentation as in [3]. In this setting, the players follow a fundamentally quantum strategy. They each have part of a multipartite quantum state as advice (e.g., provided by an external mediator or by the players interacting before playing the game), and can then perform measurements on the shared quantum system to correlate their actions.

► **Definition 15** (Quantum strategy). *A quantum strategy for a player i is a set of positive-operator-valued measures (POVMs), $\mathcal{M}^{(i)} = \{M_{t_i}^{(i)}\}_{t_i \in T_i}$, where each POVM $M_{t_i}^{(i)} = \{M_{a_i|t_i}^{(i)}\}_{a_i \in A_i}$ satisfies, for all t_i , $\sum_{a_i} M_{a_i|t_i}^{(i)} = \mathbb{1}$ and, for all a_i, t_i , $M_{a_i|t_i}^{(i)} \geq 0$.*

A quantum solution is thus formed by the POVMs of each player and the shared quantum state representing the advice.

► **Definition 16** (Quantum solution). *A quantum solution (\mathcal{M}, ρ) consists of a quantum state $\rho \in \mathcal{S}(\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n)$ and a set of POVMs $\mathcal{M} = (\mathcal{M}^{(1)}, \dots, \mathcal{M}^{(n)})$ for every player. Each player has access to their part of ρ in the corresponding Hilbert space \mathcal{H}_i . A quantum solution induces a probability distribution P via the Born rule:*

$$P(a|t) = \text{tr} \left[\rho \left(M_{a_1|t_1}^{(1)} \otimes \dots \otimes M_{a_n|t_n}^{(n)} \right) \right]. \quad (12)$$

Crucially, compared to the classical solutions with correlated advice discussed previously, the notion of an equilibrium must be adjusted to take into account the quantum nature of the strategies and corresponding greater freedom of players. In this setting, a quantum solution is a *quantum equilibrium* if no player can improve their payoff by changing their (local) choice of quantum strategy, i.e., by choosing different POVMs.

► **Definition 17** (Quantum equilibrium). *A quantum solution (\mathcal{M}, ρ) , is a quantum equilibrium if for every player i , for any type t_i and any POVM $N^{(i)} = \{N_{a_i}^{(i)}\}_{a_i \in A_i}$:*

$$\begin{aligned} & \sum_{t_{-i}, a} u_i(a, t) \operatorname{tr} \left[\rho (M_{a_1|t_1}^{(1)} \otimes \cdots \otimes M_{a_n|t_n}^{(n)}) \right] \Pi(t) \\ & \geq \sum_{t_{-i}, a} u_i(a, t) \operatorname{tr} \left[\rho (M_{a_1|t_1}^{(1)} \otimes \cdots \otimes M_{a_{i-1}|t_{i-1}}^{(i-1)} \otimes N_{a_i}^{(i)} \otimes M_{a_{i+1}|t_{i+1}}^{(i+1)} \otimes \cdots \otimes M_{a_n|t_n}^{(n)}) \right] \Pi(t). \end{aligned} \quad (13)$$

For a game G , we denote by $Q(G)$ the set of correlations induced by quantum equilibria. This set was previously studied in [3], where it was compared to the types of classical advice presented in Section 2.3.¹

Note that because no constraint is placed on the dimension of Hilbert spaces for strategies in defining $Q(G)$, it will generally be sufficient to consider only “pure” solutions, composed of pure states and projective measurements, but we will not make use of this assumption here. Also, in contrast to the case of correlated advice, there is no clear notion of a canonical quantum solution.

Let us finally note that, as also pointed out in [3], one can verify that a solution is a quantum equilibrium using semidefinite programming. Indeed, we can maximise the average payoff of an individual player while keeping the strategies of the others, as well as the state ρ , fixed. If their mean payoff is already maximised by their current POVM $\{M_{a_i|t_i}^{(i)}\}_{a_i}$ for each input t_i , it means that they have no incentive to unilaterally change their strategy. This can be verified for each of the n players separately.

4 Quantum correlations

In the type of quantum solutions introduced above, players share a potentially entangled quantum state and perform their own measurements. This supposes that they each have access to some local quantum device. However, because of the complexity of such devices, it is also interesting to consider the case where they access classically to multipartite quantum correlations either via a trusted mediator (who prepares and measures a multipartite quantum state), or via black-box measuring devices that are provided to them by the mediator. In either case, in this setting the players do not need any direct quantum capabilities.

This setting of classical access to quantum resources is extensively studied in device-independent approaches to quantum information [1, 9], and many quantum advantages or protocols can be adapted for it, such as the ability to perform blind quantum computation [8], where a player delegates their quantum computations to a server in a way that keeps its input, output and computation private.

Here we study how such resources are useful in non-cooperative games, and in particular in comparison to the fully-quantum strategies presented in the previous section.

