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# Homogeneity Based Finite/Fixed-Time Observers for Linear MIMO Systems<sup>†</sup>

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## Summary

The paper is devoted to the problem of finite-time and fixed-time observation of linear multiple input multiple output control systems. The proposed dynamic observers do not require system transformation to a canonical form and guarantee convergence of the observation error to zero in a finite or in a fixed time. It is shown that the observers are robust (in input-to-state sense) against input disturbances and measurement noises. The results are supported with simulation examples.

## KEYWORDS:

State estimation, nonlinear observer, homogeneity, robustness

## 1 | INTRODUCTION

The problem of nonlinear observer design has been widely investigated over the past decades (see, for example, <sup>2,3,4,5,6,7,8</sup>). It is worth noting that a finite/fixed-time convergence of observed states to the real ones is preferable specially for the cases when observers are used for systems with processes strongly restricted by time or for fault detection. A kind of separation principle can be recovered for nonlinear systems with such a converging observer. To achieve finite-time stability of the observation error equation, sliding-mode differentiators <sup>9,10,11</sup> and homogeneity based observers <sup>12,4,6,13,14</sup> may be used.

The present paper, for which <sup>1</sup> is a preliminary version, addresses the problem of generalized homogeneity based finite/fixed-time observers design for linear MIMO plants. In comparison with existing approaches the presented one has the following advantages:

- it is applicable for high-order systems;
- it does not require special canonical forms of the system and output matrices or block decomposition as in <sup>12</sup> (in some cases block decomposition can be accompanied by significant computational errors);
- parameter tuning is based on solution of a system of linear matrix equations and inequalities, which allows to adjust the settling time.

Moreover, due to application of the generalized linear homogeneity the proposed observers are robust (in an input-to-state sense) in the presence of input disturbances and measurement noises.

The paper is organized as follows. The problem statement is introduced in Section 2. Section 3 considers preliminaries used in the paper. The results on finite/fixed-time observer design are presented in Section 4. Finally, illustrative example and conclusions are given in Sections 5 and 6, respectively.

*Notation:*  $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ , where  $\mathbb{R}$  is the field of real numbers; the order relation  $P > 0$  for  $P \in \mathbb{R}^{n \times n}$  means that  $P$  is symmetric and positive definite;  $\|\cdot\|$  denotes a norm in  $\mathbb{R}^n$  and  $\|\cdot\|_P$  denotes weighted Euclidean norm (i.e.  $\|x\|_P = \sqrt{x^T P x}$ ).

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with  $x \in \mathbb{R}^n$  and  $P > 0$ );  $\|A\|_A = \sup_{x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|}$  and  $[A]_A = \inf_{x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|}$  for  $A \in \mathbb{R}^{n \times n}$ ;  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix;  $\exp(A)$  denotes the matrix exponential for  $A \in \mathbb{R}^{n \times n}$  (i.e.,  $\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$ ) and it is the ordinary exponential if  $A$  is a scalar;  $\lambda_i(P)$  denotes  $i^{\text{th}}$  eigenvalue of a matrix  $P \in \mathbb{R}^{n \times n}$ ,  $\lambda_{\max}(P)$  and  $\lambda_{\min}(P)$  denote its maximum and minimum eigenvalues, respectively;  $\Re(\lambda)$  denotes the real part of a complex number  $\lambda$ ;  $\mathcal{L}_{\infty}(\mathbb{R}^p)$  denotes the set of essentially bounded measurable functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}^p$ ; a continuous function  $\sigma : \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+ \cup \{0\}$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ . It belongs to class  $\mathcal{K}_{\infty}$  if it is also unbounded; a continuous function  $\beta : \mathbb{R}_+ \cup \{0\} \times \mathbb{R}_+ \cup \{0\} \rightarrow \mathbb{R}_+ \cup \{0\}$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is decreasing to zero for any fixed  $r \in \mathbb{R}_+$ .

## 2 | PROBLEM STATEMENT

Let us consider the system in the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + d_x(t), \\ y(t) = Cx(t) + d_y(t), \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state variable,  $y \in \mathbb{R}^k$  is the measured output,  $u : \mathbb{R} \rightarrow \mathbb{R}^s$  is the control input,  $d_x, d_y$  denote disturbances and measurement noises, respectively,  $A \in \mathbb{R}^{n \times n}$  is the system matrix,  $B \in \mathbb{R}^{n \times s}$  is the matrix of input gains and the matrix  $C \in \mathbb{R}^{k \times n}$  is the output matrix which links the measured outputs to the state variables. The matrix pair  $(A, C)$  is assumed to be observable and  $\text{rank}(C) = k$ .

The main goal of the paper is to propose homogeneity-based dynamic observers for the system (1) with constructive tuning rules. For non-perturbed case the observers must guarantee finite/fixed-time stability of the error equation. In the presence of  $\mathcal{L}_{\infty}$ -bounded disturbances and measurement noises the observers have to ensure robustness property in the sense of input-to-state stability.

## 3 | PRELIMINARIES

### 3.1 | Stability Notions

Consider the following system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (2)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a vector field.

According to<sup>15</sup> an absolutely continuous function  $x(t, x_0)$  is called a solution of the Cauchy problem associated to (2) if  $x(0, x_0) = x_0$  and it satisfies the following differential inclusion

$$\dot{x} \in K[f](x) = co \bigcap_{\epsilon > 0} \bigcap_{\mu(N)=0} f(B(x, \epsilon) \setminus N), \quad (3)$$

where  $co(M)$  defines the convex closure of the set  $M$ ,  $B(x, \epsilon)$  is the ball with the center at  $x \in \mathbb{R}^n$  and the radius  $\epsilon$ , the equality  $\mu(N) = 0$  means that the measure of  $N \subset \mathbb{R}^n$  is zero. Let the origin be an equilibrium point of (3).

Note that the system (2) may have non-unique solutions for a given  $x_0 \in \mathbb{R}^n$  and may admit both weak (a property holds for a solution) and strong stability (a property holds for **all solutions**) (see, for example, <sup>15,16</sup>). This paper deals only with the strong stability properties, which ask for stable behavior of all solutions of the system (2).

**Definition 1** (See the works <sup>17,18</sup>). The origin of (2) is said to be globally finite-time stable if it is globally uniformly asymptotically stable and there exists a locally bounded function  $T : \mathbb{R}^n \rightarrow [0, +\infty)$  such that any solution  $x(t, x_0)$  of the system (2) satisfies  $x(t, x_0) = 0, \forall t \geq T(x_0)$ .

The function  $T$  is called the settling-time estimate (it can be not unique).

**Definition 2** (See the work <sup>19</sup>). The origin of (2) is said to be fixed-time stable if it is globally finite-time stable and there exists a bounded function  $T(x_0)$ , i.e.,  $\exists T_{\max} > 0 : T(x_0) \leq T_{\max}, \forall x_0 \in \mathbb{R}^n$ .

Let us give definitions of input-to-state stability notions that are widely used for robustness analysis of nonlinear systems. Consider the system

$$\dot{x}(t) = \tilde{f}(x(t), d(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (4)$$

where  $x \in \mathbb{R}^n$ ,  $d \in \mathcal{L}_\infty(\mathbb{R}^p)$  is a disturbance and  $\tilde{f} \in \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  is a continuous or discontinuous vector field that satisfies Filippov conditions<sup>15</sup>.

**Definition 3** (See the work of Dashkovskiy et al<sup>20</sup>). The system (4) is called

- **Input-to-state stable** (ISS), if there exist functions  $\zeta \in \mathcal{KL}$  and  $\vartheta \in \mathcal{K}$  such that for any  $d \in \mathcal{L}_\infty(\mathbb{R}^p)$  and any  $x_0 \in \mathbb{R}^n$

$$\|x(t, t_0, d)\| \leq \zeta(\|x_0\|, t) + \vartheta(\|d\|_{[0,t]}), \quad \forall t \geq 0.$$

- **Integral Input-to-state stable** (iISS), if there are some functions  $\vartheta_1, \vartheta_2 \in \mathcal{K}_\infty$  and  $\zeta \in \mathcal{KL}$  such that for any  $d \in \mathcal{L}_\infty(\mathbb{R}^p)$  and any  $x_0 \in \mathbb{R}^n$  the following estimate holds:

$$\vartheta_1(\|x(t, t_0, d)\|) \leq \zeta(\|x_0\|, t) + \int_0^t \vartheta_2(\|d(s)\|) ds, \quad \forall t \geq 0.$$

### 3.2 | Implicit Lyapunov Function Method

The theorems given below present the Implicit Lyapunov Function (ILF) method for finite-time and fixed-time stability analysis.

**Theorem 1** (See the work<sup>21</sup>). If there exists a continuous function

$$\begin{aligned} Q &: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R} \\ (V, x) &\mapsto Q(V, x) \end{aligned}$$

such that

C1)  $Q(V, x)$  is continuously differentiable for  $\forall x \in \mathbb{R}^n \setminus \{0\}$  and  $\forall V \in \mathbb{R}_+$ ;

C2) for any  $x \in \mathbb{R}^n \setminus \{0\}$  there exist  $V^- \in \mathbb{R}_+$  and  $V^+ \in \mathbb{R}_+$ :

$$Q(V^-, x) < 0 < Q(V^+, x); \quad (5)$$

C3) for  $\Omega = \{(V, x) \in \mathbb{R}^{n+1} : Q(V, x) = 0\}$

$$\lim_{\substack{x \rightarrow 0 \\ (V, x) \in \Omega}} V = 0^+, \quad \lim_{\substack{V \rightarrow 0^+ \\ (V, x) \in \Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\| \rightarrow \infty \\ (V, x) \in \Omega}} V = +\infty;$$

C4) the inequality

$$-\infty < \frac{\partial Q(V, x)}{\partial V} < 0$$

holds for  $\forall V \in \mathbb{R}_+$  and  $\forall x \in \mathbb{R}^n \setminus \{0\}$ ;

C5) the inequality

$$\frac{\partial Q(V, x)}{\partial x} f(x) \leq \sigma V^{1-\mu} \frac{\partial Q(V, x)}{\partial V}$$

holds  $\forall (V, x) \in \Omega$ , where  $0 < \mu \leq 1$  and  $\sigma \in \mathbb{R}_+$  are some constants.

Then the origin of the system (2) is globally finite-time stable with the following settling time estimate

$$T(x_0) \leq \frac{V_0^\mu}{\sigma \mu},$$

where  $V_0 \in \mathbb{R}_+ : Q(V_0, x_0) = 0$ .

The conditions C1, C2, C4 and the implicit function theorem<sup>22</sup> imply that the equation  $Q(V, x) = 0$  implicitly defines a unique function  $V : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+$  such that  $Q(V(x), x) = 0$  for all  $x \in \mathbb{R}^n \setminus \{0\}$ . Due to the condition C3 the function  $V$  can be continuously prolonged at the origin by setting  $V(0) = 0$ . In addition, it is radially unbounded and positive definite.

The next extension allows to determine fixed-time stability using the ILF method.

**Theorem 2** (See the work<sup>21</sup>). Let there exist two functions  $Q_1 : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$  and  $Q_2(V, x) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}$  that satisfy the conditions C1-C4 of Theorem 1. If

C6) the equality

$$Q_1(1, x) = Q_2(1, x)$$

holds for all  $x \in \mathbb{R}^n \setminus \{0\}$ ;

C7) there exist  $c_1$  and  $\mu_1 \in (0, 1]$  such that the inequality

$$\frac{\partial Q_1(V, x)}{\partial x} f(x) \leq c_1 V^{1-\mu_1} \frac{\partial Q_1(V, x)}{\partial V}$$

holds for all  $V \in (0, 1]$  and all  $x \in \{z \in \mathbb{R}^n \setminus \{0\} : Q_1(V, z) = 0\}$ ;

C8) there exist  $c_2, \mu_2 \in \mathbb{R}_+$  such that the inequality

$$\frac{\partial Q_2(V, x)}{\partial x} f(x) \leq c_2 V^{1+\mu_2} \frac{\partial Q_2(V, x)}{\partial V}$$

holds for all  $V \geq 1$  and all  $x \in \{z \in \mathbb{R}^n \setminus \{0\} : Q_2(V, z) = 0\}$ ,

then the origin of the system (2) is globally fixed-time stable with the following settling time estimate

$$T(x_0) \leq \frac{1}{c_1 \mu_1} + \frac{1}{c_2 \mu_2}.$$

### 3.3 | Generalized Homogeneity

The (standard) homogeneity is a symmetry of an object with respect to a group of dilations ( $x \rightarrow e^s x$ , where  $s \in \mathbb{R}$ ,  $x \in \mathbb{R}^n$ ). The type of homogeneity dealing with non-uniform dilations is called generalized homogeneity<sup>23</sup>.

**Definition 4** (See the works<sup>24,25</sup>). A map  $\mathbf{d} : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  is called a linear dilation in the space  $\mathbb{R}^n$  if it satisfies:

- **group property:**  $\mathbf{d}(0) = I_n$  and  $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s) = \mathbf{d}(s)\mathbf{d}(t)$  for  $t, s \in \mathbb{R}$ ;
- **continuity property:**  $\mathbf{d}$  is a continuous map;
- **limit property:**  $\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)x\| = 0$  and  $\lim_{s \rightarrow +\infty} \|\mathbf{d}(s)x\| = +\infty$  uniformly on the unit sphere  $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ .

In this paper we deal only with linear dilations and omit the word "linear" for shortness. The dilation  $\mathbf{d}$  is a uniformly continuous group. Its generator is a matrix  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  defined by<sup>26</sup>

$$G_{\mathbf{d}} = \lim_{s \rightarrow 0} \frac{\mathbf{d}(s) - I_n}{s}.$$

The generator  $G_{\mathbf{d}}$  satisfies the following properties

$$\frac{d}{ds} \mathbf{d}(s) = G_{\mathbf{d}} \mathbf{d}(s) = \mathbf{d}(s) G_{\mathbf{d}}, \quad (6)$$

$$\mathbf{d}(s) = e^{G_{\mathbf{d}} s} = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \quad (7)$$

where  $s \in \mathbb{R}$ .

**Definition 5** (See the works<sup>24,27</sup>). The dilation  $\mathbf{d}$  is said to be strictly monotone if  $\exists \beta \in \mathbb{R}_+$  such that  $\|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\beta s}$  for  $s \leq 0$ .

Thus, monotonicity means that  $\mathbf{d}(s)$  is a strong contraction for  $s < 0$  (strong expansion for  $s > 0$ ) and implies that for any  $x \in \mathbb{R} \setminus \{0\}$  there exists a unique pair  $(s_0, x_0) \in \mathbb{R}_+ \times S$  such that  $x = \mathbf{d}(s_0)x_0$ .

Note, that monotonicity property may depend on a norm  $\|\cdot\|$ <sup>24</sup>.

**Theorem 3** (See the work of Polyakov<sup>27</sup>). If  $\mathbf{d}$  is a dilation in  $\mathbb{R}^n$ , then

- the generator matrix  $G_{\mathbf{d}}$  is anti-Hurwitz, i.e.

$$\Re(\lambda_i(G_{\mathbf{d}})) > 0, \quad i = 1, \dots, n$$

and there exists a matrix  $P \in \mathbb{R}^{n \times n}$  such that

$$P G_{\mathbf{d}} + G_{\mathbf{d}}^T P > 0, \quad P > 0. \quad (8)$$

- the dilation  $\mathbf{d}$  is strictly monotone with respect to the weighted Euclidean norm  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$  induced by the inner product  $\langle x, z \rangle = x^T P z$  with  $P$  satisfying (8), i.e.,

$$\begin{aligned} e^{\alpha s} &\leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\beta s} & \text{if } s \leq 0, \\ e^{\beta s} &\leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\alpha s} & \text{if } s \geq 0, \end{aligned} \quad (9)$$

where  $\alpha = \frac{1}{2}\lambda_{\max}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^TP^{\frac{1}{2}}\right)$ ,  $\beta = \frac{1}{2}\lambda_{\min}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^TP^{\frac{1}{2}}\right)$ .

**Definition 6** (See the work of Kawski<sup>28</sup>). A vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ ) is said to be **d**-homogeneous of degree  $\nu \in \mathbb{R}$  if

$$\begin{aligned} f(\mathbf{d}(s)x) &= e^{\nu s} \mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall s \in \mathbb{R}. \\ (\text{resp. } g(\mathbf{d}(s)x) &= e^{\nu s} g(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall s \in \mathbb{R}). \end{aligned} \quad (10)$$

A special case of homogeneous function is a homogeneous norm<sup>28</sup>: a continuous positive definite **d**-homogeneous function of degree 1. Define the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \rightarrow \mathbb{R}_+$  as  $\|x\|_{\mathbf{d}} = e^{s_x}$ , where  $s_x \in \mathbb{R}$  such that  $\|\mathbf{d}(-s_x)x\| = 1$ . Note that  $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s \|x\|_{\mathbf{d}}$  and

$$\|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x\| = 1. \quad (11)$$

The following lemma gives the necessary and sufficient condition of **d**-homogeneity of linear systems with nonzero degree.

**Lemma 1** (See the work<sup>29</sup>). Let  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  be a generator of the dilation  $\mathbf{d}(s) = e^{G_{\mathbf{d}}s}$ ,  $s \in \mathbb{R}$ . Then the linear system  $\dot{x} = Cx$ ,  $x \in \mathbb{R}^n$ ,  $C \in \mathbb{R}^{n \times n}$  is **d**-homogeneous of degree  $\nu \in \mathbb{R}$  if and only if

$$CG_{\mathbf{d}} - G_{\mathbf{d}}C = \nu C. \quad (12)$$

The **d**-homogeneity property used in Definition 6 is introduced for some generator  $G_{\mathbf{d}}$  and all  $s \in \mathbb{R}$ . Restricting the set of admissible values for  $s$ , the local homogeneity can be introduced.

**Definition 7.** Let  $d_0$  be a dilation in  $\mathbb{R}^n$ ,  $s_0 \in \mathbb{R} \cup_{\pm\infty}$  and  $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a  $d_0$ -homogeneous vector field in  $\mathbb{R}^n$ . A vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $(\mathbf{d}_0, s_0, f_0)$ -homogeneous with degree  $\nu_0$  if

$$\lim_{s \rightarrow s_0} [e^{-\nu s} \mathbf{d}_0(-s)f(\mathbf{d}_0(s)x) - f_0(x)] = 0,$$

for all  $x \in S_0 = \{z \in \mathbb{R}^n : \|z\|_{\mathbf{d}_0} = 1\}$  if  $s_0 \in \mathbb{R}$  and uniformly on  $S_0$  if  $s_0 \in \{-\infty, +\infty\}$ .

The sign of homogeneity degree of asymptotically stable systems determines the type of convergence: negative homogeneity degree implies finite-time stability but positive homogeneity degree guarantees nearly fixed-time stability<sup>30</sup>.

The homogeneity theory provides many other advantages to analysis and design of nonlinear control system. For instance, some results about ISS of homogeneous systems can be found in<sup>31,32,33</sup>. Let us develop the result of<sup>32</sup> about ISS stability for generalized homogeneity case.