The set of correlations we are interested in is thus that of n -partite quantum correlations, which can be obtained by performing individual measurements on each part of an n -partite quantum state.

► **Definition 18** (Quantum correlations). *A quantum correlation is a probability distribution $C(s|r)$ that can be written in the form*

$$C(s|r) = \operatorname{tr} \left[\rho (M_{s_1|r_1}^{(1)} \otimes \cdots \otimes M_{s_n|r_n}^{(n)}) \right] \quad (14)$$

¹ Note, however, that Auletta et al. [3] denoted it Quantum(G).

for some quantum state $\rho \in S(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ and POVMs $\mathcal{M}^{(i)} = \{M_{r_i}^{(i)}\}_{r_i \in R_i}$ with $M_{r_i}^{(i)} = \{M_{s_i|r_i}^{(i)}\}_{s_i \in S_i}$ in some Hilbert spaces \mathcal{H}_i .

We write \mathcal{C}_Q the family of quantum correlations and $Q_{\text{corr}}(G)$ the set of induced equilibrium correlations defined as in Eq. (7).

► **Proposition 19.** *For any game G we have $Q_{\text{corr}}(G) \subseteq \mathcal{C}_Q$, furthermore, for any probability distribution $P \in Q_{\text{corr}}(G)$, the canonical solution (id_T, id_A, P) is a \mathcal{C}_Q -Nash equilibrium.*

Proof. Let $P \in Q_{\text{corr}}(G)$ be an induced equilibrium correlation. Then, by definition, there exists a \mathcal{C}_Q -Nash equilibrium (f, g, π, C) inducing P . Because $C \in \mathcal{C}_Q$, there exist (ρ, \mathcal{M}) such that for all $(s, r) \in S \times R$:

$$C(s|r) = \text{tr} [\rho (M_{s_1|r_1}^{(1)} \otimes \cdots \otimes M_{s_N|r_N}^{(n)})]. \quad (15)$$

Consider then the new POVMs

$$\tilde{M}_{a_i|t_i}^{(i)} = \sum_{\lambda} M_{s_i|f_i(t_i, \lambda_i)}^{(i)} \pi(\lambda) \delta_{g(t, s, \lambda), a}. \quad (16)$$

Then the induced distribution P as defined in Eq. (2) can be written

$$P(a|t) = \text{tr} [\rho (\tilde{M}_{a_1|t_1}^{(1)} \otimes \cdots \otimes \tilde{M}_{a_n|t_n}^{(n)})], \quad (17)$$

and thus $P \in \mathcal{C}_Q$. Finally, because (f, g, π, C) is a Nash equilibrium, it follows that (id_T, id_A, P) is a \mathcal{C}_Q -Nash equilibrium. ◀

Because quantum mechanics obeys relativistic causality, it is well known that n -partite quantum correlations are non-signalling, and thus $Q_{\text{corr}}(G) \subset B.I.(G)$. Since local correlations can be obtained from quantum correlations, we also have that $\text{Corr}(G) \subset Q_{\text{corr}}(G)$. Finally, Proposition 19 allows us to restrict the study of \mathcal{C}_Q -Nash equilibria to that of canonical \mathcal{C}_Q -Nash equilibria, and to consider the simplified definition of Nash equilibrium for pure solutions as in Proposition 9.

5 Comparing different quantum resources in non-cooperative games

We have introduced two quantum settings, one where each player has their own quantum device and shares an entangled quantum state, and another where players receive a quantum correlation as advice, either by delegating their quantum devices to a mediator or by being afforded black-box access to quantum devices. In the former setting, the players have quantum resources, while in the latter they can be considered to be fully classical, with only indirect access to quantum devices. For a game G , the Nash equilibria accessible in each setting are fully characterised by the induced equilibrium correlations $Q(G)$ and $Q_{\text{corr}}(G)$, respectively. They are both subsets of the quantum correlations \mathcal{C}_Q . Therefore, a natural way of comparing the two quantum settings is to compare $Q(G)$ and $Q_{\text{corr}}(G)$.

Naively, one may expect fully quantum players to be able to achieve more equilibria than players with only classical access to quantum devices. However, the reality is actually precisely the opposite: the notion of a quantum equilibrium is more restrictive, since the type of deviations they can make (here, choosing an alternative local POVM), is more general, making it more difficult to obtain an equilibrium. Formally, we have the following result.

► **Theorem 20.** *For any game G , $Q(G) \subseteq Q_{\text{corr}}(G)$.*

Question $t_1 t_2 t_3$	Winning conditions
100	$a_1 \oplus a_2 \oplus a_3 = 0$
010	$a_1 \oplus a_2 \oplus a_3 = 0$
001	$a_1 \oplus a_2 \oplus a_3 = 0$
111	$a_1 \oplus a_2 \oplus a_3 = 1$

■ **Table 3** Winning conditions for the $\text{NC}(C_3)$ games.

Proof. We proceed by contradiction. Let $C \in Q(G)$ be a correlation obtained by an equilibrium strategy (ρ, \mathcal{M}) , and suppose that $C \notin Q_{\text{corr}}(G)$. Then $C \in \mathcal{C}_Q$ and by assumption, the canonical solution (id_T, id_A, C) is not a \mathcal{C}_Q -Nash equilibrium, so there exists a player i who can increase their payoff by unilaterally deviating from their local strategy. However, this deviation can easily be expressed as a change in the POVMs in the quantum equilibrium solution, as in Proposition 19. Therefore, player i in the quantum solution could also increase their payoff by unilaterally modifying their POVMs, which contradicts the fact that (ρ, \mathcal{M}) is an equilibrium. ◀

To show that this inclusion is strict, one would need to exhibit a game G and a correlation $C \in Q_{\text{corr}}(G)$ that is not in $Q(G)$, meaning that no quantum equilibrium solution would induce the correlation C . Proving that a given correlation cannot be obtained by any such solution is, unfortunately difficult, as one must consider all possible quantum solutions, in any dimension, that reproduce C . While we were unable to find such a general argument, we will see below that we were able to find correlations $C \in Q_{\text{corr}}(G)$ for which the natural quantum solution is not a quantum equilibrium, suggesting the inclusion may indeed be strict.

5.1 A family of three-player games: $\text{NC}(C_3)$

To find such an example, we introduce a family of three-player games for which the questions and winning conditions are summarised in Table 3. Like the games specified in Tables 1 and 2, these games are a form of multipartite binary graph game [2], and are based on the C_3 graph. We assume a uniform distribution Π over the questions and the payoff function is defined as in Equation (1) from the corresponding winning set \mathcal{W} . The $\text{NC}(C_3)$ family is parametrised by the two values v_0, v_1 determining the precise payoff function, and each new pair (v_0, v_1) specifies a particular game within this family.

A striking feature of these games is that they have a *pseudo-telepathic* solution allowing the players to win all the time (i.e., with probability 1), while this is impossible when they only have access to classical solutions and local correlations (or shared random variables) as advice [7]. The pseudo-telepathic solution involves the players sharing a C_3 graph state $CZ^{(1,2)}CZ^{(2,3)}CZ^{(3,1)}|+\rangle^{\otimes 3}$ (i.e., a GHZ state [11]) and measuring either in the computational $\{|0\rangle, |1\rangle\}$ basis or the $\{|+\rangle, |-\rangle\}$ basis depending on whether their type is 0 or 1, respectively. Writing P the induced correlation of this quantum solution, then, for any value of (v_0, v_1) , the solution (id_T, id_A, P) is a \mathcal{C}_Q -Nash equilibrium and $P \in Q_{\text{corr}}(G)$ [12]. Similarly, one can readily check using semidefinite programming (as described in Sec. 3) that this solution gives a quantum equilibrium and hence $P \in Q(G)$ as well. The social welfare of this solutions is $\frac{v_0 + v_1}{2}$ [12].

However, by modifying slightly the pseudo-telepathic solution, we can find solutions that still are \mathcal{C}_Q -Nash equilibria but which are themselves no longer quantum equilibria.

To this end, let us take an angle θ which will parameterise the “deviated” state and measurements and define $|\psi_\theta\rangle = \cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle$. The players now share the quantum state $|\Psi_{\text{dev}(\theta)}\rangle = CZ^{(1,2)}CZ^{(2,3)}CZ^{(3,1)}|\psi_\theta\rangle^{\otimes 3}$ and each perform measurements of the POVMs $\mathcal{M}_{\text{dev}(\theta)}^{(i)} = \{\{M_{a_i|t_i}^{(i)}\}_{a_i}\}_{t_i}$ with elements

$$M_{a_i|t_i=0}^{(i)} = |a_i\rangle\langle a_i|, \quad M_{0|t_i=1}^{(i)} = |\psi_\theta\rangle\langle\psi_\theta|, \quad M_{1|t_i=1}^{(i)} = |\psi_\theta^\perp\rangle\langle\psi_\theta^\perp|, \quad (18)$$

where $|\psi_\theta^\perp\rangle$ is orthogonal to $|\psi_\theta\rangle$. We write $C_{\text{dev}}(\theta)$ the correlation induced by this strategy.

Because $C_{\text{dev}}(\theta) \in \mathcal{C}_Q$, we know from Proposition 19 that $C_{\text{dev}}(\theta) \in C_{\text{corr}}(G)$ if and only if $(id_T, id_A, C_{\text{dev}}(\theta))$ is a \mathcal{C}_Q -Nash equilibrium. Which can be verified using the canonical Nash equilibrium conditions of Proposition 9.