Let

$$\begin{aligned} \|\tilde{f}(y, d) - \tilde{f}(y, 0)\| &\leq \sigma(\|d\|), \quad \sigma(s) = \begin{cases} cs^{\varrho_{\max}} & \text{if } s \geq 1, \\ cs^{\varrho_{\min}} & \text{if } s < 1, \end{cases} \\ \forall y \in S, \quad d \in \mathbb{R}^p \end{aligned} \quad (13)$$

be satisfied for some  $c \in \mathbb{R}_+$ ,  $\varrho_{\max} \geq \varrho_{\min} > 0$ . Then following<sup>32</sup> one may obtain:

**Theorem 4.** Define  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ ,  $\Re(\lambda_i(G_{\mathbf{d}})) > 0$ ,  $i = 1, \dots, n$  and  $G_{\tilde{\mathbf{d}}} \in \mathbb{R}^{p \times p}$ ,  $\Re(\lambda_j(G_{\tilde{\mathbf{d}}})) \geq 0$ ,  $j = 1, \dots, p$ . Let  $\tilde{f}(\mathbf{d}(s)x, \tilde{\mathbf{d}}(s)d) = \exp(\eta s)\mathbf{d}(s)\tilde{f}(x, d)$  for all  $x \in \mathbb{R}^n$ ,  $d \in \mathbb{R}^p$  and all  $s \in \mathbb{R}$  with degree  $\eta \geq -\min_{1 \leq i \leq n} \Re(\lambda_i(G_{\mathbf{d}}))$ . Assume that the system (4) is globally asymptotically stable for  $d = 0$ , then the system (4) is

$$\text{ISS if } \min_{1 \leq j \leq p} \Re(\lambda_j(G_{\tilde{\mathbf{d}}})) > 0$$

$$\text{iISS if } \min_{1 \leq j \leq p} \Re(\lambda_j(G_{\tilde{\mathbf{d}}})) = 0 \text{ and } \eta \leq 0.$$

*Proof.* Under the introduced conditions  $\tilde{f}(\mathbf{d}(s)x, 0) = \exp(\eta s)\mathbf{d}(s)\tilde{f}(x, 0)$  and the system  $\dot{x} = \tilde{f}(x, 0)$  is globally asymptotically stable, therefore according to<sup>34,27</sup> there exists a continuously differentiable, positive definite and radially unbounded Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$  such that  $V(\mathbf{d}(s)x) = \exp(\kappa s)V(x)$  for any  $s \in \mathbb{R}$  and  $\kappa > \max_{1 \leq i \leq n} \Re(\lambda_i(G_{\mathbf{d}}))$  and

$$\frac{\partial V(y)}{\partial y} \tilde{f}(y, 0) \leq -a, \quad \left\| \frac{\partial V(y)}{\partial y} \right\| \leq b \quad \forall y \in S,$$

where  $a, b \in \mathbb{R}_+$ . Note that in this case  $\eta + \kappa > 0$ .

Let us use the coordinate transformation  $x = \mathbf{d}(\ln \|x\|_{\mathbf{d}})y$ , which connects any  $x \in \mathbb{R}^n \setminus \{0\}$  with the corresponding point  $y \in S$ . Note that

$$\|\tilde{\mathbf{d}}(-\ln \|x\|_{\mathbf{d}})d\| \leq \rho(\|x\|_{\mathbf{d}})\|d\| \quad \rho(s) = \begin{cases} s^{-\tilde{\beta}} & \text{if } s \geq 1, \\ s^{-\tilde{\alpha}} & \text{if } s < 1, \end{cases}$$

where  $\tilde{\alpha} \geq \tilde{\beta} \geq 0$  are obtained similarly to  $\alpha, \beta$  in (9) with  $\|d\|_{\tilde{P}} = \sqrt{d^T \tilde{P} d}$ ,  $\tilde{P} \in \mathbb{R}^{p \times p}$  and

$$\tilde{P} G_{\tilde{d}} + G_{\tilde{d}}^T \tilde{P} \geq 0, \quad \tilde{P} > 0.$$

The case  $\tilde{\beta} = 0$  corresponds to  $\min_{1 \leq j \leq p} \Re(\lambda_j(G_{\tilde{d}})) = 0$ . Then let us consider the time derivative of the Lyapunov function  $V$  computed for the system (4) for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{R}^p$ :

$$\begin{aligned} \dot{V} &= \|x\|_{\tilde{d}}^{\eta+\kappa} \frac{\partial V(y)}{\partial y} \tilde{f}(y, \tilde{d}(-\ln \|x\|_{\tilde{d}})d) \\ &= \|x\|_{\tilde{d}}^{\eta+\kappa} \frac{\partial V(y)}{\partial y} \tilde{f}(y, 0) + \|x\|_{\tilde{d}}^{\eta+\kappa} \frac{\partial V(y)}{\partial y} [\tilde{f}(y, \tilde{d}(-\ln \|x\|_{\tilde{d}})d) - \tilde{f}(y, 0)] \\ &\leq -a\|x\|_{\tilde{d}}^{\eta+\kappa} + b\|x\|_{\tilde{d}}^{\eta+\kappa} \sigma(\|\tilde{d}(-\ln \|x\|_{\tilde{d}})d\|) \\ &\leq -a\|x\|_{\tilde{d}}^{\eta+\kappa} + b\|x\|_{\tilde{d}}^{\eta+\kappa} \sigma(\rho(\|x\|_{\tilde{d}})\|d\|). \end{aligned}$$

Therefore if  $\tilde{\beta} > 0$  ( $\min_{1 \leq j \leq p} \Re(\lambda_j(G_{\tilde{d}})) > 0$ ), then  $\rho(s)^{-1} \in \mathcal{K}_{\infty}$  and for  $\|d\| \leq \frac{\sigma^{-1}[a/(2bc)]}{\rho(\|x\|_{\tilde{d}})}$  we have  $\dot{V} \leq -0.5a\|x\|_{\tilde{d}}^{\eta+\kappa}$  that implies ISS. If  $\varrho_{\min} \tilde{\alpha} - \kappa \leq \eta \leq \varrho_{\max} \tilde{\beta}$ , then

$$\begin{aligned} \|x\|_{\tilde{d}}^{\eta+\kappa} \sigma(\rho(\|x\|_{\tilde{d}})\|d\|) &\leq \sigma(\|d\|)v(\|x\|_{\tilde{d}}), \\ v(s) &= \begin{cases} 1 & \text{if } s < 1, \\ s^{\kappa} & \text{if } s \geq 1 \end{cases} \end{aligned}$$

and for the Lyapunov function  $W(x) = \ln[1 + V(x)]$  (it is continuously differentiable, positive definite and radially unbounded) we obtain

$$\begin{aligned} \dot{W} &\leq \frac{-a\|x\|_{\tilde{d}}^{\eta+\kappa} + bv(\|x\|_{\tilde{d}})\sigma(\|d\|)}{1+V(x)} \\ &\leq -a \frac{\|x\|_{\tilde{d}}^{\eta+\kappa}}{1+V(x)} + \tilde{b}\sigma(\|d\|), \quad \tilde{b} > 0, \end{aligned}$$

which implies iISS for (4). Since  $\kappa > \max_{1 \leq i \leq n} \Re(\lambda_i(G_{\tilde{d}}))$  can be chosen arbitrary, then we may assume that the inequality  $\varrho_{\min} \tilde{\alpha} - \kappa \leq \eta$  is always satisfied. Since for  $\min_{1 \leq j \leq p} \Re(\lambda_j(G_{\tilde{d}})) > 0$  the system is ISS (and iISS by the definition), the only interesting admissible value is  $\min_{1 \leq j \leq p} \Re(\lambda_j(G_{\tilde{d}})) = 0$ , then  $\eta$  has to be non positive.  $\square$

For the case  $\min_{1 \leq j \leq p} \Re(\lambda_j(G_{\tilde{d}})) = 0$  a relaxed constraint  $\varrho_{\min} \geq 0$  has to be satisfied, which follows the continuity of  $f$ .

In the case  $f$  is continuous with respect to both arguments, the condition (13) is satisfied. Note that only continuity with respect to the second argument is not enough. For example, the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 \arctan \left( d \begin{cases} 1/x_2, & x_2 \neq 0 \\ 0, & x_2 = 0 \end{cases} \right), \\ \dot{x}_2 &= -x_2 \end{aligned}$$

is continuous with respect to  $d$ , globally asymptotically stable for  $d = 0$ , homogeneous with  $G_{\tilde{d}} = I_2$ ,  $G_{\tilde{d}} = 1$ , and not ISS.

In the case of locally homogeneous systems, ISS stability can be asserted through their homogeneous approximations at  $-\infty$  and at  $+\infty$  similarly to<sup>35</sup>.

**Theorem 5.** Let the vector field  $\tilde{f}$  be continuous and  $((\tilde{d}_{+\infty}, \tilde{d}_{+\infty}), +\infty, \tilde{f}_{+\infty})$ -homogeneous with the generators  $G_{\tilde{d}_{+\infty}}, G_{\tilde{d}_{+\infty}}$  and  $((\tilde{d}_{-\infty}, \tilde{d}_{-\infty}), -\infty, \tilde{f}_{-\infty})$ -homogeneous with the the generators  $G_{\tilde{d}_{-\infty}}, G_{\tilde{d}_{-\infty}}$ . If the origins of the systems  $\dot{x} = \tilde{f}_{-\infty}(x, 0)$ ,  $\dot{x} = \tilde{f}(x, 0)$  and  $\dot{x} = \tilde{f}_{+\infty}(x, 0)$  are globally asymptotically stable then the system (4) is ISS.

*Proof.* The proof repeats the arguments of<sup>35</sup>.  $\square$

## 4 | OBSERVER DESIGN

### 4.1 | Finite-time observer

Let us consider the following observer

$$\frac{d}{dt} \hat{x}(t) = A\hat{x}(t) + Bu(t) - g_{FT}(y(t) - C\hat{x}(t)), \quad (14)$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is the observer state vector and

$$g_{FT}(y(t) - C\hat{x}(t)) = L_0 \zeta(t) + \|\tilde{P} \zeta(t)\|^{(v-\epsilon)/\epsilon} \tilde{d} (\ln \|\tilde{P} \zeta(t)\|^{1/\epsilon}) L_{FT} \zeta(t),$$

where  $\varsigma(t) = y(t) - C\hat{x}(t)$ ,  $\mathbf{d}$  is a dilation with the generator  $G_{\mathbf{d}}$ , and  $\tilde{P} \in \mathbb{R}^{k \times k}$ ,  $L_{FT} \in \mathbb{R}^{n \times k}$ ,  $L_0 \in \mathbb{R}^{n \times k}$ ,  $\nu \in [-1, 0)$ ,  $\epsilon \in \mathbb{R}_+$  are parameters of the observer, to be determined. Then the error equation in the disturbance-free case ( $d_x = 0$ ,  $d_y = 0$ ) has the following form

$$\dot{e} = (A + L_0 C + \|\tilde{P} C e\|^{(\nu-\epsilon)/\epsilon} \mathbf{d} (\ln \|\tilde{P} C e\|^{1/\epsilon}) L_{FT} C) e, \quad (15)$$

where  $e = x - \hat{x}$ .

Notice that homogeneous finite-time observers from<sup>13</sup> and<sup>14</sup> are particular cases of (14). Indeed, selecting  $\tilde{P} = I_k$  we derive the observer studied in<sup>14</sup>. If, additionally, for  $L_{FT} = 0.5\gamma P^{-1} C^T$  for some  $\gamma \in \mathbb{R}_+$ ,  $\mathbb{R}^{n \times n} \ni P > 0$  then (14) becomes the observer given in<sup>13</sup>. Such a generalization of the homogeneous observer yields some modifications in stability analysis of the error equation and less conservative stability conditions.