Furthermore, as the correlation and the game are symmetric: for any pair $(a_i, t_i) \in \{0, 1\}^2$ of advice and type, every player i has the same probability $p_{\text{win}}^{a_i, t_i}(\theta)$ of winning the game. Here the advice is denoted by a_i , as in canonical solutions the advice and the answer are identical. We then obtain the following condition for this the solution $(id_T, id_A, C_{\text{dev}}(\theta))$ to be a \mathcal{C}_Q -Nash equilibrium:

► **Proposition 21.** *Let $G \in \text{NC}(C_3)$ be a game whose payoff function is defined by (v_0, v_1) . Then, the solution $(id_T, id_A, C_{\text{dev}}(\theta))$ is a \mathcal{C}_Q -Nash equilibrium if and only if for all $(a_i, t_i) \in \{0, 1\}^2$*

$$\frac{1 - p_{\text{win}}^{a_i, t_i}(\theta)}{p_{\text{win}}^{a_i, t_i}(\theta)} \leq \frac{v_{a_i}}{v_{1-a_i}}. \quad (19)$$

Proof. The payoff of a player receiving the pair of advice and type $(a_i, t_i) \in \{0, 1\}^2$ is $v_{a_i} p_{\text{win}}^{a_i, t_i}(\theta)$ when they follow the advice (i.e., output a_i), and $(1 - p_{\text{win}}^{a_i, t_i}(\theta))v_{1-a_i}$ when they alter their behaviour and instead output $1 - a_i$. Therefore, the solution is an equilibrium if and only if for all $(a_i, t_i) \in \{0, 1\}^2$

$$(1 - p_{\text{win}}^{a_i, t_i}(\theta))v_{1-a_i} \leq v_{a_i} p_{\text{win}}^{a_i, t_i}(\theta). \quad (20)$$

The claim follows immediately from this equation. ◀

For given values of v_0, v_1, θ , one can check the conditions of Proposition 21 and, using SDPs, whether the corresponding quantum solution is a quantum equilibrium. For example, for $v_0 = 1/2, v_1 = 3/2$ and $\theta = 1.7$, the deviated solution is a \mathcal{C}_Q -Nash equilibrium (i.e., $C_{\text{dev}}(\theta) \in Q_{\text{corr}}(G)$) but the quantum solution $(|\Psi_{\text{dev}(\theta)}\rangle\langle\Psi_{\text{dev}(\theta)}|, \mathcal{M}_{\text{dev}(\theta)})$ is not a quantum equilibrium. Furthermore, we can note that the social welfare of the solution is 1.0107 which is above that of the pseudo-telepathic solution (of $\frac{v_0+v_1}{2} = 1$).

As we already discussed, this does not necessarily imply that $C_{\text{dev}}(\theta) \notin Q(G)$; to show this one would also need to show that no other quantum equilibrium strategy induces $C_{\text{dev}}(\theta)$.

6 Optimal social welfare in the two quantum settings

Another way of comparing the two different quantum settings is to study whether equilibria in $Q_{\text{corr}}(G)$ can provide *better* equilibria than those in $Q(G)$. Indeed, even if $Q_{\text{corr}}(G)$ and $Q(G)$ differ, the *quality* of attainable equilibria—in terms of social welfare—might not. More generally, the social welfare is a more relevant measure of how useful different resources are in non-cooperative games. By studying it in these two quantum settings, we gain better insight into the power of quantum resources depending on what kind of access is provided—direct quantum access, or indirect classical access.

In this section we study the problem of optimising the social welfare over \mathcal{C}_Q -Nash equilibria and quantum equilibria solutions, said otherwise, to optimise the social welfare over equilibrium correlations in $Q_{\text{corr}}(G)$ and $Q(G)$, respectively. More formally, we wish to solve the optimisation problem

$$\max_P SW_G(P) = \frac{1}{n} \sum_{a,t} \sum_i u_i(a,t) P(a|t) \Pi(t), \quad (21)$$

where the maximisation is either over $P \in Q_{\text{corr}}(G)$ or $Q(G)$, respectively.

One can check directly whether a given distribution P is in $Q_{\text{corr}}(G)$ by verifying the (finite) set of linear constraints defined by the Nash equilibrium condition of Proposition 9. However, there is no simple characterisation of the set \mathcal{C}_Q of quantum correlations, let alone of $Q_{\text{corr}}(G)$, making solving exactly this optimisation problem difficult in this case. Instead, here we make use of the NPA hierarchy [19] which provides a convergent set of necessary conditions, each which can be expressed as a semidefinite program, leading to increasingly better approximations of the set of quantum correlations.

By optimising over distributions P compatible with a given “level” of the NPA hierarchy and additionally imposing the linear constraints for P to be a Nash equilibrium, we can obtain increasingly better upper bounds on the social welfare of correlations in $Q_{\text{corr}}(G)$ using semidefinite programming.

For the second setting, determining whether a quantum solution (ρ, \mathcal{M}) is an equilibrium—and thus whether the induced distribution $P \in Q(G)$ —is itself an SDP, as discussed earlier. The full optimisation problem of Eq. (21) nonetheless remains highly nonlinear, and moreover one can no longer simply impose the equilibrium conditions at the level of the correlations on the NPA hierarchy. Finding nontrivial upper bounds on the social welfare of correlations in $Q(G)$ is thus a complicated problem, and remains one of the main barriers to proving a separation between the two settings.

Instead, we can study the tightness of the upper bounds we obtain for $Q_{\text{corr}}(G)$ and the potential separation between the settings by optimising explicit quantum strategies. In particular, we can use a see-saw iterative optimisation approach [23], alternating between optimising the social welfare over the state ρ and the measurements of different individual parties. More precisely, we can rewrite Eq. (21) as an optimisation over the solutions as

$$\max_{\mathcal{M}^{(1)}} \cdots \max_{M^{(n)}} \max_{\rho} SW_G(P) = \frac{1}{n} \sum_{a,t} \sum_i u_i(a,t) \text{tr} \left[\rho (M_{a_1|t_1}^{(1)} \otimes \cdots \otimes M_{a_n|t_n}^{(n)}) \right] \Pi(t). \quad (22)$$

Starting from different, randomly chosen solutions (for a given dimension for each Hilbert space \mathcal{H}_i), we iteratively solve one maximisation problem at a time while keeping the other parts of the solution fixed. Each iteration is itself an SDP, and while the procedure is not guaranteed to converge, in practice, by iterating sufficiently many times, such see-saw algorithms perform well. Note that while the algorithm optimises the social welfare over solutions whose induced probability distribution is in \mathcal{C}_Q , it does not guarantee that these solutions are equilibria in either $Q(G)$ or $Q_{\text{corr}}(G)$.

To ensure that an equilibrium condition is verified, this initial see-saw optimisation can be complemented by further rounds of iterative optimisation in which equilibrium constraints are imposed. For $Q_{\text{corr}}(G)$, one can add directly the additional condition that no player can increase their payoff by applying a deterministic function to their output. For $Q(G)$, such constraints, however, are themselves expressed as SDPs, and thus cannot be added to each round of optimisation. Instead, one can modify the objective function so that each player optimises their own payoff. While this may lower the overall social welfare, it seeks to bring

the solution towards a quantum equilibrium, and in practice we find the algorithm generally converges to such an equilibrium solution.

This approach is evidently not guaranteed to find a global maximum of the social welfare, but it provides lower bounds on the social welfare of correlations in $Q(G)$ and $Q_{\text{corr}}(G)$. By repeating it on a large number of randomly chosen initial points, we can gain significant insight into the attainable social welfare in the two different settings.

6.1 Numerical results for three families of quantum games

We apply these methods to three families of games: $\text{NC}(C_3)$ as defined in Table 3, as well as $\text{NC}_{00}(C_5)$ and $\text{NC}_{01}(C_5)$ whose questions and winning conditions are summarised in Tables 1 and 2. As elsewhere, we assume the priors Π are uniform, and the payoff functions are defined as in Eq. (1), thus parameterising the families by v_0 and v_1 . Since the overall scale of the social welfare is unimportant and determined by $v_0 + v_1$, there is effectively one free parameter of interest, $v_0/(v_0 + v_1)$, and in practice we fix $v_0 + v_1 = 2$.

The lower bounds obtained via the see-saw approach are obtained by running the algorithm five times with five different, randomly chosen, initial solutions (ρ, \mathcal{M}) , where we took ρ to be an n -qubit state, and keeping the highest social welfare amongst the results. In most cases, the best solution we obtained is equivalent to that found by taking, as an initial seed, measurements such that each player measures in the computational basis when receiving the type $t_i = 0$ and in the Hadamard basis when $t_i = 1$.