**Theorem 6.** Let  $d_x = 0$ ,  $d_y = 0$  and

- for some  $\nu \in [-1, 0)$  the system of matrix equations and inequalities

$$AH - HA - A - Y_0 C = 0, \quad (16a)$$

$$CH = 0, \quad (16b)$$

$$H + H^T + 2I_n > 0, \quad (16c)$$

be feasible with  $H \in \mathbb{R}^{n \times n}$ ,  $Y_0 \in \mathbb{R}^{n \times k}$ ;

- for  $L_0 = (H + I_n)^{-1} Y_0$  and some  $\xi > \tau > 0$ ,  $\epsilon, \theta \in \mathbb{R}_+$  the system of matrix inequalities

$$\begin{pmatrix} PA + A^T P + PL_0 C + C^T L_0^T P + YC + C^T Y^T + \xi P & P \\ P & -Z \end{pmatrix} \leq 0, \quad (17a)$$

$$P > 0, \quad Z > 0, \quad X > 0, \quad (17b)$$

$$\begin{pmatrix} \tau X & Y^T \\ Y & \theta P \end{pmatrix} \geq 0, \quad (17c)$$

$$P \geq C^T X C, \quad (17d)$$

$$\nu PH + \nu H^T P + 2\epsilon P > 0, \quad (17e)$$

$$\lambda^{2\epsilon} \Xi^T(\lambda) Z \Xi(\lambda) \leq \frac{1}{\theta} P, \quad \forall \lambda \in [0, 1], \quad (17f)$$

be feasible with  $P, Z \in \mathbb{R}^{n \times n}$ ,  $X \in \mathbb{R}^{k \times k}$ ,  $Y \in \mathbb{R}^{n \times k}$  and  $\Xi(\lambda) = \exp((\nu H + \nu I_n) \ln \lambda) - I_n$ .

Then the error equation (15) with  $L_{FT} = P^{-1} Y$ ,  $\tilde{P} = X^{1/2}$ ,  $G_{\mathbf{d}} = \nu H + \epsilon I_n$  is globally finite-time stable.

*Proof.* **I.** Let us show that the system  $\dot{e}(t) = Ae(t) + L_0 Ce(t)$  is  $\mathbf{d}$ -homogeneous of degree  $\nu$ . According to Lemma 1, the system is  $\mathbf{d}$ -homogeneous of degree  $\nu$  if and only if

$$(A + L_0 C)G_{\mathbf{d}} - G_{\mathbf{d}}(A + L_0 C) = \nu(A + L_0 C). \quad (18)$$

Taking into account (16b) and  $L_0 = (H + I_n)^{-1} Y_0$  (the matrix  $(H + I_n)$  is invertible due to (16c)), the equation (16a) is equivalent to  $(A + L_0 C)H - H(A + L_0 C) = A + L_0 C$ . The matrix  $\tilde{H} \in \mathbb{R}^{n \times n}$ ,  $\tilde{H} = H + \frac{\epsilon}{\nu} I_n$  is also its solution for any  $\epsilon \in \mathbb{R}_+$ , i.e.  $(A + L_0 C)(H + \frac{\epsilon}{\nu} I_n) - (H + \frac{\epsilon}{\nu} I_n)(A + L_0 C) = (A + L_0 C)H - H(A + L_0 C)$ . Choose the generator in the form  $G_{\mathbf{d}} = \nu H + \epsilon I_n$ , where the parameter  $\epsilon$  is such that  $G_{\mathbf{d}}$  is anti-Hurwiz according to (17e). Then the equation (18) holds and

$$\mathbf{d}(-s)(A + L_0 C)e = \exp(\nu s)(A + L_0 C)\mathbf{d}(-s)e. \quad (19)$$

**II.** The equality

$$L_{FT} C e = \exp(-\epsilon s) L_{FT} C \mathbf{d}(s) e \quad (20)$$



holds. To prove this, it is enough to show that

$$C(\exp(-\epsilon s)\mathbf{d}(s) - I_n) = 0. \quad (21)$$

Since

$$\begin{aligned} C(\exp(-\epsilon s)\mathbf{d}(s) - I_n) &= C(\exp((G_{\mathbf{d}} - \epsilon I_n)s)\mathbf{d}(s) - I_n) \\ &= C \sum_{i=1}^{+\infty} \frac{s^i (G_{\mathbf{d}} - \epsilon I_n)^i}{i!} \end{aligned}$$

the equality (21) holds if  $C(G_{\mathbf{d}} - \epsilon I_n) = 0$  which is implied by (16b) with  $G_{\mathbf{d}} = \nu H + \epsilon I_n$ . Then, using (21) it is easy to show that (20) holds.

**III.** Choose ILF in the form  $Q = \|\mathbf{d}(-\ln V)e\|_P^2 - 1$  with  $0 < P = P^T \in \mathbb{R}^{n \times n}$ . Note that for  $(V, e) \in \Omega$  we have  $V = \|e\|_{\mathbf{d}}$ . Let us denote

$$\lambda = \frac{\|\tilde{P}Ce\|^{1/\epsilon}}{V} \quad (22)$$

and show that  $0 \leq \lambda \leq 1$  for  $(V, e) \in \Omega$  if (17d) holds. Since  $\mathbf{d}(s)\mathbf{d}(-s) = I_n$  and  $C\mathbf{d}(s) = C \exp(\epsilon s)$  (according to (21)) for any  $s \in \mathbb{R}$  and (21), then taking into account (17d), we have

$$\begin{aligned} \|\tilde{P}Ce\| &= \|\tilde{P}C\mathbf{d}(\ln V)\mathbf{d}(-\ln V)e\| \\ &= V^\epsilon \|\tilde{P}C\mathbf{d}(-\ln V)e\| \\ &= V^\epsilon (e^T \mathbf{d}^T(-\ln V) C^T X C \mathbf{d}(-\ln V)e)^{1/2} \\ &\leq V^\epsilon (e^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V)e)^{1/2} \\ &= V^\epsilon. \end{aligned}$$

Then substituting this estimate to (22) we obtain  $\lambda \leq 1$ .

**IV.** Let us show that the function  $Q$  satisfies the conditions C1-C5 of Theorem 1. It is continuously differentiable on  $\mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\}$ . According to (9) we have

$$\min\{V^{-\alpha}, V^{-\beta}\} \|z\|_P \leq Q(V, z) + 1 \leq \max\{V^{-\alpha}, V^{-\beta}\} \|z\|_P$$

that imply for any  $z \in \mathbb{R}^n \setminus \{0\}$  there exist  $V^- \in \mathbb{R}_+$  and  $V^+ \in \mathbb{R}_+$  :  $Q(V^-, z) < 0 < Q(V^+, z)$ . Moreover, if  $Q(V, z) = 0$  then, obviously, the condition C3 of Theorem 1 holds. Since  $\frac{\partial Q}{\partial V} = -V^{-1} z^T \mathbf{d}^T(-\ln V)(PG_{\mathbf{d}} + G_{\mathbf{d}}^T P)\mathbf{d}(-\ln V)z$  then (17e) implies  $\frac{\partial Q}{\partial V} < 0$  for  $\forall V \in \mathbb{R}_+$  and  $z \in \mathbb{R}^n \setminus \{0\}$ . So the condition C4 of Theorem 1 also holds. Therefore, the equation  $Q(V, z) = 0$  implicitly defines a positive definite Lyapunov function candidate  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ .

Denote by  $\partial Q^e$  the partial derivative of  $Q$  along (15). Taking into account (19) and (20) we derive:

$$\begin{aligned} \partial Q^e &= 2e^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V) (A + L_0 C + \|\tilde{P}Ce\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln \|\tilde{P}Ce\|^{1/\epsilon}) L_{FT} C) e \\ &= V^\nu \left( \begin{pmatrix} \mathbf{d}(-\ln V)e \\ \Xi(\lambda) L_{FT} C \mathbf{d}(-\ln V)e \end{pmatrix}^T \mathcal{Y} \begin{pmatrix} \mathbf{d}(-\ln V)e \\ \Xi(\lambda) L_{FT} C \mathbf{d}(-\ln V)e \end{pmatrix} \right. \\ &\quad \left. - \xi e^T \mathbf{d}(-\ln V)^T P \mathbf{d}(-\ln V)e + e^T \mathbf{d}(-\ln V)^T C^T L_{FT}^T \Xi(\lambda)^T Z \Xi(\lambda) L_{FT} C \mathbf{d}(-\ln V)e \right), \end{aligned} \quad (23)$$

where  $\Xi(\lambda) = \lambda^{\nu-\epsilon} \mathbf{d}(\ln \lambda) - I_n = \exp((\nu H + \nu I_n) \ln \lambda) - I_n$  and

$$\mathcal{Y} = \begin{pmatrix} P(A+L_0C+L_{FT}C)+(A+L_0C+L_{FT}C)^T P + \xi P & P \\ P & -Z \end{pmatrix}$$

is negative semi-definite according to (17a) with  $L_{FT} = P^{-1}Y$ .

Note that for  $\alpha$  defined in Theorem 3 we have

$$2\alpha P \geq PG_{\mathbf{d}} + G_{\mathbf{d}}^T P.$$

Then for  $(V, e) \in \Omega$  and using (17f), (17d), (17c) and (21) we can obtain

$$\begin{aligned} \partial Q^e &\leq V^\nu \left( -\xi + \lambda^{-2\epsilon} V^{-2\epsilon} \frac{1}{\theta} e^T C^T Y^T P^{-1} Y C e \right) \\ &\leq V^\nu \left( -\xi + \lambda^{-2\epsilon} V^{-2\epsilon} \tau e^T C^T X C e \right) \\ &= V^\nu \left( -\xi + \lambda^{-2\epsilon} V^{-2\epsilon} \tau \|\tilde{P}Ce\|^2 \right) \\ &= V^\nu (\tau - \xi) \\ &\leq \frac{\xi - \tau}{2\alpha} V^{1+\nu} \frac{\partial Q}{\partial V}. \end{aligned} \quad (24)$$

Finally, applying Theorem 1 we finish the proof.  $\square$

**Remark 1** According to Theorem 1 the following settling time estimate holds:

$$T(e_0) \leq \frac{2\alpha}{(\tau - \xi)\nu} V_0^{-\nu},$$

where  $V_0 \in \mathbb{R}_+ : Q(V_0, e_0) = 0$ .

**Remark 2** The presented observer is an extension of the results proposed in<sup>12</sup> and based on the use of weighted homogeneity (the special case of generalized homogeneity). Indeed, applying the presented approach for the system in the block form

$$\tilde{A} = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & A_{m-1\ m} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \tilde{C} = [I_k \ 0], \quad (25)$$

where  $m$  is an integer,  $A_{j-1\ j} \in \mathbb{R}^{n_{j-1} \times n_j}$ ,  $n_j = \text{rank}(A_{j-1\ j})$ ,  $j = 2, \dots, m$ ,  $n_1 = k$ ,  $\sum_{i=1}^m n_i = n$ , one obtains an observer with the same structure as in the paper<sup>12</sup>. Therefore the main advantage of this paper is not in the better transients or robustness properties, but in the fact that the presented approach does not require block decomposition. Note that block decomposition in some cases may be accompanied by errors associated with the calculation of right inverse matrices. It can be viewed also as a nonlinear generalization of Luenberger observer, having non-asymptotic convergence rates for  $\nu \neq 0$  and coinciding with it for  $\nu = 0$  and  $\epsilon = 1$ .

In order to apply Theorem 6 in practice we need to solve the nonlinear matrix inequality (17f) together with (16a)-(17e). Due to the smoothness of  $\lambda^\epsilon \Xi(\lambda)$  with respect to  $\lambda \in (0, 1]$ , this can be done on a proper grid constructed over this interval. The following proposition provides sufficient feasibility condition of the inequality (17f).

**Proposition 1.** The parametric inequality (17f) holds if there exists  $N > 0$  such that

$$\begin{pmatrix} 2(\nu + \epsilon)Z + \nu H^T Z + \nu Z H & \nu Z (I_n + H) \\ \nu (I_n + H^T) Z & M \end{pmatrix} \geq 0, \quad (26a)$$

$$P > 0, \quad M > 0, \quad Z > 0, \quad (26b)$$

$$q_i^{2\epsilon} \Xi^T(q_i) Z \Xi(q_i) + q_i^{2\epsilon-1} (q_i - q_{i-1}) M \leq \frac{1}{\theta} P, \quad i = 1, \dots, N, \quad (26c)$$

where  $0 = q_0 < q_1 < \dots < q_N = 1$ ,  $H, P, M, Z \in \mathbb{R}^{n \times n}$ .

*Proof.* Denote  $W(\lambda) = \lambda^{2\epsilon} z^T \Xi^T(\lambda) Z \Xi(\lambda) z$  for some  $z \in \mathbb{R}^n$ . Since  $(\lambda^\epsilon \Xi(\lambda))' = \lambda^{\epsilon-1} [(G_d + \nu I_n) \Xi(\lambda) + G_d + (\nu - \epsilon) I_n]$ , then

$$\begin{aligned} W'(\lambda) &= \lambda^\epsilon z^T ((\lambda^\epsilon \Xi(\lambda))')^T Z \Xi(\lambda) + \Xi(\lambda) Z (\lambda^\epsilon \Xi(\lambda))' z \\ &= \lambda^{2\epsilon-1} z^T \begin{pmatrix} \Xi(\lambda) \\ I_n \end{pmatrix}^T \mathcal{W} \begin{pmatrix} \Xi(\lambda) \\ I_n \end{pmatrix} z - \lambda^{2\epsilon-1} z^T M z, \end{aligned}$$

where

$$\mathcal{W} = \begin{pmatrix} 2\nu Z + G_d^T Z + Z G_d & Z(G_d + (\nu - \epsilon) I_n) \\ (G_d^T + (\nu - \epsilon) I_n) Z & M \end{pmatrix}.$$

The inequality  $\mathcal{W} \geq 0$  is equivalent to (26a). Then  $W'(\lambda) \geq -\lambda^{2\epsilon-1} z^T M z$  and

$$W(\lambda) \leq W(q_i) + \lambda^{2\epsilon-1} (q_i - \lambda) z^T M z$$

for any  $\lambda \in [q_{i-1}, q_i]$ ,  $i = 1, \dots, N$ . Hence, the set of inequalities (26a)-(26c) imply (17f).  $\square$

The result of Proposition 1 allows to solve the parametrized system of matrix equations and inequalities (16), (17) using the following algorithm with fixed  $\nu, \xi, \theta, \tau$ :

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#### Algorithm 1

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**Initialization:**  $N = 1$ ,  $q_0 = 0$ ,  $q_N = 1$ ,  $\Sigma = \{q_0, q_N\}$ .

**Loop:** While the system (16a)-(17e), (26a)-(26c) is not feasible, do  $\Sigma \leftarrow \Sigma \cup \left\{ \frac{q_{i-1} + q_i}{2} \right\}_{i=1}^N$  and  $N \leftarrow 2N$ .

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**Corollary 1.** The system of matrix equations and matrix inequalities (16), (17a)-(17d), (26a)-(26c) is feasible provided that  $|\nu|$  is sufficiently close to zero.

*Proof. I.* Let us use the known block decomposition procedure discussed in<sup>12</sup>. After transformation we obtain the system and output matrices in the form (25). Returning to (16a), (16b) with corresponding transformed matrices  $\tilde{A}$  and  $\tilde{C}$  it can be easily shown that

$$\tilde{H} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & I_{n_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & (m-1)I_{n_m} \end{pmatrix}$$

is a solution of (16) with  $\tilde{L}_0 = 0$ .

**II.** Observability of the pair  $(A, C)$  implies feasibility of the matrix inequality (17a) with some positive definite matrices  $P, Z \in \mathbb{R}^{n \times n}$  and  $Y \in \mathbb{R}^{k \times n}$ . The matrix inequalities (17c) and (17d) are also feasible for some positive definite  $X \in \mathbb{R}^{k \times k}$  and sufficiently large  $\theta$ .

**III.** Finally, we have that  $\|\Xi(\lambda)\| \rightarrow 0$  uniformly on  $\lambda \in [0, 1]$  as  $\nu \rightarrow 0$ . Then, since (17e), (26a) is feasible for sufficiently small  $|\nu|$ , the presented Algorithm 1 always finds the solution.  $\square$

**Remark 3** According to Corollary 1 the proposed algorithm allows to tune the parameters for sufficiently small  $|\nu|$ . Note that according to Remark 1 the settling time is inversely proportional to  $|\nu|$ . Thus, one can use Algorithm 1 multiple times with a gradual increase of  $|\nu|$  in order to tune the parameters that provide smaller settling time. The relaxation of the conditions to ensure feasibility for any  $\nu \in [-1, 0)$  is one of directions for future research.

## 4.2 | Fixed-time observer

Consider the observer

$$\frac{d}{dt} \hat{x}(t) = A\hat{x}(t) + Bu(t) - g_{Fx}(y(t) - C\hat{x}(t)), \quad (27)$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is as before and

$$g_{Fx}(y(t) - C\hat{x}(t)) = L_0 \varsigma(t) + \frac{1}{2} \left[ \|\tilde{P}_1 \varsigma(t)\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln \|\tilde{P}_1 \varsigma(t)\|^{1/\epsilon}) + \|\tilde{P}_2 \varsigma(t)\|^{(\epsilon-\nu)/\epsilon} \mathbf{d}(-\ln \|\tilde{P}_2 \varsigma(t)\|^{1/\epsilon}) \right] L_{Fx} \varsigma(t),$$

where  $\varsigma(t) = y(t) - C\hat{x}(t)$ ,  $\mathbf{d}$  is a dilation with the generator  $G_{\mathbf{d}}$ , and  $\tilde{P}_i \in \mathbb{R}^{k \times k}$ ,  $i = 1, 2$ ,  $L_{Fx} \in \mathbb{R}^{n \times k}$ ,  $L_0 \in \mathbb{R}^{n \times k}$ ,  $\nu \in [-1, 0)$ ,  $\epsilon \in \mathbb{R}_+$  are parameters of the observer, to be determined.

The error equation with  $d_x = 0$ ,  $d_y = 0$  has the form

$$\dot{e} = \left( A + L_0 C + \frac{1}{2} \|\tilde{P}_1 C e\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln \|\tilde{P}_1 C e\|^{1/\epsilon}) L_{Fx} C + \frac{1}{2} \|\tilde{P}_2 C e\|^{(\epsilon-\nu)/\epsilon} \mathbf{d}(-\ln \|\tilde{P}_2 C e\|^{1/\epsilon}) L_{Fx} C \right) e, \quad (28)$$

where  $e = x - \hat{x}$ .

Let us define

$$\Xi_i^\delta(\lambda) = \frac{1}{2} \left\{ (\delta_1 \lambda_1 \lambda_2^{1-i})^{\nu-\epsilon} \mathbf{d}(\ln(\delta_1 \lambda_1 \lambda_2^{1-i})) + (\delta_2 \lambda_1 \lambda_2^{2-i})^{\epsilon-\nu} \mathbf{d}(-\ln(\delta_2 \lambda_1 \lambda_2^{2-i})) - 2I_n \right\},$$

$\delta = (\delta_1, \delta_2)$  and  $\lambda = (\lambda_1, \lambda_2)$  for  $i = 1, 2$ .