The results obtained from the two approaches (upper bounds from the NPA hierarchy² and see-saw lower bounds) for each family of games are shown in Figure 1 as a function of $v_0/(v_0 + v_1)$. For reference, we also show the best social welfare attainable by “classical” equilibria, i.e., those obtainable with local correlations as advice and thus inducing equilibrium correlations in $\text{Corr}(G)$, as well as the social welfare of the pseudo-telepathic solutions on the range of parameters for which they remain quantum equilibria.³

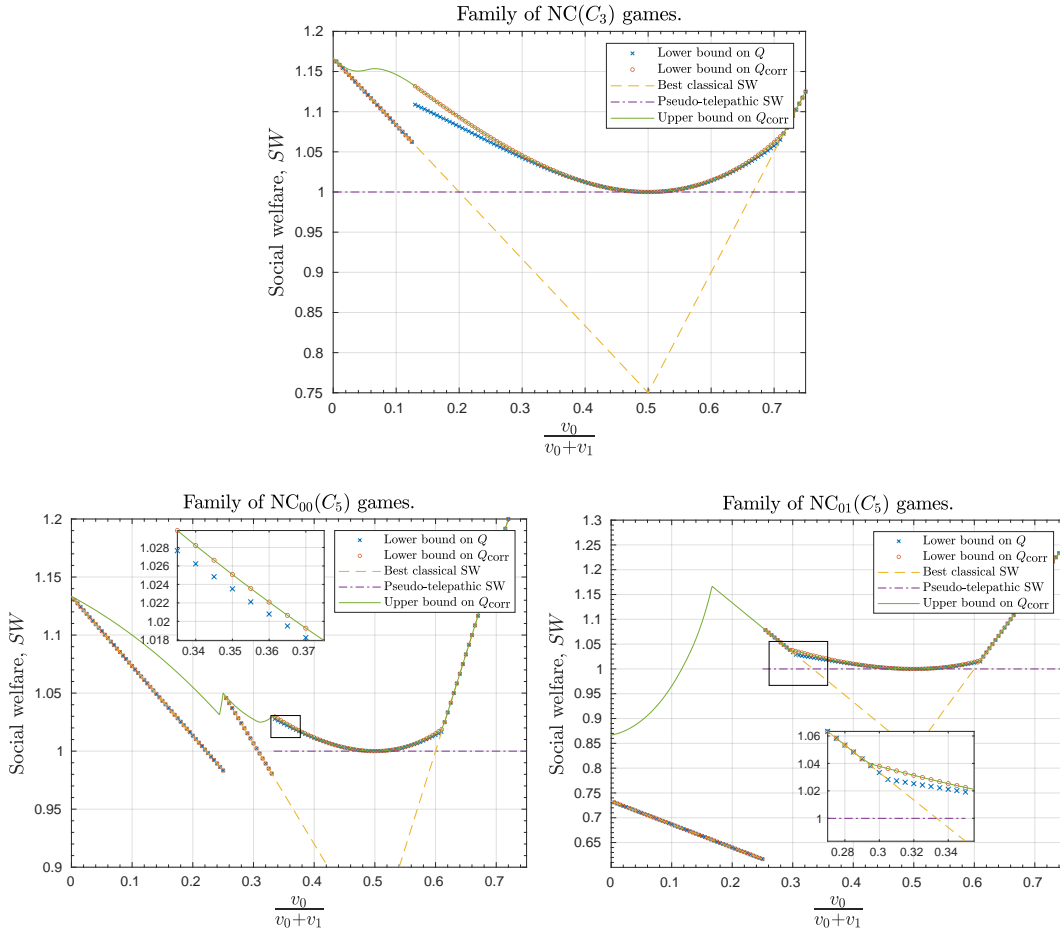
Firstly, note that the social welfare of a given classical equilibrium solution is a linear function of v_0 . The changes of slope in the social welfare of optimal classical strategies correspond to a particular strategy no longer being an equilibrium for certain values of $v_0/(v_0 + v_1)$. For example, for the family $\text{NC}(C_3)$ the best classical equilibrium in the range $[0, 0.5]$ is $(id, id, 1)$ (and its symmetric equivalents), with 1 representing the strategy where a player always answers 1. However, this solution is no longer an equilibrium when the ratio $v_0/(v_0 + v_1)$ goes above 0.5, and the optimal equilibrium is then $(0, 0, 0)$, where 0 represents the strategy of always answering 0.

The first interesting feature we observe is that when the games are biased, i.e., when $v_0 \neq v_1$, the see-saw approach allows us to find explicit equilibria in both $Q_{\text{corr}}(G)$ and $Q(G)$ that have better social welfare than that of the pseudo-telepathic ones, showing that players can obtain a higher payoff if they accept losing the game sometimes.

In general, we can discern different behaviours for the upper and lower bounds, which we can separate into three regimes. In the extremes, when the ratio $v_0/(v_0 + v_1)$ is big enough, the best social welfare is reached by a classical equilibrium and is tight with the NPA upper

² We used a natural generalisation of the “ $1 + AB$ ” intermediate level of the hierarchy to consider all monomials of order 1 and all products of zero or one measurement operator from each party [19]. E.g., for $n = 3$ one might call this level “ $1 + AB + AC + BC + ABC$ ”.