**Theorem 7.** Let  $d_x = 0$ ,  $d_y = 0$  and

- for some  $\nu \in [-1, 0)$  the system of matrix equations

$$AH - HA - A - Y_0 C = 0, \quad (29a)$$

$$CH = 0, \quad (29b)$$

$$H + H^T + 2I_n > 0, \quad (29c)$$

$$2\epsilon I_n - \nu H - \nu H^T > 0 \quad (29d)$$

be feasible with  $H \in \mathbb{R}^{n \times n}$ ,  $\epsilon \in \mathbb{R}_+$ ;

- for  $L_0 = (H + I_n)^{-1}Y_0$  and some  $\xi_i > \tau > 0$ ,  $\delta = (\delta_1, \delta_2)$ ,  $\delta_i \in \mathbb{R}_+$ ,  $i = 1, 2$ ,  $\theta \in \mathbb{R}_+$  the system of matrix inequalities

$$\begin{pmatrix} \tau X & Y^T \\ Y & \theta P \end{pmatrix} \geq 0, \quad (30a)$$

$$P \geq C^T X C, \quad (30b)$$

$$\nu P H + \nu H^T P + 2\epsilon P > 0, \quad (30c)$$

$$\begin{pmatrix} PA + A^T P + PL_0 C + C^T L_0^T P + Y C + C^T Y^T + \xi_i P & P \\ P & -Z_i \end{pmatrix} \leq 0, \quad (30d)$$

$$P > 0, \quad Z_i > 0, \quad X > 0, \quad (30e)$$

$$\lambda_1^{2\epsilon} \Xi_i^{\delta^T}(\lambda) Z_i \Xi_i^\delta(\lambda) \leq \frac{1}{\theta} P, \quad \forall \lambda \in [0, 1] \times [0, 1] \quad (30f)$$

be feasible with  $P, Z_i \in \mathbb{R}^{n \times n}$ ,  $X \in \mathbb{R}^{k \times k}$ ,  $Y \in \mathbb{R}^{n \times k}$ .

Then the error equation (28) with  $L_{Fx} = P^{-1}Y$ ,  $\tilde{P}_i = \delta_i X^{1/2}$ ,  $G_d = \nu H + \epsilon I_n$  is globally fixed-time stable.

*Proof.* **I.** According to the proof of Theorem 6 the system  $\dot{e}(t) = Ae(t) + L_0 Ce(t)$  is  $\mathbf{d}$ -homogeneous of degree  $\nu$ . Note that it is  $\mathbf{d}_2$ -homogeneous with degree  $-\nu$  due to

$$(A + L_0 C)G_{\mathbf{d}_2} - G_{\mathbf{d}_2}(A + L_0 C) = G_{\mathbf{d}}(A + L_0 C) - (A + L_0 C)G_{\mathbf{d}},$$

where  $\mathbf{d}_2(s) = e^{G_{\mathbf{d}_2}s}$ ,  $G_{\mathbf{d}_2} = 2\epsilon I_n - G_{\mathbf{d}}$  and the inequality (29d) guarantees  $G_{\mathbf{d}_2}$  to be anti-Hurwitz matrix.

**II.** Let us define  $Q_1(V, e) = \|\mathbf{d}(-\ln V)e\|_P^2 - 1$  and  $Q_2(V, e) = V^{-4\epsilon} \|\mathbf{d}(\ln V)e\|_P^2 - 1$ . Note that for  $V \in \mathbb{R}_+$ :  $Q_i(V, e) = 0$  we have  $V = \|e\|_{\mathbf{d}_i}$ ,  $i = 1, 2$ , where  $\mathbf{d}_1(s) = \mathbf{d}(s)$  and  $\mathbf{d}_2(s) = \exp(2\epsilon s)\mathbf{d}_1(-s)$  (the inequality (29d) guarantees that  $\mathbf{d}_2$  is strictly monotone dilation).

The function  $Q_i$ ,  $i = 1, 2$  satisfies the conditions C1-C4 of Theorem 1 (see the proof of Theorem 6). Note that  $Q_1(1, z) = Q_2(1, z)$  for all  $z \in \mathbb{R}^n$ . In order to complete the proof we need to show that the conditions C7-C8 of Theorem 2 hold.

**III.** Let  $\tilde{P}_i = \delta_i^\epsilon \tilde{P}$ ,  $\delta_i \in \mathbb{R}_+$ ,  $i = 1, 2$  with  $\tilde{P} = X^{1/2}$ . Analogously to the proof of Theorem 6 we have  $0 \leq \lambda_1 = \frac{\|\tilde{P}Ce\|^{1/\epsilon}}{V} \leq 1$  with  $Q_1(V, e) = 0$  if (29b), (30b) hold. The same result remains true for  $Q_2(V, e) = 0$  since:

$$\|\tilde{P}Ce\| = \|\tilde{P}C\mathbf{d}(-\ln V)\mathbf{d}(\ln V)e\| = V^{-\epsilon} \|\tilde{P}C\mathbf{d}(\ln V)e\| \leq V^{-\epsilon} (e^T \mathbf{d}^T(\ln V) P \mathbf{d}(\ln V)e)^{1/2} = V^\epsilon.$$

**IV.** Let  $\partial Q_i^e$  be the derivative of  $Q_i$  along (28), then we have

$$\partial Q_i^e = V^{(-1)^{i+1}\nu} \begin{pmatrix} \mathbf{d}_i(-\ln V)e \\ \Xi_i^\delta(\lambda) L_{Fx} C \mathbf{d}_i(-\ln V)e \end{pmatrix}^T \mathcal{Y} \begin{pmatrix} \mathbf{d}_i(-\ln V)e \\ \Xi_i^\delta(\lambda) L_{Fx} C \mathbf{d}_i(-\ln V)e \end{pmatrix}$$

where

$$\mathcal{Y} = \begin{pmatrix} P(A+L_0C+L_{Fx}C) + (A+L_0C+L_{Fx}C)^T P & P \\ P & 0 \end{pmatrix},$$

$\lambda = (\lambda_1, \lambda_2)$ ,  $\lambda_2 = V^2 \in (0, 1]$  if  $i = 1$  and  $\lambda_2 = 1/V^2 \in (0, 1]$  if  $i = 2$ .

Repeating the considerations of the proof of Theorem 6 we derive that (29), (30) implies  $\partial Q_1^e \leq \frac{\xi_1 - \tau}{2\alpha_1} V^{1+\nu} \frac{\partial Q_1}{\partial V}$ , where  $\alpha_1$  corresponds to Theorem 3 for the generator  $G_{\mathbf{d}_1}$ . Due to  $\tau X \geq \frac{1}{\theta} Y^T P^{-1} Y$  according to (30a), then for  $Q_2^e$  we have

$$\begin{aligned} \partial Q_2^e &\leq V^{-\nu} \left( -\xi_2 + \lambda_1^{-2\epsilon} V^{-2\epsilon} \frac{1}{\theta} e^T C^T Y^T P^{-1} Y C e \right) \\ &\leq V^{-\nu} \left( -\xi_2 + \lambda_1^{-2\epsilon} V^{-2\epsilon} \tau e^T C^T X C e \right) \\ &= V^{-\nu} \left( -\xi_2 + \lambda_1^{-2\epsilon} V^{-2\epsilon} \tau \|\tilde{P}Ce\|^2 \right) \\ &= V^{-\nu} (\tau - \xi_2) \\ &\leq \frac{\xi_2 - \tau}{2\alpha_2} V^{1-\nu} \frac{\partial Q_2}{\partial V}, \end{aligned} \quad (31)$$

where  $\alpha_2$  corresponds to Theorem 3 for the generator  $G_{\mathbf{d}_2}$ . Thus, all conditions of Theorem 2 hold and the error equation (28) is fixed-time stable.  $\square$

**Remark 4** According to Theorem 2 for  $\mu_1 = \mu_2 = -\nu$ ,  $c_1 = \frac{\xi_1 - \tau}{2\alpha_1}$ ,  $c_2 = \frac{\xi_2 - \tau}{2\alpha_2}$  we have

$$T_{\max} \leq \frac{2\alpha_1}{(\tau - \xi_1)\nu} + \frac{2\alpha_2}{(\tau - \xi_2)\nu}.$$

**Proposition 2.** The parametric inequality (30f) holds if there exists  $N > 0$  and  $0 = q_0 < q_1 < \dots < q_N = 1$  such that

$$\begin{pmatrix} 2(\nu + \epsilon)Z_i + \nu H^T Z_i + \nu Z_i H & -R_i & R_i \\ -R_i^T & M_i & 0 \\ R_i^T & 0 & S_i \end{pmatrix} \geq 0, \quad (32)$$

$$-2\nu M_i - \nu H^T M_i - \nu M_i H > 0, \quad (33)$$

$$-2\nu Z_i - \nu H^T Z_i - \nu Z_i H \geq 0, \quad (34)$$

$$P > 0, \quad M_i > 0, \quad S_i > 0, \quad Z_i > 0, \quad (35)$$

$$2\varpi_i^T(q_j)Z_i\varpi_i(q_j) + 2\kappa_i^T(q_j)Z_i\kappa_i(q_j) + q_j^{2\epsilon-1}(q_j - q_{j-1})(\tilde{M}_i + S_i) \leq \frac{1}{\theta}P, \quad j = 1, \dots, N, \quad (36)$$

where  $H, P, M_i, Z_i, S_i \in \mathbb{R}^{n \times n}$ ,  $R_i = \nu Z_i(H + I_n)$ ,  $i = 1, 2$  and

$$\begin{aligned} \varpi_i(q_j) &= 0.5q_j^\epsilon(q_j\delta_{3-i})^{(-1)^i(\nu-\epsilon)}\mathbf{d}((-1)^i \ln(q_j\delta_{3-i})), \\ \kappa_i(q_j) &= 0.5q_j^\epsilon(q_j\delta_i)^{(-1)^i(\epsilon-\nu)}\mathbf{d}((-1)^{i+1} \ln(q_j\delta_i)), \\ \tilde{M}_i &= \delta_2^{2(\epsilon-\nu)}\mathbf{d}^T(-\ln \delta_2)M_i\mathbf{d}(-\ln \delta_2). \end{aligned}$$

*Proof.* **I.** Consider the function

$$W_i(\lambda) = \lambda_1^{2\epsilon} z^T \Xi_i^{\delta T}(\lambda) Z_i \Xi_i^{\delta}(\lambda) z,$$

where  $\lambda \in [0, 1]^2$  and  $z \in \mathbb{R}^n$  is an arbitrary non-trivial vector. Since

$$\frac{\partial \Xi_i^{\delta}}{\partial \lambda_1} = \lambda_1^{-1} ((\nu - \epsilon)I_n + G_{\mathbf{d}}) [\Xi_i^{\delta} - (\lambda_1 \delta_2 \lambda_2^{2-i})^{\epsilon-\nu} \mathbf{d}(-\ln(\lambda_1 \delta_2 \lambda_2^{2-i})) + I_n]$$