³ For these families of games, the pseudo-telepathic solutions can readily be checked to be quantum equilibria for the same range of parameters as they lead to quantum correlated equilibria.



■ **Figure 1** Numerically obtained upper and lower bounds on the social welfare attainable with quantum resources, as well as the social welfare of the best classical equilibria and of pseudo-telepathic solutions. The plots are truncated to the regions where the classical equilibria are not optimal.

bound, meaning that access to a quantum device of any kind cannot be used to improve the social welfare of the players.

A second regime is observed in the region around $v_0/(v_0 + v_1) = 1/2$. While at $v_0/(v_0 + v_1) = 1/2$ the pseudo-telepathic solution is optimal (and better than classical ones), as one moves away from this point, the see-saw approach provided equilibrium correlations in $Q_{\text{corr}}(G)$ that coincide with the upper bound from the NPA hierarchy, and with a social welfare better than that of the pseudo-telepathic solution. We thus found, in this region, the best possible quantum correlated equilibrium solutions in terms of social welfare. These solutions are not, however, quantum equilibria, and the optimal solutions found by the see-saw approach for $Q(G)$ were slightly lower, with a gap to those for $Q_{\text{corr}}(G)$ increasing the further one goes from $v_0/(v_0 + v_1) = 1/2$. The maximal gaps we found for each game are 0.023 for $G = \text{NC}(C_3)$ at $v_0/(v_0 + v_1) = 0.13$, 0.0021 for $G = \text{NC}(C_5)$ at 0.335, and 0.008 for $G = \text{NC}_{01}(C_5)$ at 0.305. These gaps support the idea that the two settings are not equivalent for the players, although confirming this – e.g., by providing tighter upper bounds on $Q(G)$ – remains an open problem.

The last regime appears when the ratio $v_0/(v_0 + v_1)$ is small. Then, the best solutions

obtained with the see-saw approach collapse to the classical ones and a large gap is found between the social welfare of these explicit solutions and the upper bound from the NPA hierarchy. This raises the question of whether the classical solutions are optimal in this regime, and whether we might tighten the upper bound by increasing the level of the NPA hierarchy used to obtain the upper bounds.

7 Conclusions and perspectives

It was previously shown that for a specific type of non-cooperative multipartite game, pseudo-telepathic quantum solutions can increase the social welfare of players beyond what is obtainable with classical strategies [12]. Here, we showed that the social welfare can be further improved in games with unbalanced (i.e., biased) payoffs if we relax the pseudo-telepathic condition, meaning that the players can win more if they accept losing in some rounds of the game.

To understand further the utility of quantum resources in such games, we also introduced a new class of correlated advice based on quantum correlations. This allowed us to compare quantum resources in two settings: one in which they have direct access to quantum devices and can freely choose what measurement to perform, and another in which the players delegate their quantum measurement and have essentially only classical access to quantum resources. These two settings induce, for a game G , two sets of equilibrium correlations $Q_{\text{corr}}(G)$ and $Q(G)$ respectively, which fully characterise what equilibrium correlations (and thus also social welfare) are attainable. We showed that $Q(G) \subseteq Q_{\text{corr}}(G)$ and thus can extend the known inclusion structure to: $\text{Nash}(G) \subset \text{Corr}(G) \subset Q(G) \subseteq Q_{\text{corr}}(G) \subset \text{B.I.}(G) \subset \text{Comm}(G)$.

We studied more closely the relation between these two types of quantum resources, showing that there exist quantum solutions that are not quantum equilibria but which induce a correlation C in $Q_{\text{corr}}(G)$.

By applying two optimisation techniques based on SDPs to three families of non-cooperative games we found, for a range of ratios $v_0/(v_0 + v_1)$, explicit quantum correlated equilibria inducing correlations in $Q_{\text{corr}}(G)$ with provably optimal social welfare, and which outperform the best quantum equilibria (inducing correlations in $Q(G)$) found with these techniques. This gap suggests that the two settings are indeed inequivalent for the players, in the sense that they can achieve better social welfare when they do not have the freedom that comes with each controlling fully their own quantum device. Thus, perhaps counterintuitively, restricted “classical” access to quantum resources is actually a stronger resource for non-cooperative games than full quantum access.

Still, further research is needed to fully differentiate the two settings. To rigorously show that the inclusion $Q(G) \subseteq Q_{\text{corr}}(G)$ is indeed strict, one must find a correlation in $Q_{\text{corr}}(G)$ that provably cannot be induced by a quantum equilibrium. One possible approach would be to explore ways to generalise the NPA hierarchy to place upper bounds on correlations in $Q(G)$ directly.