Then

$$\frac{\partial W_i}{\partial \lambda_1} = \lambda_1^{2\epsilon-1} z^T \omega^T \begin{pmatrix} 2\nu Z_i + G_{\mathbf{d}}^T Z_i + Z_i G_{\mathbf{d}} & -R & R \\ -R^T & 0 & 0 \\ R^T & 0 & 0 \end{pmatrix} \omega z,$$

$$\text{where } \omega = \begin{pmatrix} \Xi_i^{\delta}(\lambda) \\ (\lambda_1 \delta_2 \lambda_2^{2-i})^{\epsilon-\nu} \mathbf{d}(-\ln(\lambda_1 \delta_2 \lambda_2^{2-i})) \\ I_n \end{pmatrix}, \quad R = Z_i(G_{\mathbf{d}} + (\nu - \epsilon)I_n) = \nu Z_i(H + I_n).$$

Using the inequality (32) we derive

$$\frac{\partial W_i}{\partial \lambda_1} \geq -\lambda_1^{2\epsilon-1} \Psi_i(\lambda) - \lambda_1^{2\epsilon-1} z^T S_i z,$$

where

$$\Psi_i(\lambda) = (\lambda_1 \delta_2 \lambda_2^{2-i})^{2(\epsilon-\nu)} z^T \mathbf{d}^T(-\ln(\lambda_1 \delta_2 \lambda_2^{2-i})) M_i \mathbf{d}(-\ln(\lambda_1 \delta_2 \lambda_2^{2-i})).$$

On the other hand, since

$$\frac{\partial \Psi_i}{\partial \lambda_1} = \lambda_1^{-1} z^T \Psi_i^T (2(\epsilon - \nu)M_i - G_{\mathbf{d}}^T M_i - M_i G_{\mathbf{d}}) \Psi_i z,$$

where  $\Psi_i = (\lambda_1 \delta_2 \lambda_2^{2-i})^{\epsilon-\nu} \mathbf{d}(-\ln(\lambda_1 \delta_2 \lambda_2^{2-i}))$ , then the inequality (33) implies the estimates  $\frac{\partial \Psi_i(\lambda)}{\partial \lambda_1} > 0$ ,  $\frac{\partial \Psi_i(\lambda)}{\partial \lambda_2} \geq 0$ . Hence we conclude

$$\frac{\partial W_i}{\partial \lambda_1} \geq -\lambda_1^{2\epsilon-1} z^T \delta_2^{2(\epsilon-\nu)} \mathbf{d}^T(-\ln \delta_2) M_i \mathbf{d}(-\ln \delta_2) z - \lambda_1^{2\epsilon-1} z^T S_i z$$

for  $\lambda \in [0, 1] \times [0, 1]$  and

$$W_i(\lambda) \leq W_i(q_j, \lambda_2) + \lambda_1^{2\epsilon-1} (q_j - \lambda_1) z^T \left[ \delta_2^{2(\epsilon-\nu)} \mathbf{d}^T(-\ln \delta_2) M_i \mathbf{d}(-\ln \delta_2) + S_i \right] z \quad (37)$$

for all  $\lambda_1 \in [q_{j-1}, q_j]$ .

**II.** Rewrite  $W_i(\lambda)$  in the form

$$W_i(\lambda) = z^T B_i(\lambda_1)^T F_i(\lambda_2) B_i(\lambda_1) z,$$

where

$$F_i(\lambda_2) = \begin{pmatrix} \lambda_2^{\epsilon-\nu} \mathbf{d}^T(-\ln \lambda_2) \\ I_n \end{pmatrix} Z_i \begin{pmatrix} \lambda_2^{\epsilon-\nu} \mathbf{d}(-\ln \lambda_2) \\ I_n \end{pmatrix},$$

$$B_i(\lambda_1) = \begin{pmatrix} 0.5 \lambda_1^\epsilon (\lambda_1 \delta_{3-i})^{(-1)^i(\nu-\epsilon)} \mathbf{d}((-1)^i \ln(\lambda_1 \delta_{3-i})) \\ 0.5 \lambda_1^\epsilon (\lambda_1 \delta_i)^{(-1)^i(\epsilon-\nu)} \mathbf{d}((-1)^{i+1} \ln(\lambda_1 \delta_i)) \end{pmatrix}.$$

It is easy to check that

$$W_i(\lambda) \leq \tilde{W}_i(\lambda) := z^T B_i(\lambda_1)^T \begin{pmatrix} 2\lambda_2^{2(\epsilon-\nu)} \mathbf{d}^T(-\ln \lambda_2) Z_i \mathbf{d}(-\ln \lambda_2) & 0 \\ 0 & 2Z_i \end{pmatrix} B_i(\lambda_1) z.$$

Due to (34) we have  $\frac{\partial \tilde{W}_i}{\partial \lambda_2} \geq 0$  and  $W_i(\lambda) \leq \tilde{W}_i(\lambda_1, 1)$  for all  $\lambda_2 \in [0, 1]$ . Then we can rewrite (37) in the form

$$W_i(\lambda) \leq \tilde{W}_i(q_j, 1) + \lambda_1^{2\epsilon-1} (q_j - \lambda_1) z^T \left[ \delta_2^{2(\epsilon-\nu)} \mathbf{d}^T(-\ln \delta_2) M_i \mathbf{d}(-\ln \delta_2) + S_i \right] z.$$

Therefore, the inequality (36) implies  $W(\lambda) \leq \theta^{-1} z^T P z$  for all  $\lambda \in [0, 1] \times [0, 1]$ .  $\square$

Based on this proposition, an algorithm for solving the parametrized system (29), (30) can be presented, analogously with the finite-time case.

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### Algorithm 2

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**Initialization:**  $N = 1, q_0 = 0, q_N = 1, \Sigma = \{q_0, q_N\}$ .

**Loop:** While the system (29a)-(30e), (32)-(36) is not feasible, do  $\Sigma \leftarrow \Sigma \cup \left\{ \frac{q_{i-1} + q_i}{2} \right\}_{i=1}^N$  and  $N \leftarrow 2N$ .

---

**Corollary 2.** The system of matrix equations and matrix inequalities (29a)-(30e), (32)-(36) is feasible provided that  $|\nu|$  is sufficiently close to zero.

*Proof.* The proof repeats the statements of the proof of Corollary 1. Indeed, according to the proof of Corollary 1 (29a)-(29c) and (30a)-(30e) are feasible, and  $\|\Xi_i^\delta(\lambda)\| \rightarrow 0$  on  $\lambda \in [0, 1]^2$  as  $\nu \rightarrow 0$ . Then, since (29d), (32)-(34) are feasible for sufficiently small  $|\nu|$ , Algorithm 2 always finds the solution.  $\square$

Note that Remark 3 is also valid for Algorithm 2.

## 5 | ROBUSTNESS ANALYSIS

Note, that the system (15) is  $\mathbf{d}$ -homogeneous of degree  $\nu$ . Indeed, due to (16b) we have:

$$\begin{aligned} (A + L_0 C + \|\tilde{P} C \mathbf{d}(s) e\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln \|\tilde{P} C \mathbf{d}(s) e\|^{1/\epsilon}) L_{FT} C) \mathbf{d}(s) e = \\ (A + L_0 C) \mathbf{d}(s) e + (\exp(\nu s - \epsilon s) \|\tilde{P} C e\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln(\exp(s) \|\tilde{P} C e\|^{1/\epsilon})) L_{FT} C) \mathbf{d}(s) e = \\ (A + L_0 C) \mathbf{d}(s) e + (\exp(\nu s - \epsilon s) \|\tilde{P} C e\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(s) \mathbf{d}(\ln \|\tilde{P} C e\|^{1/\epsilon}) L_{FT} C) \mathbf{d}(s) e = \\ \exp(\nu s) \mathbf{d}(s) (A + L_0 C + \|\tilde{P} C e\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln \|\tilde{P} C e\|^{1/\epsilon}) L_{FT} C) e. \end{aligned}$$

In the similar way it is easy to show that (28) is locally homogeneous at  $-\infty$  and at  $+\infty$ , namely  $(\mathbf{d}_1, -\infty, f_{-\infty})$ -homogeneous with negative degree  $\nu$ , and  $(\mathbf{d}_2, +\infty, f_{+\infty})$ -homogeneous with positive degree  $-\nu$ ,

$$\begin{aligned} f_{-\infty} &= \left( A + L_0 C + \frac{1}{2} \|\tilde{P}_1 C e\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln \|\tilde{P}_1 C e\|^{1/\epsilon}) L_{Fx} C \right) e, \\ f_{+\infty} &= \left( A + L_0 C + \frac{1}{2} \|\tilde{P}_2 C e\|^{(\epsilon-\nu)/\epsilon} \mathbf{d}(-\ln \|\tilde{P}_2 C e\|^{1/\epsilon}) L_{Fx} C \right) e. \end{aligned}$$

Using homogeneity property a qualitative assessment of robustness can be presented for the observers (14) and (27). Consider the system (1) with nonzero  $d_x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  and  $d_y : \mathbb{R}_+ \rightarrow \mathbb{R}^k$ . In this case the perturbed error equations take the form

$$\dot{e} = (A + L_0 C) e + L_0 d_y + d_x + \|\tilde{P}(C e + d_y)\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln \|\tilde{P}(C e + d_y)\|^{1/\epsilon}) L_{FT}(C e + d_y) \quad (38)$$

and

$$\begin{aligned} \dot{e} &= (A + L_0 C) e + L_0 d_y + d_x \\ &+ \frac{1}{2} (\|\tilde{P}_1 C e + d_y\|^{(\nu-\epsilon)/\epsilon} \mathbf{d}(\ln \|\tilde{P}_1 C e + d_y\|^{1/\epsilon}) + \|\tilde{P}_2 C e + d_y\|^{(\epsilon-\nu)/\epsilon} \mathbf{d}(-\ln \|\tilde{P}_2 C e + d_y\|^{1/\epsilon})) L_{Fx}(C e + d_y). \end{aligned} \quad (39)$$

Then, the following result may be obtained.

**Corollary 3.** Consider the perturbed error equation (38) and assume that all conditions of Theorem 6 are satisfied. Then the system (38) is ISS if  $-0.5\frac{1}{\nu}\lambda_{\min}(\nu H + \nu H^T + 2\epsilon I_n) < 1$  and iISS if  $-0.5\frac{1}{\nu}\lambda_{\min}(\nu H + \nu H^T + 2\epsilon I_n) = 1$ .

*Proof.* The proof of Corollary 3 is a direct consequence of Theorem 4.  $\square$

A similar result can be provided for the fixed-time observer:

**Corollary 4.** Consider the perturbed error equation (39) and assume that all conditions of Theorem 7 are satisfied. Then the system (39) is ISS with respect to additive disturbances  $d_x$  and measurement noises  $d_y$ .

*Proof.* The proof of Corollary 4 is a direct consequence of Theorem 5.  $\square$

The quantitative analysis is out the scope of this paper and needs further research developments (for example, with the use of results<sup>36,37,38</sup>).

## 6 | EXAMPLE

In the following simulations we use the same linear model as in<sup>12</sup>, where finite- and fixed- time observers were designed by preliminary applying the block decomposition. Consider the system (1), where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1818 & 2.672 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4545 & -31.181 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1.818 \\ 0 \\ -4.545 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

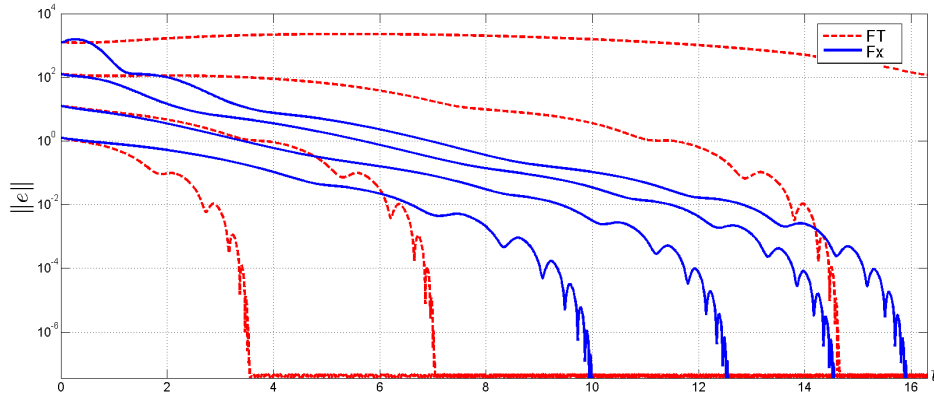
the pair  $(A, C)$  is observable.

Define the observers in the form (14), (27) with the parameter  $\nu = -0.25$ . The parameters of the finite-time observer are obtained applying Algorithm 1, where  $L_0 = \begin{pmatrix} 0.1818 & -0.0331 & -0.4545 & 0.0826 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T$ ,  $L_{FT} = \begin{pmatrix} -1.219 & -0.5925 & 0.0012 & -0.5583 \\ 0.001 & -0.0086 & -1.2183 & -0.8237 \end{pmatrix}^T$ ,  $\tilde{P} = \begin{pmatrix} 0.1608 & -0.0079 \\ -0.0079 & 0.1467 \end{pmatrix}$  are the gain matrices,  $G_d = \begin{pmatrix} -0.0455 & 0.75 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.1136 & 0 & 0 & 0.75 \end{pmatrix}$  is the generator, the dilation  $\mathbf{d}(s) = \exp(G_d s)$  for  $s \in \mathbb{R}$ , and the parameters  $(\epsilon, \tau, \theta, \xi) = (1, 1, 1.5, 1.01)$  are used. The parameters of the fixed-time observer are obtained applying Algorithm 2, where  $L_0 = \begin{pmatrix} 0.1818 & -0.0331 & -0.4545 & 0.0826 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^T$ ,  $L_{Fx} = \begin{pmatrix} -2.4381 & -2.0756 & 0.0015 & -1.1281 \\ 0.0017 & -0.0159 & -2.4359 & -2.5546 \end{pmatrix}^T$ ,  $\tilde{P} = \begin{pmatrix} 2.5730 & -0.2022 \\ -0.2022 & 2.4168 \end{pmatrix}$  are the gain matrices,  $G_d = \begin{pmatrix} -0.0455 & 0.75 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.1136 & 0 & 0 & 0.75 \end{pmatrix}$  is the generator, and the parameters  $(\epsilon, \tau, \theta, \xi_1, \xi_2, \delta_1, \delta_2) = (1, 1, 6, 1.01, 1.01, 9, 0.1)$  are used.

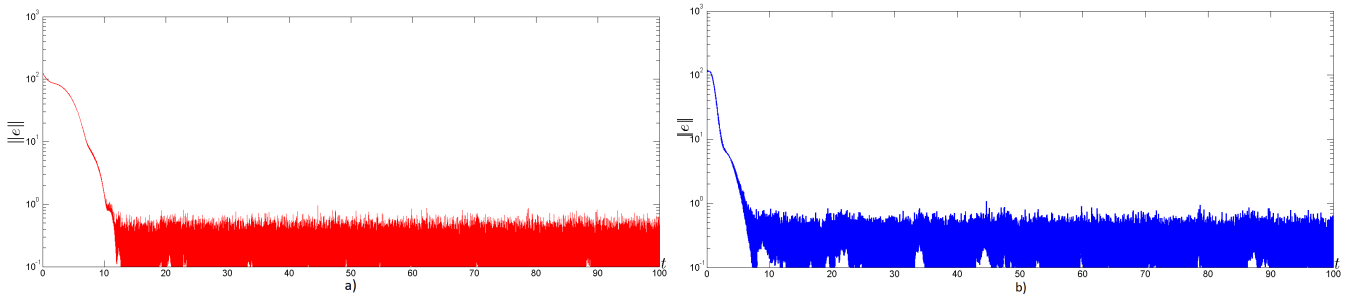
Numerical simulations (Fig. 1, 2, 3) have been done using explicit Euler method with the step size  $h = 0.001$ . Since any generalized homogeneous system is diffeomorphic to a standard (weighted) homogeneous one, the performance of the presented observers is expected to be comparable to the work of<sup>12</sup>. Indeed, according to the comparison results between the finite- and fixed-time observers one can draw conclusions similar to<sup>12</sup>: for the fixed-time algorithm the convergence time sensitivity is low with respect to initial conditions; for the finite-time observer it is possible to see that the convergence time increases drastically as the norm of the initial conditions increases; since locally (close to the origin of the error system) the fixed-time algorithm almost coincides with the finite-time one, it has almost the same sensitivity with respect to measurement noises. Note that in a particular realization, the sensitivity with respect to (bias) disturbances may vary significantly for the finite-time and fixed-time algorithms (see Fig. 3). This confirms that we can obtain comparable transients and robustness properties as in<sup>12</sup> without the use of block decomposition that is the main advantage of the proposed results.

The simulation results are shown with using the logarithmic scale in order to demonstrate fast convergence rate of the observers.

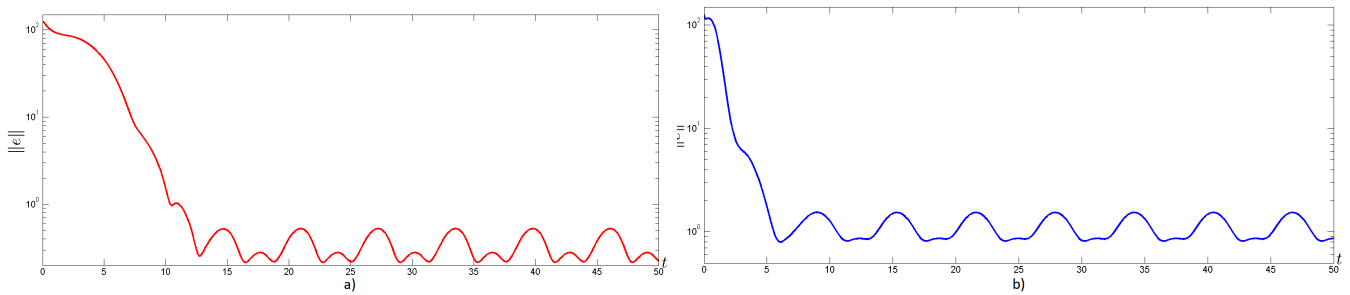
Note that settling time estimates (according to Remark 1 and Remark 4) may be quite conservative. For example, for  $x(0) = 10^2(0, 1, \pi/4, 0)$  the obtained estimates are  $T(e_0) \leq 246.3$  (finite-time estimation) and  $T_{\max} \leq 1841.9$  (fixed-time estimation) while according to simulations we have  $T(e_0) \approx 14.6$  for both cases. Relaxation of LMIs and less conservative settling time estimates are possible directions for further research.



**FIGURE 1** Simulation plot of  $\|e\|$  for four different initial conditions  $x(0) = (0, 1, \pi/4, 0)$ ,  $x(0) = 10(0, 1, \pi/4, 0)$ ,  $x(0) = 10^2(0, 1, \pi/4, 0)$ ,  $x(0) = 10^3(0, 1, \pi/4, 0)$ : FT – finite-time observer, Fx – fixed-time observer



**FIGURE 2** Simulation plot of  $\|e\|$  for the finite-time a) and fixed-time b) observers with measurement band limited noise of power  $10^{-5}$  and initial conditions  $x(0) = 10^2(0, 1, \pi/4, 0)^T$



**FIGURE 3** Simulation plot of  $\|e\|$  for the finite-time a) and fixed-time b) observers with  $d_x(t) = (0, 0.5 + 0.2 \sin t, 0, 0.5 \sin t)^T$  and initial conditions  $x(0) = 10^2(0, 1, \pi/4, 0)^T$

## 7 | CONCLUSIONS

The paper is devoted to finite/fixed-time observer design for linear MIMO systems. The key feature of the presented method is that there is no requirements on transformation of the system to a canonical form. The parameters tuning is based on solution of linear matrix equations and inequalities. The qualitative analysis of observers' robustness against bounded measurement noises and disturbances has been studied, namely, it is shown that the observers are iISS/ISS stable. Numerical examples demonstrate effectiveness of the proposed observers.



Note that the proposed observer provides the finite-time convergence of the error variables without using delayed or discontinuous terms. In addition, as it is shown in<sup>12</sup>, homogeneity-based observers have a better precision in the presence of measurement noise comparing with high gain observers as in<sup>39</sup>. These advantages and simple tuning in the LMI form make it promising to use the proposed results in practice, e.g., in real-time observer-based control systems.

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## CONFLICT OF INTEREST

The authors declared that they have no conflicts of interest to this work.

## DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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