Finally, a natural extension of this work is to consider a third intermediate setting where some players have their own quantum devices, while others have only classical access to some quantum resources. This setting may be particularly relevant for applications in quantum cryptography, where an adversary may have different capabilities than the trusted parties. In the scenario we considered, we found that even though players with quantum devices have more freedom and more strategies, their equilibria are generally worse than classical players with quantum correlated advice. However, in intermediate settings, it may be the case that owning a quantum device is an advantage.

References

- 1 Antonio Acín, Nicolas Brunner, Nicolas Gisin, Serge Massar, Stefano Pironio, and Valerio Scarani. Device-independent security of quantum cryptography against collective attacks. *Physical Review Letters*, 98:230501, 2007.
- 2 Anurag Anshu, Peter Høyer, Mehdi Mhalla, and Simon Perdrix. Contextuality in multipartite pseudo-telepathy graph games. *Journal of Computer and System Sciences*, 107:156–165, 2020.
- 3 Vincenzo Auletta, Diodato Ferraioli, Ashutosh Rai, Giannicola Scarpa, and Andreas Winter. Belief-invariant and quantum equilibria in games of incomplete information. *Theoretical Computer Science*, 895:151–177, 2021.
- 4 Robert J. Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1:67–96, 1974.
- 5 John S. Bell. On the Einstein Podolsky Rosen paradox. *Physics Physique Fizika*, 1:195, 1964.
- 6 Katarzyna Bolonek-Lasoń. Three-player conflicting interest games and nonlocality. *Quantum Information Processing*, 16(8):186, 2017.
- 7 Gilles Brassard, Anne Broadbent, and Alain Tapp. Quantum pseudo-telepathy. *Foundations of Physics*, 35(11):1877–1907, 2005.
- 8 Anne Broadbent, Joseph Fitzsimons, and Elham Kashefi. Universal blind quantum computation. In *2009 50th Annual IEEE Symposium on Foundations of Computer Science*, pages 517–526, Atlanta, GA, USA, October 2009. IEEE.
- 9 Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. Bell nonlocality. *Reviews of Modern Physics*, 86:419, 2014.
- 10 John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. Proposed experiment to test local hidden-variable theories. *Physical Review Letters*, 23:880, 1969.
- 11 Daniel M. Greenberger, Michael A. Horne, and Anton Zeilinger. Going beyond Bell’s theorem. In M. Kafatos, editor, *Bell’s Theorem, Quantum Theory, and Conceptions of the Universe*, pages 69–72. Kluwer, Dordrecht, 1989.
- 12 Berry Groisman, Michael Mc Gettrick, Mehdi Mhalla, and Marcin Pawłowski. How quantum information can improve social welfare. *IEEE Journal on Selected Areas in Information Theory*, 1(2):445–453, 2020.
- 13 Ryszard Horodecki, Pawel Horodecki, Michal Horodecki, and Karol Horodecki. Quantum entanglement. *Reviews of Modern Physics*, 81:865, 2009.
- 14 Yael Tauman Kalai, Ran Raz, and Ron D. Rothblum. How to delegate computations: the power of no-signaling proofs. In *Proceedings of the forty-sixth annual ACM symposium on Theory of computing*, pages 485–494, New York New York, May 2014. ACM.
- 15 Mamoru Kaneko and Kenjiro Nakamura. The Nash social welfare function. *Econometrica*, 47(2):423–435, 1979.
- 16 Pierfrancesco La Mura. Correlated equilibria of classical strategic games with quantum signals. *International Journal of Quantum Information*, 03(01):183–188, March 2005.
- 17 Lluís Masanes, Antonio Acín, and Nicolas Gisin. General properties of nonsignaling theories. *Physical Review A*, 73:012112, 2006.
- 18 Roger B. Myerson. Nash equilibrium and the history of economic theory. *Journal of Economic Literature*, 37:1067–1082, 1999.
- 19 Miguel Navascués, Stefano Pironio, and Antonio Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New Journal of Physics*, 10(7):073013, July 2008.
- 20 Anna Pappa, Niraj Kumar, Thomas Lawson, Miklos Santha, Shengyu Zhang, Eleni Diamanti, and Iordanis Kerenidis. Nonlocality and conflicting interest games. *Physical Review Letters*, 114(2):020401, 2015.
- 21 Arup Roy, Amit Mukherjee, Tamal Guha, Sibasis Ghosh, Some Sankar Bhattacharya, and Manik Banik. Nonlocal correlations: Fair and unfair strategies in bayesian games. *Physical Review A*, 94:032120, 2016.

- 22 Richard Steinberg and Willard I. Zangwill. The prevalence of Braess' paradox. *Transportation Science*, 17(3):301–318, 1983.
- 23 Reinhard F. Werner and Michael M. Wolf. Bell inequalities and entanglement. *Quantum Information and Computation*, 1(3):1–25, 2001.

